A Practical Algorithm for Designing H∞ Control Laws

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A Practical Algorithm for Designing Nonlinear $\mathcal{H}_\infty$ Control Laws

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Abstract

In this paper we describe a novel approximation method for the Hamilton-Jacobi-Isaacs (HJI) equation that results in feedback control. The approximation is accomplished via a two-step successive Galerkin approximation scheme. An application of the technique to the control of the forward motion of an underwater vehicle is described.

1 Introduction

While linear $\mathcal{H}_\infty$ control theory has been successfully applied to numerous applications, nonlinear $\mathcal{H}_\infty$ theory has not. The reason is that there do not exist efficient methods for solving, or even approximating, the Hamilton-Jacobi-Isaacs equation. The objective of this paper is to describe preliminary results on using a successive Galerkin approximation [1, 2] to approximate the Hamilton-Jacobi-Isaacs equation. The approximation algorithm results in control laws that approximate the $\mathcal{H}_\infty$ control law on a subset Ω of the stability region of an initial stabilizing control.

2 The Main Result

Consider the nonlinear system given by the following equations:

$$\dot{x} = f(x) + g_1(x)u + g_2(x)w, \qquad f(0) = 0 \tag{1}$$

$$y = h(x), \qquad h(0) = 0, \tag{2}$$

where $x \in I\!R^n$ is the state of the system, $u \in I\!R^m$ is the control variable, $w \in I\!R^q$ is the disturbance signal and $y \in I\!R^p$ is the system output. Given $\gamma > 0$, the system is said to have L_2 gain from w to $(y^T, u^T)^T$ less than γ if for all $T > 0$

$$\int_0^T (|y|^2 + |u|^2)dt \leq \gamma^2 \int_0^T |w|^2 \; dt \tag{3}$$

for all $w \in L_2(0,T)$ and $x(0) = 0$, where $|\cdot|$ is the Euclidean norm. The nonlinear $\mathcal{H}_\infty$ optimal control problem is to find the smallest $\gamma^* > 0$ and an associated control u^* such that the L_2 gain of system (1) from w to $(y^T, u^{*T})^T$is equal to γ^*. It is shown in [3], that the control $u_\gamma = -g_1^T(x)\frac{\partial V_\gamma}{\partial x}(x)$, where V_γ satisfies

$$\frac{\partial V_\gamma^T}{\partial x} f + \frac{1}{2\gamma^2}\frac{\partial V_\gamma^T}{\partial x} kk^T \frac{\partial V_\gamma}{\partial x} - \frac{1}{2}\frac{\partial V_\gamma^T}{\partial x} gg^T \frac{\partial V_\gamma}{\partial x} + \frac{1}{2}h^T h = 0, V(0) = 0, \tag{4}$$

renders the L_2 gain from w to (y^T, u_γ^T) less than or equal to γ. Assume that γ is such that the solution of equation (4) exists. Now consider the following algorithm.

Initialization: Let $u^{(0)}(x)$ be a state-feedback control that stabilizes the system $\dot{x} = f + g_1 u$ on Ω.

For $i = 0 : \infty$, **Set** $w_\gamma^{(i,0)} = 0$

For $j = 0 : \infty$, **Solve**

$$\frac{\partial V_\gamma^{(i,j)T}}{\partial x}(f + g_1 u_\gamma^{(i)} + g_2 w_\gamma^{(i,j)}) + |h|^2 + \left|u_\gamma^{(i)}\right|^2 - \gamma^2 \left|w_\gamma^{(i,j)}\right|^2 = 0 \tag{5}$$

Set $w_\gamma^{(i,j+1)} = \frac{1}{2\gamma^2} g_2^T \frac{\partial V_\gamma^{(i,j)}}{\partial x}$,

Set $u_\gamma^{(i+1)} = -\frac{1}{2} g_1^T \frac{\partial V_\gamma^{(i,\infty)}}{\partial x}$

This algorithm constitutes two nested iterations in policy space corresponding to the $\min - \max$ problem associated with the nonlinear $\mathcal{H}_\infty$ control paradigm [4]. The inner loop updates the disturbance variable for a given control law, so that $w_\gamma^{(i,\infty)}$ is the worst possible disturbance for the control $u_\gamma^{(i)}$. The outer loop updates the control law to improve the performance for a given worst case disturbance. The key is that the nonlinear HJI equation has been reduced to a sequence of linear partial differential equations (5) which we call Generalized-Hamilton-Jacobi-Isaacs (GHJI) equations.

In order to obtain an implementable algorithm we must approximate the solution of the GHJI equations such that the control $u_\gamma^{(i)}$ can be practically implemented in feedback form. The GHJI equation can be approximated via a global Galerkin approximation scheme. Assuming that $V_\gamma^{(i,j)}(x) = \sum_{l=1}^{\infty} c_l^{(i,j)}\phi_l(x)$, where $\{\phi_j\}_{j=1}^{\infty}$ is a complete set of basis functions on $I\!R^n$, let $V_{\gamma,N}^{(i,j)}(x) = \sum_{l=1}^{N} c_l^{(i,j)}\phi_l(x)$. The Galerkin approximation to equation (5) is given by $V_{\gamma,N}^{(i,j)}$ where the coefficients $c_l^{(i,j)}$ are the solution to the linear algebraic equations

$$\int_\Omega \left(\frac{\partial V_{\gamma,N}^{(i,j)T}}{\partial x}(f + g_1 u_\gamma^{(i)} + g_2 w_\gamma^{(i,j)}) + |h|^2 + \left|u_\gamma^{(i)}\right|^2 - \gamma^2 \left|w_\gamma^{(i,j)}\right|^2 \right) \phi_l \, dx = 0, \quad (6)$$

where $l = 1, \ldots, N$ and Ω is a compact subset of the stability region of the initial stabilizing control $u^{(0)}$. The resulting feedback control can be written as $u_{\gamma,N}^{(i,j)} = -\frac{1}{2}\sum_{\ell=1}^{N} c_\ell^{(i,j)} g_1^T \frac{\partial \phi_\ell}{\partial x}$.

3 Illustrative Example

In this section the algorithm is used to synthesize a robust controller to control the forward motion of a underwater vehicle in the presence of significant disturbances and model uncertainty. The equations of motion which describe the forward motion of the vehicle are given by:

$$(m + m_A)\ddot{x} + b\dot{x}\,|\dot{x}| = F + d.$$

where x is the position of the vehicle, m is the mass (in air) of the vehicle, m_A is added mass of the water surrounding the vehicle, b is the square-law drag coefficient, F is the applied thrust, and d is an external disturbance due to currents or forces from an attached tether. Typically, values for m_A and b are not known accurately due to the difficulty in measuring them and due to the fact that they are dependent on the operating conditions of the vehicle. In this study, the values for m_A is varied up to $\pm 100\%$ of its nominal value, while b is varied up to $\pm 50\%$. Letting $\hat{m}_A$ and $\hat{b}$ represent the nominal values of m_A and b and letting $\delta m_A = \hat{m}_A - m_A$ and $\delta b = \hat{b} - b$, the equations of motion can be written

$$(m + \hat{m}_A)\ddot{x} + \hat{b}\dot{x}\,|\dot{x}| = F + d + \delta b\dot{x}\,|\dot{x}|\,.$$

Putting the system into the standard form of equation (1) we get

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ -\frac{b}{m+\hat{m}_A} v\,|v| \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m+\hat{m}_A} \end{pmatrix} F + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left(\frac{\delta b v\,|v|}{m + \hat{m}_A} + d \right),$$

$$y = x.$$

Values for the model parameters used to design the control and simulate the performance of the closed-loop system are based on those of the NEROV vehicle [5].

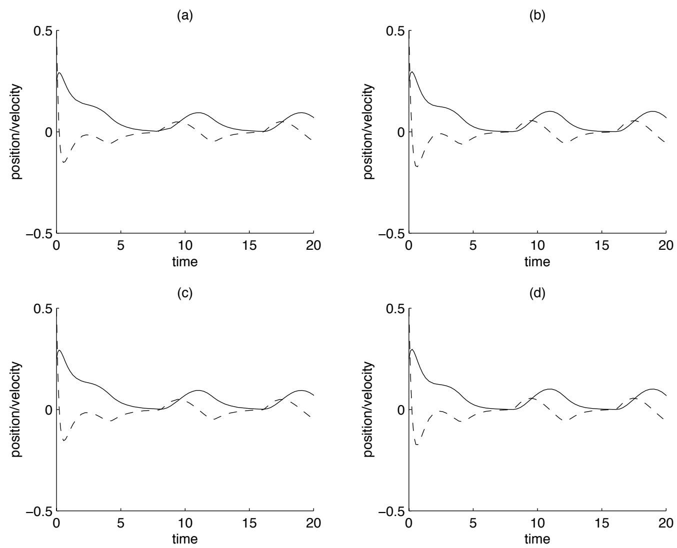

Figure 1: State trajectories at the limits of parameter uncertainty with the disturbance input.

In the ocean environment, disturbances are commonplace. The disturbance that a tether might apply under actual operating conditions is $d(t) = \text{clip}(100\sin(2\pi t))$ where $\text{clip}(x(t))$ clips $x(t)$ at zero. Figure 1 shows the response of the underwater vehicle system to disturbances (with initial condition 0.5 m/sec on the velocity state and 0.25 m on the position state) for four different parameter perturbations: (a) $[-\delta m_{Amax}, -\delta b_{max}]$, (b) $[+\delta m_{Amax}, -\delta b_{max}]$, (c) $[+\delta m_{Amax}, +\delta b_{max}]$, and (d) $[-\delta m_{Amax}, +\delta b_{max}]$. Though the parameter values are varied significantly, the similarity of results demonstrate the robustness and the ability of the controller to reject significant disturbances.

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