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Successive Galerkin Approximations to the Nonlinear Optimal Control of an Underwater Robotic Vehicle

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Nonlinear Optimal Control of an Underwater Robotic Vehicle

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Abstract

The application of a new nonlinear optimal control strategy to the station-keeping control of an underwater robotic vehicle is considered. The control approach described and demonstrated here is based upon the numerical approximation of solutions to the Hamilton-Jacobi-Bellman equation. These solutions are computed by an iterative application of a Galerkin-type method. Preliminary simulation results demonstrating the application of this approach to the control of an underwater vehicle in the horizontal plane are presented.

1 Introduction

The dynamics and control of underwater robotic vehicles poses significant challenges to the control designer due primarily to the nonlinearity and uncertainty associated with their dynamics. Since the mid-1980's, the most widely researched approach for controlling underwater vehicles has been sliding-mode control. This method was first implemented on underwater vehicles by Yoerger and Slotine [1] and later by others (e.g., see [2, 3]). The most attractive feature

of the sliding method pertaining to underwater vehicles is its inherent robustness to parametric uncertainty. As with other robust methods, the robustness of sliding-mode control often comes at the expense of performance.

More recently, adaptive extensions to the sliding-mode technique have been implemented [4, 5, 6]. These adaptive sliding-mode controllers effectively reduce model uncertainty and allow robustness to be achieved with less performance degradation.

In addition to the sliding-mode techniques, several other approaches to underwater vehicle control have been studied including neural networks [7, 8], and adaptive control [9, 10].

In this paper, a novel approach for synthesizing optimal controllers for nonlinear systems is presented and applied to the control of an underwater vehicle in the horizontal plane. This approach is based on the formulation of nonlinear optimal control problem which leads to the Hamilton-Jacobi-Bellman (HJB) equation and its approximate numerical solution by iterative application of a Galerkin-type method. This approach, which was originated by Beard [11], has been applied to a variety of nonlinear systems (e.g., see [12, 13]).

2 Control Approach

For a system modeled by the nonlinear state equations

$$\dot{x} = f(x) + g(x)u(x) \quad (1)$$

“optimal” performance can be obtained by determining the control $u(x)$ such that a predetermined performance index of the form

$$V(x) = \int_t^\infty (l(x) + \|u(x)\|_R^2) dt \quad (2)$$

is minimized, where $l(x)$ is a positive definite, zero-state observable function and $R = R^T > 0$. The optimum value of the performance index $V^*(x)$ can be found by solving the Hamilton-Jacobi-Bellman (HJB) equation:

$$\frac{\partial V^{*T}}{\partial x} f(x) + l(x) - \frac{1}{4} \frac{\partial V^{*T}}{\partial x} g(x) R^{-1} g^T(x) \frac{\partial V^*}{\partial x} = 0. \quad (3)$$

With $V^*(x)$ determined, the corresponding optimal control is given by

$$u^*(x) = -\frac{1}{2} R^{-1} g^T(x) \frac{\partial V^*}{\partial x}. \quad (4)$$

Obtaining a closed-form solution of Equation 3 for systems of the complexity of the underwater vehicle considered here is impossible. The approach for solving the HJB

equation presented here is to begin with a known control, $u^{(0)}$ (such as PD control), that is stabilizing over a bounded domain of the state space Ω and to numerically compute successive approximations to the optimal control given by Equation 4.

With the system operating under the initial feedback control law $u^{(0)}(x)$, the performance index of Equation 2 becomes

$$V^{(0)}(x) = \int_t^\infty \left(l(x) + \|u^{(0)}(x)\|_R^2 \right) dt. \quad (5)$$

Taking the derivative of both sides of Equation 5 with respect to time and invoking the chain rule results in the partial differential equation

$$\frac{\partial V^{(0)T}}{\partial x} [f + gu^{(0)}] + l + \|u^{(0)}\|_R^2 = 0. \quad (6)$$

Equations 5 and 6 show that $V^{(0)}$ is a Lyapunov function for the control $u^{(0)}$. Given $V^{(0)}$, the stability and robustness of $u^{(0)}$ can be improved by choosing a new control $u^{(1)}$ that makes

$$\dot{V}^{(0)} = \frac{\partial V^{(0)}}{\partial x} (f + gu^{(0)}) \quad (7)$$

more negative. This is done by carrying out the following optimization:

$$\begin{aligned} u^{(1)} &= \arg \min_u \left[\frac{\partial V^{(0)T}}{\partial x} (f + gu) + l + \|u\|_R^2 \right] \\ &= -\frac{1}{2} R^{-1} g^T \frac{\partial V^{(0)}}{\partial x}. \end{aligned} \quad (8)$$

Equations 6 and 8 can be generalized to form the basis for an iterative solution process where

$$\frac{\partial V^{(i)T}}{\partial x} [f + gu^{(i)}] + l + \|u^{(i)}\|_R^2 = 0 \quad (9)$$

$$u^{(i+1)} = -\frac{1}{2} R^{-1} g^T \frac{\partial V^{(i)}}{\partial x}. \quad (10)$$

The problem of determining the optimal control is simplified to solving Equation 9, which is known as the Generalized Hamilton-Jacobi-Bellman (GHJB) equation, in conjunction with the optimal control law of Equation 10. The GHJB equation differs from the HJB equation in that it is linear and therefore more easily solved numerically than the HJB equation.

The solution of Equations 9 and 10 is carried out by first starting with a known stabilizing control, $u^{(0)}$, and then iterating to find successive values for $V^{(i)}$ and $u^{(i)}$, which for sufficiently large i , have been shown to converge monotonically to the optimal values V^* and u^* [14].

The solution of the GHJB equation is carried out using a computational Galerkin method [15]. Using this approach, it is first assumed that the cost function $V^{(i)}$ can

be approximated using an infinite series of known basis functions $\phi_j(x)$ that are continuous and defined everywhere on the domain of the state space of interest (Ω):

$$V^{(i)} = \sum_{j=1}^{\infty} c_j^{(i)} \phi_j^{(i)}(x).$$

From an implementation standpoint, using an infinite number of terms in the approximation is not practical. Therefore, the approximation for $V^{(i)}$ is carried one step further by considering only the first N terms of the infinite series:

$$V_N^{(i)} = \sum_{j=1}^N c_j^{(i)} \phi_j^{(i)}(x). \quad (11)$$

The accuracy of this approximation to the assumed solution $V^{(i)}$ depends both on the number and characteristics of the basis functions chosen to form the approximation.

Examining Equation 9, it can be seen that for $V^{(i)}$ satisfying the GHJB equation, the left-hand side evaluates to zero. Similarly, when the approximation to $V_N^{(i)}$ given by Equation 11 is substituted into Equation 9, a non-zero left-hand side is obtained. The error $e^{(i)}$ resulting from this approximation can be expressed as

$$e^{(i)}(x) = \left(\sum_{j=1}^N c_j^{(i)} \frac{\partial \phi_j^T}{\partial x} \right) [f + gu^{(i)}] + l + \|u^{(i)}\|_R^2. \quad (12)$$

Using the Galerkin approach, values for the basis function coefficients $c_j^{(i)}$ are determined such that the error $e^{(i)}$ is minimized. This is done by projecting the error onto the N basis functions $\phi_1, \phi_2, \dots, \phi_N$. The error is projected onto the basis functions by taking the inner product of the two on Ω and setting that to zero, where the inner product of two functions is defined as

$$\langle f(x), g(x) \rangle_\Omega \triangleq \int_\Omega f(x) g(x) dx.$$

The result is N linear equations in the N unknown coefficients $c_1^{(i)}, c_2^{(i)}, \dots, c_N^{(i)}$ which are expressed compactly as

$$\langle e^{(i)}(x), \phi_j(x) \rangle_\Omega = 0, \quad j = 1 \dots N. \quad (13)$$

Equation 13 can be expressed in more detail in matrix form as

$$c^{(i)} = \left(A_1 + A_2^{(i)} \right)^{-1} \left(b_1 + b_2^{(i)} \right) \quad (14)$$

where

$$A_1 = \begin{bmatrix} \left\langle \frac{\partial \phi_1^T}{\partial x} f, \phi_1 \right\rangle_\Omega & \cdots & \left\langle \frac{\partial \phi_N^T}{\partial x} f, \phi_1 \right\rangle_\Omega \\ \vdots & \ddots & \vdots \\ \left\langle \frac{\partial \phi_1^T}{\partial x} f, \phi_N \right\rangle_\Omega & \cdots & \left\langle \frac{\partial \phi_N^T}{\partial x} f, \phi_N \right\rangle_\Omega \end{bmatrix}$$

$$A_2^{(i)} = \begin{bmatrix} \left\langle \frac{\partial \phi_1^T}{\partial x} g u^{(i)}, \phi_1 \right\rangle_\Omega & \dots & \left\langle \frac{\partial \phi_N^T}{\partial x} g u^{(i)}, \phi_1 \right\rangle_\Omega \\ \vdots & \ddots & \vdots \\ \left\langle \frac{\partial \phi_1^T}{\partial x} g u^{(i)}, \phi_N \right\rangle_\Omega & \dots & \left\langle \frac{\partial \phi_N^T}{\partial x} g u^{(i)}, \phi_N \right\rangle_\Omega \end{bmatrix}$$

$$b_1 = - \begin{bmatrix} \langle l, \phi_1 \rangle_\Omega \\ \vdots \\ \langle l, \phi_N \rangle_\Omega \end{bmatrix}$$

$$b_2^{(i)} = - \begin{bmatrix} \left\langle \|u^{(i)}\|_R^2, \phi_1 \right\rangle_\Omega \\ \vdots \\ \left\langle \|u^{(i)}\|_R^2, \phi_N \right\rangle_\Omega \end{bmatrix}.$$

With these coefficients $c^{(i)} = [c_1^{(i)} \ c_2^{(i)} \ \dots \ c_N^{(i)}]^T$ identified, the control associated with the $(i+1)^{th}$ iteration can be calculated:

$$u_N^{(i+1)} = -\frac{1}{2} R^{-1} g^T \left(\sum_{j=1}^N c_j^{(i)} \frac{\partial \phi_j}{\partial x} \right). \quad (15)$$

Expressions for $A_2^{(i)}$ and $b_2^{(i)}$ above indicate that their integrals require recalculation at each iteration step due to their dependence on $u^{(i)}$. This is not necessarily the case. By taking advantage of Equation 15, $A_2^{(i)}$ and $b_2^{(i)}$ can be evaluated in an iterative fashion as well. $A_2^{(0)}$ and $b_2^{(0)}$ are calculated using the expressions above, while values for subsequent iterations can be calculated as follows:

$$A_2^{(i)} = -\frac{1}{2} \sum_{j=1}^N c_j^{(i-1)} M_j$$

$$b_2^{(i)} = -\frac{1}{4} \begin{bmatrix} c^{(i-1)T} M_1 c^{(i-1)} \\ c^{(i-1)T} M_2 c^{(i-1)} \\ \vdots \\ c^{(i-1)T} M_N c^{(i-1)} \end{bmatrix}$$

where

$$\{M_i\}_1^N = \begin{bmatrix} m_i(1,1) & \dots & m_i(1,N) \\ \vdots & \ddots & \vdots \\ m_i(N,1) & \dots & m_i(N,N) \end{bmatrix}$$

$$m_i(j,k) = \left\langle \frac{\partial \phi_k^T}{\partial x} g R^{-1} g^T \frac{\partial \phi_i}{\partial x}, \phi_j \right\rangle_\Omega.$$

Calculation of the basis function coefficients as outlined above can be quite intense from a computational standpoint. It should be noted that these computations are carried out prior to implementation. The calculations carried out during operation are straightforward and easily done in real time.

3 Underwater Vehicle System Model

The vehicle model chosen to demonstrate the control technique presented here is that of the NEROV vehicle [10, 16]. Figure 1 shows a schematic representation of the vehicle and the inertial and body-fixed reference frames used to represent its motion.

Figure 1: Underwater Vehicle Reference Frames

The equations of motion which describe the dynamics of the NEROV vehicle are given by

$$\dot{u} = \frac{1}{186} (-119u|u| + 268vr + T_1 + T_2) \quad (16)$$

$$\dot{v} = \frac{1}{268} (-208v|v| - 213ur + T_3 - T_4) \quad (17)$$

$$\dot{r} = \frac{1}{29} (-15r|r| + 0.4T_1 - 0.4T_2) \quad (18)$$

$$\dot{x} = u \cos(\psi) - v \sin(\psi) \quad (19)$$

$$\dot{y} = u \sin(\psi) + v \cos(\psi) \quad (20)$$

$$\dot{\psi} = r \quad (21)$$

where u , v , and r are the surge velocity, the sway velocity, and the yaw rate of the vehicle. Inertial positions and heading are represented by x , y , and ψ . T_1 , T_2 , T_3 , and T_4 are the forces applied by the thrusters whose configuration is shown in Figure 1.

As implemented here, the control law calculates the control forces and torque in the inertial frame: $u = [F_x \ F_y \ T_z]^T$. To calculate the thruster forces, two mappings must be performed: (1) a mapping of inertial control to body-fixed control, and (2) a mapping of body-fixed

control to thruster forces. The first mapping is a simple rotation transformation:

$$u_b = Cu$$

where $u_b = [F_{xb} \ F_{yb} \ T_{zb}]^T$ and

$$C = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The basis for the second mapping is the physical configuration of the thrusters on the vehicle. Body-fixed control effort related to the thruster forces by

$$u_b = RT$$

where $T = [T_1 \ T_2 \ T_3 \ T_4]^T$ and

$$R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0.4 & -0.4 & 0 & 0 \end{bmatrix}.$$

For a given body-frame control u_b , the thruster forces are calculated using

$$T = R^\dagger u_b$$

where

$$R^\dagger = R^T(RR^T)^{-1}.$$

Considering both transformations together,

$$T = R^\dagger Cu. \quad (22)$$

Equations 16 through 22 are of the general form of Equation 1, and from them the functions $f(x)$ and $g(x)$ required for the synthesis of the control can be determined.

4 Control Synthesis and Implementation

With the model described above, the optimal feedback control can be synthesized using the control synthesis algorithm. With $f(x)$ and $g(x)$ coming directly from the equations of motion and $l(x)$ and R being determined by the control designer, the domain of the states Ω , the basis functions $\{\phi_j(x)\}_1^N$, and the initial control law $u^{(0)}$ remain to be determined.

The domain of possible values for the states is determined both by the physical capabilities of the system and the likely deviation of the states from their nominal value of zero. For the results presented here, Ω was defined to be

$$-2 \text{ m/sec} \leq u \leq 3 \text{ m/sec}$$

$$-2 \text{ m/sec} \leq v \leq 2 \text{ m/sec}$$

$$-4 \text{ rad/sec} \leq r \leq 4 \text{ rad/sec}$$

$$-5 \text{ m} \leq x \leq 5 \text{ m}$$

$$-5 \text{ m} \leq y \leq 5 \text{ m}$$

$$-\pi \text{ rad} \leq \psi \leq \pi \text{ rad}.$$

Proper selection of the basis functions $\{\phi_j(x)\}_1^N$ is critical to the design of the optimal controller. The basis functions must be able to capture the nonlinear dynamics behavior of the system in order to approximate the cost function $V^{(i)}$ with sufficient accuracy. By doing so, a control law can be synthesized that will compensate for the nonlinear dynamics of the system. For this problem, the following basis functions were used:

$$\phi_1 = u^2$$

$$\phi_2 = v^2$$

$$\phi_3 = r^2$$

$$\phi_4 = (u \cos \psi - v \sin \psi + x)^2$$

$$\phi_5 = (u \sin \psi + v \cos \psi + y)^2$$

$$\phi_6 = (r + \psi)^2$$

$$\phi_7 = uvr.$$

The initial stabilizing control was chosen to be PD control on each degree of freedom with respect to the inertial reference frame:

$$u^{(0)} = \begin{bmatrix} -K_{px}x - K_{dx}(u \cos \psi - v \sin \psi) \\ -K_{py}y - K_{dy}(u \sin \psi + v \cos \psi) \\ -K_{p\psi}\psi - K_{d\psi}r \end{bmatrix}$$

5 Results

Using the synthesis algorithm outlined above with the inputs as described, three different optimal controllers were designed and implemented. Table 1 shows the weighting functions for each of the three controllers. The response of each controller to initial conditions of $x(0) = -3 \text{ m}$, $y(0) = 1 \text{ m}$, $\psi(0) = 2 \text{ rad}$ are presented below.

Figure 2 shows the response of the vehicle system using the initial choice of weighting functions. The heading response of the vehicle is reasonably quick and well-damped, while the x and y position response is somewhat slower and slightly underdamped. As shown in Figure 3 the control effort F_x is the largest of the three inputs to the system. This is expected because of the larger initial error in x . Each of the control efforts is relatively small compared to the vehicle's thrust capability.

The response of the system is tuned by adjusting the weighting function $l(x)$ and the weighting matrix R . To improve the speed of response and reduce the settling time for the x and y positions and the heading angle, the cost associated with errors in those states is increased significantly in the second choice of weighting function $l(x)$.

Table 1: Weighting Functions

Initial Design
$l = 5000u^2 + 5000v^2 + 5000x^2 + 5000y^2 + 5000\psi^2$
$R = F_x^2 + F_y^2 + T_z^2$
High-Performance Design
$l = 10^3u^2 + 10^3v^2 + 10^5x^2 + 10^5y^2 + 10^6\psi^2$
$R = F_x^2 + F_y^2 + T_z^2$
Moderate-Control Design
$l = 10^3u^2 + 10^3v^2 + 10^5x^2 + 10^5y^2 + 10^6\psi^2$
$R = 20F_x^2 + 20F_y^2 + 200T_z^2$

Figure 2: Transient Response — Initial

Figure 3: Control Effort — Initial

Figure 4 shows the response of the states when the second choice of $l(x)$ and R are used to formulate the control. As expected, the settling time for each of the states is reduced significantly. To bring about this improved performance, more control effort is required as Figure 5 shows. Most notable are the increases in F_y and T_z .

Figure 4: Transient Response — High Performance

Figure 5: Control Effort — High Performance

Control effort can be easily moderated by increasing the cost of the control. For the final set of results, the weighting function $l(x)$ was left unchanged from the second case, while the weighting on the control R was increased. Figure 6 shows the response of the system with the final set of weights, while Figure 7 shows the control effort used. With the increased weight on the control usage, F_y and T_z drop significantly. Corresponding to this moderation of control usage, the regulation response of the system is more sluggish.

The results demonstrate the ability to develop and tune

optimal controllers for multi-input multi-output nonlinear systems. Through the proper selection of basis functions and appropriate tuning of $l(x)$ and R , the behavior of the controlled system can be altered to meet the demands of a specific application. This can be done by considering the full nonlinear dynamics of the system without the need for linearization or decoupling of the equations of motion.

Figure 6: Transient Response — Moderate Control

Figure 7: Control Effort — Moderate Control

6 Conclusions

A nonlinear optimal control approach based on the numerical solution of the Hamilton-Jacobi-Bellman equation has been applied to the station-keeping control of an underwater robotic vehicle. Using this approach, nonlinear controllers were designed taking into account the full

nonlinear dynamics of the system. Preliminary simulation results demonstrate the potential of the approach for application to a real underwater vehicle system.

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