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Four-body Problem with Collision Singularity

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FOUR-BODY PROBLEM WITH COLLISION SINGULARITY

by

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ABSTRACT

Four-body Problem with Collision Singularity

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In this dissertation, regularization of simultaneous binary collision, existence of a Schubart-like periodic orbit, existence of a planar symmetric periodic orbit with multiple simultaneous binary collisions, and their linear stabilities are studied. The detailed background of those problems is introduced in chapter 1.

The singularities of simultaneous binary collision in the collinear four-body problem is regularized in chapter 2. We use canonical transformations to collectively analytically continue the singularities of the simultaneous binary collision solutions in both the decoupled case and the coupled case. All the solutions are found and more importantly, we find a crucial first integral which describes the relationship between the decoupled solutions and the coupled solutions.

In chapter 3, we show the existence of a Schubart-like orbit, a periodic orbit with singularities in the symmetric collinear four-body problem. In each period of the orbit, there is a binary collision (BC) between the inner two bodies and a simultaneous binary collision (SBC) of the two clusters on both sides of the origin. The system is regularized and the existence is proven by using a "turning point" technique and a continuity argument on differential equations of the regularized Hamiltonian.

Analytical methods are used in chapter 4 to prove the existence of a periodic, symmetric solution with singularities in the planar 4-body problem. A numerical calculation and simulation are used to generate the orbit. The analytical method can be extended to any even number of bodies. Multiple simultaneous binary collisions are a key feature of the orbits generated.

In chapter 5, we apply the analytic-numerical method of Roberts to determine the linear stability of time-reversible periodic simultaneous binary collision orbits in the symmetric collinear four body problem with masses 1, m, m , 1, and also in a symmetric planar four-body problem with equal masses. For the collinear problem, this verifies the earlier numerical results of Sweatman for linear stability.

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Chapter 1. Introduction

The N-body problem of celestial mechanics considers the motion of a system of N points with masses m_1, m_2, ..., m_N governed by a Newtonian gravitational force. The familiar differential equation

$$m_i\ddot{\rho}_i = \sum_{j\neq i} -\frac{m_i m_j(\rho_i - \rho_j)}{|\rho_i - \rho_j|^3} \tag{1.1}$$

gives a mathematical description of the problem, where $\rho_i \in \mathbb{R}^3$ denotes the position of the ith body with mass m_i. All derivatives are taken with respect to time. In particular, the potential energy of the system is given by

$$U = \sum_{1\leq i<j\leq n} \frac{m_i m_j}{|\rho_i - \rho_j|} \tag{1.2}$$

and the kinetic energy is

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i|\dot{\rho}_i|^2 \tag{1.3}$$

"The N-body problem consists of describing the complete behavior of all solutions to these equations of motion for arbitrary preassigned initial conditions. Despite efforts by outstanding mathematicians for over 200 years, the problem for $N > 2$ remains unsolved to this day." [25]

To understand the dynamics of the N-body problem, singularity is one of the most important topics to investigate. From the equations of motions for the N-body problem, it is clear that if some mutual distance $\rho_i - \rho_j$ tends to zero, the differential equation 1.1 describing the dynamics of the system becomes singular. This type of singularity is called collision singularity. Collision singularity contains binary collisions, simultaneous binary collisions, total collapse, etc. A binary collision occurs when only two bodies collide. A simultaneous binary collision (SBC) occurs when two or more pairs of binary collisions happen at the same time. A total collapse occurs when three or more bodies collide. However, collisions are not the only source of singularities. A noncollision singularity (or pseudocollision) happens at time $t = t_c$ if the position vector ρ_i is unbounded as t tends to t_c. Actually, a well-know example in Xia's dissertation [31] tells us that noncollision singularity exists for $N \geq 5$.

In this dissertation, we mainly focus on the collinear and planar four-body problem with collision singularity. The occurrence of a collision is a very difficult subject to handle, since the equations of motion cease to be valid at the singularity. As a consequence of the conservation of the energy, since the potential function is infinite at collision, the velocity becomes itself infinite. The description of the motion fails at the singularity, but what is even worse, it is rather difficult to investigate the dynamics in a neighborhood of the singularity.

A way to overcome these difficulties has been explored by several mathematicians at the end of the 19th century and at the beginning of the 20th century. Among others, T. Levi-Civita, G.D. Birkhoff, P. Kustaanheimo, E.L. Stiefel, K.F. Sundman, C.L. Siegel, J.K. Moser, and J. Waldvogel contributed to develop a theory of regularization for the study of the motion at a collision. Binary collisions and total collapse of planar and spatial three-body problems have been investigated.

From the theory of regularization, one knows that the solution of a binary collision can be written as a convergent power series in terms of $(t-t_c)^{1/3}$. It is impossible to holomorphically extend the solution up to t_c, but Sundman found a real analytic continuation for $t > t_c$, using complex analytic continuation around t_c. A general definition of regularization was given by R. Easton [6], who developed the so-called block regularization in order to investigate whether nearby orbits provide an extension for an orbit ending into a collision. This procedure of pasting orbits is denoted as Easton′s method.

Another important regularization we have to mention is the LeviCivita regularization, which is based upon three main steps:

(i) A suitable change of coordinates, usually called the Levi-Civita transformation;

(ii) A new time scale, to remove the singularity, namely the introduction of a so-called fictitious time;

(iii) the conservation of the energy, to transform the singular differential equations into regular ones, i.e. the study of the Hamiltonian system in the extended phase space.

Comparing with binary collisions, SBC is much harder to analyze. The differential equations 1.1 have at least two zero denominators which gives us a big challenge when we try to apply the Levi-Civita regularization. Although many results ([22],[27], [7], [8], [12]) have been obtained, the study of the simultaneous binary collision is far from complete.

The degree of difficulty in dealing with more complicated models increases immediately as soon as we are concerned with triple collisions. Indeed, one finds an extremely chaotic behavior, such that a small variation of the initial conditions leads to large effects on the successive dynamics. While regularization always works for binary collisions, triple collisions cannot be regularized, except for a very small set of masses [25]. In 1974, McGehee [13] showed that there is no regularization for the triple collision in the collinear three-body problem. Instead, he devised a system of coordinates and a change of time scale under which motions that previously ended in triple collision at some finite time, now approach an equilibrium point as the rescaled time tends to infinity.

1.1 REGULARIZATION OF SBC

A binary collision occurs when two bodies collide. Simultaneous binary collision means two or more pairs of bodies collide at the same time. In 1984, D.Saari [22] showed that a solution of the N-body problem which ends in several simultaneous binary collisions is branch regularizable with time $s = t^{1/3}$.

Simó and Lacomba [27] have proved that SBC for the N-body problem and in any dimension are C^0-block regularizable in the sense of Eastons C^k-block regularization: near a SBC orbit there exists a C^k diffeomorphism connecting collision and near collision orbits with ejection and near ejection orbits and the motion can be continued beyond the SBC maintaining continuity with respect to initial conditions.

El Bialy [7], [8] has shown that SBC in one dimension are C^1-block regularizable and that the series expansion of the SBC singularity has coefficients which depends analytically on SBC initial conditions.

Martínez and Simó [12] have used a geometric approach to get more insight into the problem and numerical evidence that the degree of differentiability, in the planar four-body SBC problem, of the block regularization is exactly 8/3.

In chapter 2, we introduce a Levi-Civita type canonical transformation near SBC singularity and study the asymptotic behavior of SBC orbits. The regularization starts from the simplified, decoupled case, which assumes that the distance of the two collision pairs is infinity and the masses are all equal to 1. An important first integral C is discovered. It helps us prove that SBC is a regular singular point of the differential equations by introducing an extended implicit function theorem. Also, all the solutions of the decoupled case are found which is a one-parameter class, where the first integral C is the parameter. The nearby motion of SBC on the same energy surface is only C^0 by introducing C into the equations for the decoupled case.

The approach used in the decoupled case can also be applied to the coupled case, which is exactly the

SBC collinear four-body problem. The regularization in the coupled case is given by introducing a different singular transformation and a majorant method argument. For any initial condition leading to SBC, there is only one way to extend the solution analytically to the ejection solution. On the other hand, there exist other extensions which are $C^{1/3}$. Two constants are discovered in the solutions for the collinear four-body SBC problem: one is C, the first integral found in the decoupled case; the other is related to the distance and total momentum of the two colliding clusters.

1.2 Existence of a Schurbart-like Periodic Orbit

In chapter 3, we study a special symmetric periodic orbit with masses 1, m, m, 1, which is called Schubart-like orbit. In each period of this Schubart-like orbit, there is a binary collision (or BC for short) between the inner two bodies and then a simultaneous binary collision (or SBC for short) of the two clusters on both sides of the origin. This research is motivated by some important work on a remarkable periodic orbit in the collinear three-body problem, which is named as Schubart orbit.

Numerically in 1956, Schubart [24] found this orbit with singularities in the equal mass collinear three-body problem. The Schubart orbit has during each period two binary collisions where the inner body alternatively collides with the two outer bodies. In 1977, Hénon [10] extended the Schubart orbit to the case of unequal masses. Later, Mikkola and Hietarinta [14] and [11] numerically investigated the linear stability of the Schubart orbit when the masses are unequal. Only recently in 2008, did Moeckel [15] and Venturelli [30] give two different proofs of the existence of a Schubart orbit when the outer masses are the same. Moeckel's proof is topological and uses an idea developed by Conley [5] for the restricted three-body problem. Venturelli's proof is variational wherein he minimizes the Lagrangian action over a well chosen class of paths.

Numerically in 2001, Sweatman [28] found that the Schubart-like orbit exists in the symmetric collinear four-body problem. In this chapter, we give a theoretical proof of existence of this Schubart-like orbit. The author is not aware of any previously published existence proof by this time. The regularized Hamiltonian is analyzed where we use a continuity argument to prove the existence of a periodic orbit for any m. In order to show that the periodic orbit is exactly the Schubart-like orbit, we give an important estimate of the maximum distance of the outer bodies which guarantees that there is no extra collisions between each BC and SBC in the periodic orbit. This estimate corresponds to what we call a "turning point" for the inner bodies. More importantly, this "turning point" technique can help us prove the existence of periodic solutions other than the Schubart orbit in the collinear three-body problem, such as the ones mentioned by Saito and Tanikawa [23].

1.3 Existence of 2D Periodic Solutions with Singularities

Searching for periodic, linearly stable solutions is significant in the study of N-body problem. The most noteworthy result is that of Moore [16], who numerically developed the figure-eight choreography for three equal masses. His work was continued by Chenciner and Montgomery [4] who proved the existence of Moore's orbit. This work was further extended by Roberts [21], who developed an innovative method for studying linear stability of orbits, and used it to prove the linear stability of the figure-eight orbit. Each of these make use of the symmetries present in the orbit.

Schubart [24] was the first to combine these two concepts (singularities and periodic, stable solutions.) He produced a periodic orbit in the three-body collinear problem with binary collision singularities in which the center body regularly alternates between binary collisions with each of the outer two masses. His work was subsequently extended to the unequal mass case by both Hénon [9] and Hietarinta and Mikkola [11]. Sweatman [28] later extended this work to a four-body periodic solution in one dimension, with bodies alternating between SBC of the outer mass pairs and binary collision of the inner two masses.

The orbit that is presented in chapter 4 is another combination of these ideas in two dimensions. We present a family of configurations that are symmetric in both initial positions and velocities. These initial conditions will lead to arbitrarily many simultaneous binary collisions, with each body alternating between collisions with its two nearest neighbors. Due to the abundance of symmetries present in the configurations, we can reduce the number of variables that need to be studied to four–two representing position and two representing momentum. In contrast to its one-dimensional counterparts, the proof for existence of this orbit is surprisingly simple.

After precisely defining the symmetries that are present in the regularized coordinates, it is shown that the group of symmetries in the orbit is isomorphic to the dihedral group D_4. Further, as a consequence of Robert's technique, we [2] show that the four-body orbit presented in this work is linearly stable.

In this chapter, we first present a technique for generating a periodic orbit in the two-dimensional four-body problem with singularities. We begin in section **4.1.1** by giving a description of the proposed orbit and prove its existence. Section **4.1.2** will present the numerical methods used to produce the initial conditions that will lead to this orbit. Following this, in section **4.2**, we consider variants on the orbit we generate, giving a family of orbits with singularities with an even number of masses.

Further work has also been done on orbits in this family with alternating un-equal masses. Rather than a single mass parameter, the bodies have masses m_1, m_2, m_1, m_2 as numbered moving counterclockwise through the plane. Since some symmetry has been lost by this change in masses, it is necessary to choose two initial condition parameters as well as two initial velocities. Although numerically this is not a difficult problem, an analytical technique will require much more work.

1.4 LINEAR STABILITY ANALYSIS

Recently, Roberts [21] described an analytic-numerical method for determining the linear stability of a symmetric periodic orbit of a Hamiltonian system. He applied this method to the time-reversible collision-free figure-eight orbit in the equal mass three body problem numerically discovered by Moore [16] and whose existence was proven by Chenciner and Montgomery [4]. (Other such choreographic solutions were found numerically by Simó [26]). Roberts' method shows that the figure eight orbit is linearly stable. The method uses the symmetries to factor a matrix similar to the monodromy matrix for the periodic orbit into an integer power of the product of two involutions. One of the two involutions depends on the linearized dynamics along only a part of the periodic orbit. For the figure eight this part is one-twelfth of the full orbit since it has a symmetry group isomorphic to the group $D_3 \times \mathbb{Z}_2$ of order 12. (Here the dihedral group D_k is the group of symmetries of the regular k-gon.) The eigenvalues of the product of the two involutions are then reduced to the numerical computation of a few real numbers.

Schubart [24] numerically discovered a singular periodic orbit in the collinear equal mass three-body problem. The orbit alternates between binary collisions. Hénon [9] extended Schubart's numerical investigations to the case of unequal masses. Only recently did Venturelli [30] and Moeckel [15] prove the existence of the Schubart orbit when the outer masses are equal and the inner mass is arbitrary. The linear stability of the Schubart orbit was determined numerically by Hietarinta and Mikkola [11] revealing that linear stability occurs for some but not all of the choices of the three masses. Sweatman ([28] and [29]) numerically found and determined the linear stability of a Schubart-like orbit in the symmetric collinear four body problem with masses 1, m, m, and 1. This Schubart-like periodic orbit alternates between simultaneous binary collisions (SBC) and inner binary collisions. Ouyang and Yan [18] proved the existence of this orbit. In the regularized setting, this periodic orbit has a symmetry group isomorphic to D_2, of which both of the generators are time-reversing symmetries. Ouyang, Yan, and Simmons [17] numerically found and then proved the existence of a singular periodic orbit in a symmetric planar four-body problem with equal masses in which the four bodies alternate between different simultaneous binary collisions. In the regularized setting, this periodic orbit has a symmetry group isomorphic to D_4, of which one of the generators is a time-reversing symmetry. The regularization of these singular periodic orbits is achieved by a generalized Levi-Civita type transformation and an appropriate scaling of time, as adapted from Aarseth and Zare [1].

In chapter 5, we apply the method of Roberts to prove the linear stability of the Schubart-like orbit in the symmetric collinear four body 1, m, m, 1 problem for certain values of m, and of the singular periodic orbit in the symmetric planar equal mass problem. In both settings, the linear stability is determined for the regularized equations only and is reduced to the rigorous numerical computation of a single real number. Our linear stability analysis determines values of m in the interval $[0, 50]$ in the collinear problem for which the singular periodic orbit is linear stable, and also shows that the $2D$ singular periodic orbit is linear stable. These examples support and extend the conjecture made by Roberts [21] that the only linearly stable periodic orbits in the equal mass n-body problem are those that exhibit a time-reversing symmetry.

Our linear stability analysis confirms Sweatman's linear stability analysis [29] for the singular periodic orbit in the collinear four-body problem. Sweatman used a numerical perturbation technique to assess the stability of the singular periodic orbit when the masses are arranged from left to right as m_1, m_2, m_2, and m_1 with the condition that $m_1 + m_2 = 2$. Our mass parameter m is related to his mass parameter m_1 by $m = (2 - m_1)/m_1$. In terms of our mass parameter m, Sweatman's numerical results indicate that linear stability occurs when the value of m is smaller than approximately 2.83 and when it is larger than approximately 35.4, and does not occur otherwise.

Chapter 2. Simultaneous Binary Collisions for the Collinear Four-Body Problem

The question of the regularization for a simultaneous binary collision (SBC) solution is not completely understood although many results about it have been obtained. In 1984, Saari [22] showed that a SBC solution can be analytically continued by rescaling the time $s = t^{\frac{1}{3}}$. In 1992, Lacomba and Simo [27] gave a different approach and also they showed that "simultaneous binary collisions in the classical n-body problem are C^0 block-regularizable". In 1996, Elbialy [8] proved that "collision and ejection orbits can be collectively analytically continued, i.e. each collision-ejection orbit can be written as a convergent power series in $t^{\frac{1}{3}}$, with coefficients that depend real analytically on the initial conditions". In 2000, Martínez and Simó [12] also discussed the block regularization and the result is "this regularization is differentiable but the map passing from initial to final conditions is exactly $C^{8/3}$". In 2005, Punosevac and Wang [20] constructed coordinate transforms that removed the singularities of simultaneous binary collisions in a pair of decoupled Kepler problems and in a restricted collinear four-body problem. In 2009, Ouyang and Xie [19] introduced a new transformation which regularizes the SBC solution up to C^2.

2.1 Preliminaries

In Chapter 2, we study the collinear four-body problem. Let the four bodies lie on a line and the positions of body 1 to body 4 are q_1, q_2, q_3, q_4 respectively. Assume that $q_1 < q_2 < q_3 < q_4$ at time $t = \tau$. we assume that the two pairs: body 1 and 2, body 3 and 4 will have a collision at the same time $t = t_1$, which is called Simultaneous Binary Collision (SBC for short). For $t \in (\tau, t_1)$, no collision happens.

2.1.1 Simplified Hamiltonian form. We will use the center of mass and total momentum first integrals to eliminate one pair of variables p_k, q_k from these 8 differential equations, and we will achieve this by taking all the p_k, q_k into new variables x_k, y_k via a suitable canonical transformation. We set

$$p_k = W_{q_k}, \quad x_k = W_{y_k} \quad (k = 1, 2, 3, 4) \tag{2.1}$$

where $W(q,y)$ is a generating function whose Jacobian determinant $| W_{y_k q_l} |$ is not 0. We wish to set up the canonical transformation so that x_1 becomes the distance between P_1 and P_2, x_2 becomes the distance between P_3 and P_4 and x_3 becomes the distance between P_2 and P_3, while x_4 remains to be the coordinate of P_4, i.e.

$$x_1 = q_2 - q_1, \;\; x_2 = q_4 - q_3, \;\; x_3 = q_3 - q_2, \;\; x_4 = q_4 \tag{2.2}$$

and set the generating function as

$$W(q,y) = (q_2 - q_1)y_1 + (q_4 - q_3)y_2 + (q_3 - q_2)y_3 + q_4 y_4.$$

Then $| W_{y_k q_l} | = 1$, and it defines a canonical transformation. From the generating function, we have

$$p_1 = -y_1, \;\; p_2 = y_1 - y_3, \;\; p_3 = y_3 - y_2, \;\; p_4 = y_2 + y_4.$$

Therefore,

$$y_1 = -p_1, \;\; y_2 = -p_1 - p_2 - p_3, \;\; y_3 = -p_1 - p_2, \;\; y_4 = p_1 + p_2 + p_3 + p_4 \tag{2.3}$$

Note that if $p_1 + p_2 + p_3 + p_4 = 0$, then $y_4 = 0$, $y_2 = p_4$. For $t \in (0,t_1)$, x_1, x_2, x_3, x_4 are nonnegative but y_1 and y_2 are negative.

In (2.2), (2.3) we have the desired transformation, which we see is linear. Since, in addition, it does not depend on t, the new Hamiltonian system is

$$\dot{x_k} = E_{y_k}, \qquad \dot{y_k} = -E_{x_k} \qquad (k = 1,2,3,4) \tag{2.4}$$

where $E = T - U$ is regarded as a function of the x_k and y_k. Then

$$T = \frac{1}{2}\sum_{k=1}^{4}\frac{p_k^2}{m_k} = \frac{1}{2}[\frac{y_1^2}{m_1} + \frac{(y_1 - y_3)^2}{m_2} + \frac{(y_3 - y_2)^2}{m_3} + \frac{(y_2 + y_4)^2}{m_4}],$$

$$\begin{aligned} U &= \frac{m_1 m_2}{q_2 - q_1} + \frac{m_1 m_3}{q_3 - q_1} + \frac{m_1 m_4}{q_4 - q_1} + \frac{m_2 m_3}{q_3 - q_2} + \frac{m_2 m_4}{q_4 - q_2} + \frac{m_3 m_4}{q_4 - q_3} \\ &= \frac{m_1 m_2}{x_1} + \frac{m_1 m_3}{x_1 + x_3} + \frac{m_1 m_4}{x_1 + x_2 + x_3} + \frac{m_2 m_3}{x_3} + \frac{m_2 m_4}{x_2 + x_3} + \frac{m_3 m_4}{x_2}. \end{aligned}$$

We know $y_4 = 0$ and

$$0 = m_1 q_1 + m_2 q_2 + m_3 q_3 + m_4 q_4$$

$$= m_1(x_4 - x_3 - x_2 - x_1) + m_2(x_4 - x_3 - x_2) + m_3(x_4 - x_2) + m_4 x_4,$$

so that

$$x_4 = \frac{x_1 m_1 + x_3(m_1 + m_2) + x_2(m_1 + m_2 + m_3)}{m_1 + m_2 + m_3 + m_4}.$$

Therefore, we only have to consider the system

$$\dot{x_k} = E_{y_k}, \qquad \dot{y_k} = -E_{x_k} \qquad (k = 1, 2, 3) \tag{2.5}$$

with

$$E = T - U,$$

$$T = \frac{1}{2}[\frac{y_1^2}{m_1} + \frac{(y_1 - y_3)^2}{m_2} + \frac{(y_3 - y_2)^2}{m_3} + \frac{y_2^2}{m_4}],$$

$$U = \frac{m_1 m_2}{x_1} + \frac{m_1 m_3}{x_1 + x_3} + \frac{m_1 m_4}{x_1 + x_2 + x_3} + \frac{m_2 m_3}{x_3} + \frac{m_2 m_4}{x_2 + x_3} + \frac{m_3 m_4}{x_2}.$$

2.1.2 Binary Collision. Let $t = t_1$ be the time of SBC. Then, $x_1 \to 0$ and $x_2 \to 0$ simultaneously as $t \to t_1$.

The following result is from the work of Belbruno([3]).

Lemma 2.1.

$$\lim_{t \to t_1} \frac{x_1}{x_2} = \alpha, \qquad \textit{where } \alpha = (\frac{m_1 + m_2}{m_3 + m_4})^{\frac{1}{3}} \tag{2.6}$$

and

$$\lim_{t \to t_1}(q_2 - q_1)(\dot{q_2} - \dot{q_1})^2 = 2(m_1 + m_2) \tag{2.7}$$

$$\lim_{t \to t_1}(q_4 - q_3)(\dot{q_4} - \dot{q_3})^2 = 2(m_3 + m_4) \tag{2.8}$$

Lemma 2.2. *$x_1 y_1^2$ and $x_2 y_2^2$ both are finite when $t \to t_1$. Furthermore,* $\lim_{t \to t_1} x_1 y_1^2$ *and* $\lim_{t \to t_1} x_2 y_2^2$ *exist,* *and also*

$$\lim_{t\to t_1} x_1 y_1^2 = \lim_{t\to t_1} x_1 p_1^2 = \frac{2(m_1m_2)^2}{m_1+m_2},$$

$$\lim_{t\to t_1} x_2 y_2^2 = \lim_{t\to t_1} x_2 p_4^2 = \frac{2(m_3m_4)^2}{m_3+m_4}.$$

Proof. First, we will show both $x_1y_1^2$ and $x_2y_2^2$ are finite.

By the formula of U,

$$x_1U = m_1m_2 + x_1\frac{m_1m_3}{x_1+x_3} + x_1\frac{m_1m_4}{x_1+x_2+x_3} + x_1\frac{m_2m_3}{x_3} + x_1\frac{m_2m_4}{x_2+x_3} + x_1\frac{m_3m_4}{x_2}.$$

As $t \to t_1$, $x_1 \to 0$, $x_2 \to 0$, x_1+x_3, $x_1+x_2+x_3$, x_3, x_2+x_3, are all positive and finite.

$$\therefore \lim_{t\to t_1} x_1U = \lim_{t\to t_1}[m_1m_2 + x_1\frac{m_3m_4}{x_2}] = m_1m_2 + \alpha m_3m_4.$$

Note that on the phase space of Hamiltonian system, $E = T - U = h$, where h is the Hamiltonian constant.

Therefore, when $t \to t_1$,

$$x_1T = x_1(U+h) \to m_1m_2 + \alpha m_3m_4,$$

that is,

$$\frac{1}{2}x_1[\frac{y_1^2}{m_1} + \frac{(y_1-y_3)^2}{m_2} + \frac{(y_3-y_2)^2}{m_3} + \frac{y_2^2}{m_4}] \to m_1m_2 + \alpha m_3m_4 \tag{2.9}$$

In particular, $x_1y_1^2$ and $x_1y_2^2$ are bounded at SBC. Then, by Lemma 2.1, $x_1y_1^2$ and $x_2y_2^2$ are bounded at SBC.

Next, we will use the boundedness of $x_1y_1^2$ and $x_2y_2^2$ and Lemma 2.1 to show the existence of the limits of them.

Note that from (2.7),(2.8),(2.9):

$$\lim_{t\to t_1} x_1(\frac{p_1}{m_1} - \frac{p_2}{m_2})^2 = 2(m_1+m_2) \tag{2.10}$$

$$\lim_{t\to t_1} x_2\left(\frac{p_3}{m_3}-\frac{p_4}{m_4}\right)^2 = 2(m_3+m_4) \tag{2.11}$$

$$\lim_{t\to t_1} x_1\left[\frac{y_1^2}{m_1}+\frac{(y_1-y_3)^2}{m_2}+\frac{(y_3-y_2)^2}{m_3}+\frac{y_2^2}{m_4}\right] = 2(m_1m_2+\alpha m_3m_4) \tag{2.12}$$

and

$$p_1+p_2+p_3+p_4 = 0 \tag{2.13}$$

By the boundedness, we know that $x_1p_1^2,\ x_1p_2^2,\ \ x_1p_3^2,\ \ x_1p_4^2$ are all finite when $t\to t_1$.

Because, $p_1+p_2=-y_3$, and by the Hamiltonian,

$$y_3' = -E_{x_3} = \frac{m_1m_3}{(x_1+x_3)^2}+\frac{m_1m_4}{(x_1+x_2+x_3)^2}+\frac{m_2m_3}{(x_3)^2}+\frac{m_2m_4}{(x_2+x_3)^2}$$

For $\tau<t<t_1$, since x_3 is strictly positive, there exists a positive constant B, such that $x_3>B>0$. Integrate the above identity from τ to t_1,

$$y_3(t_1)-y_3(\tau) < \frac{1}{B^2}(t_1-\tau)\cdot(m_1m_3+m_1m_4+m_2m_3+m_2m_4).$$

Since the right hand side of the inequality is finite, $y_3(t_1)$ is bounded above.That is, p_1+p_2 is finite as t approaches t_1.

Then

$$\lim_{t\to t_1} x_1(p_1+p_2)^2 = 0, \qquad \lim_{t\to t_1} x_1p_1(p_1+p_2) = 0.$$

Consider 2.10:

$$2(m_1+m_2) = \lim_{t\to t_1} x_1\left(\frac{p_1}{m_1}-\frac{p_2}{m_2}\right)^2 = \lim_{t\to t_1} x_1\left(\frac{p_1}{m_1}+\frac{p_1}{m_2}-\frac{p_1}{m_2}-\frac{p_2}{m_2}\right)^2$$

$$= \lim_{t\to t_1} x_1p_1^2\left(\frac{1}{m_1}+\frac{1}{m_2}\right)^2+\frac{1}{m_2^2}\lim_{t\to t_1} x_1(p_1+p_2)^2-\frac{2}{m_2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\lim_{t\to t_1} x_1p_1(p_1+p_2)$$

$$= \lim_{t\to t_1} x_1p_1^2\left(\frac{1}{m_1}+\frac{1}{m_2}\right)^2$$

Therefore, $\lim_{t\to t_1} x_1 p_1^2$ exists and the value is $\frac{2(m_1m_2)^2}{m_1+m_2}$.

Similarly, by considering 2.11, we can get the existence of $\lim_{t\to t_1} x_2 p_4^2$ and also

$$\lim_{t\to t_1} x_2 y_2^2 = \lim_{t\to t_1} x_2 p_4^2 = \frac{2(m_3m_4)^2}{m_3+m_4}.$$

□

2.2 Decoupled case with all masses equal to 1

Define a new independent time variable

$$s = \int_\tau^t \left(\frac{m_1m_2}{x_1} + \frac{m_3m_4}{x_2}\right)dt, \qquad (\tau \le t < t_1).$$

Let $s = s_1$ be the corresponding collision time in the new time variable. Siegel and Moser([25]) have shown that $\int_\tau^{t_1} U dt$ is finite, so $s_1 = \int_\tau^{t_1} (\frac{m_1m_2}{x_1} + \frac{m_3m_4}{x_2})dt$ is also finite.

Denote $\frac{dx_k}{ds}$ by x_k' and $\frac{dy_k}{ds}$ by y_k'.Then the Hamiltonian system 2.5 becomes

$$x_k' = \frac{1}{\frac{m_1m_2}{x_1} + \frac{m_3m_4}{x_2}} E_{y_k}, \qquad y_k' = -\frac{1}{\frac{m_1m_2}{x_1} + \frac{m_3m_4}{x_2}} E_{x_k} \qquad (k = 1,2,3) \tag{2.14}$$

Set $F = \frac{1}{\frac{m_1m_2}{x_1}+\frac{m_3m_4}{x_2}}(E-h) = \frac{1}{\frac{m_1m_2}{x_1}+\frac{m_3m_4}{x_2}}(T-U-h)$, where $E = T-U = h$. Then for the solution of Hamiltonian system 2.5 on the energy surface $E = h$, we have

$$F_{x_k} = \frac{1}{\frac{m_1m_2}{x_1} + \frac{m_3m_4}{x_2}} E_{x_k}, \quad F_{y_k} = \frac{1}{\frac{m_1m_2}{x_1} + \frac{m_3m_4}{x_2}} E_{y_k}.$$

Consequently, for the solution of Hamiltonian system 2.5 on the energy surface $E = h$, (2.14) can be written as

$$x_k' = F_{y_k}, \qquad y_k' = -F_{x_k} \qquad k = 1,2,3$$

with $F = \frac{1}{\frac{m_1m_2}{x_1}+\frac{m_3m_4}{x_2}}(T-U-h)$.

If x_k and y_k are solutions of 2.5 on the energy surface $E = h$, F is a constant with respect to s because

$$F' = \sum_{k=1}^{3} (F_{x_k} x_k' + F_{y_k} y_k') = 0.$$

For the decoupled case, assume $y_3 = 0$, $x_3 = \infty$, $h = 0$, and $m_1 = m_2 = m_3 = m_4 = 1$.

We choose

$$F = \frac{y_1^2 + y_2^2}{\frac{1}{x_1} + \frac{1}{x_2}} \tag{2.15}$$

To solve the new Hamiltonian system,

$$x_k' = F_{y_k}, \qquad y_k' = -F_{x_k} \qquad\qquad (k = 1, 2) \tag{2.16}$$

with the Hamiltonian $F = \frac{y_1^2+y_2^2}{\frac{1}{x_1}+\frac{1}{x_2}} = 1$, we are going to introduce a canonical transformation.

2.2.1 Relationship between x_k and y_k.

First, let's write (2.16) into explicit forms:

$$x_1' = \frac{2y_1}{\frac{1}{x_1} + \frac{1}{x_2}} = \frac{2y_1 x_1 x_2}{x_1 + x_2}, \tag{2.17}$$

$$y_1' = -\frac{y_1^2 + y_2^2}{(\frac{1}{x_1} + \frac{1}{x_2})^2} \frac{1}{x_1^2} = -\frac{x_2}{x_1(x_1 + x_2)} F, \tag{2.18}$$

$$x_2' = \frac{2y_2}{\frac{1}{x_1} + \frac{1}{x_2}} = \frac{2y_2 x_1 x_2}{x_1 + x_2}, \tag{2.19}$$

$$y_2' = -\frac{y_1^2 + y_2^2}{(\frac{1}{x_1} + \frac{1}{x_2})^2} \frac{1}{x_2^2} = -\frac{x_1}{x_2(x_1 + x_2)} F. \tag{2.20}$$

Lemma 2.3. *If $\{x_1, x_2, y_1, y_2\}$ is the solution for the above system 2.17 to 2.20, there exists a constant C such that*

$$y_1^2 = \frac{F}{x_1} + C, \qquad \textit{and} \qquad y_2^2 = \frac{F}{x_2} - C,$$

$$\textit{or} \quad C = \frac{x_1 y_1^2 - x_2 y_2^2}{x_1 + x_2}.$$

Proof. By equations 2.17 and 2.18:

$$\frac{dy_1}{dx_1} = \frac{y_1'}{x_1'} = -\frac{F}{2y_1 x_1^2}$$

Then separate the variables:

$$\int 2y_1 dy_1 = \int \frac{-F}{x_1^2} dx_1$$

Therefore,

$$y_1^2 = \frac{F}{x_1} + C, \ \ or \ \ x_1 y_1^2 - Cx_1 = F \tag{2.21}$$

where C is a constant, which depends on the initial conditions.

By a similar process, we have

$$y_2^2 = \frac{F}{x_2} + C_1, \ \ or \ \ x_2 y_2^2 - C_1 x_2 = F \tag{2.22}$$

where C_1 is another constant, which depends on the initial conditions, too.

Add 2.21 and 2.22 together, by using :

$$y_1^2 + y_2^2 = \frac{F}{x_1} + \frac{F}{x_2} + C + C_1$$

$$= y_1^2 + y_2^2 + C + C_1$$

Therefore, $C_1 = -C$. Then we can rewrite (22) as

$$y_2^2 = \frac{F}{x_2} - C, \ \ or \ \ x_2 y_2^2 + Cx_2 = F. \tag{2.23}$$

2.21-2.23:

$$x_1 y_1^2 - x_2 y_2^2 - C(x_1 + x_2) = 0,$$

$$C = \frac{x_1 y_1^2 - x_2 y_2^2}{x_1 + x_2}.$$

□

2.2.2 Canonical Transformation. As we know, at the collision time t_1, $y_1 \to \infty$ and $y_2 \to \infty$. It would be nice if we can introduce a canonical transformation and remove the singularity at $t = t_1$.

From the two-body problem, we have a transformation as $\eta_k = \frac{1}{y_k}$. Similarly, we want to use this part to generate our new canonical transformation.

$$y_1 = \frac{1}{\eta_1}, \qquad y_2 = \frac{1}{\eta_2}.$$

Let $y = (y_1, y_2)^T$, $b(\eta) = (\frac{1}{\eta_1}, \frac{1}{\eta_2})^T$ and $x = (x_1, x_2)^T$. Assume the generating function $V = V(x, \eta)$. The canonical transformation is given by

$$y = V_x(x, \eta), \qquad \xi = V_\eta(x, \eta).$$

Hence,

$$y = V_x(x, \eta) = b(\eta)$$

Therefore,

$$V(x, \eta) = < b(\eta), x > + g(\eta)$$

Then

$$\xi = V_\eta(x, \eta) = b_\eta^T(\eta) \cdot x + g_\eta(\eta)$$

So

$$x = [b_\eta^T(\eta)]^{-1} \cdot (\xi - g_\eta(\eta)).$$

In particular, let $g(\eta) = 0$.

Since

$$b_\eta(\eta) = \begin{pmatrix} -\frac{1}{\eta_1^2} & 0 \\ 0 & -\frac{1}{\eta_2^2} \end{pmatrix} = b_\eta^T(\eta),$$

we can write down the canonical transformation as

$$\xi_1 = -x_1 y_1^2, \quad \xi_2 = -x_2 y_2^2, \quad \eta_1 = \frac{1}{y_1}, \quad \eta_2 = \frac{1}{y_2};$$

$$x_1 = -\xi_1 \eta_1^2, \quad x_2 = -\xi_2 \eta_2^2, \quad y_1 = \frac{1}{\eta_1}, \quad y_2 = \frac{1}{\eta_2}.$$

And the new hamiltonian system is going to be

$$\xi_k' = F_{\eta_k}, \qquad \eta_k' = -F_{\xi_k} \qquad k = 1, 2 \tag{2.24}$$

with

$$F=-\frac{\xi_1\xi_2(\eta_1^2+\eta_2^2)}{\xi_1\eta_1^2+\xi_2\eta_2^2}=-\frac{\eta_1^2+\eta_2^2}{\frac{\eta_1^2}{\xi_2}+\frac{\eta_2^2}{\xi_1}}. \tag{2.25}$$

2.2.3 Meaning of C. Since we only know the behavior at time $t=t_1$ or $s=s_1$, we may think $s=s_1$ as the initial time for the Hamiltonian system2.16. Without loss of generality, let $s_1=0$. We consider the following two differential equations

$$\eta_1=-F_{\xi_1},\qquad \eta_2=-F_{\xi_2}$$

with initial conditions:

$$\eta_1(0)=\eta_2(0)=0,\qquad \xi_1(0)=\xi_2(0)=-F.$$

2.21 and 2.23 can be rewritten in terms of ξ_k and η_k:

$$-\xi_1+C\xi_1\eta_1^2=-\xi_2-C\xi_2\eta_2^2=F,$$

then

$$\xi_1=\frac{F}{-1+C\eta_1^2},\qquad \xi_2=\frac{F}{-1-C\eta_2^2}.$$

Differentiate ,

$$\begin{aligned}F_{\xi_1}&=\frac{-\xi_2(\eta_1^2+\eta_2^2)}{\xi_1\eta_1^2+\xi_2\eta_2^2}+\frac{\xi_1\xi_2(\eta_1^2+\eta_2^2)\eta_1^2}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}\\&=\frac{F}{\xi_1}+\frac{F^2}{\xi_1\xi_2}\cdot\frac{\eta_1^2}{\eta_1^2+\eta_2^2}=-\frac{F^2}{\xi_1^2}\cdot\frac{\eta_2^2}{\eta_1^2+\eta_2^2}\\&=-(-1+C\eta_1^2)^2\cdot\frac{\eta_2^2}{\eta_1^2+\eta_2^2},\end{aligned}$$

and similarly,

$$\begin{aligned}F_{\xi_2}&=\frac{-\xi_1(\eta_1^2+\eta_2^2)}{\xi_1\eta_1^2+\xi_2\eta_2^2}+\frac{\xi_1\xi_2(\eta_1^2+\eta_2^2)\eta_2^2}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}\\&=\frac{F}{\xi_2}+\frac{F^2}{\xi_1\xi_2}\cdot\frac{\eta_2^2}{\eta_1^2+\eta_2^2}=-\frac{F^2}{\xi_2^2}\cdot\frac{\eta_1^2}{\eta_1^2+\eta_2^2}\end{aligned}$$

$$= -(-1 - C\eta_2^2)^2 \cdot \frac{\eta_1^2}{\eta_1^2 + \eta_2^2}.$$

Therefore,

$$\eta_1' = (-1 + C\eta_1^2)^2 \cdot \frac{\eta_2^2}{\eta_1^2 + \eta_2^2}, \tag{2.26}$$

$$\eta_2' = (-1 - C\eta_2^2)^2 \cdot \frac{\eta_1^2}{\eta_1^2 + \eta_2^2}, \tag{2.27}$$

where C is an arbitrary constant.

When $C = 0$, it is the most special case. The solution for the equations (2.26) and (2.27) with initial conditions $\eta_1(0) = \eta_2(0) = 0$ is

$$\eta_1 = \eta_2 = \frac{s}{2}.$$

Then

$$\xi_1 = \frac{F}{-1 + C\eta_1^2} = -F, \quad \xi_2 = \frac{F}{-1 - C\eta_2^2} = -F.$$

Therefore, $x_1 = x_2$ and $y_1 = y_2$.

So when $C = 0$, the motions of these two decoupled pairs are exactly the same.

When $C > 0$, that is $C = \frac{x_1 y_1^2 - x_2 y_2^2}{x_1 + x_2} > 0$, hence $x_1 y_1^2 > x_2 y_2^2$.

Consider the initial conditions. Assume x_1, x_2, y_1 and y_2 are the initial values for the decoupled system. If $x_1 < x_2$, which means the distance between the two object P_1 and P_2 is less than the distance between the two object P_3 and P_4, then $y_1 > y_2$ because $x_1 y_1^2 > x_2 y_2^2$. That is, the initial velocity of P_1 is also greater than the initial velocity of P_2. Note that for each collision system, the force between the objects only depends on the relative distance, and then by Newton's second law, the acceleration for P_1 or P_2 is greater than the acceleration for P_3 or P_4. So it is impossible for these two collisions to happen at the same time. Contradiction!

Therefore, when $C > 0$, we can get

$$x_1 > x_2 \quad \text{and} \quad y_1 > y_2.$$

By a similar argument, when $C < 0$, we have

$$x_1 < x_2 \quad \text{and} \quad y_1 < y_2.$$

2.2.4 Apply C to the decoupled system. First, we want to show that the constant C can also be derived directly from the following initial value problem:

$$\xi_1' = -\frac{2\eta_1\eta_2^2(\frac{1}{\xi_1}-\frac{1}{\xi_2})}{(\frac{\eta_1^2}{\xi_2}+\frac{\eta_2^2}{\xi_1})^2} = \frac{2\xi_1\xi_2\eta_1\eta_2^2(\xi_1-\xi_2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2},$$

$$\xi_2' = -\frac{2\eta_2\eta_1^2(\frac{1}{\xi_2}-\frac{1}{\xi_1})}{(\frac{\eta_1^2}{\xi_2}+\frac{\eta_2^2}{\xi_1})^2} = -\frac{2\xi_1\xi_2\eta_2\eta_1^2(\xi_1-\xi_2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2},$$

$$\eta_1' = \frac{F^2}{\xi_1^2}\cdot\frac{\eta_2^2}{\eta_1^2+\eta_2^2},$$

$$\eta_2' = \frac{F^2}{\xi_2^2}\cdot\frac{\eta_1^2}{\eta_1^2+\eta_2^2},$$

$$\xi_1(0)=\xi_2(0)=-F, \qquad \eta_1(0)=\eta_2(0)=0.$$

Let $f(s) = \frac{\frac{F}{\xi_1}-\frac{F}{\xi_2}}{\eta_1^2+\eta_1^2}$, because $F = -\frac{\eta_1^2+\eta_2^2}{\frac{\eta_1^2}{\xi_2}+\frac{\eta_2^2}{\xi_1}}$,

$$f(s) = -\frac{\eta_1^2+\eta_2^2}{\frac{\eta_1^2}{\xi_2}+\frac{\eta_2^2}{\xi_1}}\cdot\frac{\frac{1}{\xi_1}-\frac{1}{\xi_2}}{\eta_1^2+\eta_1^2} = \frac{\xi_1-\xi_2}{\xi_1\eta_1^2+\xi_2\eta_2^2}.$$

Lemma 2.4. *If $\{\xi_1,\xi_2,\eta_1,\eta_2\}$ is the solution for the above system with the initial conditions, then $f(s)$ is a constant with respect to s.*

Proof.

$$\frac{df}{ds} = \frac{\partial f}{\partial \xi_1}\cdot\xi_1' + \frac{\partial f}{\partial \xi_2}\cdot\xi_2' + \frac{\partial f}{\partial \eta_1}\cdot\eta_1' + \frac{\partial f}{\partial \eta_2}\cdot\eta_2'$$

$$= \frac{\xi_2(\eta_1^2+\eta_2^2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}\cdot\frac{2\xi_1\xi_2\eta_1\eta_2^2(\xi_1-\xi_2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2} + \frac{-\xi_1(\eta_1^2+\eta_2^2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}\cdot\frac{-2\xi_1\xi_2\eta_2\eta_1^2(\xi_1-\xi_2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}$$

$$+\frac{-2\xi_1\eta_1(\xi_1-\xi_2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}\cdot\frac{\xi_2^2\eta_2^2(\eta_1^2+\eta_2^2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2} + \frac{-2\xi_2\eta_2(\xi_1-\xi_2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}\cdot\frac{\xi_1^2\eta_1^2(\eta_1^2+\eta_2^2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}$$

$$= 0.$$

Therefore, $f(s)$ is a constant. $\square$

Lemma 2.5. *Let $f(s) = C_2$, then $C_2 = C$, where C is the constant in Lemma 2.3.*

Proof. From the initial value problem,

$$\frac{d\xi_1}{d\eta_1} = \frac{\xi_1'}{\eta_1'} = \frac{2\xi_1\xi_2\eta_1\eta_2^2(\xi_1-\xi_2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2} \cdot \frac{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}{\xi_2^2\eta_2^2(\eta_1^2+\eta_2^2)} = \frac{2\xi_1\eta_1(\xi_1-\xi_2)}{\xi_2(\eta_1^2+\eta_2^2)}$$

Note that $F = -\frac{\eta_1^2+\eta_2^2}{\frac{\eta_1^2}{\xi_2}+\frac{\eta_2^2}{\xi_1}}$, then $\eta_1^2+\eta_2^2 = -F\cdot(\frac{\eta_1^2}{\xi_2}+\frac{\eta_2^2}{\xi_1})$, so

$$\frac{d\xi_1}{d\eta_1} = \frac{2\xi_1\eta_1(\xi_1-\xi_2)}{\xi_2(\eta_1^2+\eta_2^2)}$$

$$= -\frac{1}{F}\cdot\frac{2\xi_1\eta_1(\xi_1-\xi_2)}{\xi_2(\frac{\eta_1^2}{\xi_2}+\frac{\eta_2^2}{\xi_1})}$$

$$= -\frac{1}{F}\cdot 2\xi_1^2\eta_1\cdot\frac{\xi_1-\xi_2}{\xi_1\eta_1^2+\xi_2\eta_2^2} = -\frac{C_2}{F}\cdot 2\xi_1^2\eta_1$$

Separate the variable,

$$\frac{d\xi_1}{-\xi_1^2} = \frac{C_2}{F}\cdot 2\eta_1 d\eta_1$$

integrate both sides,

$$\frac{1}{\xi_1} = \frac{C_2}{F}\eta_1^2 + C_3,$$

where C_3 is a constant.

By the initial condition,

$$-\frac{1}{F} = C_3.$$

Therefore,

$$\frac{1}{\xi_1} = \frac{C_2}{F}\eta_1^2 - \frac{1}{F}$$

that is

$$F = -\xi_1 + C_2\xi_1\eta_1^2.$$

But from the definition of transformation, we have

$$F = -\xi_1 + C\xi_1\eta_1^2,$$

then

$$C = C_2. \quad \square$$

$\square$

Therefore, we can use the constant C in the new hamiltonian system. And note that C acts as a first integral in the new hamiltonian system, which does not depend on the constant F. Rewrite the system:

$$\xi_1' = -\frac{2\eta_1\eta_2^2(\frac{1}{\xi_1} - \frac{1}{\xi_2})}{(\frac{\eta_1^2}{\xi_2} + \frac{\eta_2^2}{\xi_1})^2} = -2CF\frac{\eta_1\eta_2^2}{\eta_1^2+\eta_2^2},$$

$$\xi_2' = -\frac{2\eta_2\eta_1^2(\frac{1}{\xi_2} - \frac{1}{\xi_1})}{(\frac{\eta_1^2}{\xi_2} + \frac{\eta_2^2}{\xi_1})^2} = 2CF\frac{\eta_1^2\eta_2}{\eta_1^2+\eta_2^2},$$

$$\eta_1' = \frac{F^2}{\xi_1^2} \cdot \frac{\eta_2^2}{\eta_1^2+\eta_2^2},$$

$$\eta_2' = \frac{F^2}{\xi_2^2} \cdot \frac{\eta_1^2}{\eta_1^2+\eta_2^2}.$$

By observation, we have

$$\eta_1'\xi_1 + \eta_2'\xi_2 = -F, \tag{2.28}$$

$$\eta_1\xi_1' + \eta_2\xi_2' = 0, \tag{2.29}$$

$$\eta_1'\xi_1^2 + \eta_2'\xi_2^2 = F^2, \tag{2.30}$$

$$\eta_1'\eta_1^2\xi_1^2 - \eta_2'\eta_2^2\xi_2^2 = 0, \tag{2.31}$$

$$\frac{\xi_1'}{\eta_1} - \frac{\xi_2'}{\eta_2} = -2CF. \tag{2.32}$$

By (2.28) and (2.29),

$$(\eta_1\xi_1 + \eta_2\xi_2)' = -F$$

Since when $s=0$, $\eta_1=\eta_2=0$, therefore by integrating both sides,

$$\eta_1\xi_1+\eta_2\xi_2=-Fs$$

By the identities $-\xi_1+C\xi_1\eta_1^2=-\xi_2-C\xi_2\eta_2^2=F$,

$$\frac{\eta_1}{1-C\eta_1^2}+\frac{\eta_2}{1+C\eta_2^2}=s, \tag{2.33}$$

and also the differential equations of η_1 and η_2 will be

$$\eta_1'=(-1+C\eta_1^2)^2\cdot\frac{\eta_2^2}{\eta_1^2+\eta_2^2}$$

$$\eta_2'=(-1-C\eta_2^2)^2\cdot\frac{\eta_1^2}{\eta_1^2+\eta_2^2}$$

For $C<0$, the equations are

$$\eta_1'=(-1-\mid C\mid\eta_1^2)^2\cdot\frac{\eta_2^2}{\eta_1^2+\eta_2^2}$$

$$\eta_2'=(-1+\mid C\mid\eta_2^2)^2\cdot\frac{\eta_1^2}{\eta_1^2+\eta_2^2}$$

Note that the solutions $\{\eta_1,\eta_2\}$ of the above two equations are the same as the solutions $\{\eta_2,\eta_1\}$ of (2.26) and (2.27) with positive C.

Without loss of generality, we can assume that $C>0$.

Lemma 2.6. *Let $\{\eta_1,\eta_2\}$ be the solution for*

$$\eta_1'=(-1+C\eta_1^2)^2\cdot\frac{\eta_2^2}{\eta_1^2+\eta_2^2},$$

$$\eta_2'=(-1-C\eta_2^2)^2\cdot\frac{\eta_1^2}{\eta_1^2+\eta_2^2}.$$

Define $N_1(s)=C^{\frac{1}{2}}\eta_1(\frac{s}{C^{\frac{1}{2}}})$ and $N_2(s)=C^{\frac{1}{2}}\eta_2(\frac{s}{C^{\frac{1}{2}}})$. Then

$$\tanh^{-1}(N_1)+\tan^{-1}(N_2)=s,$$

$$\frac{N_1}{1-N_1^2}+\frac{N_2}{1+N_2^2}=s.$$

Proof. Consider the ratio between the above the differential equations for η_1 and η_2:

$$\frac{\eta_1'}{\eta_2'}=\frac{(-1+C\eta_1^2)^2}{(-1-C\eta_2^2)^2}\cdot\frac{\eta_2^2}{\eta_1^2}$$

Separating the variables and integrating both sides:

$$\frac{\eta_1^2}{(-1+C\eta_1^2)^2}d\eta_1=\frac{\eta_2^2}{(1+C\eta_2^2)^2}d\eta_2$$

$$-\frac{1}{2C}\frac{\eta_1}{-1+C\eta_1^2}-\frac{1}{2}\frac{\tanh^{-1}(C^{\frac{1}{2}}\eta_1)}{C^{\frac{3}{2}}}=-\frac{1}{2C}\frac{\eta_2}{1+C\eta_2^2}+\frac{1}{2}\frac{\tan^{-1}(C^{\frac{1}{2}}\eta_2)}{C^{\frac{3}{2}}}+D$$

where D is a constant.

By the initial condition $\eta_1(0)=\eta_2(0)=0$,

$$D=0.$$

Simplify the above identity of η_1 and η_2:

$$\frac{C^{\frac{1}{2}}\eta_1}{-1+C\eta_1^2}+\tanh^{-1}(C^{\frac{1}{2}}\eta_1)=\frac{C^{\frac{1}{2}}\eta_2}{1+C\eta_2^2}-\tan^{-1}(C^{\frac{1}{2}}\eta_2) \tag{2.34}$$

Therefore, combine (2.33) and (2.34),

$$\tanh^{-1}(C^{\frac{1}{2}}\eta_1)+\tan^{-1}(C^{\frac{1}{2}}\eta_2)=\frac{C^{\frac{1}{2}}\eta_2}{1+C\eta_2^2}+\frac{C^{\frac{1}{2}}\eta_1}{1-C\eta_1^2}=C^{\frac{1}{2}}s. \tag{2.35}$$

Then it is easy to express η_2 in terms of η_1:

$$\eta_2=C^{-\frac{1}{2}}\tan[C^{\frac{1}{2}}s-\tanh^{-1}(C^{\frac{1}{2}}\eta_1)].$$

Let $N_1(s)=C^{\frac{1}{2}}\eta_1(\frac{s}{C^{\frac{1}{2}}})$, $N_2(s)=C^{\frac{1}{2}}\eta_2(\frac{s}{C^{\frac{1}{2}}})$.

Then the equations of η_1 and η_2 can be changed to equations of N_1 and N_2:

$$N_1' = (1-N_1^2)^2 \cdot \frac{N_2^2}{N_1^2+N_2^2}$$

$$N_2' = (1+N_2^2)^2 \cdot \frac{N_1^2}{N_1^2+N_2^2}$$

Therefore, by (2.35),

$$\tanh^{-1}(N_1) + \tan^{-1}(N_2) = s, \tag{2.36}$$

and

$$\frac{N_1}{1-N_1^2} + \frac{N_2}{1+N_2^2} = s. \tag{2.37}$$

□

2.2.5 Existence, Uniqueness and Analytic Properties. Consider the equations

$$N_1' = (1-N_1^2)^2 \cdot \frac{N_2^2}{N_1^2+N_2^2},$$

$$N_2' = (1+N_2^2)^2 \cdot \frac{N_1^2}{N_1^2+N_2^2},$$

with the initial conditions

$$N_1(0) = N_2(0) = 0.$$

Theorem 2.7. *The above system has analytic solutions $(N_1(s), N_2(s))$ as s approaches 0.*

Consider the ratio of the differential equations of N_1 and N_2, we can get the following equation:

$$\tanh^{-1}(N_1) + \tan^{-1}(N_2) = \frac{N_1}{1-N_1^2} + \frac{N_2}{1+N_2^2},$$

or

$$-\tanh^{-1}(N_1) - \tan^{-1}(N_2) + \frac{N_1}{1-N_1^2} + \frac{N_2}{1+N_2^2} = 0.$$

To prove the above theorem, we need to introduce several propositions.

Proposition 2.8. *Assume N_1 and N_2 satisfy*

$$\tanh^{-1}(N_1) + \tan^{-1}(N_2) = \frac{N_1}{1-N_1^2} + \frac{N_2}{1+N_2^2}.$$

Then N_1 is a real analytic function of N_2 in a small neighborhood of $N_2 = 0$.

Basically, we will apply the implicit function theorem.

Let $G(N_1,N_2) = -\tanh^{-1}(N_1) - \tan^{-1}(N_2) + \frac{N_1}{1-N_1^2} + \frac{N_2}{1+N_2^2}$. Of course there exists a small neighborhood V of (0,0), such that $G(N_1,N_2)$ is analytic in V with respect to (N_1,N_2). The Taylor series of $G(N_1,N_2)$ at $(0,0)$ is

$$\sum_{n=1}^{\infty} \frac{2n}{2n+1} N_1^{2n+1} + \sum_{n=1}^{\infty} \frac{2n(-1)^n}{2n+1} N_2^{2n+1}$$

We know N_1 and N_2 satisfy $G(N_1,N_2) = 0$. When applying the implicit theorem, we will see that $\frac{\partial G}{\partial N_1}(0,0) = 0$. So we introduce an extended implicit function theorem.

Proposition 2.9. *(Extended Implicit Function Theorem)*
Denote $\frac{\partial G}{\partial N_1}$ by G'_{N_1}, the second partial derivative $\frac{\partial^2 G}{\partial N_1^2}$ by G''_{N_1}, and the third partial derivative $\frac{\partial^3 G}{\partial N_1^3}$ by G'''_{N_1}. Then there exist intervals $I = (-\delta_1, \delta_1)$ and $J = (-\delta_2, \delta_2)$ and a unique function g, such that

$$g : J \longrightarrow I, \qquad N_2 \mapsto N_1 = g(N_2).$$

Proof. Differentiating the Taylor series of $G(N_1,N_2)$ with respect to N_1 three times, we can find some good properties of the partial derivatives of $G(N_1,N_2)$:

$$G(0,0) = 0,$$

$$G'_{N_1}(0,N_2) = 0,$$

$$G''_{N_1}(0,N_2) = 0,$$

$$G'''_{N_1}(0,N_2) = 4 \neq 0.$$

Because $G'''_{N_1}(0,N_2) = 4 > 0$ and G'''_{N_1} is continuous, there exist a rectangular area R: $\mid N_1 \mid < \delta_1$, $\mid N_2 \mid <$

δ_2', such that the closure $\overline{R} \subset V$ and

$$m = \min_{(N_1,N_2)\in R} G'''_{N_1}(N_1,N_2) > 0.$$

Since $G''_{N_1}(0,N_2) = 0$ and $G''_{N_1}(N_1,N_2)$ is continuous and strictly increasing with respect to N_1,

$$G''_{N_1}(N_1,N_2) > 0 \quad for \;\; 0 < N_1 < \delta_1$$

and

$$G''_{N_1}(N_1,N_2) < 0 \quad for \;\; -\delta_1 < N_1 < 0.$$

By the above result, $G'_{N_1}(N_1,N_2)$ is strictly increasing with respect to N_1 when $0 < N_1 < \delta_1$ and $G'_{N_1}(N_1,N_2)$ is strictly decreasing with respect to N_1 in $(-\delta_1,0)$. Note that $G'_{N_1}(0,N_2) = 0$,

$$G'_{N_1}(N_1,N_2) > 0 \quad for \;\; 0 < N_1 < \delta_1$$

$$G'_{N_1}(N_1,N_2) > 0 \quad for \;\; -\delta_1 < N_1 < 0$$

that is

$$G'_{N_1}(N_1,N_2) > 0 \quad for \;\; -\delta_1 < N_1 < \delta_1, \;\; N_1 \neq 0.$$

Because $G(0,0) = 0$, $G'_{N_1}(N_1,N_2) > 0$ when $N_1 \neq 0$, so

$$G(-\delta_1,0) < 0, \qquad G(\delta_1,0) > 0.$$

By the continuity of $G(N_1,N_2)$, there exists $0 < \delta_2 < \delta_2'$, such that when $| N_2 |< \delta_2$,

$$G(-\delta_1,N_2) < 0, \qquad G(\delta_1,N_2) > 0.$$

Consider the intervals $I = (-\delta_1,\delta_1)$ and $J = (-\delta_2,\delta_2)$. For any point N_2 in J, the function $G(N_1,N_2)$ is strictly increasing in I, so by the intermediate value theorem for continuous function, there exists exactly one $N_1 \in I$ such that $G(N_1,N_2) = 0$. That means, for any given $N_2 \in J$, according to $G(N_1,N_2) = 0$, we can always find exactly one $N_1 \in I$ corresponds to N_2. By the definition of function, there exist a unique function g such that

$$g : J \longrightarrow I, \qquad N_2 \mapsto N_1 = g(N_2).$$

Hence, so far we have proven the existence and uniqueness of N_1 as a function of N_2 which satisfy $G(N_1, N_2) = 0$. □

Proof of Proposition 2.8: By Proposition 2.9, the existence and uniqueness are guaranteed. If we can show N_1 is an analytic function of N_2, then we are done.

Consider

$$-\tanh^{-1}(N_1) - \tan^{-1}(N_2) + \frac{N_1}{1-N_1^2} + \frac{N_2}{1+N_2^2} = 0$$

Since each function on the left hand side of the equality is analytic close to 0, we can find their Taylor expansions for (N_1, N_2) in a small interval of $(0,0)$:

$$\sum_{n=1}^{\infty} \frac{2n}{2n+1} N_1^{2n+1} + \sum_{n=1}^{\infty} \frac{2n(-1)^n}{2n+1} N_2^{2n+1} = 0,$$

that is

$$\sum_{n=1}^{\infty} \frac{2n}{2n+1} N_1^{2n+1} = \sum_{n=1}^{\infty} \frac{2n(-1)^{n+1}}{2n+1} N_2^{2n+1}$$

$$N_1{}^3\left(\frac{2}{3} + \sum_{n=2}^{\infty} \frac{2n}{2n+1} N_1^{2n-2}\right) = N_2{}^3\left(\frac{2}{3} + \sum_{n=2}^{\infty} \frac{2n(-1)^{n+1}}{2n+1} N_2^{2n-2}\right).$$

For simplicity, let

$$h_1(N_1) = \frac{2}{3} + \sum_{n=2}^{\infty} \frac{2n}{2n+1} N_1^{2n-2}$$

and

$$h_2(N_2) = \frac{2}{3} + \sum_{n=2}^{\infty} \frac{2n(-1)^{n+1}}{2n+1} N_2^{2n-2},$$

By the ratio test, we can see that $h_1(N_1)$ and $h_2(N_2)$ both are analytic in a neighborhood of 0 and the radius of convergence is 1.

In calculus, we know that when $r \neq 0$, $(1+x)^r$ is analytic for $x \in (-1,1)$ and the Taylor series at 0 is

$$(1+x)^r = \sum_{k=0}^{\infty} \frac{r[r-1][r-2]...[r-(k-1)]}{k!} x^k.$$

Denote $\frac{3}{2}\sum_{n=2}^{\infty}\frac{2n}{2n+1}N_1^{2n-2}$ by $u_1(N_1)$ and note that $\frac{3}{2}\sum_{n=2}^{\infty}\frac{2n(-1)^{n+1}}{2n+1}N_1^{2n-2}$ by $u_2(N_2)$,

$$[\frac{3}{2}h_1(N_1)]^{\frac{1}{3}} = [1+u_1]^{\frac{1}{3}}$$

is an analytic function of u_1. Because the composition of two analytic functions is still analytic, then $[\frac{3}{2}h_1(N_1)]^{\frac{1}{3}}$ is analytic for N_1 in a small neighborhood of 0, and so is $[h_1(N_1)]^{\frac{1}{3}}$.

Similarly, $[h_2(N_2)]^{\frac{1}{3}}$ is analytic for N_2 in a small neighborhood of 0.

Because

$$N_1^3 \cdot h_1(N_1) = N_2^3 \cdot h_2(N_2),$$

take the cube roots of both sides,

$$N_1 \cdot [h_1(N_1)]^{\frac{1}{3}} = N_2 \cdot [h_2(N_2)]^{\frac{1}{3}}.$$

By the above argument, both sides are analytic. Let

$$\Gamma(N_1,N_2) = N_1 \cdot [h_1(N_1)]^{\frac{1}{3}} - N_2 \cdot [h_2(N_2)]^{\frac{1}{3}},$$

then $\Gamma(N_1,N_2)$ is analytic with respect to (N_1,N_2) in a small neighborhood of $(0,0)$.

In order to apply the analytic implicit function theorem, we need to check the conditions:

$$\Gamma(0,0) = 0,$$

$$\begin{aligned}\frac{\partial\Gamma}{\partial N_1}(0,0) &= [h_1(N_1)]^{\frac{1}{3}} + N_1 \cdot \frac{1}{3}[h_1(N_1)]^{-\frac{2}{3}} \cdot h_1'(N_1) \mid_{N_1=0} \\ &= (\frac{2}{3})^{\frac{1}{3}} + 0 \cdot \frac{1}{3} \cdot (\frac{2}{3})^{-\frac{2}{3}} \cdot 0 \\ &= (\frac{2}{3})^{\frac{1}{3}} \neq 0,\end{aligned}$$

by Cauchy's analytic implicit function theorem, there exists $r_0 > 0$,and a power series

$$N_1(N_2) = \sum_{i=0}^{\infty} a_i N_2^i$$

such that $N_1(N_2) = \sum_{i=0}^{\infty} a_i N_2^i$ is absolutely convergent for $| N_2 |< r_0$ and $\Gamma(N_1(N_2), N_2) = 0$. That is, N_1 is an analytic function of N_2 when $| N_2 |< r_0$. $\square$

Proof of the Theorem 2.7: Since (2.36) and (2.37) are true if N_1 and N_2 satisfy the system, by Prop. 2.8 and Prop. 2.9, N_1 is an analytic function of N_2 for N_2 close to 0.

By setting,

$$N_1(N_2) = a_0 + a_1 N_1 + a_2 N_2^2 + \ldots$$

we will show that $a_0 = 0$, $a_1 = 1$.

$a_0 = 0$ since $N_1(0) = 0$.

Because

$$\sum_{n=1}^{\infty} \frac{2n}{2n+1} N_1^{2n+1} = \sum_{n=1}^{\infty} \frac{2n(-1)^{n+1}}{2n+1} N_2^{2n+1},$$

$$\frac{2}{3} N_1^3 + \frac{4}{5} N_1^5 + \ldots = \frac{2}{3} N_2^3 - \frac{4}{5} N_2^5 + \ldots.$$

Substitute N_1 by $\sum_{i=0}^{\infty} a_i N_2^i$ and compare the coefficient of N_2^3 on both sides:

$$\frac{2}{3} a_1^3 = \frac{2}{3}$$

then

$$a_1 = 1.$$

Comments: Since $a_1 = 1$, and when $s \to 0$, $N_2 \to 0$,

$$\lim_{s\to 0} \frac{N_1}{N_2} = \lim_{N_2 \to 0} \frac{N_1}{N_2} = 1.$$

At the end of this part, we will use the analytic property of N_1 with respect to N_2 to show that both N_1 and N_2 are analytic functions of s.

Rewrite the differential equation corresponding to N_2':

$$N_2' = (1 + N_2^2)^2 \cdot \frac{N_1^2}{N_1^2 + N_2^2}$$

$$= (1+N_2^2)^2(1-\frac{N_2^2}{N_1^2+N_2^2})$$

$$= (1+N_2^2)^2(1-\frac{1}{1+(\frac{N_1}{N_2})^2})$$

Set $N_1 = \sum_{n=0}^{\infty} b_n N_2^n$. By the claim $\frac{N_1}{N_2}$ approaches to 1 as s get close to zero, and when $s \to 0$, N_2 also approaches to 0, so

$$\frac{N_1}{N_2} = \frac{b_0}{N_2} + b_1 + \sum_{n=2}^{\infty} b_n N_2^{n-1}$$

$$1 = \lim_{s\to 0} \frac{N_1}{N_2} = \lim_{s\to 0}(\frac{b_0}{N_2} + b_1)$$

thus, $b_0 = 0$, $b_1 = 1$ and $\frac{N_1}{N_2} = 1 + \sum_{n=2}^{\infty} b_n N_2^{n-1}$.

Set

$$(\frac{N_1}{N_2})^2 = 1 + \sum_{n=1}^{\infty} d_n N_2^n \equiv 1 + \phi(N_2),$$

where $\phi(N_2) = \sum_{n=1}^{\infty} d_n N_2^n$ is an analytic function of N_2 in a small neighborhood of 0.

$$\frac{1}{1+(\frac{N_1}{N_2})^2} = \frac{1}{2+\phi(N_2)}$$

$$= \frac{1}{2} \cdot \frac{1}{1+\frac{\phi(N_2)}{2}}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (\frac{\phi(N_2)}{2})^n$$

which is obviously an analytic function of N_2 in a neighborhood of 0 with radius of convergence 1.

Therefore, the right hand side of the above differential equation

$$(1+N_2^2)^2(1-\frac{1}{1+(\frac{N_1}{N_2})^2})$$

is also analytic with respect to N_2 in a small neighborhood of 0.

By Cauchy's theorem, $N_2' = (1+N_2^2)^2 \cdot \frac{N_1^2}{N_1^2+N_2^2}$, $N_2(0) = 0$ has a unique analytic solution $N_2 = N_2(s)$ in a small neighborhood of 0.

And because N_1 is an analytic function of N_2, then $N_1 = N_1(s)$ is also analytic when s is close to zero.

Because $\eta_i(s) = C^{-\frac{1}{2}} N_i(C^{\frac{1}{2}} s)$ for $i = 1, 2$, then η_1 and η_2 both are analytic in a neighborhood of 0.

Since $\xi_1 = \frac{F}{-1+C\eta_1^2}$ is a analytic function of η_1 in a neighborhood of 0 and η_1 is also analytic in a neighborhood of 0, then ξ_1 is analytic in a neighborhood of 0. And by the same argument, ξ_2 is also analytic in a neighborhood of 0. $\square$

In fact, we can write down the first few terms of the power series solutions of N_1 and N_2 for the differential system:

$$N_1' = (1-N_1^2)^2 \cdot \frac{N_2^2}{N_1^2+N_2^2},$$

$$N_2' = (1+N_2^2)^2 \cdot \frac{N_1^2}{N_1^2+N_2^2},$$

with the initial conditions

$$N_1(0) = N_2(0) = 0.$$

By an easy calculation, we can get

$$N_1(s) = \frac{1}{2}s - \frac{1}{20}s^3 + \frac{1}{160}s^5 - \frac{29}{36000}s^7 + \dots$$

$$N_2(s) = \frac{1}{2}s + \frac{1}{20}s^3 + \frac{1}{160}s^5 + \frac{29}{36000}s^7 + \dots$$

Then we can write down the solutions $\{\xi_1, \xi_2, \eta_1, \eta_2\}$ for the decoupled system on the energy surface $E = 0$ or $F = 1$:

$$\xi_1(s,C) = -1 - \frac{1}{4}Cs^2 - \frac{1}{80}C^2s^4 + \frac{1}{1600}C^3s^6 + \frac{7}{288000}C^4s^8 + \dots$$

$$\xi_2(s,C) = -1 + \frac{1}{4}Cs^2 - \frac{1}{80}C^2s^4 - \frac{1}{1600}C^3s^6 + \frac{7}{288000}C^4s^8 + \dots$$

$$\eta_1(s,C) = \frac{1}{2}s - \frac{C}{20}s^3 + \frac{C^2}{160}s^5 - \frac{29C^3}{36000}s^7 + \dots$$

$$\eta_2(s,C) = \frac{1}{2}s + \frac{C}{20}s^3 + \frac{C^2}{160}s^5 + \frac{29C^3}{36000}s^7 + \dots$$

2.2.6 Solution on the energy surface $E = h$. To consider the solution on the energy surface $E = h$, define the new Hamiltonian

$$F = \frac{y_1^2 + y_2^2 - h}{\frac{1}{x_1} + \frac{1}{x_2}}$$

The differential equations become:

$$x_1' = \frac{2y_1}{\frac{1}{x_1}+\frac{1}{x_2}}, \qquad x_2' = \frac{2y_2}{\frac{1}{x_1}+\frac{1}{x_2}},$$

$$y_1' = -\frac{y_1^2+y_2^2-h}{(\frac{1}{x_1}+\frac{1}{x_2})^2 x_1^2} = -\frac{x_2}{x_1(x_1+x_2)}F,$$

$$y_2' = -\frac{y_1^2+y_2^2-h}{(\frac{1}{x_1}+\frac{1}{x_2})^2 x_2^2} = -\frac{x_1}{x_2(x_1+x_2)}F.$$

By a similar argument, we have

$$y_1^2 = \frac{F}{x_1}+C, \qquad y_2^2 = \frac{F}{x_2}+C_1.$$

Then $C+C_1 = h$ and

$$C = \frac{x_1y_1^2 - x_2y_2^2 + x_2h}{x_1+x_2},$$

$$C_1 = \frac{x_1y_1^2 - x_2y_2^2 + x_1h}{x_1+x_2}.$$

After the same canonical transformation,the new Hamiltonian is $F = -\frac{\xi_1\xi_2(\eta_1^2+\eta_2^2)-h\xi_1\xi_2\eta_1^2\eta_2^2}{\xi_1\eta_1^2+\xi_2\eta_2^2}$ and the equations become:

$$\eta_1' = (C\eta_1^2-1)^2\frac{\eta_2^2}{\eta_1^2+\eta_2^2-h\eta_1^2\eta_2^2}$$

$$\eta_2' = [(h-C)\eta_2^2-1]^2\frac{\eta_1^2}{\eta_1^2+\eta_2^2-h\eta_1^2\eta_2^2}$$

Let $D = C-\frac{1}{2}h$, then the equations become

$$\eta_1' = [(D+\frac{1}{2}h)\eta_1^2-1]^2\frac{\eta_2^2}{\eta_1^2+\eta_2^2-h\eta_1^2\eta_2^2}$$

$$\eta_2' = [(-D+\frac{1}{2}h)\eta_2^2-1]^2\frac{\eta_1^2}{\eta_1^2+\eta_2^2-h\eta_1^2\eta_2^2}$$

with initial conditions

$$\eta_1(0) = \eta_2(0) = 0.$$

Also $\xi_1 = \frac{F}{(D+\frac{h}{2})\eta_1^2-1}$, $\xi_2 = \frac{F}{(-D+\frac{h}{2})\eta_2^2-1}$.

By a similar argument, we can see that η_1 and η_2 have analytic solutions.

Then at $s = 0$, there are power series expansions for η_1 and η_2. From the differential equations, we can see both η_1 and η_2 are even functions. By using MATLAB, we can get the first few terms:

$$\eta_1(s,D) = \frac{1}{2}s + (-\frac{1}{48}h - \frac{1}{20}D)s^3 + (\frac{1}{960}h^2 + \frac{1}{160}D^2 + \frac{3}{560}hD)s^5 +$$
$$+(-\frac{17}{322560}h^3 - \frac{19}{44800}h^2D - \frac{139}{134400}hD^2 - \frac{29}{36000}D^3)s^7 \ldots$$
$$\eta_2(s,D) = \frac{1}{2}s + (-\frac{1}{48}h + \frac{1}{20}D)s^3 + (\frac{1}{960}h^2 + \frac{1}{160}D^2 - \frac{3}{560}hD)s^5 +$$
$$+(-\frac{17}{322560}h^3 + \frac{19}{44800}h^2D - \frac{139}{134400}hD^2 + \frac{29}{36000}D^3)s^7 \ldots$$

$$\xi_1(s,D) = -1 + \left(-\frac{1}{8}h - \frac{1}{4}D\right)s^2 + \left(-\frac{1}{192}h^2 - \frac{1}{60}hD - \frac{1}{80}D^2\right)s^4 +$$
$$+\left(-\frac{1}{11520}h^3 - \frac{1}{4032}Dh^2 + \frac{11}{67200}D^2h + \frac{1}{1600}D^3\right)s^6 \ldots$$
$$\xi_2(s,D) = -1 + \left(-\frac{1}{8}h + \frac{1}{4}D\right)s^2 + \left(-\frac{1}{192}h^2 + \frac{1}{60}hD - \frac{1}{80}D^2\right)s^4 +$$
$$+\left(-\frac{1}{11520}h^3 + \frac{1}{4032}Dh^2 + \frac{11}{67200}D^2h - \frac{1}{1600}D^3\right)s^6 \ldots$$

Recall the Hamiltonian $F = -\frac{\xi_1\xi_2(\eta_1^2+\eta_2^2)-h\xi_1\xi_2\eta_1^2\eta_2^2}{\xi_1\eta_1^2+\xi_2\eta_2^2}$ is a constant along the solution. By plugging into the power series forms, we can get: $\lim_{s\to 0} -\frac{\xi_1\xi_2(\eta_1^2+\eta_2^2)-h\xi_1\xi_2\eta_1^2\eta_2^2}{\xi_1\eta_1^2+\xi_2\eta_2^2} = 1$, which has nothing to do with the choice of D. So F is always 1 no matter what D is. And then the above one parameter solution family$\{\xi_1(s,D), \xi_2(s,D), \eta_1(s,D), \eta_2(s,D)\}$ are always on the same energy surface.

2.2.7 The initial conditions leading to SBC in the decoupled case. Consider the system:

$$\eta_1' = (-1 + C\eta_1^2)^2 \cdot \frac{\eta_2^2}{\eta_1^2 + \eta_2^2}$$

$$\eta_2' = (1+C\eta_2^2)^2 \cdot \frac{\eta_1^2}{\eta_1^2+\eta_2^2}$$

with initial conditions $\eta_1(0) = \varepsilon_1 < 0, \eta_2(0) = \varepsilon_2 < 0$.

For the above initial value problem, we know the solution $\{\eta_1(s,C), \eta_2(s,C)\}$ exists and it is unique. And also $\eta_1(s,C)$ and $\eta_2(s,C)$ are analytic with respect to s.

Actually, we only need to consider two cases since the case $|\varepsilon_1| < |\varepsilon_2|$ and the case $|\varepsilon_1| > |\varepsilon_2|$ are exactly the same argument.

If $\varepsilon_1 = \varepsilon_2$, from the physical sense, we know that SBC happens if and only if $x_1 = x_2$, that is $C = 0$.

If $|\varepsilon_1| < |\varepsilon_2|$, We will show there exists some $C_0 = C_0(\varepsilon_1, \varepsilon_2)$ such that SBC occurs.

Lemma 2.10. *Assume $|\varepsilon_1| < |\varepsilon_2|$, $C \geq 0$ and $-1+C\varepsilon_1^2 < 0$. There exist unique s_1 and s_2, such that*

$$\eta_1(s_1, C) = \eta_2(s_2, C) = 0.$$

Proof. Assume $C \geq 0$, $-1+C\varepsilon_1^2 < 0$. From the equations, we can see that $\eta_1'(0) > 0$, $\eta_2'(0) > 0$.

As time s increases, $-1+C\eta_1^2 < -1+C\varepsilon_1^2 < 0$. Then $\eta_1' > 0$ and $\eta_2' > 0$ whenever $\eta_1 \neq 0$ and $\eta_2 \neq 0$.

First, we will show the existence of s_1.

If the claim is not true, then for any s, $\eta_1(s,C) < 0$, and

$$\lim_{s\to\infty} \eta_1'(s,C) = \lim_{s\to\infty} \eta_1(s,C) = 0.$$

On the other hand, from the equations,

$$\lim_{s\to\infty} \frac{\eta_1'(s,C)}{\eta_2'(s,C)} = \lim_{s\to\infty} \frac{(-1+C\eta_1^2)^2}{(1+C\eta_2^2)^2} \lim_{s\to\infty} \frac{\eta_2^2(s,C)}{\eta_1^2(s,C)} = \lim_{s\to\infty} \frac{\eta_2^2(s,C)}{\eta_1^2(s,C)}$$

$$\lim_{s\to\infty} [\eta_1'(s,C) + \eta_2'(s,C)] = \lim_{s\to\infty} \frac{\eta_2^2}{\eta_1^2+\eta_2^2} + \lim_{s\to\infty} \frac{\eta_1^2}{\eta_1^2+\eta_2^2} = 1.$$

Then $\lim_{s\to\infty} \eta_1'(s,C) = 0$, $\lim_{s\to\infty} \eta_2'(s,C) = 1$ and $\lim_{s\to\infty} \frac{\eta_1'(s,C)}{\eta_2'(s,C)} = 0$. Therefore

$$\lim_{s\to\infty} \frac{\eta_2^2(s,C)}{\eta_1^2(s,C)} = 0,$$

and

$$\lim_{s\to\infty}\eta_2^2(s,C)=0.$$

By L'Hospital's rule,

$$\lim_{s\to\infty}\frac{\eta_1^3(s,C)}{\eta_2^3(s,C)}=\lim_{s\to\infty}\frac{\eta_1^2(s,C)\eta_1'(s,C)}{\eta_2^2(s,C)\eta_2'(s,C)}=\lim_{s\to\infty}\frac{\eta_1'(s,C)}{\eta_2'(s,C)}\Bigg/\lim_{s\to\infty}\frac{\eta_2^2(s,C)}{\eta_1^2(s,C)}=1.$$

So

$$1=\lim_{s\to\infty}\frac{\eta_1(s,C)}{\eta_2(s,C)}=\lim_{s\to\infty}\frac{\eta_1'(s,C)}{\eta_2'(s,C)}=0.$$

Contradiction!

Therefore, there exists some s_1 such that $\eta_1(s_1,C)=0$.

By the same argument, we can show that there also exists another s_2 such that $\eta_2(s_2,C)=0$.

Uniqueness can be seen from the equations. Since η_1 and η_2 are nondecreasing, and also if $s_1\neq s_2$,

$$\eta_1'(s_1,C)=1>0, \qquad \eta_2'(s_2,C)=1>0;$$

if $s_1=s_2$, by the L'Hospital's rule,

$$\lim_{s\to s_1}\frac{\eta_1^3}{\eta_2^3}=\lim_{s\to s_1}\frac{\eta_1^2\eta_1'}{\eta_2^2\eta_2'}=\lim_{s\to s_1}\frac{\eta_1^2}{\eta_2^2}\cdot\frac{(-1+C\eta_1^2)^2}{(1+C\eta_2^2)^2}\cdot\frac{\eta_2^2}{\eta_1^2}=1$$

Then

$$\lim_{s\to s_1}\frac{\eta_1}{\eta_2}=\lim_{s\to s_1}\frac{\eta_1'}{\eta_2'}=1.$$

Because $\lim_{s\to s_1}\eta_1'(s_1,C)+\eta_2'(s_1,C)=1$, therefore

$$\eta_1'(s_1,C)=\eta_2'(s_1,C)=\frac{1}{2}>0.$$

In both cases, $\eta_1'(s_1,C)>0$ and $\eta_2'(s_2,C)>0$, which means $\eta_1>0$, $\eta_2>0$ for s big enough. Thus, there exist unique s_1 and s_2 such that $\eta_1(s_1,C)=\eta_2(s_2,C)=0$. □

Lemma 2.11. *With the same assumption as the previous lemma, there exists a unique constant $C_0=$*

$C_0(\varepsilon_1, \varepsilon_2)$, such that $s_1 = s_2$.

Proof. From Lemma 2.10, we know that there exist $s_1(C_f, \varepsilon_1, \varepsilon_2)$ and $s_2(C_f, \varepsilon_1, \varepsilon_2)$ for any fixed C_f such that

$$\eta_1(s_1, C_f, \varepsilon_1, \varepsilon_2) = 0$$

$$\eta_2(s_2, C_f, \varepsilon_1, \varepsilon_2) = 0.$$

And

$$\eta_1'(s_1, C) > 0$$

$$\eta_2'(s_2, C) > 0.$$

By the implicit function theorem, $s_1 = s_1(C, \varepsilon_1, \varepsilon_2)$ and $s_2 = s_2(C, \varepsilon_1, \varepsilon_2)$ exists and also s_1, s_2 are continuously differentiable with respect to C in a small neighborhood of C_f.

Consider $s_1(C, \varepsilon_1, \varepsilon_2) - s_2(C, \varepsilon_1, \varepsilon_2)$, which is a continuous function of C if we fix ε_1 and ε_2.

We know $|\varepsilon_1| < |\varepsilon_2|$, which means the momentum of the 1st body $|y_1|$ is greater than the momentum of the 4th body $|y_2|$.

Note that at the initial time, $C = \frac{x_1 y_1^2 - x_2 y_2^2}{x_1 + x_2} = \frac{x_{10}\varepsilon_1^{-2} - x_{20}\varepsilon_2^{-2}}{x_{10} + x_{20}}$, where x_{10} represents the distance between the 1st and 2nd bodies at the initial time and x_{20} represents the distance between the 3rd and 4th bodies at the initial time. If it is a SBC, $x_{10} > x_{20}$.

If $C = 0$, that is $x_{10}\varepsilon_1^{-2} = x_{20}\varepsilon_2^{-2}$, then $x_{10} < x_{20}$. So for $C = 0$, the time s_1 must be less than s_2.

$$\therefore \qquad s_1(C, \varepsilon_1, \varepsilon_2) - s_2(C, \varepsilon_1, \varepsilon_2) < 0, \qquad \text{when} \quad C = 0.$$

On the other hand, we can choose $x_{10} \gg x_{20}$ and they satisfy $\frac{1}{x_{10}} + \frac{1}{x_{20}} = \varepsilon_1^{-2} + \varepsilon_2^{-2}$, such that the collision time for the first two bodies s_1 is greater than the collision time for the last two bodies s_2. In that situation, $C = \frac{x_{10}\varepsilon_1^{-2} - x_{20}\varepsilon_2^{-2}}{x_{10} + x_{20}}$ will be big enough. So we can choose some C big enough so that $s_1(C, \varepsilon_1, \varepsilon_2) - s_2(C, \varepsilon_1, \varepsilon_2) > 0$.

By the continuity of the function $s_1(C, \varepsilon_1, \varepsilon_2) - s_2(C, \varepsilon_1, \varepsilon_2)$, there exists some C_0, such that $s_1(C, \varepsilon_1, \varepsilon_2) = s_2(C, \varepsilon_1, \varepsilon_2) = s_0$.

To prove the uniqueness, let's consider

$$C_1 - C_0 = \frac{x_{11}\varepsilon_1^{-2} - x_{21}\varepsilon_2^{-2}}{x_{11}+x_{21}} - \frac{x_{10}\varepsilon_1^{-2} - x_{20}\varepsilon_2^{-2}}{x_{10}+x_{20}},$$

where $\{x_{11}, x_{21}, \varepsilon_1^{-1}, \varepsilon_2^{-1}\} = \{x_1, x_2, y_1, y_2\}$ is another set of initial conditions.

$$C_1 - C_0 = \frac{(x_{11}\varepsilon_1^{-2} - x_{21}\varepsilon_2^{-2})(x_{10}+x_{20}) - (x_{10}\varepsilon_1^{-2} - x_{20}\varepsilon_2^{-2})(x_{11}+x_{21})}{(x_{11}+x_{21})(x_{10}+x_{20})}$$
$$= \frac{(\varepsilon_1^{-2}+\varepsilon_2^{-2})(x_{11}x_{20} - x_{10}x_{21})}{(x_{11}+x_{21})(x_{10}+x_{20})}.$$

Note that $\frac{1}{x_{11}} + \frac{1}{x_{21}} = \frac{1}{x_{10}} + \frac{1}{x_{20}} = \varepsilon_1^{-2} + \varepsilon_2^{-2}$, so
if $x_{21} = x_{20}$, then $x_{11} = x_{10}$, which means $C_1 = C_0$;
if $x_{21} > x_{20}$, then $x_{11} < x_{10}$, which means $C_1 < C_0$;
if $x_{21} < x_{20}$, then $x_{11} > x_{10}$, which means $C_1 > C_0$.
These are the only three possible cases for the relationship between the two sets of initial conditions. So when $C_1 > C_0$, we have $x_{21} < x_{20}$ and $x_{11} > x_{10}$, hence $s_1 > s_2$. Similarly, when $C_1 < C_0$, $s_1 < s_2$. Therefore, given $|\varepsilon_1| < |\varepsilon_2|$, there exists only one C_0 leading to a SBC.

□

So far, we have proven that for any given $\varepsilon_1 < 0$, $\varepsilon_2 < 0$, there always exist a unique $C_0 = C_0(\varepsilon_1, \varepsilon_2)$ and $s_0 = s_0(C_0)$, such that

$$\eta_1(s_0) = \eta_2(s_0) = 0.$$

Lemma 2.12. *If $C_0 > 0$,*

$$s_0 = \frac{\varepsilon_1}{-1 + C_0\varepsilon_1^2} + \frac{\varepsilon_2}{-1 - C_0\varepsilon_2^2},$$

$$C_0^{1/2} s_0 = -\tanh^{-1}(C_0^{1/2}\varepsilon_1) - \tan^{-1}(C_0^{1/2}\varepsilon_2).$$

Proof. From section 4(d), we have

$$\xi_1 = \frac{1}{-1 + C\eta_1^2}, \qquad \xi_2 = \frac{1}{-1 - C\eta_2^2}$$

$$(\eta_1\xi_1 + \eta_2\xi_2)' = -1$$

Choose $C = C_0$ and integrate the above equation from 0 to s_0:

$$(\eta_1\xi_1 + \eta_2\xi_2)\mid_0^{s_0} = -s_0$$

Therefore,

$$\frac{\varepsilon_1}{-1 + C_0\varepsilon_1^2} + \frac{\varepsilon_2}{-1 - C_0\varepsilon_2^2} = s_0$$

On the other hand,

$$\tanh^{-1}(C^{1/2}\eta_1) + \tan^{-1}(C^{1/2}\eta_2) = C^{1/2}s + DD$$

where DD is a constant determined by the initial conditions.

Let $s = 0$ and $C = C_0$,

$$\tanh^{-1}(C^{1/2}\eta_1) + \tan^{-1}(C^{1/2}\eta_2) = C_0^{1/2}s + \tanh^{-1}(C^{1/2}\varepsilon_1) + \tan^{-1}(C^{1/2}\varepsilon_2)$$

When $s = s_0$, $\eta_1(s_0) = \eta_2(s_0) = 0$, then

$$0 = C_0^{1/2}s_0 + \tanh^{-1}(C^{1/2}\varepsilon_1) + \tan^{-1}(C^{1/2}\varepsilon_2)$$

Therefore,

$$C_0^{1/2}s_0 = -\tanh^{-1}(C_0^{1/2}\varepsilon_1) - \tan^{-1}(C_0^{1/2}\varepsilon_2).$$

□

In fact, from Lemma 2.12, we can solve for C_0 and s_0 in terms of ε_1 and ε_2.

$$C_0^{1/2}\left[\frac{-C_0\varepsilon_1\varepsilon_2(\varepsilon_1 - \varepsilon_2) + (\varepsilon_1 + \varepsilon_2)}{C_0^2\varepsilon_1\varepsilon_2 + C_0(\varepsilon_1 - \varepsilon_2) - 1}\right] = -\tanh^{-1}(C_0^{1/2}\varepsilon_1) - \tan^{-1}(C_0^{1/2}\varepsilon_2).$$

Let $C_0^{1/2}\varepsilon_1 = d_1$, $C_0^{1/2}\varepsilon_2 = d_2$, and $S_0 = C_0^{1/2}s_0$, then Lemma 2.12 becomes

$$S_0 = \frac{d_1}{-1 + d_1^2} + \frac{d_2}{-1 - d_2^2},$$

$$S_0 = -\tanh^{-1} d_1 - \tan^{-1} d_2.$$

Recall that the solution for

$$N_1' = (-1+N_1^2)^2 \frac{N_2^2}{N_1^2+N_2^2},$$

$$N_2' = (1+N_2^2)^2 \frac{N_1^2}{N_1^2+N_2^2}$$

with $N_1(0) = N_2(0) = 0$ is

$$N_1(s) = \frac{1}{2}s - \frac{1}{20}s^3 + \frac{1}{160}s^5 - \frac{29}{36000}s^7 + \ldots$$

$$N_2(s) = \frac{1}{2}s + \frac{1}{20}s^3 + \frac{1}{160}s^5 + \frac{29}{36000}s^7 + \ldots$$

So if the initial conditions are changed to $N_1(0) = d_1$, $N_2(0) = d_2$ such that $N_1(S_0) = N_2(S_0) = 0$, the new solution will be $\{N_1(s-S_0), N_2(s-S_0)\}$. Then when $s = 0$,

$$d_1 = N_1(-S_0), \qquad d_2 = N_2(-S_0).$$

Perturb the initial condition in the following way: $N_1(0) = d_1$, $N_2(0) = d_2 + \varepsilon$, where ε is small enough. Assume the solution under this perturbed initial condition is $\{\widetilde{N}_1(s,\varepsilon), \widetilde{N}_2(s,\varepsilon)\}$.
When $\varepsilon = 0$, the solution for the initial value problem is

$$\widetilde{N}_1(s,0) = N_1(s-S_0), \qquad \widetilde{N}_2(s,0) = N_2(s-S_0).$$

Consider the two identities under the new initial condition:

$$\frac{\widetilde{N}_1}{1-\widetilde{N}_1^2} + \frac{\widetilde{N}_2}{1+\widetilde{N}_2^2} = s + \frac{d_1}{1-d_1^2} + \frac{d_2+\varepsilon}{1+(d_2+\varepsilon)^2}$$

$$\tanh^{-1}\widetilde{N}_1 + \tan^{-1}\widetilde{N}_2 = s + \tanh^{-1} d_1 + \tan^{-1}(d_2+\varepsilon)$$

Without loss of generality, let $\varepsilon > 0$. Then the collision of the second pair happens first, which means the path of (η_1, η_2) intersects with the η_1 axis first at $(-a_1, 0)$ as in the picture.

Therefore,

$$\frac{-a_1}{1-a_1^2} = s_1 + \frac{d_1}{1-d_1^2} + \frac{d_2+\varepsilon}{1+(d_2+\varepsilon)^2}$$

$$-\tanh^{-1} a_1 = s_1 + \tanh^{-1} d_1 + \tan^{-1}(d_2+\varepsilon),$$

so

$$\frac{-a_1}{1-a_1^2} + \tanh^{-1} a_1 = \frac{d_1}{1-d_1^2} + \frac{d_2+\varepsilon}{1+(d_2+\varepsilon)^2} - \tanh^{-1} d_1 - \tan^{-1}(d_2+\varepsilon).$$

On the other hand,

$$\frac{d_1}{1-d_1^2} + \frac{d_2}{1+d_2^2} - \tanh^{-1} d_1 - \tan^{-1} d_2 = 0,$$

so

$$\frac{-a_1}{1-a_1^2} + \tanh^{-1} a_1 = \frac{d_2+\varepsilon}{1+(d_2+\varepsilon)^2} - \tan^{-1}(d_2+\varepsilon) - \frac{d_2}{1+d_2^2} + \tan^{-1} d_2$$

Expand both sides in Taylor series:

$$-\frac{2}{3}a_1^3 + O(a_1^5) = -\varepsilon\frac{2d_2^2}{(1+d_2^2)^2} + O(\varepsilon^2)$$

Hence for small enough ε close to 0,

$$a_1 = O(\varepsilon^{1/3}).$$

Actually, for any time s, $\widetilde{N}_1$ and $\widetilde{N}_2$ always satisfy the following identities:

$$\frac{\widetilde{N}_1}{1-\widetilde{N}_1^2} + \frac{\widetilde{N}_2}{1+\widetilde{N}_2^2} - \tanh^{-1}\widetilde{N}_1 - \tan^{-1}\widetilde{N}_2 = \frac{d_2+\varepsilon}{1+(d_2+\varepsilon)^2} - \tan^{-1}(d_2+\varepsilon) - \frac{d_2}{1+d_2^2} + \tan^{-1} d_2$$

when $\varepsilon = 0$:

$$\frac{\widetilde{N}_1(s,0)}{1-\widetilde{N}_1^2(s,0)} + \frac{\widetilde{N}_2(s,0)}{1+\widetilde{N}_2^2(s,0)} - \tanh^{-1}\widetilde{N}_1(s,0) - \tan^{-1}\widetilde{N}_2(s,0) = 0$$

For any s_1, choose s_2, such that $\widetilde{N}_1(s_1,0) = \widetilde{N}_1(s_2,\varepsilon)$. Consider the difference of the above identities:

$$\frac{\widetilde{N}_2(s_2,\varepsilon)}{1+\widetilde{N}_2^2(s_2,\varepsilon)} - \tan^{-1}\widetilde{N}_2(s_2,\varepsilon) - \frac{\widetilde{N}_2(s,0)}{1+\widetilde{N}_2^2(s,0)} + \tan^{-1}\widetilde{N}_2(s,0)$$

$$= \frac{d_2+\varepsilon}{1+(d_2+\varepsilon)^2} - \tan^{-1}(d_2+\varepsilon) - \frac{d_2}{1+d_2^2} + \tan^{-1} d_2$$

Let $f(x) = \frac{x}{1+x^2} - \tan^{-1} x$ and $\widetilde{N}_2(s_2,\varepsilon) = \widetilde{N}_2(s_1,0) + \Delta_s$, then

$$f(\widetilde{N}_2(s_1,0)+\Delta_s) - f(\widetilde{N}_2(s_1,0)) = f(d_2+\varepsilon) - f(d_2)$$

or

$$f(N_2(s_1-S_0)+\Delta_s) - f(N_2(s_1-S_0)) = f(d_2+\varepsilon) - f(d_2)$$

If $\widetilde{N}_1(s_1,0) = N_1(s_1-S_0) \neq 0$, then $N_2(s_1-S_0) \neq 0$ and

$$f(N_2(s_1-S_0)+\Delta_s) - f(N_2(s_1-S_0)) = -\Delta_s \frac{2N_2(s_1-S_0)^2}{[1+N_2(s_1-S_0)^2]^2} + O(\Delta_s^2)$$

$$f(d_2+\varepsilon) - f(d_2) = -\varepsilon \frac{2d_2^2}{(1+d_2^2)^2} + O(\varepsilon^2)$$

where $\frac{2N_2(s_1-S_0)^2}{[1+N_2(s_1-S_0)^2]^2} \neq 0$ and $\frac{2d_2^2}{(1+d_2^2)^2} \neq 0$. By the implicit function theorem, Δ_s is an analytic function of ε for $s_1 \neq S_0$, and $\Delta_s = O(\varepsilon)$.

But when $s_1 = S_0$, denote Δ_s by Δ,

$$f(N_2(0)+\Delta) - f(N_2(0)) = -\frac{4}{3}\Delta^3 + O(\Delta^5)$$

$$\therefore \qquad -\frac{4}{3}\Delta^3 + O(\Delta^5) = -\varepsilon \frac{2d_2^2}{(1+d_2^2)^2} + O(\varepsilon^2)$$

Therefore when $s_1 = S_0$, $\Delta = O(\varepsilon^{1/3})$ for ε close to 0 and small enough.

From the above argument, we can see that only at the two points $(-a_1,0)$ and $(0,\Delta)$, the solution $\widetilde{N}_1 = \widetilde{N}_1(\widetilde{N}_2)$ does not approach $N_1 = N_1(N_2)$ analytically.

Consider the initial value problem

$$N_1' = (-1+N_1^2)^2 \frac{N_2^2}{N_1^2+N_2^2}$$

$$N_2' = (1+N_2^2)^2 \frac{N_1^2}{N_1^2+N_2^2}$$

$$N_1(0) = 0, \qquad N_2(0) = \Delta.$$

By the uniqueness of the solution of the initial value problem, we may denote the solution of the above system as $\{\widetilde{N}_1(s,\Delta), \widetilde{N}_2(s,\Delta)\}$.

Proposition 2.13. *For any given s,*

$$\lim_{\Delta \to 0} \widetilde{N}_1(s,\Delta) = \widetilde{N}_1(s,0),$$

$$\lim_{\Delta \to 0} \widetilde{N}_2(s,\Delta) = \widetilde{N}_2(s,0).$$

Proof. We will show $\lim_{\Delta\to 0} \widetilde{N}_1(s,\Delta) = \widetilde{N}_1(s,0)$, and the limit for $\widetilde{N}_2(s,\Delta)$ follows exactly the same argument.

Consider the two identities

$$\frac{\widetilde{N}_1(s,\Delta)}{1-\widetilde{N}_1^2(s,\Delta)} + \frac{\widetilde{N}_2(s,\Delta)}{1+\widetilde{N}_2^2(s,\Delta)} = s + \frac{\Delta}{1+\Delta^2}$$

$$\tanh^{-1} \widetilde{N}_1(s,\Delta) + \tan^{-1} \widetilde{N}_2(s,\Delta) = s + \tan^{-1} \Delta$$

Let $g(x,\Delta) = \frac{x}{1-x^2} + \frac{\tan(s+\tan^{-1}\Delta - \tanh^{-1}x)}{1+\tan^2(s+\tan^{-1}\Delta-\tanh^{-1}x)}$, so $\lim_{\Delta\to 0} g(x,\Delta) = g(x,0)$, and $g(x,\Delta)$ is analytic for $-1 < x < 1$, $-1 < \Delta < 1$. Also, the derivative

$$\frac{dg(x,0)}{dx} = 2\frac{x^2}{(1-x^2)^2} + 2\frac{\tan^2(s-\tanh^{-1}x)}{(1+\tan^2(s-\tanh^{-1}x)^2)(1-x^2)}$$

$$\frac{dg(x,0)}{dx}\Big|_{x=0} = \frac{2\tan^2 s}{1+\tan^2 s}$$

then for any given $s \neq 0$, $\frac{dg(x,0)}{dx}\big|_{x=0} > 0$.

Therefore,when $s \neq 0$, by the inverse function theorem, $g(x,0)$ has a uniquely analytic inverse $g^{-1}(x)$ for x close to 0, such that $g^{-1}(g(x,0)) = x$.

From the above two identities, for fixed $s \neq 0$, since g is continuous,

$$s = \lim_{\Delta\to 0} g(\widetilde{N}_1(s,\Delta),\Delta) = g(\lim_{\Delta\to 0}\widetilde{N}_1(s,\Delta), 0)$$

On the other hand,

$$s = g(\widetilde{N}_1(s,0),0)$$

Then

$$g(\lim_{\Delta\to 0}\widetilde{N}_1(s,\Delta),0) = g(\widetilde{N}_1(s,0),0)$$

Applying g^{-1} to both sides:

$$g^{-1}(g(\lim_{\Delta\to 0}\widetilde{N}_1(s,\Delta),0)) = \lim_{\Delta\to 0}\widetilde{N}_1(s,\Delta) = g^{-1}(g(\widetilde{N}_1(s,0),0)) = \widetilde{N}_1(s,0).$$

Therefore, $\lim_{\Delta\to 0}\widetilde{N}_1(s,\Delta) = \widetilde{N}_1(s,0)$ for any fixed $s \neq 0$.

If $s = 0$, $\widetilde{N}_1(0,\Delta) = 0 = \widetilde{N}_1(0,0)$, so

$$\lim_{\Delta\to 0}\widetilde{N}_1(s,\Delta) = \widetilde{N}_1(s,0)$$

is true for any given s. $\square$

Proposition 2.14. *For $s \in [-1,1]$, $\widetilde{N}_i(s,\Delta)$ approaches $\widetilde{N}_i(s,0)(i = 1,2)$ uniformly as Δ goes to 0.*

Proof. Consider the derivative of $g(x,s,\Delta) = \frac{x}{1-x^2} + \frac{\tan(s+\tan^{-1}\Delta-\tanh^{-1}x)}{1+\tan^2(s+\tan^{-1}\Delta-\tanh^{-1}x)}$.

From the previous argument, for $s \neq 0$, g has a unique inverse. At $(x,s,\Delta) = (0,0,0)$,

$$\frac{\partial g}{\partial x}(0,0,0) = \frac{\partial^2 g}{\partial x^2}(0,0,0) = 0, \qquad \text{but} \qquad \frac{\partial^3 g}{\partial x^3}(0,0,0) = 8 \neq 0.$$

By the extended implicit theorem, $y = g(x,s,\Delta)$ has a unique inverse g^{-1} on a small neighborhood of $(0,0,0)$. So for $s \in [-1,1]$, and Δ close to 0, the inverse g^{-1} always exists.

On the other hand,

$$\lim_{\Delta\to 0} g(\widetilde{N}_i(s,\Delta),s,\Delta) = g(\widetilde{N}_1(s,0),0) \quad \text{uniformly.}$$

Then since g^{-1} is analytic, there exists M, such that

$$\mid \widetilde{N}_i(s,\Delta) - \widetilde{N}_i(s,0) \mid = \mid g^{-1}\big(g(\widetilde{N}_i(s,\Delta),s,\Delta)\big) - g^{-1}\big(g(\widetilde{N}_1(s,0),0)\big) \mid$$

$$\leq M \mid g(\widetilde{N}_i(s,\Delta),s,\Delta) - g(\widetilde{N}_1(s,0),0) \mid .$$

Therefore, $\lim_{\Delta\to 0}\widetilde{N}_i(s,\Delta) = \widetilde{N}_i(s,0)(i = 1,2)$ uniformly.

$\square$

Proposition 2.15. *For any fixed $s \in [-1,1]$ and $s \neq 0$,*

$$\lim_{\Delta \to 0} \widetilde{N}_i'(s,\Delta) = \widetilde{N}_i'(s,0),$$

but the above limit is not uniform with respect to s.

Proof. We will show the proof for $\widetilde{N}_1'(s,\Delta)$, and $\widetilde{N}_2'(s,\Delta)$ follows the same argument. Consider the differential equation

$$\widetilde{N}_1'(s,\Delta) = (-1+\widetilde{N}_1^2(s,\Delta))^2 \frac{\widetilde{N}_2^2(s,\Delta)}{\widetilde{N}_1^2(s,\Delta)+\widetilde{N}_2^2(s,\Delta)}$$

For any fixed $s \neq 0$, $\widetilde{N}_1^2(s,0)+\widetilde{N}_2^2(s,0) \neq 0$. Then

$$\lim_{\Delta \to 0} \widetilde{N}_1'(s,\Delta) = \lim_{\Delta \to 0} (-1+\widetilde{N}_1^2(s,\Delta))^2 \frac{\widetilde{N}_2^2(s,\Delta)}{\widetilde{N}_1^2(s,\Delta)+\widetilde{N}_2^2(s,\Delta)}$$

$$= (-1+\widetilde{N}_1^2(s,0))^2 \frac{\widetilde{N}_2^2(s,0)}{\widetilde{N}_1^2(s,0)+\widetilde{N}_2^2(s,0)} = \widetilde{N}_1'(s,0).$$

On the other hand, choose a nonzero sequence $\{\Delta_n\}$ such that $\lim_{n\to\infty} \Delta_n = 0$ and $\Delta_n \neq 0$ for any n. And choose a sequence of s: $\{s_n\}$ such that $s_n = \Delta_n^2$.
Since $\lim_{n\to\infty} \widetilde{N}_1'(s_n,0) = \lim_{s\to 0} \widetilde{N}_1'(s,0) = \frac{1}{2}$, there exists M_1, such that

$$|\, \widetilde{N}_1'(s_n,0) \,| < \frac{2}{3} \qquad \text{for any } n > M_1.$$

On the other hand, we consider the initial value problem

$$N_1' = (-1+N_1^2)^2 \frac{N_2^2}{N_1^2+N_2^2}$$

$$N_2' = (1+N_2^2)^2 \frac{N_1^2}{N_1^2+N_2^2}$$

$$N_1(0) = 0, \qquad N_2(0) = \Delta.$$

Let $\widehat{N}_1(s,\Delta) = \frac{1}{\Delta}\widetilde{N}_1(\Delta s,\Delta)$, $\widehat{N}_2(s,\Delta) = \frac{1}{\Delta}\widetilde{N}_2(\Delta s,\Delta)$, then the initial value problem becomes

$$\widehat{N}_1' = (-1+\Delta^2\widehat{N}_1^2)^2 \frac{\widehat{N}_2^2}{\widehat{N}_1^2+\widehat{N}_2^2}$$

$$\widehat{N}_2' = (1+\Delta^2\widehat{N}_2^2)^2 \frac{\widehat{N}_1^2}{\widehat{N}_1^2+\widehat{N}_2^2},$$

$$\widehat{N}_1(0) = 0, \qquad \widehat{N}_2(0) = 1.$$

Since the right hand side of the equations are analytic for $(\widehat{N}_1, \widehat{N}_2)$ close to $(0,1)$, the above initial value problem has an analytic solution for s close to 0.The series solutions are as following:

$$\widehat{N}_1(s,\Delta) = s - \frac{s^3}{3}(2\Delta^2+1) + \frac{s^5}{15}(11\Delta^4+14\Delta^2+5)+...$$

$$\widehat{N}_2(s,\Delta) = 1 + \frac{s^3}{3}(\Delta^2+1)^2 - \frac{s^5}{15}(\Delta^2+1)^2(4\Delta^2+5)+...$$

Since $s_n = \Delta_n^2$,

$$\widetilde{N}_1'(s_n,\Delta_n) = \widehat{N}_1'(\frac{s_n}{\Delta_n},\Delta_n) = \widehat{N}_1'(\Delta_n,\Delta_n),$$

$$\lim_{n\to\infty} \widehat{N}_1'(\Delta_n,\Delta_n) = 1,$$

there exist another constant M_2, such that

$$\widetilde{N}_1'(s_n,\Delta_n) = \widehat{N}_1'(\Delta_n,\Delta_n) > \frac{5}{6} \qquad \text{for any } n > M_2.$$

Let $M = max(M_1, M_2)$, then

$$|\ \widetilde{N}_1'(s_n,\Delta_n) - \widetilde{N}_1'(s_n,0)\ | \geq |\ \widetilde{N}_1'(s_n,\Delta_n)\ | - |\ \widetilde{N}_1'(s_n,0)\ | > \frac{5}{6} - \frac{2}{3} = \frac{1}{6}$$

for any $n > M$.

Therefore, $\lim_{\Delta\to 0}\widetilde{N}_i'(s,\Delta) = \widetilde{N}_i'(s,0)$ is not uniform with respect to s. □

Proposition 2.16. *For any $1 > a > 0$, $\widetilde{N}_i(\Delta,s)$ converges to $\widetilde{N}_i(0,s)$ analytically as Δ approach 0 and uniformly for $s \in [-1,-a]\cup[a,1]$.*

Proof. From the equation, we see that when $s \in [-1,-a]\cup[a,1]$, $\widetilde{N}_1^2(\Delta,s)+\widetilde{N}_2^2(\Delta,s) \neq 0 (i=1,2)$ is always true for any Δ. Then the left right hand side of the differential equations are analytic with respect to both s and Δ, thus the solution $\{\widetilde{N}_1(\Delta,s), \widetilde{N}_2(\Delta,s)\}$ is analytic with respect to both s and Δ. Therefore, $\widetilde{N}_i(\Delta,s)$ converges to $\widetilde{N}_i(0,s)$ analytically as Δ approaches 0 and uniformly for $s \in [-1,-a]\cup[a,1]$.

$\square$

Remarks:

1. Fixing s, $\widetilde{N}_i(s,\Delta)(i=1,2)$ is a continuous function with respect to Δ.

2. The first derivative at $s=0$: $\widetilde{N}_i'(0,\Delta)(i=1,2)$ is not continuous with respect to Δ. Actually, from the differential equations,

$$\widetilde{N}_1'(0,\Delta) = (-1+\widetilde{N}_1^2)^2 \frac{\widetilde{N}_2^2}{\widetilde{N}_1^2+\widetilde{N}_2^2} \mid_{s=0} = 1,$$

$$\widetilde{N}_2'(0,\Delta) = (1+\widetilde{N}_2^2)^2 \frac{\widetilde{N}_1^2}{\widetilde{N}_1^2+\widetilde{N}_2^2} \mid_{s=0} = 0.$$

But when $\Delta = 0$, we know $\widetilde{N}_1'(0,0) = \widetilde{N}_2'(0,0) = \frac{1}{2}$.

3. $\frac{\partial \widetilde{N}_i}{\partial \Delta}(i=1,2)$ is not continuous at $(0,0)$.

Differentiate the two identities with respect to Δ and evaluate at $\Delta = 0$:

$$\frac{1+\widetilde{N}_1^2}{(1-\widetilde{N}_1^2)^2} \cdot \frac{\partial \widetilde{N}_1}{\partial \Delta} + \frac{1-\widetilde{N}_2^2}{(1+\widetilde{N}_2^2)^2} \cdot \frac{\partial \widetilde{N}_2}{\partial \Delta} = 1$$

$$\frac{1}{1-\widetilde{N}_1^2} \cdot \frac{\partial \widetilde{N}_1}{\partial \Delta} + \frac{1}{1-\widetilde{N}_2^2} \cdot \frac{\partial \widetilde{N}_2}{\partial \Delta} = 1$$

Then

$$\frac{\partial \widetilde{N}_1}{\partial \Delta} = \frac{1}{1-\widetilde{N}_1^2} \cdot \frac{1-\frac{1-\widetilde{N}_2^2}{1+\widetilde{N}_2^2}}{\frac{1+\widetilde{N}_1^2}{1-\widetilde{N}_1^2} - \frac{1-\widetilde{N}_2^2}{1+\widetilde{N}_2^2}}$$

When $\Delta = s = 0$, $\widetilde{N}_1 = \frac{s}{2} - \frac{s^3}{20} + ...$ and $\widetilde{N}_2 = \frac{s}{2} + \frac{s^3}{20} + ...$,so

$$\lim_{s\to 0} \frac{\partial \widetilde{N}_1}{\partial \Delta} = \frac{1}{2}.$$

But at $s=0$, $\widetilde{N}_1 = 0$, which has nothing to do with Δ. So $\frac{\partial \widetilde{N}_1}{\partial \Delta} = 0$. Therefore, it is not continuous at $(0,0)$.

Similarly, we can see that $\lim_{s\to 0} \frac{\partial \widetilde{N}_2}{\partial \Delta} = \frac{1}{2}$, but at $s=0$, $\frac{\partial \widetilde{N}_2}{\partial \Delta} = 1$. So $\frac{\partial \widetilde{N}_2}{\partial \Delta}$ is not continuous at $(0,0)$, either.

4. Another view of this problem is to think about the time difference of the two collisions. Assume $(N_1,N_2) = (d_1,d_2)$ would lead to simultaneous binary collision. Then at the initial condition $(N_1,N_2) = (d_1, d_2+\varepsilon)$, there will be a time difference between the two collisions as we can see from the picture below.

The intersection with N_1 axis $(-a_1, 0)$ represents the collision of the second pair and the intersection with N_2 axis $(0, \Delta)$ represents the collision of the first pair. And the time difference is $\frac{a_1}{1-a_1^2} + \frac{\Delta}{1+\Delta^2} = O(\varepsilon^{\frac{1}{3}})$. When $(N_1, N_2) = (-a_1, 0)$, $(N_1', N_2') = (0, 1)$; when $(N_1, N_2) = (0, \Delta)$, $(N_1', N_2') = (1, 0)$.

As $\Delta \to 0$, a_1 will go to 0, too. Then those two points become one and the two derivatives will change to the average: $(N_1', N_2') = \frac{1}{2}[(0,1) + (1,0)] = (\frac{1}{2}, \frac{1}{2})$, which matches the result of the SBC solution.

The picture is as the following:

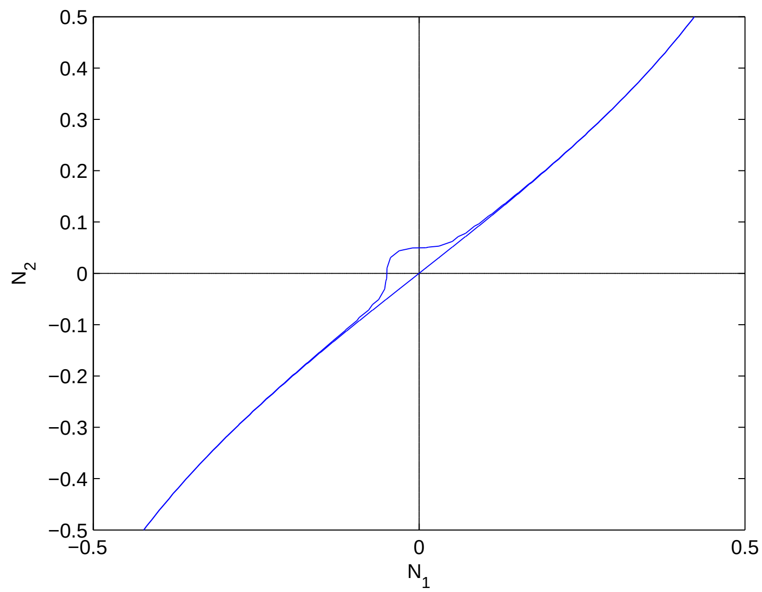

Figure 2.1: Picture of $(N1, N2)$.

2.3 Coupled system with all masses equal to 1

Consider the Hamiltonian F in the coupled case:

$$F = \frac{1}{\frac{m_1 m_2}{x_1} + \frac{m_3 m_4}{x_2}} \cdot (T - U - h)$$

$$= \frac{1}{\frac{m_1 m_2}{x_1} + \frac{m_3 m_4}{x_2}} (\frac{1}{2}[\frac{y_1^2}{m_1} + \frac{(y_1 - y_3)^2}{m_2} + \frac{(y_3 - y_2)^2}{m_3} + \frac{y_2^2}{m_4}]$$

$$-[\frac{m_1 m_2}{x_1} + \frac{m_1 m_3}{x_1 + x_3} + \frac{m_1 m_4}{x_1 + x_2 + x_3} + \frac{m_2 m_3}{x_3} + \frac{m_2 m_4}{x_2 + x_3} + \frac{m_3 m_4}{x_2}] - h)$$

For simplicity, assume $m_1 = m_2 = m_3 = m_4 = 1$, then

$$F = \frac{1}{\frac{1}{x_1}+\frac{1}{x_2}}(y_1^2+y_2^2+y_3^2-y_1y_3-y_2y_3)$$

$$-\frac{1}{\frac{1}{x_1}+\frac{1}{x_2}}(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_1+x_3}+\frac{1}{x_2+x_3}+\frac{1}{x_1+x_2+x_3}+h)$$

$$=\frac{y_1^2+y_2^2}{\frac{1}{x_1}+\frac{1}{x_2}}-\frac{y_1+y_2}{\frac{1}{x_1}+\frac{1}{x_2}}\cdot y_3-\frac{1}{\frac{1}{x_1}+\frac{1}{x_2}}[\frac{1}{x_3}+\frac{1}{x_1+x_3}+\frac{1}{x_2+x_3}+\frac{1}{x_1+x_2+x_3}+h-y_3^2]-1$$

We introduce a canonical transformation such that $\frac{y_1+y_2}{\frac{1}{x_1}+\frac{1}{x_2}}\cdot y_3$ can be absorbed into $\frac{y_1^2+y_2^2}{\frac{1}{x_1}+\frac{1}{x_2}}$. Let

$$Y_1 = y_1-\frac{1}{2}y_3, \qquad Y_2 = y_2-\frac{1}{2}y_3, \qquad Y_3 = y_3$$

and the generating function

$$W(x_1,x_2,x_3,Y_1,Y_2,Y_3) = x_1(Y_1+\frac{1}{2}Y_3)+x_2(Y_2+\frac{1}{2}Y_3)+x_3Y_3,$$

satisfying

$$\frac{\partial W}{\partial x_i} = y_i, \quad \text{and} \quad \frac{\partial W}{\partial Y_i} = X_i;$$

then

$$X_1 = \frac{\partial W}{\partial Y_1} = x_1$$

$$X_2 = \frac{\partial W}{\partial Y_2} = x_2$$

$$X_3 = \frac{\partial W}{\partial Y_3} = \frac{1}{2}x_1+\frac{1}{2}x_2+x_3.$$

Under the above transformation, the new hamiltonian is

$$F = \frac{Y_1^2+Y_2^2}{\frac{1}{X_1}+\frac{1}{X_2}}-\frac{1}{\frac{1}{X_1}+\frac{1}{X_2}}[\frac{1}{X_3-\frac{1}{2}X_1-\frac{1}{2}X_2}$$

$$+\frac{1}{X_3+\frac{1}{2}X_1-\frac{1}{2}X_2}+\frac{1}{X_3+\frac{1}{2}X_2-\frac{1}{2}X_1}+\frac{1}{X_3+\frac{1}{2}X_1+\frac{1}{2}X_2}+h-\frac{1}{2}Y_3^2]-1$$

Let

$$A = A(X_i, Y_3)$$

$$= [\frac{1}{X_3 - \frac{1}{2}X_1 - \frac{1}{2}X_2} + \frac{1}{X_3 + \frac{1}{2}X_1 - \frac{1}{2}X_2} + \frac{1}{X_3 + \frac{1}{2}X_2 - \frac{1}{2}X_1} + \frac{1}{X_3 + \frac{1}{2}X_1 + \frac{1}{2}X_2} + h - \frac{1}{2}Y_3^2];$$

then

$$F = \frac{Y_1^2 + Y_2^2}{\frac{1}{X_1} + \frac{1}{X_2}} - \frac{1}{\frac{1}{X_1} + \frac{1}{X_2}} A(X_i, Y_3) - 1.$$

From the above Hamiltonian, we can get 6 differential equations:

$$X_1' = F_{Y_1} = \frac{2Y_1X_1X_2}{X_1 + X_2}$$

$$X_2' = F_{Y_2} = \frac{2Y_2X_1X_2}{X_1 + X_2}$$

$$X_3' = F_{Y_3} = \frac{Y_3X_1X_2}{X_1 + X_2}$$

$$Y_1' = -F_{X_1} = -\frac{Y_1^2 + Y_2^2}{(\frac{1}{X_1} + \frac{1}{X_2})^2} \cdot \frac{1}{X_1^2} + \frac{1}{(\frac{1}{X_1} + \frac{1}{X_2})^2} \cdot \frac{1}{X_1^2} A + \frac{1}{\frac{1}{X_1} + \frac{1}{X_2}} A_{X_1}$$

$$= (-F - 1) \cdot \frac{1}{\frac{1}{X_1} + \frac{1}{X_2}} \cdot \frac{1}{X_1^2} + \frac{1}{\frac{1}{X_1} + \frac{1}{X_2}} A_{X_1}$$

$$Y_2' = -F_{X_2} = (-F - 1) \cdot \frac{1}{\frac{1}{X_2} + \frac{1}{X_2}} \cdot \frac{1}{X_1^2} + \frac{1}{\frac{1}{X_1} + \frac{1}{X_2}} A_{X_2}$$

$$Y_3' = -F_{X_3} = \frac{A_{X_3}X_1X_2}{X_1 + X_2}.$$

Following the idea in 2.2.1, we have

$$\frac{dY_1}{dX_1} = \frac{Y_1'}{X_1'} = \frac{(-F - 1) \cdot \frac{1}{\frac{1}{X_1} + \frac{1}{X_2}} \cdot \frac{1}{X_1^2} + \frac{1}{\frac{1}{X_1} + \frac{1}{X_2}} A_{X_1}}{\frac{2Y_1X_1X_2}{X_1 + X_2}}$$

$$= \frac{\frac{-F-1}{X_1^2} + A_{X_1}}{2Y_1}$$

Therefore, by separating the variables, and integrating on the solution surface $F = 0$,

$$\int 2Y_1 dY_1 = \int (\frac{-F - 1}{X_1^2} + A_{X_1}) dX_1$$

$$Y_1^2 = \frac{F+1}{X_1} + \int A_{X_1} dX_1 + C_1 = \frac{1}{X_1} + \int A_{X_1} dX_1 + C_1$$

where C_1 is a constant with respect to X_1.

By a similar process,

$$Y_2^2 = \frac{F+1}{X_2} + \int A_{X_2} dX_2 + C_2 = \frac{1}{X_2} + \int A_{X_2} dX_2 + C_2$$

where C_2 is a constant with respect to X_2.

2.3.1 New transformation. By a canonical transformation similar to that we defined in section 2.2.2:

$$\xi_1 = -X_1 Y_1^2, \quad \xi_2 = -X_2 Y_2^2, \quad \xi_3 = X_3, \quad \eta_1 = \frac{1}{Y_1}, \quad \eta_2 = \frac{1}{Y_2}, \quad \eta_3 = Y_3$$

$$X_1 = -\xi_1 \eta_1^2, \quad X_2 = -\xi_2 \eta_2^2, \quad X_3 = \xi_3, \quad Y_1 = \frac{1}{\eta_1}, \quad Y_2 = \frac{1}{\eta_2}, \quad Y_3 = \eta_3$$

therefore,

$$F = -\frac{\xi_1 \xi_2 (\eta_1^2 + \eta_2^2)}{\xi_1 \eta_1^2 + \xi_2 \eta_2^2} - 1$$

$$+ \frac{\xi_1 \xi_2 \eta_1^2 \eta_2^2}{\xi_1 \eta_1^2 + \xi_2 \eta_2^2} [h - \frac{1}{2}\eta_3^2 + \frac{1}{-\frac{1}{2}\xi_1\eta_1^2 - \frac{1}{2}\xi_2\eta_2^2 + \xi_3} + \frac{1}{\frac{1}{2}\xi_1\eta_1^2 - \frac{1}{2}\xi_2\eta_2^2 + \xi_3}$$

$$+ \frac{1}{-\frac{1}{2}\xi_1\eta_1^2 + \frac{1}{2}\xi_2\eta_2^2 + \xi_3} + \frac{1}{\frac{1}{2}\xi_1\eta_1^2 + \frac{1}{2}\xi_2\eta_2^2 + \xi_3}]$$

and the differential equations are

$$\xi_1' = \frac{2\xi_1\xi_2\eta_1\eta_2^2(\xi_1 - \xi_2)}{(\xi_1\eta_1^2 + \xi_2\eta_2^2)^2} + M_1$$

$$\xi_2' = \frac{-2\xi_1\xi_2\eta_2\eta_1^2(\xi_1 - \xi_2)}{(\xi_1\eta_1^2 + \xi_2\eta_2^2)^2} + M_2$$

$$\eta_1' = -F_{\xi_1} = \frac{\xi_2^2\eta_2^2(\eta_1^2 + \eta_2^2)}{(\xi_1\eta_1^2 + \xi_2\eta_2^2)^2} + G_1$$

$$\eta_2' = -F_{\xi_2} = \frac{\xi_1^2\eta_1^2(\eta_1^2 + \eta_2^2)}{(\xi_1\eta_1^2 + \xi_2\eta_2^2)^2} + G_2$$

$$\xi_3' = N_1$$

$$\eta_3' = N_2.$$

Since we know, when $s \to 0$, ξ_3, η_3, ξ_1 and ξ_2 approach nonzero constants, in the above equations, we can see M_1, M_2 , N_1 and N_2 are $O(s)$; G_1, G_2 are $O(s^2)$.

Introduce a new transformation

$$\frac{\xi_i + 1}{s} = u_i, \qquad \frac{\eta_i}{s} - \frac{1}{2} = v_i, \quad i = 1,2$$

$$\xi_3 = \widehat{\xi}_3 + u_3, \qquad \eta_3 = \widehat{\eta}_3 + v_3,$$

and

$$s = e^{-\tau}, \qquad ds = -s d\tau,$$

where $\widehat{\xi}_3$ and $\widehat{\eta}_3$ are the limits of ξ_3 and η_3 at $s = 0$.

Then we can get a differential system for u_i and v_i:

$$\frac{du_1}{d\tau} = -F_{\eta_1} + u_1, \qquad \frac{dv_1}{d\tau} = F_{\xi_1} + v_1 + \frac{1}{2},$$

$$\frac{du_2}{d\tau} = -F_{\eta_2} + u_2, \qquad \frac{dv_2}{d\tau} = F_{\xi_2} + v_2 + \frac{1}{2},$$

$$\frac{du_3}{d\tau} = -e^{-\tau} F_{\eta_3}, \qquad \frac{dv_3}{d\tau} = e^{-\tau} F_{\xi_3},$$

and

$$\frac{ds}{d\tau} = -s.$$

2.3.2 Limits of u_i and v_i at $s = 0$ ($i = 1,2$).

Lemma 2.17.

$$\lim_{s\to 0} u_1 = \lim_{s\to 0} u_2 = \lim_{s\to 0} v_1 = \lim_{s\to 0} v_2 = 0$$

Proof. According to the discussion in section 2.1.2, we know

$$\lim_{s\to 0}\frac{\eta_2^2}{\eta_1^2}=\lim_{t\to t_1}\frac{Y_1^2}{Y_2^2}=\lim_{t\to t_1}\frac{y_1^2}{y_2^2}=\lim_{t\to t_1}\frac{x_1p_1^2}{x_1p_4^2}=\frac{2(m_1m_2)^2}{(m_1+m_2)}\cdot\frac{(m_3+m_4)}{2\alpha(m_3m_4)^2}.$$

Since in our case $m_1=m_2=m_3=m_4$,

$$\lim_{s\to 0}\frac{\eta_2^2}{\eta_1^2}=1.$$

On the other hand, we know when t is very close to t_1, y_1 and y_2 are the same sign. Then $\lim_{s\to 0}\frac{\eta_2}{\eta_1}$ is positive. Therefore,

$$\lim_{s\to 0}\frac{\eta_2}{\eta_1}=1.$$

By L'Hospital rule,

$$\lim_{s\to 0}\frac{\eta_1}{s}=\lim_{s\to 0}\eta_1'=\lim_{s\to 0}\frac{\xi_2^2\eta_2^2(\eta_1^2+\eta_2^2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}+\lim_{s\to 0}G_1.$$

If the limit on the right hand side is finite, then the limit on the left hand side also exists and equals to the same value.

According to section 2.1.2, we have

$$\lim_{s\to 0}\eta_1=\lim_{s\to 0}\eta_2=0,$$

and

$$\lim_{s\to 0}\xi_1=\lim_{s\to 0}\xi_2=-1.$$

So

$$\lim_{s\to 0}\frac{\xi_2^2\eta_2^2(\eta_1^2+\eta_2^2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}=\lim_{s\to 0}\frac{\xi_2^2\frac{\eta_1^2}{\eta_2^2}(1+\frac{\eta_1^2}{\eta_2^2})}{(\xi_1+\xi_2\frac{\eta_1^2}{\eta_2^2})^2}=\frac{1\cdot 1\cdot 2}{(-1-1)^2}=\frac{1}{2}.$$

And it is obvious that

$$\lim_{s\to 0}G_1=0.$$

Therefore,

$$\lim_{s\to 0}\frac{\eta_1}{s}=\frac{1}{2},$$

and

$$\lim_{s\to 0}\frac{\eta_2}{s}=\lim_{s\to 0}\frac{\eta_2}{\eta_1}\cdot\lim_{s\to 0}\frac{\eta_1}{s}=\frac{1}{2}.$$

Here we can say

$$\lim_{s\to 0}v_1=\lim_{s\to 0}v_2=0.$$

To consider the limit of u_i, we need to go back to X_i and Y_i.

Because $\lim_{s\to 0}X_1Y_1^2=1$, and $Y_1^{-1}=\eta_1=O(s)$, then $X_1=O(s^2)$.

Since A is analytic and finite at $s=0$, A_{X_1} is also analytic at $s=0$. Consider the integral on the interval $[0,s_0]$, where s_0 is a small positive number. It is obvious that $X_1'=O(s)$ which is bounded on the interval $[0,s_0]$. And also A_{X_1} is also bounded because it is analytic at $s=0$. Hence $\int_0^{s_0}A_{X_1}X_1'ds=\lim_{\varepsilon\to 0}\int_\varepsilon^{s_0}A_{X_1}X_1'ds$ is bounded by some constant M_0.

On the other hand, we have

$$Y_1^2=\frac{1}{X_1}+\int A_{X_1}dX_1+C_1$$

$$X_1Y_1^2-1=X_1\int A_{X_1}dX_1+C_1X_1$$

that is

$$-(\xi_1+1)=X_1\int A_{X_1}dX_1+C_1X_1$$

$$-\frac{(\xi_1+1)}{s}=\frac{X_1}{s}\int A_{X_1}dX_1+C_1\frac{X_1}{s}.$$

Integrate on $[0,s_0]$, and we can see that

$$-\lim_{s\to 0}\frac{(\xi_1+1)}{s}=\lim_{s\to 0}\frac{X_1}{s}[\lim_{\varepsilon\to 0}\int_\varepsilon^{s_0}A_{X_1}X_1'ds+C_2]$$

Here the constant C_2 depends on the choice of s_0. Then $\lim_{\varepsilon\to 0}\int_\varepsilon^{s_0}A_{X_1}X_1'ds+C_2=\int_0^{s_0}A_{X_1}X_1'ds+C_2$ is bounded. Note that $X_1=O(s^2)$, therefore,

$$\lim_{s\to 0}\frac{\xi_1+1}{s}=0,\qquad\text{or}\qquad\lim_{s\to 0}u_1=0.$$

From the equations, we can see

$$\lim_{s\to 0}\frac{\xi_2'}{\xi_1'}=-\lim_{s\to 0}\frac{\eta_1}{\eta_2}=-1,$$

and

$$\lim_{s\to 0}(\xi_1+1)=\lim_{s\to 0}(\xi_2+1)=0,$$

so by L'Hospital's rule,

$$\lim_{s\to 0}\frac{\xi_2+1}{\xi_1+1}=\lim_{s\to 0}\frac{\xi_2'}{\xi_1'}=-1.$$

Therefore,

$$\lim_{s\to 0}u_2=\lim_{s\to 0}\frac{\xi_2+1}{s}=\lim_{s\to 0}\frac{\xi_2+1}{\xi_1+1}\cdot\lim_{s\to 0}\frac{\xi_1+1}{s}=(-1)\cdot 0=0.$$

$\square$

2.3.3 Analytic solutions of u_i and v_i at $s=0$. So far, we've got a system of 6 differential equations with initial conditions $u_i(0)=v_i(0)=0$.

$$s\frac{du_1}{ds}=F_{\eta_1}-u_1,\qquad s\frac{dv_1}{ds}=-F_{\xi_1}-v_1-\frac{1}{2},$$

$$s\frac{du_2}{ds}=F_{\eta_2}-u_2,\qquad s\frac{dv_2}{ds}=-F_{\xi_2}-v_2-\frac{1}{2},$$

$$\frac{du_3}{ds}=F_{\eta_3},\qquad \frac{dv_3}{ds}=-F_{\xi_3}.$$

Let $s=e^{-\tau}$, this system can be rewritten as an autonomous system with seven variable u_i, v_i and s:

$$\frac{du_1}{d\tau}=-F_{\eta_1}+u_1,\qquad \frac{dv_1}{d\tau}=F_{\xi_1}+v_1+\frac{1}{2},$$

$$\frac{du_2}{d\tau}=-F_{\eta_2}+u_2,\qquad \frac{dv_2}{d\tau}=F_{\xi_2}+v_2+\frac{1}{2},$$

$$\frac{du_3}{d\tau}=-sF_{\eta_3},\qquad \frac{dv_3}{d\tau}=sF_{\xi_3},$$

and

$$\frac{ds}{d\tau}=-s.$$

For simplification, we may use different notations:

$$\frac{d\sigma_k}{d\tau} = \Sigma_{l=1}^{7} b_{kl}\sigma_l + \varphi_k, \qquad (k = 1,...,7)$$

where $\sigma = (\sigma_1,...,\sigma_7)^T = (u_1,u_2,v_1,v_2,u_3,v_3,s)^T$.

The initial value is $\sigma_k = 0$ (k=1,...,7) and φ_k are power series in σ_1, ...,σ_7 beginning with quadratic terms, and the b_{kl} are real constants.

The seven-by-seven matrix (b_{kl}) has the structure

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \omega \\ 1 & 0 & 0 & 0 & 0 & 0 & \omega \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

where $\omega = \frac{1}{4}h - \frac{1}{8}\widehat{\eta}_3^2 + \frac{1}{\widehat{\xi}_3}$.

Theorem 2.18. *The system*

$$-s\frac{d\sigma}{ds} = B\sigma + \varphi, \qquad \varphi = (\varphi_1,...,\varphi_7)^T$$

has the initial condition $\sigma = (\sigma_1,...,\sigma_7)^T = 0$ *and*

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \omega \\ 1 & 0 & 0 & 0 & 0 & 0 & \omega \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

where $\omega = \frac{1}{4}h - \frac{1}{8}\widehat{\eta}_3^2 + \frac{1}{\xi_3}$. *And also* φ_k*(k=1,2...,7) are power series in* σ_1, ...,σ_7 *beginning with quadratic terms.*

Then this system has analytic solution σ *for s close to 0.*

The eigenvalues of B are -1, -1,0, 0, 1, 1, 3 and B can be diagonalized as

$$R = (r_{kl}) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

so by a linear transformation

$$T = \begin{bmatrix} -1 & -\omega & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which satisfies $T^{-1}BT = R$, $\sigma = T(\rho_1, ..., \rho_7)^T \equiv T\rho$ and $(\chi_k) = T^{-1}(\varphi_k)$, the system can be changed to be

$$\frac{d\rho_k}{d\tau} = r_{kk}\rho_k + \chi_k, \qquad (k = 1, ..., 7)$$

where χ_k are also power series in ρ_1, ... ρ_7 beginning with quadratic terms.

Next, we will show that the above differential system

$$\frac{d\rho_k}{d\tau} = f_k(\rho) = r_{kk}\rho_k + \chi_k, \qquad (k = 1, ..., 7) \tag{2.38}$$

has analytic solutions, where $\rho = (\rho_1, ..., \rho_7)$.

To find the solution, we will carry out substitutions of the special form

$$\mu_k = \rho_k - \phi_k(\rho_1, \rho_2) \qquad (k = 1, ..., 7) \tag{2.39}$$

where ϕ_k are formal series in the first two variables ρ_1, ρ_2 only and begin with quadratic terms. If one sets

$$j_k(\mu) = j_k(\mu_1, ..., \mu_7) = \chi_k + r_{kk}\phi_k - \phi_{k\rho_1} f_1 - \phi_{k\rho_2} f_2, \qquad (k = 1, ..., 7) \tag{2.40}$$

where on the right ρ can be expressed as a function of μ by means of the substitution inverse to 2.39, then 2.38 becomes

$$\frac{d\mu_k}{d\tau} = r_{kk}\mu_k + j_k(\mu), \qquad (k = 1, ..., 7) \tag{2.41}$$

with the power series j_k beginning again with quadratic terms. We will now determine the coefficients of the ϕ_k so that none of the series j_1, ..., j_7 contain product of powers of ρ_1, ρ_2 alone. In other words, the equations

$$j_k(\rho_1, \rho_2, 0, 0, 0, 0, 0) = 0, \qquad (k = 1, ..., 7) \tag{2.42}$$

are to hold identically.

By 2.39 the ρ_1, ρ_2 are invertible power series in the two indeterminate variables μ_1, μ_2 only, and moreover, for $\mu_3 = 0, ..., \mu_7 = 0$ we have

$$\rho_k = \phi_k(\rho_1, \rho_2), \qquad (k = 3, ..., 7). \tag{2.43}$$

Consequently, 2.42 reduces to the requirement that the equations

$$\chi_k + r_{kk}\phi_k - \phi_{k\rho_1} f_1 - \phi_{k\rho_2} f_2 = 0 \qquad (k = 1, ..., 7)$$

or

$$-r_{kk}\phi_k + \phi_{k\rho_1} r_{11}\rho_1 + \phi_{k\rho_2} r_{22}\rho_2 = \chi_k - \phi_{k\rho_1}\chi_1 - \phi_{k\rho_2}\chi_2, \qquad (k = 1, ..., 7) \tag{2.44}$$

be satisfied identically in ρ_1, ρ_2, where ρ_3,..., ρ_7 are defined by 2.43. Conversely, from 2.39, 2.43, 2.44

we again obtain 2.42. We now undertake comparison of coefficients in 2.44. If $\alpha\rho_1^{g_1}\rho_2^{g_2}$ is a term of ϕ_k with $g_1+g_2=m>1$, the comparison gives

$$(-r_{kk}+g_1r_{11}+g_2r_{22})\alpha=\gamma$$

where γ is got from a polynomial in the coefficients of the terms in ϕ_1, ... ,ϕ_7 of degree less than m. Since $r_{11}=r_{22}=-1$ and $m>1$,

$$-r_{kk}+g_1r_{11}+g_2r_{22}=-r_{kk}-m=1-m\neq 0, \qquad k=1,2$$

For $k=3,...,7$, $r_{kk}\geq 0$, then of course $-r_{kk}-m\neq 0$. So

$$-r_{kk}+g_1r_{11}+g_2r_{22}=-r_{kk}-m\neq 0, \qquad (k=1,...,7). \tag{2.45}$$

Therefore,induction shows that 2.42 has exactly one solution in power series ϕ_1, ... , ϕ_7.

Next, we need to show the convergence of ϕ_k(k=1,...,7).

2.3.4 Method of majorants.

Convergence is proved by the method of majorants.

If

$$f=\sum_l a_{l_1...l_m}x_1^{l_1}...x_m^{l_m}, \qquad g=\sum_l b_{l_1...l_m}x_1^{l_1}...x_m^{l_m}$$

are two power series, which need not converge, then g is said to be a majorant of f, symbolically $f\prec g$, if

$$|\, a_{l_1...l_m}\,|\leq b_{l_1...l_m}$$

for all the coefficients.

Let

$$\rho_1+\rho_2+...+\rho_7=\Gamma, \qquad \chi_k\prec\frac{c_1\Gamma^2}{1-c_1\Gamma}, \qquad (k=1,...7).$$

Since $r_{11}=r_{22}=-1$ and (7) is satisfied, it follows that

$$g_1 + g_2 < c_2 \mid -r_{kk} + g_1 r_{11} + g_2 r_{22} \mid \qquad (k = 1, \ldots 7). \tag{2.46}$$

Consequently for the uniquely determined solution ψ_1, ... ,ψ_7 of

$$\psi_{k\rho_1}\rho_1 + \psi_{k\rho_2}\rho_2 = c_2(1 + \psi_{k\rho_1} + \psi_{k\rho_2})\frac{c_1\Gamma^2}{1 - c_1\Gamma} \qquad (k = 1, \ldots 7) \tag{2.47}$$

$$\rho_k = \psi_k(\rho_1, \rho_2) \qquad (k = 3, \ldots 7)$$

we have the relation $\phi_k \prec \psi_k$. The reason is the following:

By the previous argument,we have

$$-r_{kk}\phi_k + \phi_{k\rho_1} r_{11}\rho_1 + \phi_{k\rho_2} r_{22}\rho_2 = \chi_k - \phi_{k\rho_1}\chi_1 - \phi_{k\rho_2}\chi_2, \qquad (k = 1, \ldots, 7).$$

If $\alpha\rho_1^{g_1}\rho_2^{g_2}$ is a term of ϕ_k with $g_1 + g_2 = m > 1$, the comparison gives

$$(-r_{kk} + g_1 r_{11} + g_2 r_{22})\alpha = \gamma$$

so

$$\mid \alpha \mid = \frac{\mid \gamma \mid}{\mid (-r_{kk} + g_1 r_{11} + g_2 r_{22}) \mid}$$

where γ comes from the right hand side of 2.44.

In the equations of ψ_k, if $\beta\rho_1^{g_1}\rho_2^{g_2}$ is a term of ϕ_k with $g_1 + g_2 = m > 1$, the comparison gives

$$(g_1 + g_2)\beta = c_2\gamma_1$$

where γ_1 comes from the right hand side of 2.47.

Since $\chi_k \prec \frac{c_1\Gamma^2}{1 - c_1\Gamma}(k = 1, \ldots 7)$, it is easy to see that $\gamma_1 > \mid \gamma \mid$. Then from 2.46,

$$\beta = \frac{c_2\gamma_1}{g_1 + g_2} > \frac{c_2\gamma_1}{c_2 \mid -r_{kk} + g_1 r_{11} + g_2 r_{22} \mid}$$

$$= \frac{\gamma_1}{| -r_{kk} + g_1 r_{11} + g_2 r_{22} |}$$

$$> \frac{| \gamma |}{| -r_{kk} + g_1 r_{11} + g_2 r_{22} |} = | \alpha | .$$

By 2.46, however, $\psi_1 = ... = \psi_7 = \psi$, and if in addition one sets $x_1 = x_2 = x$, it is evidently enough to prove the convergence for the solution $\psi(x)$ of

$$x\psi_x = (1+\psi_x)\frac{c_3(x+\psi)^2}{1-c_4(x+\psi)}.$$

On the other hand, let $\psi(x)/x = \overline{\psi}(x)$, so

$$(\overline{\psi} + x\overline{\psi}_x) = (1+\overline{\psi}+x\overline{\psi}_x)\frac{c_3 x(1+\overline{\psi})^2}{1-c_4 x(1+\overline{\psi})}$$

or

$$(\overline{\psi}+x\overline{\psi}_x)[1-c_4 x(1+\overline{\psi})] = (1+\overline{\psi}+x\overline{\psi}_x)c_3 x(1+\overline{\psi})^2$$

or

$$(\overline{\psi}+x\overline{\psi}_x) = c_4 x\overline{\psi} + c_4 x\overline{\psi}^2 + c_3 x(1+\overline{\psi})^3 + x^2\overline{\psi}_x(1+\overline{\psi})(c_4+c_3+c_3\overline{\psi}) \tag{2.48}$$

Let $\overline{\psi} = \sum_{n=1}^{\infty} a_n x^n$, from 2.48 we can get the recursion formulas for a_k($k \geq 2$):

$$\begin{aligned}
& a_k(1+k) \\
= \; & c_4 a_{k-1} + c_4 \sum_{m_1+m_2=k-1} a_{m_1} a_{m_2} + 3c_3 a_{k-1} + 3c_3 \sum_{m_1+m_2=k-1} a_{m_1} a_{m_2} \\
& + c_3 \sum_{m_1+m_2+m_3=k-1} a_{m_1} a_{m_2} a_{m_3} + (c_4+c_3)(k-1)a_{k-1} \\
& + (c_4+2c_3) \sum_{m_1+m_2=k-1} m_1 a_{m_1} a_{m_2} + c_3 \sum_{m_1+m_2+m_3=k-1} m_1 a_{m_1} a_{m_2} a_{m_3} \\
= \; & [kc_4 + (k+2)c_3]a_{k-1} + (c_4+3c_3) \sum_{m_1+m_2=k-1} a_{m_1} a_{m_2} \\
& + (c_4+2c_3) \sum_{m_1+m_2=k-1} m_1 a_{m_1} a_{m_2} \\
& + c_3 \sum_{m_1+m_2+m_3=k-1} a_{m_1} a_{m_2} a_{m_3} + c_3 \sum_{m_1+m_2+m_3=k-1} m_1 a_{m_1} a_{m_2} a_{m_3} .
\end{aligned} \tag{2.49}$$

Consider the equation

$$\Psi[1-c_4x(1+\Psi)] = c_3x(1+\Psi)^3$$

or

$$\Psi = c_4x\Psi + c_4x\Psi^2 + c_3x(1+\Psi)^3. \tag{2.50}$$

Let $\Psi = \sum_{n=1}^{\infty} b_n x^n$. From 2.49 we can get some other recursion formulas for $b_k(\ k \geq 2)$:

$$\begin{aligned} b_k &= c_4 b_{k-1} + c_4 \sum_{m_1+m_2=k-1} b_{m_1} b_{m_2} + 3c_3 b_{k-1} + 3c_3 \sum_{m_1+m_2=k-1} b_{m_1} b_{m_2} \\ &\quad + c_3 \sum_{m_1+m_2+m_3=k-1} b_{m_1} b_{m_2} b_{m_3} \\ &= (c_4+3c_3)b_{k-1} + (c_4+3c_3) \sum_{m_1+m_2=k-1} b_{m_1} b_{m_2} + c_3 \sum_{m_1+m_2+m_3=k-1} b_{m_1} b_{m_2} b_{m_3}. \end{aligned}$$

Then

$$\begin{aligned} b_k(k+1) &= (k+1)(c_4+3c_3)b_{k-1} + (k+1)(c_4+3c_3) \sum_{m_1+m_2=k-1} b_{m_1} b_{m_2} \\ &\quad + (k+1)c_3 \sum_{m_1+m_2+m_3=k-1} b_{m_1} b_{m_2} b_{m_3}. \end{aligned} \tag{2.51}$$

It is easy to see $a_1 = \frac{c_3}{2}$ and $b_1 = c_3$, so $a_1 < b_1$.

Assume $a_i < b_i$ for $i < k$, compare the terms on the right hand sides of (11) and (13):

$$(k+1)(c_4+3c_3)b_{k-1} > kc_4 + (k+2)c_3 a_{k-1}$$

$$(k+1)(c_4+3c_3) \sum_{m_1+m_2=k-1} b_{m_1} b_{m_2} > (c_4+3c_3) \sum_{m_1+m_2=k-1} a_{m_1} a_{m_2} +$$

$$(c_4+2c_3) \sum_{m_1+m_2=k-1} m_1 a_{m_1} a_{m_2}$$

$$(k+1)c_3 \sum_{m_1+m_2+m_3=k-1} b_{m_1} b_{m_2} b_{m_3} > c_3 \sum_{m_1+m_2+m_3=k-1} a_{m_1} a_{m_2} a_{m_3}$$

$$+ c_3 \sum_{m_1+m_2+m_3=k-1} m_1 a_{m_1} a_{m_2} a_{m_3}.$$

Therefore,

$$b_k(k+1) > a_k(k+1)$$

or

$$b_k > a_k.$$

By induction, $a_k < b_k$ is true for all k. So Ψ is a majorant of ψ.

From 2.50,we can see Ψ satisfies a cubic equation. Of course, it has a convergent solution.

Therefore, all the ϕ_k are convergent series. ■

By 2.41 and 2.42, one obtains for the given differential equation the particular solutions

$$\mu_k = \begin{cases} d_k e^{r_{kk}\tau}, & (k = 1, 2) \\ 0, & (k = 3, ..., 7). \end{cases}$$

Since $r_{11} = r_{22} = -1$ and $s = e^{-\tau}$,

$$\mu_k = \begin{cases} d_1 e^{-\tau}, \\ e^{-\tau}, \\ 0, & (k = 3, ..., 7) \end{cases}$$

where d_1 is an arbitrary constant.

Therefore, the solution for ρ_k is

$$\rho_k = \omega_k(\mu_1, \mu_2) = \omega_k(d_1 e^{-\tau}) = \omega_k(d_1 s) \qquad (k = 1, ..., 7),$$

where ω_k are convergent power series in the variables μ_1 and μ_2 without a constant term, and d_1 is an arbitrary constant.

That is, ρ_k are convergent series of s in a sufficiently small neighborhood of $s = 0$.

Then σ_k are also convergent power series of s in a sufficiently small neighborhood of $s = 0$ for $k = 1, ..., 7$.

2.4 Properties of the power series solutions

$$\xi_1 = -1 + u_1 s = -1 + ds^2 + (\frac{1}{15}d\omega - \frac{1}{5}d^2)s^4 +$$

$$(-\frac{23}{6300}d\omega^2 - \frac{26}{525}d^2\omega - \frac{1}{25}d^3)s^6 + O(s^8)$$

$$\xi_2 = -1 + u_2 s = -1 + (-\omega - d)s^2 + (-\frac{1}{5}d^2 - \frac{7}{15}d\omega - \frac{4}{15}\omega^2)s^4 +$$

$$(-\frac{37}{6300}\omega^3 + \frac{31}{1260}d\omega^2 + \frac{37}{525}d^2\omega + \frac{1}{25}d^3)s^6 + O(s^8)$$

$$\eta_1 = \frac{s}{2} + v_1 s = \frac{s}{2} + \frac{3d + \frac{1}{4}\omega}{15}s^3 + (\frac{1}{10}d^2 + \frac{1}{70}d\omega - \frac{1}{840}\omega^2)s^5 +$$

$$(\frac{1}{10500}\omega^3 - \frac{1}{2625}d\omega^2 + \frac{39}{3500}d^2\omega + \frac{58}{1125}d^3)s^7 + O(s^9)$$

$$\eta_2 = \frac{s}{2} + v_2 s = \frac{s}{2} + \frac{-\frac{11}{4}\omega - 3d}{15}s^3 + (\frac{1}{10}d^2 + \frac{13}{70}d\omega + \frac{71}{840}\omega^2)s^5 +$$

$$(-\frac{629}{15750}\omega^3 - \frac{33}{250}d\omega^2 - \frac{1507}{10500}d^2\omega - \frac{58}{1125}d^3)s^7 + O(s^9)$$

$$\xi_3 = \widehat{\xi}_3 + u_3 = \widehat{\xi}_3 + \frac{1}{24}\widehat{\eta}_3 s^3 + \frac{1}{240}\omega\widehat{\eta}_3 s^5 - \frac{1}{288\widehat{\xi}_3^2}s^6 + \frac{\widehat{\eta}_3}{7}(-\frac{1}{900}\omega^2 - \frac{1}{100}d\omega - \frac{1}{100}d^2)s^7 + \ldots$$

$$\eta_3 = \widehat{\eta}_3 + v_3 = \widehat{\eta}_3 - \frac{1}{6\widehat{\xi}_3^2}s^3 - \frac{\omega}{60\widehat{\xi}_3^2}s^5 + \frac{\widehat{\eta}_3}{144\widehat{\xi}_3^3}s^6 + \frac{1}{7\widehat{\xi}_3^2}(-\frac{611}{14400}\omega^2 + \frac{1}{25}d\omega + \frac{1}{25}d^2)s^7 \ldots$$

where

$$\omega = \frac{1}{4}h - \frac{1}{8}\widehat{\eta}_3^2 + \frac{1}{\widehat{\xi}_3} = \frac{1}{4}\lim_{s\to 0} A,$$

and d is an arbitrary constant.

If we set $\widetilde{d} = d + \frac{\omega}{2}$, then the first four power series solutions can be rewritten as:

$$\xi_1 = -1 + (\widetilde{d} - \frac{\omega}{2})s^2 + (-\frac{1}{5}\widetilde{d}^2 + \frac{4}{15}\widetilde{d}\omega - \frac{1}{12}\omega^2)s^4 +$$

$$(-\frac{1}{180}\omega^3 + \frac{1}{63}\widetilde{d}\omega^2 + \frac{11}{1050}\widetilde{d}^2\omega - \frac{1}{25}\widetilde{d}^3)s^6 + O(s^8)$$

$$\xi_2 = -1 + (-\tilde{d} - \frac{\omega}{2})s^2 + (-\frac{1}{5}\tilde{d}^2 - \frac{4}{15}\tilde{d}\omega - \frac{1}{12}\omega^2)s^4 +$$
$$(-\frac{1}{180}\omega^3 - \frac{1}{63}\tilde{d}\omega^2 + \frac{11}{1050}\tilde{d}^2\omega + \frac{1}{25}\tilde{d}^3)s^6 + O(s^8)$$

$$\eta_1 = \frac{s}{2} + (\frac{1}{5}\tilde{d} - \frac{\omega}{12})s^3 + (\frac{1}{10}\tilde{d}^2 - \frac{3}{35}\tilde{d}\omega + \frac{1}{60}\omega^2)s^5 +$$
$$(-\frac{17}{5040}\omega^3 + \frac{19}{700}\tilde{d}\omega^2 - \frac{139}{2100}\tilde{d}^2\omega + \frac{58}{1125}\tilde{d}^3)s^7 + O(s^9)$$

$$\eta_2 = \frac{s}{2} + (-\frac{1}{5}\tilde{d} - \frac{\omega}{12})s^3 + (\frac{1}{10}\tilde{d}^2 + \frac{3}{35}\tilde{d}\omega + \frac{1}{60}\omega^2)s^5 +$$
$$(-\frac{17}{5040}\omega^3 - \frac{19}{700}\tilde{d}\omega^2 - \frac{139}{2100}\tilde{d}^2\omega - \frac{58}{1125}\tilde{d}^3)s^7 + O(s^9).$$

This tells us that each pair $\xi_1 = \xi_2$, $\eta_1 = \eta_2$ when $\tilde{d} = 0$ or $d = -\frac{\omega}{2}$.

In fact, by the symmetry of the equation, if $(\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3)$ is a solution of the differential system, then $(\xi_2, \xi_1, \xi_3, \eta_2, \eta_1, \eta_3)$ is also a solution of the same system.

Basically, in the solutions, the coefficients of ξ_3 and η_3 have nothing to do with d up to the power s^6. And ξ_1, ξ_2, η_1, η_2 has no mixed term $d\omega$ for the first two nonzero terms in the power series solutions.

Compare this solution with the solution for the decoupled case,

$$\xi_1^0(s,C) = -1 - \frac{1}{4}Cs^2 - \frac{1}{80}C^2s^4 + \frac{1}{1600}C^3s^6 + \frac{7}{288000}C^4s^8 + \ldots$$

$$\xi_2^0(s,C) = -1 + \frac{1}{4}Cs^2 - \frac{1}{80}C^2s^4 - \frac{1}{1600}C^3s^6 + \frac{7}{288000}C^4s^8 + \ldots$$

$$\eta_1^0(s,C) = \frac{1}{2}s - \frac{C}{20}s^3 + \frac{C^2}{160}s^5 - \frac{29C^3}{36000}s^7 + \ldots$$

$$\eta_2^0(s,C) = \frac{1}{2}s + \frac{C}{20}s^3 + \frac{C^2}{160}s^5 + \frac{29C^3}{36000}s^7 + \ldots$$

where η_1^0 and η_2^0 are odd functions of s, and $\eta_1^0(0,C) = \eta_2^0(0,-C)$;
ξ_1^0 and ξ_2^0 are even functions of s, and $\xi_1^0(0,C) = \xi_2^0(0,-C)$;
and F approaches 1 as s approaches 0, which means that $F = 1$ on the phase space of the solutions for the decoupled case.

If we let $C=-4\widetilde{d}$, then

$$\xi_1^0=-1+\widetilde{d}s^2-\frac{1}{5}\widetilde{d}^2s^4-\frac{1}{25}\widetilde{d}^3s^6+\frac{7}{1125}\widetilde{d}^4s^8+\ldots$$

$$\xi_2^0=-1-\widetilde{d}s^2-\frac{1}{5}\widetilde{d}^2s^4+\frac{1}{25}\widetilde{d}^3s^6+\frac{7}{1125}\widetilde{d}^4s^8+\ldots$$

$$\eta_1^0=\frac{1}{2}s+\frac{\widetilde{d}}{5}s^3+\frac{\widetilde{d}^2}{10}s^5+\frac{58\widetilde{d}^3}{1125}s^7+\ldots$$

$$\eta_2^0=\frac{1}{2}s-\frac{\widetilde{d}}{5}s^3+\frac{\widetilde{d}^2}{10}s^5-\frac{58\widetilde{d}^3}{1125}s^7+\ldots.$$

Therefore,

$$\xi_1=\xi_1^0-\frac{\omega}{2}s^2+O(s^4)$$

$$\xi_2=\xi_2^0-\frac{\omega}{2}s^2+O(s^4)$$

$$\eta_1=\eta_1^0-\frac{1}{12}\omega s^3+O(s^5)$$

$$\eta_2=\eta_2^0-\frac{1}{12}\omega s^3+O(s^5).$$

The above results tell us that

1. In each of the decoupled case and the coupled case, there is a parameter $\widetilde{d}$, which is an arbitrary constant. From the comparison, we can see those two constants are the same. Recall the meaning of C in section 2.2.3,

$$\widetilde{d}=-\frac{C}{4}=-\frac{1}{4}\lim_{s\to 0}\frac{\xi_1-\xi_2}{\xi_1\eta_1^2+\xi_2\eta_2^2};$$

2.The motion of the decoupled case and the coupled case are very similar. Up to the power s^4, the coupled case can be considered as a decoupled case adding another motion which is related to the initial conditions: h, $\widehat{\xi}_3$ and $\widehat{\eta}_3$;

3. Because of the mixed term $d\omega$, the coupled solution can NOT be considered exactly as the sum of a decoupled solution and a special solution which has nothing to do with d;

4. Up to the power s^7, the solution ξ_1, ξ_2, η_1, and η_2 is still symmetric with respect to the new constant $\widetilde{d}$: $\xi_1(\widetilde{d})=\xi_2(-\widetilde{d})$ and $\eta_1(\widetilde{d})=\eta_2(-\widetilde{d})$;

5. In the solution of the coupled case, basically there are two constants: $\widetilde{d}=-\frac{1}{4}C$, where C is the

constant in the decoupled case; another one ω is given by the initial conditions and it shows the effect of the coupling terms to the solutions, and also the coupling term $\widetilde{d}\omega$ will start appearing from the term s^6 in the power series form of X_1 and X_2;

6. ω is fixed for given initial values, but $\widetilde{d}$ will make the solution to be a one-parameter family which is similar to the decoupled case. And the analytic solution can ONLY happen if we choose the same common constant $\widetilde{d}$ on both negative and positive sides of s.

2.5 THE SYSTEM WITH GENERAL MASSES

By definition the new Hamiltonian F is

$$F = \frac{1}{\frac{m_1m_2}{x_1}+\frac{m_3m_4}{x_2}}\cdot(T-U-h)$$

$$= \frac{1}{\frac{m_1m_2}{x_1}+\frac{m_3m_4}{x_2}}\Big(\frac{1}{2}\Big[\frac{y_1^2}{m1}+\frac{(y_1-y_3)^2}{m_2}+\frac{(y_3-y_2)^2}{m_3}+\frac{y_2^2}{m_4}\Big]$$
$$-\Big[\frac{m_1m_2}{x_1}+\frac{m_1m_3}{x_1+x_3}+\frac{m_1m_4}{x_1+x_2+x_3}+\frac{m_2m_3}{x_3}+\frac{m_2m_4}{x_2+x_3}+\frac{m_3m_4}{x_2}\Big]-h\Big)$$

$$= \frac{1}{\frac{m_1m_2}{x_1}+\frac{m_3m_4}{x_2}}\Big\{\Big(\frac{1}{2m_1}+\frac{1}{2m_2}\Big)y_1^2+\Big(\frac{1}{2m_3}+\frac{1}{2m_4}\Big)y_2^2-\Big(\frac{y_1}{m_2}+\frac{y_2}{m_3}\Big)y_3$$

$$+\Big(\frac{1}{2m_2}+\frac{1}{2m_3}\Big)y_3^2-\Big[\frac{m_1m_2}{x_1}+\frac{m_1m_3}{x_1+x_3}+\frac{m_1m_4}{x_1+x_2+x_3}+\frac{m_2m_3}{x_3}+\frac{m_2m_4}{x_2+x_3}+\frac{m_3m_4}{x_2}\Big]-h\Big\}.$$

Let

$$Y_1 = y_1-\frac{m_1}{m_1+m_2}y_3, \qquad Y_2 = y_2-\frac{m_4}{m_3+m_4}y_3, \qquad Y_3 = y_3.$$

Let the generating function be

$$W(x_i,Y_i) = x_1\Big(Y_1+\frac{m_1}{m_1+m_2}Y_3\Big)+x_2\Big(Y_2+\frac{m_4}{m_3+m_4}Y_3\Big)+x_3Y_3.$$

So

$$X_1 = W_{Y_1} = x_1, \qquad X_2 = W_{Y_2} = x_2, \qquad X_3 = W_{Y_3} = \frac{m_1}{m_1+m_2}x_1+\frac{m_4}{m_3+m_4}x_2+x_3.$$

And the new Hamiltonian is

$$F = \frac{1}{\frac{m_1 m_2}{X_1} + \frac{m_3 m_4}{X_2}} \{ \frac{m_1 + m_2}{2m_1 m_2} Y_1^2 + \frac{m_3 + m_4}{2m_3 m_4} Y_2^2$$

$$+ [\frac{m_2 + m_3}{2m_2 m_3} - \frac{m_1}{2m_2(m_1 + m_2)} - \frac{m_4}{2m_3(m_3 + m_4)}] Y_3^2$$

$$-[\frac{m_1 m_2}{X_1} + \frac{m_1 m_3}{X_3 + \frac{m_2}{m_1+m_2} X_1 - \frac{m_4}{m_3+m_4} X_2} + \frac{m_1 m_4}{X_3 + \frac{m_2}{m_1+m_2} X_1 + \frac{m_3}{m_3+m_4} X_2}$$

$$+ \frac{m_2 m_3}{X_3 - \frac{m_1}{m_1+m_2} X_1 - \frac{m_4}{m_3+m_4} X_2} + \frac{m_2 m_4}{X_3 - \frac{m_1}{m_1+m_2} X_1 + \frac{m_3}{m_3+m_4} X_2} + \frac{m_3 m_4}{X_2}] - h \}$$

$$= \frac{1}{\frac{m_1 m_2}{X_1} + \frac{m_3 m_4}{X_2}} (\frac{m_1 + m_2}{2m_1 m_2} Y_1^2 + \frac{m_3 + m_4}{2m_3 m_4} Y_2^2)$$

$$+ \frac{1}{\frac{m_1 m_2}{X_1} + \frac{m_3 m_4}{X_2}} \{ [\frac{m_2 + m_3}{2m_2 m_3} - \frac{m_1}{2m_2(m_1 + m_2)} - \frac{m_4}{2m_3(m_3 + m_4)}] Y_3^2$$

$$-[\frac{m_1 m_3}{X_3 + \frac{m_2}{m_1+m_2} X_1 - \frac{m_4}{m_3+m_4} X_2} + \frac{m_1 m_4}{X_3 + \frac{m_2}{m_1+m_2} X_1 + \frac{m_3}{m_3+m_4} X_2}$$

$$+ \frac{m_2 m_3}{X_3 - \frac{m_1}{m_1+m_2} X_1 - \frac{m_4}{m_3+m_4} X_2} + \frac{m_2 m_4}{X_3 - \frac{m_1}{m_1+m_2} X_1 + \frac{m_3}{m_3+m_4} X_2}] - h \} - 1$$

Follow the canonical transformation similar to that in section 2.2.3:

$$\xi_1 = -X_1 Y_1^2, \quad \xi_2 = -X_1 Y_1^2, \quad \xi_3 = X_3, \quad \eta_1 = \frac{1}{Y_1}, \quad \eta_2 = \frac{1}{Y_2}, \quad \eta_3 = Y_3$$

$$X_1 = -\xi_1 \eta_1^2, \quad X_2 = -\xi_1 \eta_1^2, \quad X_3 = \xi_3, \quad Y_1 = \frac{1}{\eta_1}, \quad Y_2 = \frac{1}{\eta_2}, \quad Y_3 = \eta_3$$

and the new Hamiltonian is

$$F = -\frac{\xi_1 \xi_2 (\frac{m_1+m_2}{2m_1 m_2} \eta_2^2 + \frac{m_3+m_4}{2m_3 m_4} \eta_1^2)}{m_3 m_4 \xi_1 \eta_1^2 + m_1 m_2 \xi_2 \eta_2^2}$$

$$+ \frac{\xi_1 \xi_2 \eta_1^2 \eta_2^2}{m_3 m_4 \xi_1 \eta_1^2 + m_1 m_2 \xi_2 \eta_2^2} \{ [-\frac{m_2 + m_3}{2m_2 m_3} + \frac{m_1}{2m_2(m_1 + m_2)} + \frac{m_4}{2m_3(m_3 + m_4)}] \eta_3^2 + h$$

$$+ \frac{m_1 m_3}{\xi_3 - \frac{m_2}{m_1+m_2} \xi_1 \eta_1^2 + \frac{m_4}{m_3+m_4} \xi_2 \eta_2^2} + \frac{m_1 m_4}{\xi_3 - \frac{m_2}{m_1+m_2} \xi_1 \eta_1^2 - \frac{m_3}{m_3+m_4} \xi_2 \eta_2^2} + \frac{m_2 m_3}{\xi_3 + \frac{m_1}{m_1+m_2} \xi_1 \eta_1^2 + \frac{m_4}{m_3+m_4} \xi_2 \eta_2^2}$$

$$+ \frac{m_2 m_4}{\xi_3 + \frac{m_1}{m_1+m_2} \xi_1 \eta_1^2 - \frac{m_3}{m_3+m_4} \xi_2 \eta_2^2} \} - 1$$

The equations corresponding to ξ_1, ξ_2, η_1 and η_2 are

$$\xi_1' = F_{\eta_1} = \frac{2\xi_1\xi_2\eta_1\eta_2^2(\frac{m_3m_4(m_1+m_2)}{2m_1m_2}\xi_1 - \frac{m_1m_2(m_3+m_4)}{2m_3m_4}\xi_2)}{(m_3m_4\xi_1\eta_1^2+m_1m_2\xi_2\eta_2^2)^2} + ...$$

$$\xi_2' = F_{\eta_2} = \frac{-2\xi_1\xi_2\eta_1^2\eta_2(\frac{m_3m_4(m_1+m_2)}{2m_1m_2}\xi_1 - \frac{m_1m_2(m_3+m_4)}{2m_3m_4}\xi_2)}{(m_3m_4\xi_1\eta_1^2+m_1m_2\xi_2\eta_2^2)^2} + ...$$

$$\eta_1' = -F_{\xi_1} = \frac{m_1m_2\xi_2^2\eta_2^2(\frac{m_1+m_2}{2m_1m_2}\eta_2^2 + \frac{m_3+m_4}{2m_3m_4}\eta_1^2)}{(m_3m_4\xi_1\eta_1^2+m_1m_2\xi_2\eta_2^2)^2} + ...$$

$$\eta_2' = -F_{\xi_2} = \frac{m_3m_4\xi_1^2\eta_1^2(\frac{m_1+m_2}{2m_1m_2}\eta_2^2 + \frac{m_3+m_4}{2m_3m_4}\eta_1^2)}{(m_3m_4\xi_1\eta_1^2+m_1m_2\xi_2\eta_2^2)^2} + ...$$

Consider the limit of ξ_i and η_i at $s=0$: Similar to the previous argument, we can see

$$\lim_{s\to 0}\frac{\eta_2^2}{\eta_1^2} = \frac{2m_1^2m_2^2}{m_1+m_2}\cdot\frac{m_3+m_4}{2m_3^2m_4^2}\cdot(\frac{m_3+m_4}{m_1+m_2})^{\frac{1}{3}},$$

$$\lim_{s\to 0}\xi_1 = -\frac{2m_1^2m_2^2}{m_1+m_2},$$

$$\lim_{s\to 0}\xi_2 = -\frac{2m_3^2m_4^2}{m_3+m_4},$$

$$\lim_{s\to 0}\eta_1 = \lim_{s\to 0}\eta_2 = 0,$$

$$\lim_{s\to 0}\frac{\xi_1+\frac{2m_1^2m_2^2}{m_1+m_2}}{s} = \lim_{s\to 0}\xi_1' = 0, \qquad \lim_{s\to 0}\frac{\xi_2+\frac{2m_3^2m_4^2}{m_3+m_4}}{s} = \lim_{s\to 0}\xi_2' = 0,$$

$$\lim_{s\to 0}\frac{\eta_1}{s} = \lim_{s\to 0}\eta_1' = \frac{(m_1+m_2)(m_3+m_4)^{\frac{1}{3}}}{2m_1m_2[m_1m_2(m_3+m_4)^{\frac{1}{3}}+m_3m_4(m_1+m_2)^{\frac{1}{3}}]},$$

$$\lim_{s\to 0}\frac{\eta_1}{s} = \lim_{s\to 0}\eta_1' = \frac{(m_1+m_2)^{\frac{1}{3}}(m_3+m_4)}{2m_3m_4[m_1m_2(m_3+m_4)^{\frac{1}{3}}+m_3m_4(m_1+m_2)^{\frac{1}{3}}]}.$$

Denote $\lim_{s\to 0}\frac{\eta_1}{s} = \widehat{v}_1$ and $\lim_{s\to 0}\frac{\eta_2}{s} = \widehat{v}_2$. Do the change of variable

$$u_1 = \frac{\xi_1+\frac{2m_1^2m_2^2}{m_1+m_2}}{s}, \qquad u_2 = \frac{\xi_2+\frac{2m_3^2m_4^2}{m_3+m_4}}{s},$$

$$v_1 = \frac{\eta_1}{s} - \widehat{v}_1, \qquad v_2 = \frac{\eta_2}{s} - \widehat{v}_2,$$

$$u_3 = \xi_3 - \widehat{\xi}_3, \qquad v_3 = \eta_3 - \widehat{\eta}_3$$

where $\widehat{\xi}_3$ and $\widehat{\eta}_3$ are the limits of ξ_3 and η_3 at $s = 0$.

Then the new equations become:

$$su_1' = F_{\eta_1} - u_1, \qquad su_2' = F_{\eta_2} - u_2,$$

$$sv_1' = -F_{\xi_1} - v_1 - \widehat{v}_1, \qquad sv_2' = -F_{\xi_2} - v_2 - \widehat{v}_2,$$

$$u_3' = F_{\eta_3}, \qquad v_3' = -F_{\xi_3},$$

with the initial conditions at $s = 0$:

$$u_i(0) = v_i(0) = 0 \qquad (i = 1, 2, 3)$$

Let $s = e^{-\tau}$; the above equations can be rewritten as an autonomous system:

$$\frac{du_1}{d\tau} = -F_{\eta_1} + u_1, \qquad \frac{du_2}{d\tau} = -F_{\eta_2} + u_2,$$

$$\frac{dv_1}{d\tau} = F_{\xi_1} + v_1 + \widehat{v}_1, \qquad \frac{dv_2}{d\tau} = F_{\xi_2} + v_2 + \widehat{v}_2,$$

$$\frac{du_3}{d\tau} = -sF_{\eta_3}, \qquad v_3' = sF_{\xi_3},$$

and

$$\frac{ds}{d\tau} = -s.$$

For simplification, we may use different notations:

$$\frac{d\sigma_k}{d\tau} = \Sigma_{l=1}^{7} b_{kl}\sigma_l + \varphi_k, \qquad (k = 1, ..., 7). \tag{2.52}$$

The initial value is $\sigma_k = 0$ (k=1,...,7) and φ_k are power series in σ_1, ...,σ_7 beginning with quadratic terms, and the b_{kl} are real constants.

The seven-by-seven matrix (b_{kl}) has the structure

$$B = \begin{bmatrix} b_{11} & b_{12} & 0 & 0 & 0 & 0 & b_{17} \\ b_{21} & b_{22} & 0 & 0 & 0 & 0 & b_{27} \\ 0 & 0 & b_{33} & b_{34} & 0 & 0 & 0 \\ 0 & 0 & b_{43} & b_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

where

$$b_{11} = 1 - \frac{2m_3m_4\sqrt[3]{m_1+m_2}}{m_1m_2\sqrt[3]{m_3+m_4}+m_3m_4\sqrt[3]{m_1+m_2}};$$

$$b_{12} = \frac{2m_1^2m_2^2(m_3+m_4)}{m_3m_4(m_1+m_2)^{\frac{2}{3}}} \cdot \frac{1}{m_1m_2\sqrt[3]{m_3+m_4}+m_3m_4\sqrt[3]{m_1+m_2}};$$

$$b_{21} = \frac{2m_3^2m_4^2(m_1+m_2)}{m_1m_2(m_3+m_4)^{\frac{2}{3}}} \cdot \frac{1}{m_1m_2\sqrt[3]{m_3+m_4}+m_3m_4\sqrt[3]{m_1+m_2}};$$

$$b_{22} = 1 - \frac{2m_1m_2\sqrt[3]{m_3+m_4}}{m_1m_2\sqrt[3]{m_3+m_4}+m_3m_4\sqrt[3]{m_1+m_2}};$$

$$b_{33} = 1 + \frac{2m_3m_4\sqrt[3]{m_1+m_2}}{m_1m_2\sqrt[3]{m_3+m_4}+m_3m_4\sqrt[3]{m_1+m_2}};$$

$$b_{34} = \frac{-2m_3^2m_4^2(m_1+m_2)}{m_1m_2(m_3+m_4)^{\frac{2}{3}}} \cdot \frac{1}{m_1m_2\sqrt[3]{m_3+m_4}+m_3m_4\sqrt[3]{m_1+m_2}};$$

$$b_{43} = \frac{-2m_1^2m_2^2(m_3+m_4)}{m_3m_4(m_1+m_2)^{\frac{2}{3}}} \cdot \frac{1}{m_1m_2\sqrt[3]{m_3+m_4}+m_3m_4\sqrt[3]{m_1+m_2}};$$

$$b_{44} = 1 + \frac{2m_1m_2\sqrt[3]{m_3+m_4}}{m_1m_2\sqrt[3]{m_3+m_4}+m_3m_4\sqrt[3]{m_1+m_2}};$$

$$b_{17} = [h - \frac{m_1+m_2+m_3+m_4}{2m_2m_3(m_1+m_2)(m_3+m_4)}\widehat{\eta}_3^2 + (m_1m_3+m_2m_3+m_1m_4+m_2m_4)\frac{1}{\widehat{\xi}_3}]$$
$$\cdot \frac{2m_1^2m_2^2(m_3+m_4)}{(m_1m_2\sqrt[3]{m_3+m_4}+m_3m_4\sqrt[3]{m_1+m_2})^3};$$

$$b_{27} = [h - \frac{m_1+m_2+m_3+m_4}{2m_2m_3(m_1+m_2)(m_3+m_4)}\widehat{\eta}_3^2 + (m_1m_3+m_2m_3+m_1m_4+m_2m_4)\frac{1}{\widehat{\xi}_3}]$$
$$\cdot \frac{2m_3^2m_4^2(m_1+m_2)}{(m_1m_2\sqrt[3]{m_3+m_4}+m_3m_4\sqrt[3]{m_1+m_2})^3}.$$

To find the eigenvalues of B, we only need to find the eigenvalues for the two different 2 by 2 matrices:

$$B_1 = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix}.$$

By careful calculation, we find out that the eigenvalues for B_1 are 1 and -1; the eigenvalues for B_2 are 1 and 3. Fortunately, they are exactly the same as the case with equal masses. And also B is similar to the same diagonal matrix R:

$$R = (r_{kl}) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then the previous argument works. Therefore we have the analytic properties of the solutions of u_i and v_i in a neighborhood of $s = 0$.

Chapter 3. Periodic Solutions with Alternating Singularities in the Collinear Four-body Problem

The collinear four-body problem considers a system of four points with masses m_1, m_2, m_3, m_4 on a real line attracting each other by newtonian gravitational law. In this chapter, we study a special symmetric periodic orbit with masses 1, m, m, 1, which is called Schubart-like orbit later. In each period of this Schubart-like orbit, there is a binary collision (or BC for short) between the inner two bodies and then a simultaneous binary collision (or SBC for short) of the two clusters on both sides of the origin. This research is motivated by some important work on a remarkable periodic orbit in the collinear three-body problem, which is named as Schubart orbit.

3.1 The Setting and the orbit

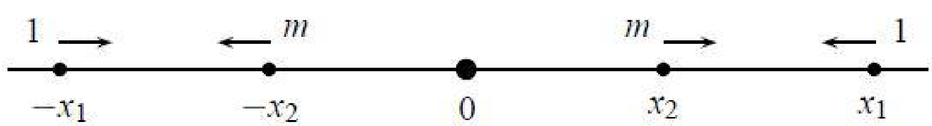

Figure 3.1: Problem Setting.

3.1.1 The setting in Cartesian Coordinate System. From right to left, let's number the four bodies from 1 to 4. As in Figure 1, the masses for body 1 to 4 are 1, m, m, and 1 respectively. The system remains symmetrically distributed about the center of mass. The coordinates for the four bodies are x_1, x_2, $-x_2$ and $-x_1$, and the velocities are $\dot{x}_1$, $\dot{x}_2$, $-\dot{x}_2$, $-\dot{x}_1$ respectively.

The Netowanian equations are

$$\ddot{x}_1 = -\frac{1}{4x_1^2} - \frac{m}{(x_1+x_2)^2} - \frac{m}{(x_1-x_2)^2}, \tag{3.1}$$

$$\ddot{x}_2 = -\frac{m}{4x_2^2} - \frac{1}{(x_1+x_2)^2} + \frac{1}{(x_1-x_2)^2}. \tag{3.2}$$

In this paper, we are interested in proving the existence of a special periodic orbit with singularities. The orbit alternates between binary collision (BC) between the inner two bodies 2 and 3, and SBC between bodies 1 and 2 and bodies 3 and 4. By introducing a new set of transformations, the singularities of BC and SBC can be regularized in this case.

3.1.2 Regularization. We will adopt Sweatman's[29] work to regularize the system. The system has Hamiltonian

$$H = \frac{1}{4}w_1^2 + \frac{1}{4m}w_2^2 - \frac{1}{2x_1} - \frac{m^2}{2x_2} - \frac{2m}{x_1+x_2} - \frac{2m}{x_1-x_2},$$

where $w_1 = 2\dot{x}_1$ and $w_2 = 2m\dot{x}_2$ are the conjugate momenta to x_1 and x_2. In order to describe the behavior at collision, we introduce a canonical transformation

$$q_1 = x_1 - x_2, \qquad q_2 = 2x_2, \qquad p_1 = w_1, \qquad p_2 = \frac{1}{2}(w_1 + w_2).$$

This results in a new form for the Hamiltonian

$$H = (1+\frac{1}{m})\frac{p_1^2}{4} - \frac{p_1p_2}{m} + \frac{p_2^2}{m} - \frac{2m}{q_1} - \frac{m^2}{q_2} - \frac{2m}{q_1+q_2} - \frac{1}{2q_1+q_2}.$$

To regularize the equations of motion, we introduce a Levi-Civita type of canonical transformation

$$Q_i^2 = q_i, \qquad P_i = 2Q_ip_i \ \ (i = 1,2),$$

and we also replace time t by the new independent variable s which is given by $\frac{dt}{ds} = q_1q_2$. In the extended phase space, this produces a regularized Hamiltonian

$$\begin{aligned}\Gamma &= \frac{dt}{ds}(H-E)\\ &= \frac{1}{16}Q_2^2P_1^2 + \frac{Q_2^2P_1^2 - 4Q_1Q_2P_1P_2 + 4Q_1^2P_2^2}{16m} - m^2Q_1^2 - 2mQ_2^2 - \frac{2mQ_1^2Q_2^2}{Q_1^2+Q_2^2} - \frac{Q_1^2Q_2^2}{2Q_1^2+Q_2^2} - Q_1^2Q_2^2E,\end{aligned}$$

where E is the total energy. Without loss of generality, let $E = -1$.

We start at BC with initial conditions

$$x_1(0) = A, \qquad x_2(0) = 0, \qquad \dot{x}_1(0) = 0, \qquad \dot{x}_2(0) = +\infty$$

which is a singular point. To analyze the motion, it is necessary to deal with the singularity in the regularized coordinate system. The corresponding initial conditions at $s = 0$ in this new coordinate system are:

$$Q_1(0) = R, \qquad Q_2(0) = 0, \qquad P_1(0) = 0, \qquad P_2(0) = 2m^{\frac{3}{2}},$$

where $R = \sqrt{A}$.

Note that this initial point turns out to be a regular point in the new Hamiltonian system.

3.1.3 Estimation of A. Intuitively, if A is sufficiently large, there will be multiple BCs before the first SBC happens. In order to find the desired orbit, we will have to give an estimation of A such that there is no BC for $t \in (0, t_1]$, where t_1 is the time of first SBC.

Definition: Assume the velocity of a body is 0 at time t^*, i.e. $v(t^*) = 0$. If there exists a time interval $[t_m, t_n]$, such that $t_m < t^* < t_n$, and v is positive for $t \in [t_m, t^*)$ and is negative for $t \in (t^*, t_n]$, or v is negative for $t \in [t_m, t^*)$ and is positive for $t \in (t^*, t_n]$, then we call t^* the *turning time* and the position of the body at t^* is called the *turning point*.

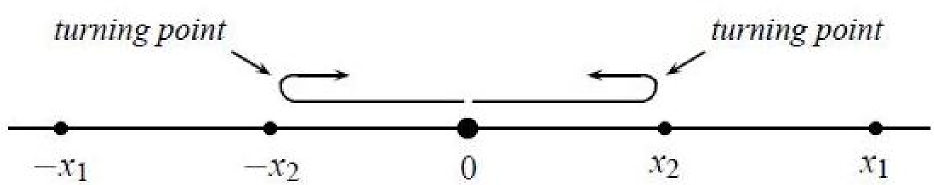

Figure 3.2: Turning Point

Theorem 3.1. *There exists an A_0, such that the second body has no turning point for $t \in (0, t_1)$ whenever $0 < A \le A_0$, where t_1 is the time when the first SBC happens. Further, at $A = A_0$, there exists some t^* such that $\dot{x}_2(t^*) = \ddot{x}_2(t^*) = 0$.*

Proof. In order to get an upper bound of A, we consider a necessary condition for the second body having a turning point. Assume $t = t^* < t_1$ is the time when $\dot{x}_2 = 0$; then $\ddot{x}_2 \le 0$ for $t \in (0, t^*]$.

Let $x_1(t^*)/x_2(t^*) = a$ with $a > 1$. Note that by the setting, $\dot{x_1}(t) < 0,\ \dot{x_2}(t) \geq 0$ for $t \in (0,t^*]$. Then

$$\frac{x_1(t)}{x_2(t)} \geq \frac{x_1(t^*)}{x_2(t^*)} = a, \qquad \text{for } t \in (0,t^*]. \tag{3.3}$$

Also $\ddot{x}_2(t^*) \leq 0$, by equation 3.2,

$$\ddot{x}_2(t^*) = \left[-\frac{m}{4} - \frac{1}{(a+1)^2} + \frac{1}{(a-1)^2}\right] \frac{1}{x_2^2(t^*)} \leq 0$$

i.e.

$$16a \leq m(a^2-1)^2. \tag{3.4}$$

Rewrite equation 3.1 as:

$$\begin{aligned} -\ddot{x}_1 &= \frac{1}{4x_1^2} + \frac{m}{(x_1+x_2)^2} + \frac{m}{(x_1-x_2)^2} \\ &= \frac{1}{x_1^2}\left[\frac{1}{4} + \frac{2m(1+\frac{x_2^2}{x_1^2})}{(1-\frac{x_2^2}{x_1^2})^2}\right]. \end{aligned}$$

For $t \in (0,t^*]$, by inequality 3.3,

$$\begin{aligned} \frac{1}{x_1^2}\left[\frac{1}{4} + \frac{2m(1+\frac{x_2^2}{x_1^2})}{(1-\frac{x_2^2}{x_1^2})^2}\right] &\leq \frac{1}{x_1^2}\left[\frac{1}{4} + \frac{2m(1+\frac{1}{a^2})}{(1-\frac{1}{a^2})^2}\right] \\ &= \frac{1}{x_1^2}\left[\frac{1}{4} + \frac{2ma^2(a^2+1)}{(a^2-1)^2}\right]. \end{aligned}$$

Therefore,

$$-\ddot{x}_1(t) \leq \frac{1}{x_1^2(t)}\left[\frac{1}{4} + \frac{2ma^2(a^2+1)}{(a^2-1)^2}\right], \quad \text{for } t \in (0,t^*]. \tag{3.5}$$

Let $x_1(0) = A$, $x_1(t^*) = A_1 < A$. Since $\dot{x}_1(t) < 0$ for $t \in [0,t^*]$, the following inequality is true for $t \in [0,t^*]$, multiplying both sides of inequality 3.5 by $-\dot{x}_1(t)$:

$$\dot{x}_1(t)\ddot{x}_1(t) \leq -\dot{x}_1(t)\frac{1}{x_1(t)^2}\left[\frac{1}{4} + \frac{2ma^2(a^2+1)}{(a^2-1)^2}\right],$$

integrate this from $t=0$ to $t=t^*$ to get:

$$\int_0^{t^*} \dot{x}_1(t)\ddot{x}_1(t)dt \leq \left[\frac{1}{4}+\frac{2ma^2(a^2+1)}{(a^2-1)^2}\right]\int_0^{t^*}\left(-\dot{x}_1(t)\frac{1}{x_1(t)^2}\right)dt,$$

$$\frac{1}{2}\dot{x}_1^2(t)|_0^{t^*} \leq \left[\frac{1}{4}+\frac{2ma^2(a^2+1)}{(a^2-1)^2}\right]\frac{1}{x_1(t)}|_0^{t^*}.$$

Note that $x_1(0)=A,\ \dot{x}_1(0)=0,\ x_1(t^*)=A_1$, then

$$\frac{1}{2}\dot{x}_1^2(t^*) \leq \left(\frac{1}{A_1}-\frac{1}{A}\right)\left[\frac{1}{4}+\frac{2ma^2(a^2+1)}{(a^2-1)^2}\right]. \tag{3.6}$$

At $t=t^*$, $\dot{x}_2(t^*)=0,\ x_1(t^*)=A_1,\ x_2(t^*)=A_1/a$. As $E=-1$, consider the energy at $t=t^*$:

$$\begin{aligned}
-1 &= \dot{x}_1^2(t^*)-\left[\frac{1}{2x_1(t^*)}+\frac{m^2}{2x_2(t^*)}+\frac{2m}{x_1(t^*)+x_2(t^*)}+\frac{2m}{x_1(t^*)-x_2(t^*)}\right]\\
&= \dot{x}_1^2(t^*)-\frac{1}{A_1}\left[\frac{1}{2}+\frac{m^2a}{2}+\frac{2m}{1+\frac{1}{a}}+\frac{2m}{1-\frac{1}{a}}\right]\\
&= \dot{x}_1^2(t^*)-\frac{1}{A_1}\left[\frac{1}{2}+\frac{m^2a}{2}+\frac{4ma^2}{a^2-1}\right].
\end{aligned}$$

Applying inequality 3.6,

$$-1 \leq 2(\frac{1}{A_1}-\frac{1}{A})\left[\frac{1}{4}+\frac{2ma^2(a^2+1)}{(a^2-1)^2}\right]-\frac{1}{A_1}\left[\frac{1}{2}+\frac{m^2a}{2}+\frac{4ma^2}{a^2-1}\right]$$

$$=\frac{1}{A_1}\left[\frac{4ma^2(a^2+1)}{(a^2-1)^2}-\frac{m^2a}{2}-\frac{4ma^2}{a^2-1}\right]-\frac{1}{A}\left[\frac{1}{2}+\frac{4ma^2(a^2+1)}{(a^2-1)^2}\right]$$

$$=\frac{1}{A_1}\frac{ma[16a-m(a^2-1)^2]}{2(a^2-1)^2}-\frac{1}{A}\left[\frac{1}{2}+\frac{4ma^2(a^2+1)}{(a^2-1)^2}\right].$$

Applying inequality 3.4 to this gives,

$$-1 \leq -\frac{1}{A}\left[\frac{1}{2}+\frac{4ma^2(a^2+1)}{(a^2-1)^2}\right].$$

Then

$$A \geq \frac{1}{2} + \frac{4ma^2(a^2+1)}{(a^2-1)^2}. \tag{3.7}$$

Hence, if body 2 has a turning point for $t \in (0,t_1)$, $A \geq \frac{1}{2} + \frac{4ma^2(a^2+1)}{(a^2-1)^2}$, where $16a \leq m(a^2-1)^2,\ a > 1$.

Define A_0= inf $\{\ A \mid$ body 2 has at least one turning point for $t \in (0,t_1)\ \}$. Since the inequalities 3.7 and 3.4 hold, $A_0 \geq \frac{1}{2} + \frac{4ma^2(a^2+1)}{(a^2-1)^2} > \frac{1}{2}$. Also, by the definition of A_0, body 2 has no turning point for $t \in (0,t_1)$ whenever $A < A_0$.

For each $A \in \{\ A \mid$ body 2 has at least one turning point for $t \in (0,t_1]\ \}$, there exists some t^* such that $\dot{x}_2(t^*) = 0$, and $\ddot{x}_2(t^*) \leq 0$.

We are going to show that when $A = A_0$, there exists some $t^* < t_1$ such that $\dot{x}_2(t^*) = \ddot{x}_2(t^*) = 0$. Since $\dot{x}_2(t)$ is continuous for $t \in (0,t_1)$, and $\lim_{t\to 0^+} \dot{x}_2(t) = +\infty$, the proof can be ended by two cases:

(i) When $A = A_0$, $\dot{x}_2(t) > 0$, for any $t \in (0,t_1)$.

Define a such that $16a = m(a^2-1)^2,\ a > 1$. Consider the function $x_1(t) - ax_2(t)$. For any given A, $x_1(t) - ax_2(t)$ is a continuous function for $t \in [0,t_1]$ according to the regularization theory of BC and SBC [22]. Note that $x_1(0) - ax_2(0) = A > 0$ and $x_1(t_1) - ax_2(t_1) = (1-a)x_2(t_1) < 0$. Then for any A, there exists some time t_a, such that

$$x_1(t_a) - ax_2(t_a) = 0. \tag{3.8}$$

Further,

$$\ddot{x}_2(t_a) = \frac{1}{x_2^2(t_a)}\left[-\frac{4}{m} - \frac{1}{(a+1)^2} + \frac{1}{(a-1)^2}\right] = \frac{16a - m(a^2-1)^2}{4(a^2-1)^2 x_2^2(t_a)} = 0.$$

Note that $\dot{x}_1(t_a) < 0$ and $\dot{x}_2(t_a) > 0$, then $\dot{x}_1(t_a) - a\dot{x}_2(t_a) < 0$. Applying the implicit function theorem to 3.8, t_a is continuous with respect to A. So is $\dot{x}_2(t_a)$. Since $\dot{x}_2(t_a) > 0$ at $A = A_0$, by continuity there must exist an open interval $[A_0 - \varepsilon_0, A_0 + \varepsilon_0]$ such that $\dot{x}_2(t_a) > 0$. Because $\dot{x}_2(t)$ realizes the minimum at some t_a, $\dot{x}_2(t) > \dot{x}_2(t_a) > 0$ for $t \in (0,t_1)$.

Therefore, for $A \in [A_0 - \varepsilon_0, A_0 + \varepsilon_0]$, the second body has no turning point. Contradiction to the definition of A_0!

(ii) When $A = A_0$, there exists some time $t^* \in (0,t_1)$ such that $\dot{x}_2(t^*) = 0$.

Assume at $A = A_0$, there is no time t such that $\dot{x}_2(t) = \ddot{x}_2(t) = 0$. So $\ddot{x}_2(t^*) \neq 0$, which implies that $\dot{x}_2(t^*)$ is not the minimum of $\dot{x}_2(t)$ for $t \in (0,t_1)$. Then there exists some time t such that $\dot{x}_2(t) < 0$, i.e., at $A = A_0$, body 2 has at least one turning point.

Case 1 tells us that when $A = A_0$, there exists some t_a, such that $\dot{x}_2(t_a) \leq 0$, $\ddot{x}_2(t_a) = 0$. Note that $\dot{x}_2$ is a continuous function of two variables t and A for $t \in (0,t_1)$ and $A > 0$.

If $\dot{x}_2(t_a) < 0$ at $A = A_0$, by continuity there exists a δ, such that $\dot{x}_2(t_a) < 0$ for $(t,A) \in (t_a - \delta, t_a + \delta) \times (A_0 - \delta, A_0 + \delta)$. Then for $A \in (A_0 - \delta, A_0 + \delta)$, body 2 has at least one turning point. Contradiction!

Therefore, when $A = A_0$, there must exist some $t^* = t_a < t_1$ such that $\dot{x}_2(t^*) = \ddot{x}_2(t^*) = 0$.

Differentiate equation 3.2 with respect to t and evaluate at t^*:

$$\dddot{x}_2(t^*) = \frac{m\dot{x}_2(t^*)}{2x_2^3(t^*)} + \frac{2(\dot{x}_1(t^*) + \dot{x}_2(t^*))}{[x_1(t^*) + x_2(t^*)]^3} - \frac{2(\dot{x}_1(t^*) - \dot{x}_2(t^*))}{[x_1(t^*) - x_2(t^*)]^3}.$$

Since $\dot{x}_2(t^*) = 0$, $\ddot{x}_2(t^*) = 0$, $\dot{x}_1(t^*) < 0$, $x_1(t^*) > x_2(t^*) > 0$,

$$\dddot{x}_2(t^*) = 2\dot{x}_1(t^*)\left[\frac{1}{[x_1(t^*) + x_2(t^*)]^3} - \frac{1}{[x_1(t^*) - x_2(t^*)]^3}\right] > 0,$$

which means body 2 will keep moving towards to body 1 when time passes $t = t^*$. Hence, when $A = A_0$, body 2 also has no turning point for $t \in (0,t_1)$.

Therefore, body 2 has no turning point for $t \in (0,t_1)$ whenever $A \leq A_0$, where t_1 is the time of the first SBC.

$\square$

Remark: When $A = A_0$, $16a = m(a^2-1)^2$, $a > 1$ and $A_0 \geq \frac{1}{2} + \frac{4ma^2(a^2+1)}{(a^2-1)^2}$. In the special case $m = 1$, a satisfies $16a = (a^2-1)^2$ with $a > 1$, $a \approx 2.766$. Then

$$A_0 \geq \frac{1}{2} + \frac{4a^2(a^2+1)}{(a^2-1)^2} \approx 6.484.$$

3.2 EXISTENCE OF THE PERIODIC ORBIT

Recall that in Section 3.1.2,

$$\begin{aligned}\Gamma &= \frac{dt}{ds}(H-E) \\ &= \frac{1}{16}Q_2^2P_1^2 + \frac{Q_2^2P_1^2 - 4Q_1Q_2P_1P_2 + 4Q_1^2P_2^2}{16m} - m^2Q_1^2 - 2mQ_2^2 - \frac{2mQ_1^2Q_2^2}{Q_1^2+Q_2^2} - \frac{Q_1^2Q_2^2}{2Q_1^2+Q_2^2} - Q_1^2Q_2^2E,\end{aligned}$$

where $E=-1$ is the total energy. The initial conditions at $s=0$ are:

$$Q_1(0)=R, \qquad Q_2(0)=0, \qquad P_1(0)=0, \qquad P_2(0)=2m^{\frac{3}{2}}.$$

By Theorem 3.1, when $0<R=\sqrt{A}\le\sqrt{A_0}$, $Q_2^2=2x_2$ increases from $s=0$ to $s=s_1$, where s_1 is the time when the first SBC happens.

The equations of motion from the regularized Hamiltonian Γ are:

$$Q_1' = \frac{1+m}{8m}Q_2^2P_1 - \frac{1}{4m}Q_1Q_2P_2, \tag{3.9}$$

$$Q_2' = \frac{1}{2m}Q_1^2P_2 - \frac{1}{4m}Q_1Q_2P_1, \tag{3.10}$$

$$P_1' = \frac{1}{4m}P_1P_2Q_2 - \frac{1}{2m}Q_1P_2^2 + 2m^2Q_1 + \frac{4mQ_1Q_2^4}{(Q_1^2+Q_2^2)^2} + \frac{2Q_1Q_2^4}{(2Q_1^2+Q_2^2)^2} - 2Q_1Q_2^2, \tag{3.11}$$

$$P_2' = \frac{1}{4m}P_1P_2Q_1 - \frac{1+m}{8m}Q_2P_1^2 + 4mQ_2 + \frac{4mQ_1^4Q_2}{(Q_1^2+Q_2^2)^2} + \frac{4Q_1^4Q_2}{(2Q_1^2+Q_2^2)^2} - 2Q_1^2Q_2. \tag{3.12}$$

where $'$ is the derivative with respect to s. The initial conditions are

$$Q_1(0)=R, \qquad Q_2(0)=0, \qquad P_1(0)=0, \qquad P_2(0)=2m^{\frac{3}{2}}.$$

At the time s_1 when the first SBC happens,

$$Q_1(s_1)=0, \qquad Q_2(s_1)=R_1>0, \qquad P_1(s_1)=-\frac{8m}{\sqrt{2m+2}}.$$

To prove the existence of the periodic orbit, we are going to find a value of R, such that $P_2(s_1) = 0$.

Theorem 3.2. *For the differential equations 3.9 to 3.12, let the initial conditions be $Q_1(0) = R$, $Q_2(0) = 0$, $P_1(0) = 0$ and $P_2(0) = 2m^{\frac{3}{2}}$, where $R \in (0, \sqrt{A_0}]$. Assume $Q_2(s_1) > 0$, $Q_1(s_1) = 0$, where s_1 is the time of first SBC. Also, assume $Q_2 > 0$ for $0 < s \leq s_1$. Then $P_2(s_1, R)$ is a continuous function of R.*

Proof. Since the Hamiltonian Γ is regularized, the solution $P_i = P_i(s,R)$ and $Q_i = Q_i(s,R)$ are continuous functions with respect to s and R. We are going to show $s_1 = s_1(R)$ is a continuous function of R. In order to apply the implicit function theorem for $Q_1 = Q_1(s_1,R) = 0$, we need to show that $(\partial Q_1/\partial s)(s_1,R) \neq 0$. By the regularized Hamiltonian Γ,

$$\frac{\partial Q_1}{\partial s}\ |_{(s_1,R)} = \Gamma_{P_1}\ |_{(s_1,R)} = \left[\frac{1+m}{8m} Q_2^2 P_1 - \frac{1}{4m} Q_1 Q_2 P_2\right] |_{(s_1,R)}$$

Note that for fixed R, $\Gamma = 0$ at any time s. At $s = s_1$, $Q_1 = Q_1(s_1,R) = 0$, then $P_1 = P_1(s_1,R) = 4$. Therefore,

$$\frac{\partial Q_1}{\partial s}\ |_{(s_1,R)} = \left[\frac{1+m}{8m} Q_2^2 P_1 - \frac{1}{4m} Q_1 Q_2 P_2\right] |_{(s_1,R)} = -\sqrt{\frac{m+1}{2}} Q_2^2(s_1,R) < 0.$$

By the implicit function theorem, s_1 is a continuous function of R. Then $P_2(s_1,R)$ is also a continuous function of R. $\square$

Corollary 3.3. *There exists R such that $P_2(s_1) = P_2(s_1,R) = 0$.*

Proof. First, we show that there exists an $R > 0$ such that $P_2(s_1) > 0$.

From equations 3.9-3.12,

$$(P_1Q_1 + P_2Q_2)' = P_1'Q_1 + P_1Q_1' + P_2'Q_2 + P_2Q_2'$$

$$= 4mQ_2^2 + 2m^2Q_1^2 + 2Q_1^2Q_2^2\left[\frac{2m}{Q_1^2+Q_2^2} + \frac{1}{2Q_1^2+Q_2^2} - 2\right]$$

Note that for $t \in [0,t_1]$, $x_1(t)$ is decreasing, $x_2(t)$ is increasing and $A \geq x_1(t) > x_2(t) \geq 0$. Then $0 \leq Q_1^2 = x_1 - x_2 \leq A$, $2x_1(t_1) = 2x_2(t_1) = Q_2^2(s_1) = R_1^2 < 2A = 2R^2$ for $s \in [0,s_1]$. Thus

$$\frac{2m}{Q_1^2+Q_2^2} + \frac{1}{2Q_1^2+Q_2^2} \geq \frac{2m}{3R^2} + \frac{1}{4R^2}.$$

Choose $R = \sqrt{\frac{m}{3}}$,

$$\frac{2m}{Q_1^2+Q_2^2} + \frac{1}{2Q_1^2+Q_2^2} - 2 \geq \frac{3}{4m} > 0,$$

so $(P_1Q_1+P_2Q_2)' \geq 0$ for $s \in [0,s_1]$, or $P_1Q_1+P_2Q_2$ is increasing for $s \in [0,s_1]$.

From the initial conditions,

$$(P_1Q_1+P_2Q_2)\mid_{s=0} = 0,$$

and $(P_1Q_1+P_2Q_2)'$ is not identically equal to 0; hence

$$0 < (P_1Q_1+P_2Q_2)\mid_{s=s_1} = R_1P_2(s_1).$$

Therefore, when $R = \sqrt{\frac{m}{3}},\ P_2(s_1) > 0$.

Next, we show that $P_2(s_1) < 0$ when $R^2 = A_0$.

At $A = A_0$, by the proof of theorem 3.1, there exists a time $t^* < t_1$, such that $\dot{x}_2(t^*) = 0$ and $\dot{x}_1(t^*) < 0$. Then $\dot{x}_1(t^*) + m\dot{x}_2(t^*) < 0$. Consider the sum of the Newtonian equations 3.1 and 3.2:

$$\ddot{x_1} + m\ddot{x_2} = -\frac{1}{4x_1^2} - \frac{m^2}{4x_2^2} - \frac{2m}{(x_1+x_2)^2} < 0,$$

which means $\dot{x}_1(t) + m\dot{x}_2(t)$ is a decreasing function with respect to t. Hence,

$$\lim_{t\to t_1}[\dot{x}_1(t)+m\dot{x}_2(t)] < \dot{x}_1(t_a)+m\dot{x}_2(t_a) < 0.$$

Note that $P_2(s_1)/[2Q_2(s_1)] = \lim_{t\to t_1} p_2(t) = \lim_{t\to t_1}[\dot{x}_1(t)+m\dot{x}_2(t)] < 0$, and $Q_2(s_1) > 0$, then $P_2(s_1) < 0$.

By continuity, there must exist an R, such that $P_2(s_1) = 0$ where s_1 is the time when the first SBC happens.

□

Theorem 3.4. *If R satisfies $P_2(s_1) = P_2(s_1,R) = 0$, then the orbit will be a Schubart-like periodic orbit.*

Proof. At time $s = 0$, a BC happens between bodies 2 and 3. At time $s = s_1$, a SBC occurs. Since the system is regularized, the solution $\{P_i, Q_i\}$ $(i = 1,2)$ is continuous.

At time $s = 0$,

$$Q_1(0) = R, \quad Q_2(0) = 0, \quad P_1(0) = 0, \quad P_2(0) = 2m^{\frac{3}{2}}.$$

At time $s = s_1$,

$$Q_1(s_1) = 0, \quad Q_2(s_1) = R_1, \quad P_1(s_1) = -\frac{8m}{\sqrt{2m+2}}, \quad P_2(s_1) = 0,$$

where R_1 is a positive number.

From the Hamiltonian Γ, we can see that $Q_1'(s_1) = \frac{1}{4}Q_2(Q_2P_1 - Q_1P_2) < 0$, $Q_2'(s_1) = \frac{1}{4}Q_1(2Q_1P_2 - Q_2P_1) = 0$, $Q_2''(s_1) < 0$. This means that $Q_2(s_1)$ is a relative maximum of Q_2. In other words, when time passes s_1, Q_2 will decrease. Similarly, Q_1 will decrease when time passes s_1.

Compare the motion for $s \in [0, s_1]$ and the motion for $s \in [s_1, s_2]$. By the uniqueness of the regularized Hamiltonian system and symmetry, the orbit from $s = s_1$ to $s = s_2$ will be the same trajectory from $s = s_1$ to $s = 0$ by reversing the direction of each velocity. Then at the time $s = s_2$ when the second BC occurs,

$$Q_1(s_2) = -R, \quad Q_2(s_2) = 0, \quad P_1(s_2) = 0, \quad P_2(s_2) = -2m^{\frac{3}{2}}.$$

Further, $s_2 = 2s_1$.

By symmetry and uniqueness again, at time $s = 3s_1$,

$$Q_1(3s_1) = 0, \quad Q_2(3s_1) = -R_1, \quad P_1(3s_1) = \frac{8m}{\sqrt{2m+2}}, \quad P_2(3s_1) = 0.$$

At time $s = 4s_1$,

$$Q_1(4s_1) = R, \quad Q_2(4s_1) = 0, \quad P_1(4s_1) = 0, \quad P_2(4s_1) = 2m^{\frac{3}{2}},$$

which is exactly the same as the initial condition at $s = 0$. Then the orbit from $s = 0$ to $s = 4s_1$ generates one period.

$\square$

The following figure is a picture of the periodic solution for $m = 1$ in terms of $\{Q_1, Q_2, P_1, P_2\}$. The initial conditions are

$$Q_1(0) = 2.295, \qquad Q_2(0) = 0, \qquad P_1(0) = 0, \qquad P_2(0) = 2.$$

The horizontal axis represents time s. At $s = 0$, it is a BC between the inner two bodies. As in the picture, this Schubart-like periodic orbit in the regularized coordinate system has a D^2 symmetry and a time-reverse symmetry.

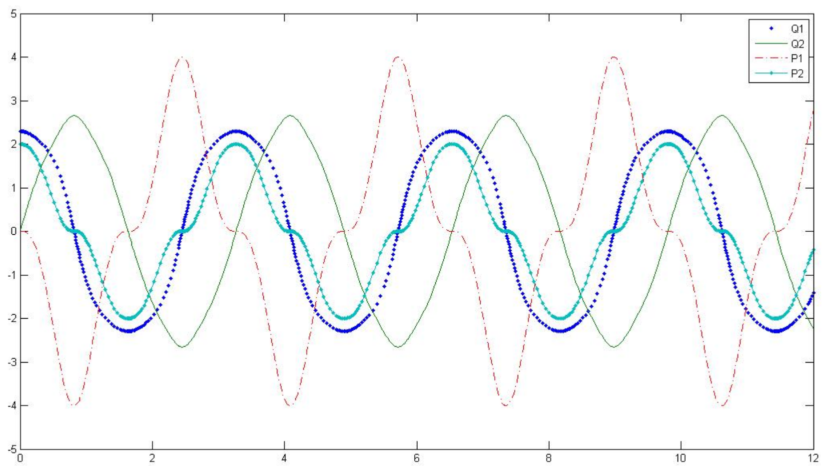

Figure 3.3: Schubart-like periodic orbit with equal masses.

Chapter 4. Periodic Solutions with Singularities in Two Dimensions in the $2N$-body Problem

In this chapter, we first present a technique for generating a periodic orbit in the two-dimensional four-body problem with singularities. We begin in section **4.1.1** by giving a description of the proposed orbit and prove its existence. Section **4.1.2** will present the numerical methods used to produce the initial conditions that will lead to this orbit. Following this, in section **4.2**, we consider variants on the orbit we generate, giving a family of orbits with singularities with an even number of masses.

We present a family of configurations that are symmetric in both initial positions and velocities. These initial conditions will lead to arbitrarily many simultaneous binary collisions, with each body alternating between collisions with its two nearest neighbors. Due to the abundance of symmetries present in the configurations, we can reduce the number of variables that need to be studied to four–two representing position and two representing momentum. In contrast to its one-dimensional counterparts, the proof for existence of this orbit is surprisingly simple.

4.1 The Proposed Orbit

4.1.1 Analytical Description. Initially we focused on finding a symmetric, periodic SBC orbit for four equal masses in two dimensions. Without loss of generality, we assume that the orbit begins with the four bodies lying at $(\pm 1, 0)$ and $(0, \pm 1)$ in the standard coordinate plane, numbered from 1 to 4 as in Figure 1. The initial velocities for each body are given as $(0, \pm v)$ and $(\mp v, 0)$, respectively, where $v \in (0, +\infty)$.

The singularity of SBC in this problem is not essential. For a better understanding of the behavior of the motion of the bodies in a neighborhood of a collision, the standard technique is to make a change of coordinates and rescale time. In the new coordinates, the orbits which approach collision can be extended across the collision in a smooth manner with respect to the new time variable. This technique is called *regularization*. In our problem, the regularization describes the behavior of the bodies approaching and escaping collisions, similar to the collisions of billiard balls.

Due to the symmetry of the initial conditions and the equations governing the motion of the bodies, the

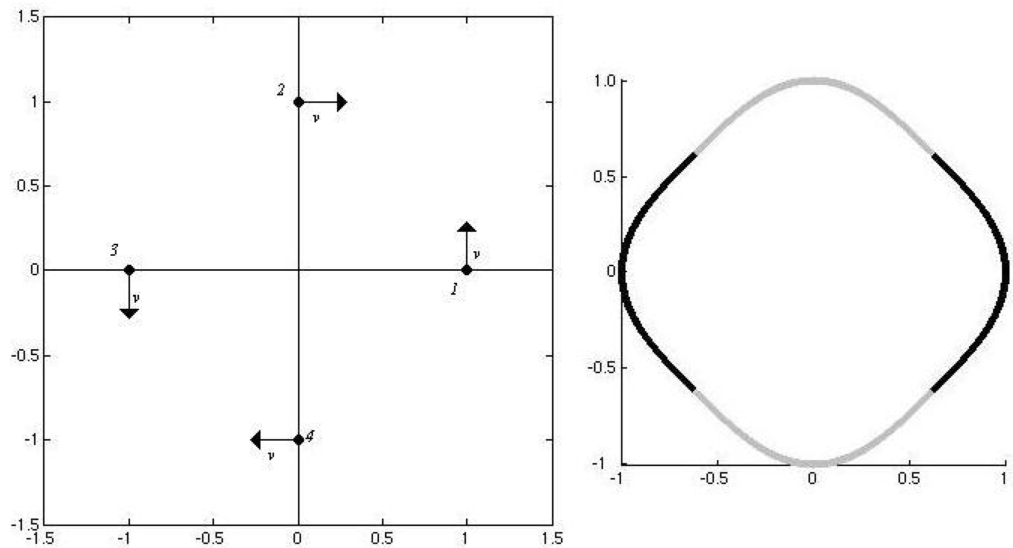

Figure 4.1: On the left, we illustrate the initial conditions leading to the four-body two-dimensional periodic SBC orbit. On the right, the orbit is shown.

symmetry that is present in the initial conditions is maintained in the regularized sense.

Main Theorem. *Let $E = T - U$ be the total energy and m be the mass for each of the four bodies. For any $E < 0$ and $m > 0$, there exists a symmetric, periodic, four-body orbit with SBC in $\mathbb{R}^2$.*

Without loss of generality, we can assume $m = 1$ and the initial positions are as illustrated in Figure 1. The proof will be given at the end of this section.

Let t_0 be the time of first SBC. For $t \in [0, t_0)$, let the coordinate of body 1 be (x_1, x_2). By symmetry, the coordinates of bodies 2, 3, and 4 are (x_2, x_1), $(-x_1, -x_2)$ and $(-x_2, -x_1)$, respectively. Using equation (1.1), the acceleration of a body at point (x_1, x_2) is given by:

$$(\ddot{x_1}, \ddot{x_2}) = -\left[\frac{(x_1 - x_2, x_2 - x_1)}{(2(x_1 - x_2)^2)^{\frac{3}{2}}} + \frac{(2x_1, 2x_2)}{(4x_1^2 + 4x_2^2)^{\frac{3}{2}}} + \frac{(x_1 + x_2, x_1 + x_2)}{(2(x_1 + x_2)^2)^{\frac{3}{2}}}\right] \tag{4.1}$$

We now perform the regularization of the system. The system has the Hamiltonian:

$$H = \frac{1}{8}(w_1^2 + w_2^2) - \frac{\sqrt{2}}{x_1 - x_2} - \frac{\sqrt{2}}{x_1 + x_2} - \frac{1}{\sqrt{x_1^2 + x_2^2}} \tag{4.2}$$

where $w_1 = 4\dot{x}_1$ and $w_2 = 4\dot{x}_2$ are the conjugate momenta to x_1 and x_2. Note that SBC happens when $x_1 = \pm x_2$. We introduce a new set of coordinates:

$$q_1 = x_1 - x_2, \qquad q_2 = x_1 + x_2.$$

Their conjugate momenta p_i are given by using a generating function $F = (x_1 - x_2)p_1 + (x_1 + x_2)p_2$:

$$w_1 = p_1 + p_2, \qquad w_2 = p_2 - p_1.$$

The Hamiltonian corresponding to the new coordinate system is

$$H = \frac{1}{4}(p_1^2 + p_2^2) - \frac{\sqrt{2}}{q_1} - \frac{\sqrt{2}}{q_2} - \frac{\sqrt{2}}{\sqrt{q_1^2 + q_2^2}}. \tag{4.3}$$

Following the work of Sweatman [28], we introduce another canonical transformation:

$$q_i = Q_i^2, \qquad P_i = 2Q_i p_i \qquad (i = 1, 2).$$

We also introduce a new time variable s, which satisfies $\frac{dt}{ds} = q_1 q_2$. This produces a regularized Hamiltonian in extended phase space:

$$\begin{aligned} \Gamma &= \frac{dt}{ds}(H - E) \\ &= \frac{1}{16}(P_1^2 Q_2^2 + P_2^2 Q_1^2) - \sqrt{2}(Q_1^2 + Q_2^2) - \frac{\sqrt{2}Q_1^2 Q_2^2}{\sqrt{Q_1^4 + Q_2^4}} - Q_1^2 Q_2^2 E \end{aligned} \tag{4.4}$$

where E is the total energy of the Hamiltonian H.

The regularized Hamiltonian gives the following differential equations of motion:

$$Q_1' = \frac{1}{8} P_1 Q_2^2 \tag{4.5}$$

$$Q_2' = \frac{1}{8} P_2 Q_1^2 \tag{4.6}$$

$$P_1' = -\frac{1}{8} P_2^2 Q_1 + 2\sqrt{2} Q_1 + \frac{2\sqrt{2} Q_1 Q_2^2}{\sqrt{Q_1^4 + Q_2^4}} - \frac{2\sqrt{2} Q_1^5 Q_2^2}{(Q_1^4 + Q_2^4)^{\frac{3}{2}}} + 2E Q_1 Q_2^2 \tag{4.7}$$

$$P_2' = -\frac{1}{8}P_1^2 Q_2 + 2\sqrt{2}Q_2 + \frac{2\sqrt{2}Q_2 Q_1^2}{\sqrt{Q_1^4+Q_2^4}} - \frac{2\sqrt{2}Q_2^5 Q_1^2}{(Q_1^4+Q_2^4)^{\frac{3}{2}}} + 2EQ_2Q_1^2 \tag{4.8}$$

with initial conditions

$$Q_1(0) = 1, \quad Q_2(0) = 1, \quad P_1(0) = -4v, \quad P_2(0) = 4v \tag{4.9}$$

where derivatives are with respect to s, and E is the total energy of the Hamiltonian H.

Theorem 4.1. *Let s_0 be the time of the first SBC in the regularized system. Then s_0 is a continuous function with respect to the initial velocity v. Furthermore,*

$$p_2(t_0) = \frac{P_2(s_0,v)}{2Q_2(s_0,v)}$$

is also continuous with respect to v.

Proof. At the first SBC, $Q_1(s_0) = 0$, and $Q_2(s_0) = \sqrt{q_2} = \sqrt{x_1+x_2} > 0$. Our goal is to show that $p_2(t_0)$ is a continuous function with respect to v.

Because $\Gamma = 0$ at $s = s_0$, $P_1(s_0) = -4\sqrt[4]{2}$ from 4.4. Since Γ is regularized, the solution $P_i = P_i(s,v)$ and $Q_i = Q_i(s,v)$ are continuous functions with respect to the two variables s and v. At time $s = s_0$,

$$0 = Q_1(s_0(v),v).$$

To apply the implicit function theorem, we need to show that

$$\frac{\partial Q_1}{\partial s}(s_0,v) \neq 0.$$

From (4.5)

$$\frac{\partial Q_1}{\partial s}(s_0,v) = \frac{1}{8}P_1Q_2^2 \mid_{(s_0,v)} = -\frac{1}{2}\sqrt[4]{2}Q_2(s_0)^2 < 0.$$

So $s_0 = s_0(v)$ is a continuous function of v. Therefore both $P_2(s_0,v)$ and $Q_2(s_0,v)$ are continuous functions of v. Further, since $Q_2(s_0,v) > 0$, $p_2(t_0)$ is also a continuous function of v. $\square$

Theorem 4.2. *There exists a $v = v_0$ such that $\dot{x}_1(t_0) + \dot{x}_2(t_0) = \frac{1}{2} p_2(t_0) = 0$, where t_0 is the time of the first SBC, i.e. the net momentum of bodies 1 and 2 at the first SBC is 0.*

The outline of this proof is as follows: We will show that there exist v_1 and v_2 such that $\dot{x}_1 + \dot{x}_2$ is negative at SBC for $v = v_1$ and positive at SBC for $v = v_2$. The result then follows by Theorem 4.1.

Proof. Consider Newton's equation before the time of the first SBC:

$$\ddot{x}_1 = \frac{x_2 - x_1}{2\sqrt{2}(x_1 - x_2)^3} - \frac{2x_1}{8(x_1^2 + x_2^2)^{3/2}} - \frac{x_1 + x_2}{2\sqrt{2}(x_1 + x_2)^3}, \tag{4.10}$$

$$\ddot{x}_2 = \frac{x_1 - x_2}{2\sqrt{2}(x_1 - x_2)^3} - \frac{2x_2}{8(x_1^2 + x_2^2)^{3/2}} - \frac{x_1 + x_2}{2\sqrt{2}(x_1 + x_2)^3}. \tag{4.11}$$

Therefore,

$$\ddot{x}_1 + \ddot{x}_2 = -\frac{x_1 + x_2}{4(x_1^2 + x_2^2)^{3/2}} - \frac{1}{\sqrt{2}(x_1 + x_2)^2} < 0, \tag{4.12}$$

which means $\dot{x}_1 + \dot{x}_2$ is decreasing with respect to t.

At the initial time $t = 0$, $x_1 = 1$, $x_2 = 0$, $\dot{x}_1 = 0$, and $\dot{x}_2 = v$. Note that for $v \in (0, \infty)$, there is no triple collision or total collision for $t \in [0, t_0]$, where t_0 is the time of the first SBC. Also, from the initial time to t_0, $0 \le x_2 \le x_1 \le 1$, $0 < x_1 + x_2 < 2$, and $x_1^2 + x_2^2 < 4$.

Let $y(t) = x_1(t) + x_2(t)$. Then for any choice of v, $\ddot{y}(t) < 0$ and $0 < y(t) < 2$ hold for any $t \in [0, t_0]$. In other words, $\dot{y}(t)$ is decreasing with respect to t.

First, we will show that there exists v_1 such that $\dot{y}(t_0) < 0$. When $v = 0$ the four bodies form a central configuration and, as a consequence, the motion of the four bodies leads to total collapse. Consider the time interval $t \in [0, t_0/2)$. In this interval, the differential equations (4.10) and (4.11) have no singularity, and $\ddot{y}(t_0/2) < 0$. By continuous dependence on initial conditions, $\dot{y}(t_0/2) = \dot{x}_1(t_0/2) + \dot{x}_2(t_0/2)$ is a continuous function with respect to the initial velocity v. When $v = 0$, $\dot{x}_1(t_0/2) < 0$, $\dot{x}_2(t_0/2) < 0$, which gives $\dot{y}(t_0/2) < 0$. Therefore, there exists a $\delta > 0$, such that $\dot{y}(t_0/2) < 0$ holds for any $v \in (-\delta, \delta)$.

Choose $v_1 = \delta/2$, then $\dot{y}(t_0/2) < 0$. Because $\dot{y}(t)$ is decreasing with respect to t, $\dot{y}(t_0) \le \dot{y}(t_0/2) < 0$.

Next we will show that there exists v_2 big enough, such that $\dot{y}(t_0) > 0$. Note that as $v \to \infty$,

$$\lim_{v\to\infty} y(t_0) = \lim_{v\to\infty} x_1(t_0) + x_2(t_0) = 2$$

and

$$\lim_{v\to\infty} \dot{y}(t_0) = \infty.$$

Therefore there exists some positive value v_2, such that $\dot{y}(t_0) > 0$. □

Proof of the Main Theorem. From Theorem 4.2, we know there exists an initial velocity $v = v_0$ such that $\dot{x}_1(t_0) + \dot{x}_2(t_0) = 0$. Let $\{P_1, P_2, Q_1, Q_2\}$ for $s \in [0, s_0]$ be the solution in the regularized system corresponding to the orbit from $t = 0$ to $t = t_0$. Following collision, consider the behavior of the first and second bodies. Assume their velocity was reflected about the $y = x$ line in the plane. In the new coordinate system, this corresponds to a new set of functions

$$\{-P_1(2s_0 - s), -P_2(2s_0 - s), -Q_1(2s_0 - s), -Q_2(2s_0 - s)\}$$

for $s \in [s_0, 2s_0]$. We can easily check that

$$\{-P_1(2s_0 - s), -P_2(2s_0 - s), -Q_1(2s_0 - s), -Q_2(2s_0 - s)\}$$

for $s \in [s_0, 2s_0]$ is also a set of solutions for equations (4.5) through (4.8) with initial conditions at $s = s_0$. Also, $\{P_1(s), P_2(s), Q_1(s), Q_2(s)\}$ for $s \in [s_0, 2s_0]$ satisfies equations (4.5) through (4.8) with the same initial conditions at $s = s_0$. Note that equations (4.5) through (4.8) with initial conditions at $s = s_0$ have a unique solution for any choice of $v \in (0, \infty)$. Then by uniqueness, the orbit for $s \in [s_0, 2s_0]$ must be the same as the orbit for $s \in [0, s_0]$ in reverse, i.e.

$$P_i(s) = -P_i(2s_0 - s), Q_i(s) = -Q_i(2s_0 - s)$$

for $s \in [0, s_0]$. Therefore at time $s = 2s_0$, bodies 1 and 2 will have returned to their initial positions with velocities $(0, -v)$ and $(-v, 0)$ respectively. Similarly, at time $s = 2s_0$, bodies 3 and 4 will have also returned to their initial positions with velocities $(0, v)$ and $(v, 0)$ respectively.

Next, we use symmetry and uniqueness to show the orbit from $s = 2s_0$ to $s = 4s_0$ and the orbit from $s = 0$ to $s = 2s_0$ will be symmetric with respect to the $y-$axis. Compare the motion of body 2 and body 3 from $s = 2s_0$ to $s = 4s_0$ with the motion of body 2 and body 1 from time $s = 0$ to $s = 2s_0$. The initial conditions of body 3 at $s = 2s_0$ and the initial conditions of body 1 at $s = 0$ are symmetric with respect to the $y-$axis. Also the initial conditions of body 2 at $s = 2s_0$ and the initial conditions of body 4 at $s = 0$ are symmetric with respect to the $x-$axis. Therefore, by uniqueness, the orbit of bodies 2 and 3 from $s = 2s_0$ to $s = 4s_0$ and the orbit of bodies 1 and 2 from $s = 0$ to $s = 2s_0$ must be symmetric with respect to $y-$axis. Therefore, the orbit of bodies 1 and 4 from $s = 2s_0$ to $s = 4s_0$ and and the orbit of bodies 3 and 4 from $s = 0$ to $s = 2s_0$ are symmetric with respect to the $y-$axis.

Hence, at $s = 4s_0$, the positions and velocities of the four bodies are exactly the same as at $s = 0$. Therefore, the orbit is periodic with period $s = 4s_0$. □

It is worth noting here that the previous proof implies a time-reversing symmetry for the periodic orbit. This provides further evidence for the conjecture made by Roberts [21], stating that linearly stable periodic orbits in the equal mass n-body problem must have a time-reversing symmetry.

4.1.2 Numerical Method. As is the case for all periodic orbits of the n-body problem, the value of the Hamiltonian needs to be negative. Using the initial positions of the four bodies described earlier, it is not hard to find the potential energy at $t = 0$:

$$U = 2\sqrt{2} + 1.$$

Then, acting under the negative Hamiltonian assumption:

$$2\sqrt{2} + 1 \geq \sum_{i=1}^{n} \frac{m_i |v_i|^2}{2}.$$

Since all masses are equal, if we require that the velocities of each body are equal in magnitude, we obtain:

$$v_{max} = \sqrt{\frac{2\sqrt{2}+1}{2}} \tag{4.13}$$

with v_{max} defined to be the value of v such that the value of the Hamiltonian is zero. Define $\frac{v}{v_{max}} = \theta$. This parameter is used in the numerical algorithm.

At this point it becomes necessary to find out just how much kinetic energy is required to obtain the periodic orbit. Since we know suitable bounds on the velocity parameter ($\theta \in (0,1)$), we can search the interval numerically. We use an n-body simulator with the initial positions previously described. The simulation is run until one SBC occurs. For simplicity, we consider only the collision between the first and second bodies in the first quadrant. Summing their velocities immediately before the collision gives a vector running along the line $y = x$ (due to symmetry), with both components having the same sign. The magnitude of this vector is given in Figure 2. Negative magnitudes represent vectors with both components less than zero.

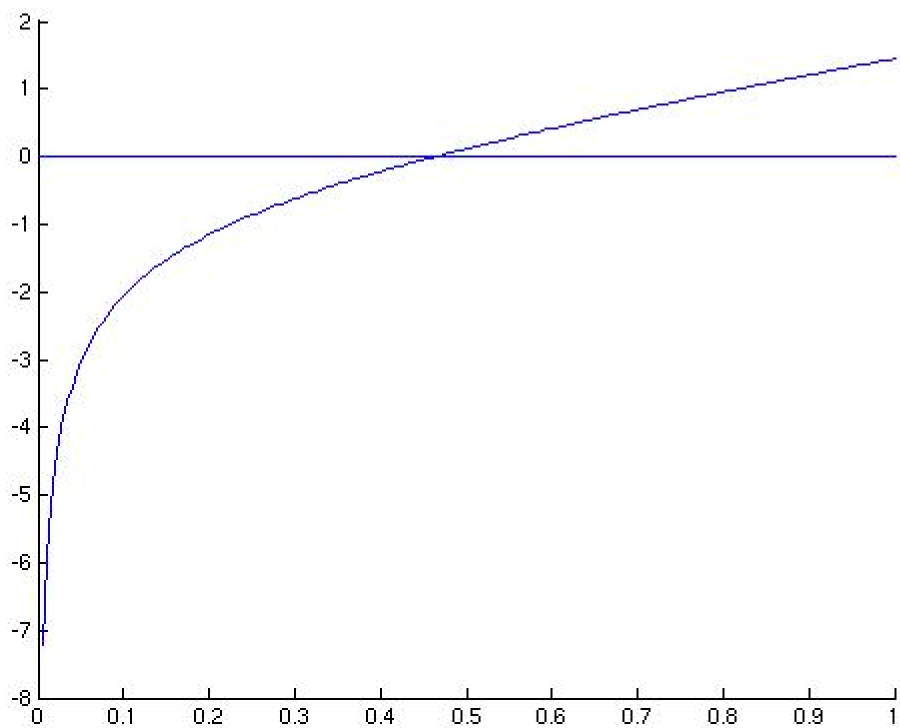

Figure 4.2: The magnitude of the net velocity of the first two bodies (vertical axis) at the time of collision for various values of θ (horizontal axis).

Next, a standard bisection method is used to find the amount of energy required to cause the net velocity at collision to be zero. Using the initial interval $\theta \in [0,1]$ and iterating to a tolerance of 10^{-7}, the correct value of θ was found to be $\theta \approx 0.4644954$.

4.2 Variants

This same technique can be adopted to find similar orbits for any arbitrary even number n. A key feature of these orbits will be higher numbers of simultaneous binary collisions. For a given value of n, initial positions are given by spacing the bodies evenly about the unit circle. The potential energy (and the value of v_{max}) is found numerically by iterating over each pair of planets and summing the reciprocal of the distances between them. (Recall that all $m_i = 1$.) Velocities are then assigned to the bodies in alternating counter-clockwise and clockwise directions, initially tangent to the circle. Again we consider the collision between the first and the second bodies. Although the net velocity of the two at collision will not lie along the $y = x$ line, the components of this vector will both have the same sign. The magnitudes of the net velocity between the first two bodies at initial collision are shown in Figure 3 for various values of n. Lower curves in the graph correspond to higher values of n. Again, negative magnitudes correspond to both components being negative.

Pictures of the orbit for $n = 6$ and $n = 8$ are shown in Figure 4. It is readily seen that as n increases, the shape of the orbit more closely approximates a circle.

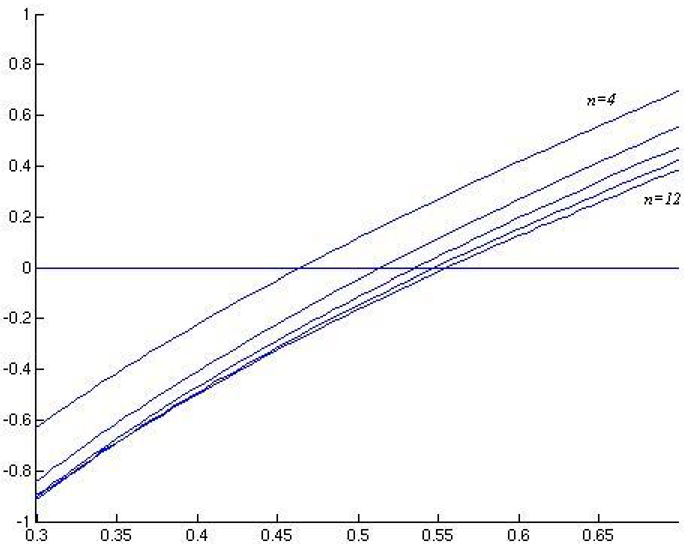

Figure 4.3: Curves showing the magnitude of the net velocity of the first two bodies (vertical axis) at the time of collision for various values of θ (horizontal axis) for $n = 4, 6, 8, 10, 12$.

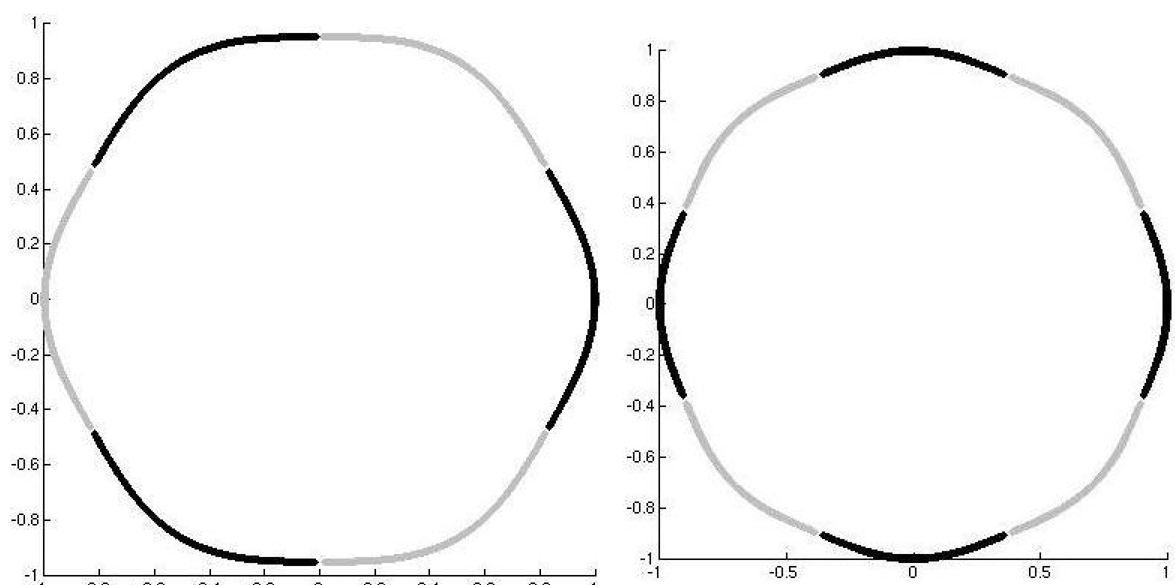

Figure 4.4: The six- and eight-body two-dimensional periodic SBC orbits.

Chapter 5. Linear Stability Analysis for the Two Periodic Orbits

In this chapter we apply the method of Roberts to prove the linear stability of the Schubart-like orbit in the symmetric collinear four body 1, m, m, 1 problem for certain values of m, and of the singular periodic orbit in the symmetric planar equal mass problem. In both settings, the linear stability is determined for the regularized equations only and is reduced to the rigorous numerical computation of a single real number. Our linear stability analysis determines values of m in the interval $[0,50]$ in the collinear problem for which the singular periodic orbit is linearly stable, and also shows that the $2D$ singular periodic orbit is linearly stable. These examples support and extend the conjecture made by Roberts [21] that the only linearly stable periodic orbits in the equal mass n-body problem are those that exhibit a time-reversing symmetry.

5.1 Linear Stability of Periodic Orbits

For a smooth function Γ defined on an open subset of $\mathbb{R}^{2n}$, suppose that $\gamma(s)$ is a T-periodic solution of a Hamiltonian system $z' = JD\Gamma(z)$ where $' = d/ds$,

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

and I is the appropriately sized identity matrix. The fundamental matrix solution $X(s)$ of the linearized equations along $\gamma(s)$,

$$\xi' = JD^2\Gamma(\gamma(s))\xi, \qquad \xi(0) = I \tag{5.1}$$

is symplectic and satisfies $X(s+T) = X(s)X(T)$ for all s. The matrix $X(T)$ is commonly called the monodromy matrix for γ, and it measures the non-periodicity of solutions to the linearized equations. The eigenvalues of $X(T)$ are the characteristic multipliers of γ, and determine the linear stability of the periodic solution γ. Linear stability therefore requires that all of the multipliers lie on the unit circle.

The characteristic multipliers may be obtained by solving (5.1) with different initial conditions. For an invertible matrix Y_0, let $Y(s)$ be the fundamental matrix solution to

$$\xi' = JD^2\Gamma(\gamma(s))\xi, \qquad \xi(0) = Y_0. \tag{5.2}$$

By definition of $X(s)$, we know that $Y(s) = X(s)Y_0$, and so $X(T) = Y(T)Y_0^{-1}$. It follows that the matrix $Y_0^{-1}Y(T)$ is similar to the monodromy matrix i.e.,

$$X(T) = Y(T)Y_0^{-1} = Y_0(Y_0^{-1}Y(T))Y_0^{-1}.$$

Thus the eigenvalues of $Y_0^{-1}Y(T)$ are identical to the characteristic multipliers.

5.1.1 Stability reduction using symmetry.

The monodromy matrix for a periodic solution with special types of symmetry can be factored using some linear algebra and standard techniques in differential equations. We begin by reviewing the relevant factorization and reduction theory that are applicable to a wide range of symmetric periodic orbits commonly found in Hamiltonian systems. Proofs of the following statements can be found in [21].

Lemma 5.1. *Suppose that $\gamma(s)$ is a symmetric $T-$periodic solution of a Hamiltonian system with Hamiltonian Γ and symmetry matrix S such that:*

(i) for some positive integer N, $\gamma(s+T/N) = S\gamma(s)$ for all s;

(ii) $\Gamma(Sz) = \Gamma(z)$;

(iii) $SJ = JS$;

(iv) S is orthogonal.

Then the fundamental matrix solution $X(s)$ to the linearization problem in (5.1) *satisfies*

$$X(s+T/N) = SX(s)S^T X(T/N).$$

Here of course, the notation S^T means the transpose of S. We mention this because we are using the letter T in two distinct ways.

Corollary 5.2. *Given the hypothesis of Lemma 5.1, the fundamental matrix solution $X(s)$ satisfies*

$$X(kT/N) = S^k\left(S^T X(T/N)\right)^k$$

for any $k \in \mathbb{N}$.

A remark here is that if $Y(s)$ is the fundamental matrix solution to Equation (5.2), then for any $k \in \mathbb{N}$, the matrix $Y(kT/N)$ factors as

$$Y(kT/N) = S^k Y_0 (Y_0^{-1} S^T Y(T/N))^k.$$

Lemma 5.3. *Suppose that $\gamma(s)$ is a $T-$periodic solution of a Hamiltonian system with Hamiltonian Γ and time-reversing symmetry S such that:*

(i) for some positive integer N, $\gamma(-s+T/N) = S\gamma(s)$ for all s;

(ii) $\Gamma(Sz) = \Gamma(z)$;

(iii) $SJ = -JS$;

(iv) S is orthogonal.

Then the fundamental matrix solution $X(s)$ to the linearization problem in (5.1) satisfies

$$X(-s+T/N) = SX(s)S^T X(T/N).$$

Corollary 5.4. *Given the hypothesis of Lemma 5.3,*

$$X(T/N) = SB^{-1}S^T B \quad \textit{where } B = X(T/2N).$$

Several more remarks about these factorizations are needed here.

(i) In the case of time-reversing symmetry matrix, S is typically block diagonal with two blocks of opposite sign, one for the position variable and one for the momenta, that is,

$$\begin{bmatrix} F & 0 \\ 0 & -F \end{bmatrix}$$

where F is orthogonal. A matrix of this form is orthogonal and anti-commutes with J.

(ii) A matrix satisfying properties 3 and 4 of Lemma 5.3 is symplectic with a multiplier of -1 since $S^T JS = -S^T SJ = -J$.

(iii) If $Y(s)$ is the fundamental matrix solution to (5.2), then a similar argument shows that $Y(-s+T/N) = SY(s)Y_0^{-1}S^T Y(T/N)$ and consequently

$$Y(T/N) = SY_0 B^{-1} S^T B, \quad \text{where } B = Y(T/2N).$$

Applying this factorization theory results in expressing the matrix $Y_0^{-1}Y(T)$, which is similar to $X(T)$, as W^k for some positive integer k, where the symplectic matrix W is the product of two involutions. If an eigenvalue of W lies on the unit circle, then so does its k^{th} power. The symplectic matrix W is called *stable* if all of its eigenvalues lie on the unit circle.

Lemma 5.5. *For a symplectic matrix W, suppose there is a matrix K such that*

$$\frac{1}{2}\left(W + W^{-1}\right) = \begin{bmatrix} K^T & 0 \\ 0 & K \end{bmatrix}. \tag{5.3}$$

Then W is stable if and only if all of the eigenvalues of K are real and have absolute value smaller than or equal to 1.

We will show for each of the periodic orbits under consideration, there is a choice of Y_0 such that W satisfies Lemma 5.5. This reduces the linear stability to the computation of the eigenvalues of K. As one of the eigenvalues of K is known to be real and have absolute value 1, the linear stability is determined by the numerical computation of one real number and showing that, within error, it lies between -1 and 1.

5.2 Linear Stability for the Schubart-like Periodic Orbit

The existence of the Schubart-like periodic orbit in the collinear four-body problem has been shown in [18]. We review it here. For $x_1 \geq x_2 \geq 0$, we assume that four masses are located at x_1, x_2, $-x_2$ and $-x_1$ with masses 1, m, m, and 1 respectively with $m > 0$. We also assume that the system remains symmetrically distributed about the center of mass located at the origin. The respective velocities of the four bodies are

$\dot{x}_1, \dot{x}_2, -\dot{x}_2, -\dot{x}_1$ where $\dot{} = d/dt$. The Netowanian equations are

$$\ddot{x}_1 = -\frac{1}{4x_1^2} - \frac{m}{(x_1+x_2)^2} - \frac{m}{(x_1-x_2)^2}$$

$$\ddot{x}_2 = -\frac{m}{4x_2^2} - \frac{1}{(x_1+x_2)^2} + \frac{1}{(x_1-x_2)^2}.$$

We recount Sweatman's approach in [28] and [29] to regularize this system. The Hamiltonian for this system is

$$H = \frac{1}{4}w_1^2 + \frac{1}{4m}w_2^2 - \frac{1}{2x_1} - \frac{m^2}{2x_2} - \frac{2m}{x_1+x_2} - \frac{2m}{x_1-x_2},$$

where $w_1 = 2\dot{x}_1$ and $w_2 = 2m\dot{x}_2$ are the conjugate momenta to x_1 and x_2. Introduce new canonical coordinates q_1, q_2, p_1, p_2 by

$$q_1 = x_1 - x_2, \quad q_2 = 2x_2, \quad p_1 = w_1, \quad p_2 = \frac{1}{2}(w_1 + w_2).$$

The Hamiltonian in the new canonical coordinates is

$$H = \frac{1}{4}\left(1 + \frac{1}{m}\right)p_1^2 - \frac{p_1 p_2}{m} + \frac{p_2^2}{m} - \frac{2m}{q_1} - \frac{m^2}{q_2} - \frac{2m}{q_1+q_2} - \frac{1}{2q_1+q_2}.$$

To regularize the equations of motion, Sweatman introduced a Levi-Civita type of canonical transformation

$$Q_i^2 = q_i, \;\; P_i = 2Q_i p_i \;\; (i = 1, 2),$$

for the the canonical coordinates Q_1, Q_2, P_1, P_2, and then replaced time t by the new independent variable s given by

$$\frac{dt}{ds} = Q_1^2 Q_2^2.$$

In the extended phase space, this produces the regularized Hamiltonian

$$\Gamma = \frac{dt}{ds}(H - E) = \frac{1}{16}\left(1 + \frac{1}{m}\right)Q_2^2 P_1^2 + \frac{-Q_1 Q_2 P_1 P_2 + Q_1^2 P_2^2}{4m}$$
$$- m^2 Q_1^2 - 2m Q_2^2 - \frac{2m Q_1^2 Q_2^2}{Q_1^2 + Q_2^2} - \frac{Q_1^2 Q_2^2}{2Q_1^2 + Q_2^2} - E Q_1^2 Q_2^2.$$

We fix the energy $E=-1$. The Hamiltonian system in the new coordinate system is

$$Q_1' = \frac{Q_2}{4}\left[\frac{1}{2}\left(1+\frac{1}{m}\right)Q_2P_1 - \frac{1}{m}Q_1P_2\right], \tag{5.4}$$

$$Q_2' = \frac{Q_1}{2m}\left[Q_1P_2 - \frac{1}{2}Q_2P_1\right], \tag{5.5}$$

$$P_1' = \frac{P_2}{4m}(Q_2P_1 - 2Q_1P_2) + 2m^2Q_1 + \frac{4mQ_1Q_2^4}{(Q_1^2+Q_2^2)^2} + \frac{2Q_1Q_2^4}{(2Q_1^2+Q_2^2)^2} - 2Q_1Q_2^2, \tag{5.6}$$

$$P_2' = \frac{P_1}{4}\left[\frac{Q_1P_2}{m} - \frac{Q_2P_1}{2}\left(1+\frac{1}{m}\right)\right] + 4mQ_2 + \frac{4mQ_1^4Q_2}{(Q_1^2+Q_2^2)^2} + \frac{4Q_1^4Q_2}{(2Q_1^2+Q_2^2)^2} - 2Q_1^2Q_2, \tag{5.7}$$

where $'$ is the derivative with respect to s.

From the proof [18] of the existence of the Schubart-like periodic orbit $Q_1(s)$, $Q_2(s)$, $P_1(s)$, $P_2(s)$ of periodic T, there is a positive constant $R(m)$ such that

$$Q_1(0) = R(m),\ Q_2(0) = 0,\ P_1(0) = 0,\ P_2(0) = 2m^{3/2}.$$

These initial conditions correspond to a binary collision of the two inner bodies. By the construction of the periodic orbit, another binary collision of the two inner bodies occurs at $s = T/2$ where the conditions are

$$Q_1(T/2) = -R(m),\ Q_2(T/2) = 0,\ P_1(T/2) = 0,\ P_2(T/2) = -2m^{3/2}.$$

Simultaneous binary collisions correspond to the conditions of the periodic solution when $s = T/4$ and $s = 3T/4$, i.e., $Q_1(s) = 0$ at these values of s. The value of $R(1)$ is approximately 2.29559. Figure 5.1 contains a plot of the coordinates Q_1, Q_2, P_1, P_2 of the periodic orbit when $m = 1$.

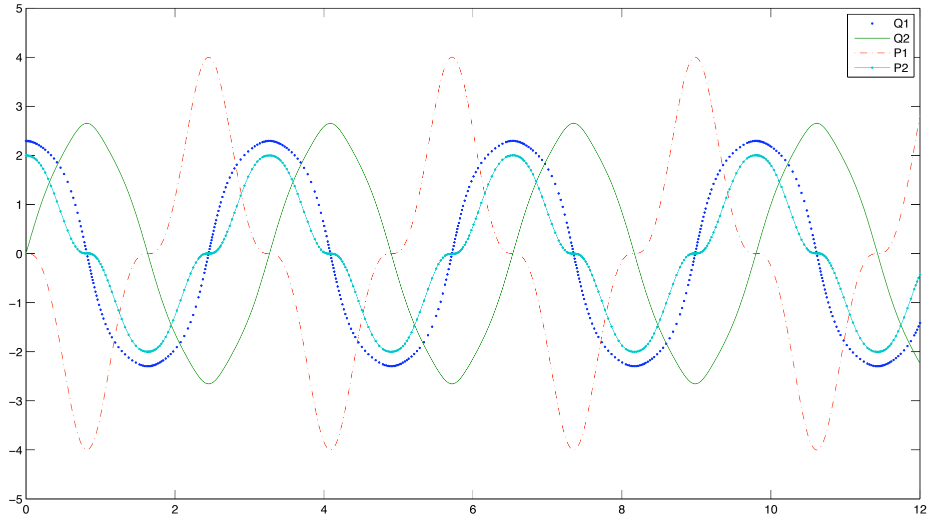

Figure 5.1: The periodic solution in the coordinate system Q_1, Q_2, P_1, P_2 when $m = 1$.

5.2.1 Stability Reductions using Symmetry. The Schubart-like periodic solution in the regularized coordinate system $\gamma(s) = (Q_1(s), Q_2(s), P_1(s), P_2(s))$ with period T in the collinear problem has two time-reversing symmetries. For

$$F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

the matrix

$$S = \begin{bmatrix} F & 0 \\ 0 & -F \end{bmatrix}$$

is orthogonal and symmetric: $S^{-1} = S^T = S$. It is also an involution, i.e., $S^2 = I$. Since $S\gamma(-s+T)$ is a solution of (5.4) through (5.7), and since this solution shares the same initial conditions as $\gamma(s)$ at $s = 0$ by T-periodicity of γ, uniqueness of solutions implies that the matrix S satisfies

$$\gamma(-s+T) = S\gamma(s) \text{ for all } s.$$

Thus S is a time-reversing symmetry of $\gamma(s)$. With $N = 1$, conditions (2), (3), and (4) in Lemma 5.3 are satisfied, and so by Corollary 5.4, the monodromy matrix for γ satisfies

$$X(T) = SX(T/2)^{-1}S^T X(T/2) = SX(T/2)^{-1}SX(T/2). \tag{5.8}$$

Consequently, from the above equation and $S^2 = I$,

$$[SX(T)]^2 = \left[X(T/2)^{-1}SX(T/2)\right]\left[X(T/2)^{-1}SX(T/2)\right] = I.$$

Since $-S\gamma(-s+T/2)$ is a solution of (5.4) through (5.7), and as $-S\gamma(T/2)$ is the same as $\gamma(0)$, uniqueness of solutions implies that the matrix $-S$ satisfies

$$\gamma(-s+T/2) = -S\gamma(s) \text{ for all } s.$$

Thus $-S$ is another time-reversing symmetry of $\gamma(s)$. For $N = 2$, conditions (2), (3), and (4) of Lemma 5.3 are satisfied, and so Corollary 5.4 implies that

$$X(T/2) = SX(T/4)^{-1}SX(T/4). \tag{5.9}$$

For

$$B = X(T/4),$$

combining equations (5.8) and (5.9) gives

$$X(T) = (SB^{-1}SB)^2$$

With $A = SB^{-1}SB$ and $D = B^{-1}SB$, then

$$X(T) = A^2 = (SD)^2,$$

where $S^2 = I$ and $D^2 = I$. The two time-reversing symmetries S and $-S$ of γ are both involutions, and together they generate a D_2 symmetry group for γ.

5.2.2 A Good Basis.

We have reduced the stability analysis to the first quarter of the periodic orbit. Let $Y(s)$ be the fundamental matrix solution to the linearized equations about Schubart-like periodic orbit $\gamma(s)$ with arbitrary initial conditions Y_0. Let

$$B = Y(T/4).$$

By the third remark following Corollary 5.4, the matrix $Y_0^{-1}Y(T)$, which is similar to the monodromy matrix $X(T) = Y(T)Y_0^{-1}$, satisfies

$$Y_0^{-1}Y(T) = \left(\left(Y_0^{-1}SY_0\right)B^{-1}SB\right)^2.$$

The question of stability reduces to showing that the eigenvalues of

$$W = \left(Y_0^{-1}SY_0\right)B^{-1}SB$$

are on the unit circle. An appropriate choice of Y_0 will simplify the factor $Y_0^{-1}SY_0$ in W. Set

$$\Lambda = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

Lemma 5.6. *There exists Y_0 such that*

(i) Y_0 is orthogonal and symplectic, and

(ii) $Y_0^{-1}SY_0 = \Lambda$.

Proof. Choose the third column of Y_0 to be $\gamma'(0)/\|\gamma'(0)\| = [0\ 1\ 0\ 0]^T = e_2$. For $e_3 = [0\ 0\ 1\ 0]^T$, the matrix

$$Y_0 = [Je_2, Je_3, e_2, e_3] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

is orthogonal and symplectic. Since $S = \mathrm{diag}\{1, -1, -1, 1\}$, it follows that $Y_0^{-1}SY_0$ has the desired form. $\square$

Setting $D = B^{-1}SB$ and choosing Y_0 as constructed in Lemma 5.6 gives

$$W = \left(Y_0^{-1}SY_0\right)B^{-1}SB = \Lambda D.$$

The matrices Λ and D are both involutions, i.e., $\Lambda^2 = I$, $D^2 = I$. From these it follows that

$$W^{-1} = D\Lambda.$$

Because B is a symplectic matrix, a short computation using the formula for the inverse of a symplectic matrix shows that D has the form

$$\begin{bmatrix} K^T & L_1 \\ -L_2 & -K \end{bmatrix}$$

for 2×2 matrices K, L_1, L_2. It follows that

$$W = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} K^T & L_1 \\ -L_2 & -K \end{bmatrix} = \begin{bmatrix} K^T & L_1 \\ L_2 & K \end{bmatrix},$$

and

$$W^{-1} = \begin{bmatrix} K^T & L_1 \\ -L_2 & -K \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = \begin{bmatrix} K^T & -L_1 \\ -L_2 & K \end{bmatrix}.$$

Hence,

$$\frac{1}{2}\left(W + W^{-1}\right) = \begin{bmatrix} K^T & 0 \\ 0 & K \end{bmatrix}.$$

We show that the first column of K is $[-1\ 0]^T$. Set $v = Y_0^{-1}\gamma'(0)$. By the choice of Y_0,

$$v = Y_0^T\gamma'(0) = \|\gamma'(0)\|e_3.$$

Since S is symmetric and Y_0 is orthogonal, then by the third remark after Corollary 5.4,

$$W = Y_0^{-1}SY_0B^{-1}SB = Y_0^{-1}SY_0B^{-1}S^TB = Y_0^TY(T/2).$$

Now $\gamma'(s)$ is a solution of $\dot{\xi} = JD^2\Gamma(\gamma(s))\xi$ and $\gamma'(0) = Y(0)Y_0^{-1}\gamma'(0) = Y(0)v$, and so $\gamma'(s) = Y(s)Y_0^{-1}\gamma'(0) = Y(s)v$. This implies that

$$Y_0^{-1}\gamma'(T/2) = Y_0^T Y(T/2)v = Wv.$$

Since $\gamma(s)$ satisfies $\gamma(-s+T/2) = -S\gamma(s)$ for all s, then $\gamma'(-s+T/2) = S\gamma'(s)$ for all s. Setting $s = 0$ in this gives $\gamma'(T/2) = S\gamma'(0)$. Since $\gamma'(0)$ is a nonzero scalar multiple of e_2 and since $Se_2 = -e_2$, then

$$Y_0^{-1}\gamma'(T/2) = Y_0^{-1}S\gamma'(0) = -Y_0^{-1}\gamma'(0) = -Y_0^T\gamma'(0) = -v.$$

Thus $Wv = -v$, implying that -1 is an eigenvalue of W and e_3 is an eigenvector of W corresponding to this eigenvalue. Thus the first column of K is as claimed. The form of the rest of K comes from the formula for the inverse of a symplectic matrix and the definition of D:

$$K = \begin{bmatrix} -1 & * \\ 0 & c_2^T(SJc_4) \end{bmatrix},$$

where c_i is the i^{th} column of $Y(T/4)$.

5.2.3 Numerical Calculations.

With an absolute error tolerance of 1×10^{-12}, our numerical results for $m = 1$ showed that the initial condition

$$Q_1(0) = R(1) = 2.295592258717, \;\; Q_2(0) = 0, \;\; P_1(0) = 0, \;\; P_2(0) = 2$$

leads to a periodic simultaneous binary collision periodic orbit (as in Figure 1) whose period T satisfies $T/4 = 0.817348080989685$. Using MATLAB and a Runge-Kutta-Fehlberg algorithm, we computed the columns of the matrix $Y(T/4)$ with an absolute error tolerance of 4×10^{-6}. From this, we got

$$c_2^T(SJc_4) = 0.598490.$$

For values of m between 0 and 50 at 0.01 increments, we numerically computed the value of $R(m)$ in the initial conditions and the value of the period T (with an absolute error tolerance of 4×10^{-6}), and the values of $c_2^T(SJc_4)$ (with an absolute error tolerance of 1×10^{-6}). The results of these computations are contained in Figure 5.2.

A closer look at the numerical data in Figure 5.2 for where the value of $c_2^T(SJc_4)$ is close to 1 gives estimates of the two values of m where the stability of the periodic orbit changes. The first critical value of m is approximately $m = 2.83$, and the second critical value of m is approximately $m = 35.4$.

The eigenvalues of K are -1 and $c_2^T(SJc_4)$. The eigenvalues of K are distinct for most values of m in $[0,50]$ because of the rigorous numerical estimates we have for $c_2^T(SJc_4)$. Lemma 5.5 now implies the following linear stability result.

Theorem 5.7. *There exists small positive constants ε_i, $i = 1,2,3,4$ such that the periodic simultaneous binary collision orbit in the collinear symmetric four body problem with masses 1, m, m, 1 is linearly stable when $m < 2.83 - \varepsilon_1$ and $35.4 + \varepsilon_2 < m \leq 50$, and is linearly unstable when $2.83 + \varepsilon_3 < m < 35.4 - \varepsilon_4$.*

This result confirms the linear stability analysis of Sweatman [29] for m between 0 and 50, asserting that the periodic orbit is unstable when m is between 2.83 and 35.4. Simulations of the periodic orbit when m is between 2.83 and 35.4 indicate that the linear instability is manifested slowly over time.

5.3 Linear Stability for the 2D Symmetric Periodic Orbit

In [17], we proved the existence of a special type of planar periodic solution of $2n$ bodies with equal masses. In this section, we are going to consider the linear stability of this periodic solution when $n = 2$. If (x_1, x_2) is the position of the first body, then the positions of the remaining three bodies are (x_2, x_1), $(-x_1, -x_2)$, and $(-x_2, -x_1)$. When each body has mass $m = 1$, the Newtonian equations for this planar four-body problem are

$$(\ddot{x}_1, \ddot{x}_2) = -\left[\frac{(x_1 - x_2, x_2 - x_1)}{2^{3/2}|x_1 - x_2|^3} + \frac{(x_1, x_2)}{4(x_1^2 + x_2^2)^{3/2}} + \frac{(x_1 + x_2, x_1 + x_2)}{2^{3/2}|x_1 + x_2|^3}\right].$$

The initial conditions for the periodic orbit, and the periodic orbit are illustrated in Figure 5.3.

We adapt Sweatman's approach ([28], [29]) to regularize this system. The Hamiltonian for this system is

$$H = \frac{1}{8}\left(w_1^2 + w_2^2\right) - \frac{\sqrt{2}}{|x_1 - x_2|} - \frac{\sqrt{2}}{|x_1 + x_2|} - \frac{1}{\sqrt{x_1^2 + x_2^2}},$$

where $w_1 = 4\dot{x}_1$ and $w_2 = 4\dot{x}_2$ are the conjugate momentum. In terms of the canonical coordinates (q_1, q_2, p_1, p_2)

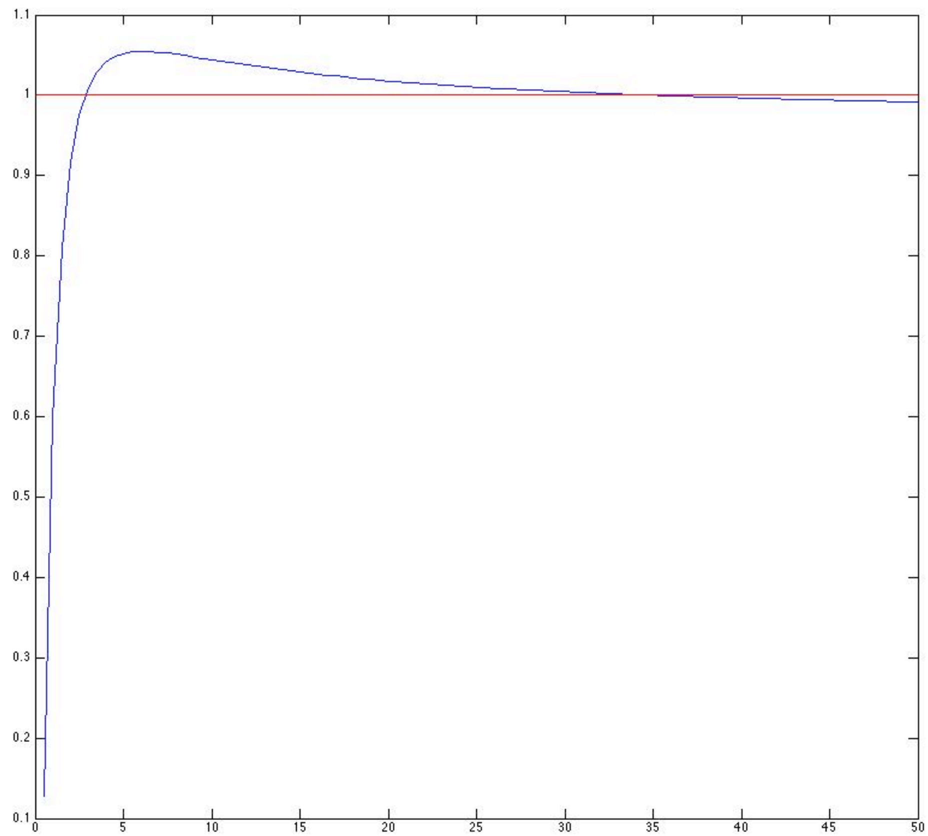

Figure 5.2: The value of $c_2^T(SJc_4)$ for values of m between 0 and 50.

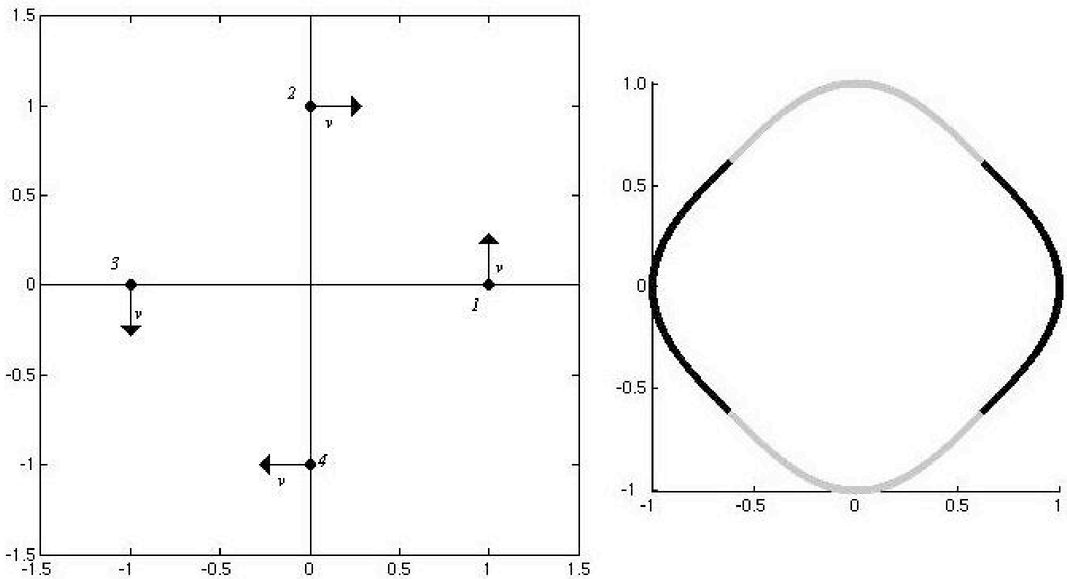

Figure 5.3: On the left are the initial conditions leading to the four-body two-dimensional periodic SBC orbit. On the right is the orbit.

defined by

$$q_1 = x_1 - x_2, \quad q_2 = x_1 + x_2, \quad w_1 = p_1 + p_2, \quad w_2 = p_2 - p_1,$$

the Hamiltonian becomes

$$H = \frac{1}{4}\left(p_1^2 + p_2^2\right) - \frac{\sqrt{2}}{|q_1|} - \frac{\sqrt{2}}{|q_2|} - \frac{\sqrt{2}}{\sqrt{q_1^2 + q_2^2}}.$$

The Levi-Civita type of canonical transformation used to regularize the collinear problem now applies to the four body equal mass 2D problem. In terms of the canonical coordinates (Q_1, Q_2, P_1, P_2) defined by

$$q_i = Q_i^2, \quad P_i = 2Q_i p_i \quad (i = 1, 2),$$

and the new time variable s defined by

$$\frac{dt}{ds} = Q_1^2 Q_2^2,$$

the Hamiltonian in extended phase space becomes

$$\Gamma = \frac{dt}{ds}(H-E) = \frac{1}{16}(P_1^2 Q_2^2 + P_2^2 Q_1^2) - \sqrt{2}(Q_1^2+Q_2^2) - \frac{\sqrt{2}Q_1^2Q_2^2}{\sqrt{Q_1^4+Q_2^4}} - EQ_1^2Q_2^2 \tag{5.10}$$

where E is the total energy of the Hamiltonian H. The differential equations in terms of the new coordinates $\{Q_1, Q_2, P_1, P_2\}$ are

$$Q_1' = \frac{1}{8}P_1 Q_2^2 \tag{5.11}$$

$$Q_2' = \frac{1}{8}P_2 Q_1^2 \tag{5.12}$$

$$P_1' = -\frac{1}{8}P_2^2 Q_1 + 2\sqrt{2}Q_1 + \frac{2\sqrt{2}Q_1Q_2^2}{\sqrt{Q_1^4+Q_2^4}} - \frac{2\sqrt{2}Q_1^5Q_2^2}{(Q_1^4+Q_2^4)^{\frac{3}{2}}} + 2EQ_1Q_2^2 \tag{5.13}$$

$$P_2' = -\frac{1}{8}P_1^2 Q_2 + 2\sqrt{2}Q_2 + \frac{2\sqrt{2}Q_2Q_1^2}{\sqrt{Q_1^4+Q_2^4}} - \frac{2\sqrt{2}Q_2^5Q_1^2}{(Q_1^4+Q_2^4)^{\frac{3}{2}}} + 2EQ_2Q_1^2. \tag{5.14}$$

Unlike the collinear problem, we do not fix the value of E here. As shown in [18], for each $\zeta > 0$ there exists $v_0 > 0$ such that the initial conditions

$$Q_1(0) = \zeta,\ \ Q_2(0) = \zeta,\ \ P_1(0) = -4v_0,\ \ P_2(0) = 4v_0, \tag{5.15}$$

lead to a periodic solution with a minimal period T. From $\Gamma = 0$, the value of E is determined by this choice of ζ and v_0. By its construction in [17], this periodic orbit satisfies

$$Q_1(T/4) = -\zeta,\ \ Q_2(T/4) = \zeta,\ \ P_1(T/4) = -4v_0,\ \ P_2(T/4) = -4v_0.$$

Simultaneous binary collisions correspond to $s = T/8, 5T/8$ i.e., when $Q_1(s) = 0$, and to $s = 3T/8, 7T/8$, i.e., when $Q_2(s) = 0$. For $\zeta = 1$, $4v_0 = 2.57486992651942$, and $T/8 = 1.62047369909693$. Figure 4 illustrates the coordinates (Q_1, Q_2, P_1, P_2) of this periodic solution.

5.3.1 Stability Reductions using Symmetry. We will reduce the stability analysis to the first eighth of the periodic orbit. The symmetric periodic 2D orbit

$$\gamma(t) = (Q_1(t), Q_2(t), P_1(t), P_2(t))$$

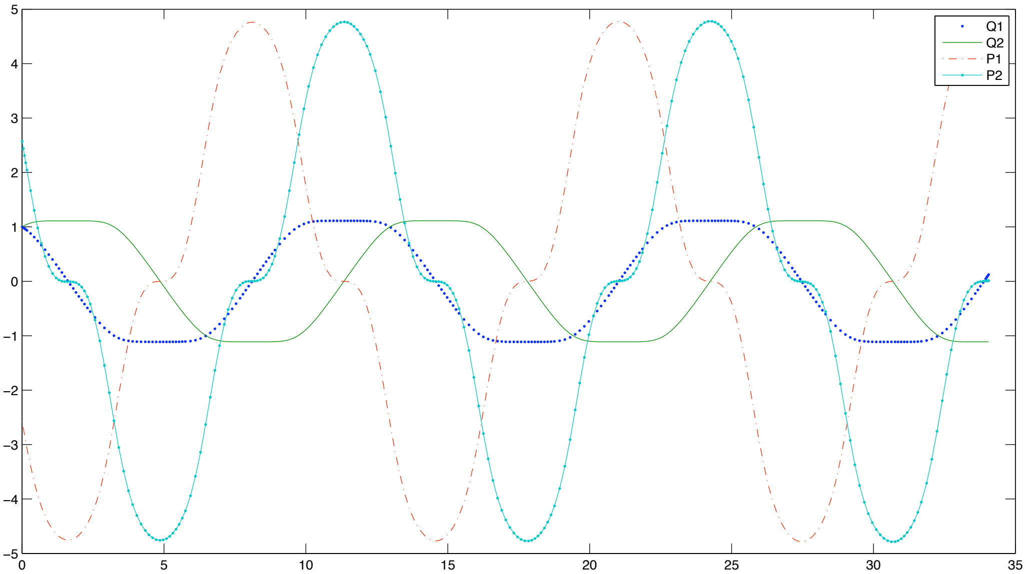

Figure 5.4: The periodic solution in the coordinate system Q_1, Q_2, P_1, P_2.

with period T has a time-reversing symmetry and a time-preserving symmetry. For

$$F = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

the matrices

$$S_F = \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}, \quad S_G = \begin{bmatrix} G & 0 \\ 0 & -G \end{bmatrix}$$

satisfy $S_F^{-1} = S_F^T$, $S_F^2 \neq I$, $S_F^3 \neq I$, $S_F^4 = I$, $S_G^2 = I$, $S_G^T = S_G$, and $(S_F S_G)^2 = I$. Since $\gamma(s+T/4)$ and $S_F\gamma(s) = (-Q_2(s), Q_1(s), -P_2(s), P_1(s))$ are solutions of (5.11) through (5.14) and share the same initial conditions when $s = 0$, uniqueness of solutions implies that

$$\gamma(s+T/4) = S_F\gamma(s) \text{ for all } s.$$

Thus S_F is a time-preserving symmetry of $\gamma(s)$. With $N = 4$, conditions (2), (3), and (4) of Lemma 5.1 are satisfied, so that Corollary 5.2 (with $k = 4$) and $S_F^4 = I$ imply that

$$X(T) = S_F^4 \left(S_F^T X(T/4)\right)^4 = \left(S_F^T X(T/4)\right)^4.$$

Since $\gamma(-s+T/4)$ and $S_G\gamma(s)$ are solutions of (5.11) through (5.14) and share the same initial conditions when $s = 0$, uniqueness of solutions implies that

$$\gamma(-s+T/4) = S_G\gamma(s) \text{ for all } s.$$

Thus S_G is a time-reversing symmetry for $\gamma(s)$. With $N = 4$, conditions (2), (3), and (4) of Lemma 5.3 are satisfied, and so Corollary 5.4 implies that

$$X(T/4) = S_G\,[X(T/8)]^{-1}\,S_G^T X(T/8) = S_G\,[X(T/8)]^{-1}\,S_G X(T/8).$$

Let

$$B = X(T/8).$$

Combining the factorization of $X(T)$ that involves S_F and the factorization of $X(T/4)$ that involves S_G gives the factorization

$$X(T) = \left(S_F^T S_G B^{-1} S_G B\right)^4.$$

Setting

$$Q = S_F^T S_G = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

and $D = B^{-1}S_G B$ results in the factorization

$$X(T) = (QD)^4$$

where Q and D are both involutions. The symmetries S_F and S_G generate a D_4 symmetry group for the periodic orbit $\gamma(s)$.

5.3.2 A Good Basis. Let $Y(s)$ be the fundamental matrix solution to the linearized equations about the 2D periodic orbit $\gamma(s)$ with arbitrary initial conditions Y_0. Let

$$B = Y(T/8).$$

By remarks following Corollaries 5.2 and 5.4, the matrix $Y_0^{-1}Y(T)$, which is similar to the monodromy matrix $X(T) = Y(T)Y_0^{-1}$, satisfies

$$Y_0^{-1}Y(T) = (Y_0^{-1}S_F^T S_G Y_0 B^{-1} S_G B)^4 = (Y_0^{-1}QY_0B^{-1}S_GB)^4.$$

The question of linear stability reduces to showing that the eigenvalues of

$$W = Y_0^{-1}QY_0B^{-1}S_GB$$

are on the unit circle. Recall that

$$\Lambda = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

Lemma 5.8. *There exists Y_0 such that*

(i) *Y_0 is orthogonal and symplectic, and*

(ii) $Y_0^{-1}QY_0 = \Lambda$.

Proof. Choose the third column of Y_0 to be

$$\frac{\gamma'(0)}{\|\gamma'(0)\|} = \frac{1}{c}\begin{bmatrix} -a & a & b & b \end{bmatrix}^T$$

where $a = v_0\zeta^2/2$, $b = E\zeta^3 = (2v_0^2 - 2\sqrt{2} - 1)\zeta$ and $c = \sqrt{2a^2 + 2b^2}$. Let $\mathrm{col}_i(Y_0)$ denote the i^{th} column of Y_0. Define

$$\mathrm{col}_1(Y_0) = J \cdot \mathrm{col}_3(Y_0) = \frac{1}{c}\begin{bmatrix} b & b & a & -a \end{bmatrix}^T.$$

We now choose $\mathrm{col}_4(Y_0)$ such that $\mathrm{col}_4(Y_0)$ is orthogonal to $\mathrm{col}_3(Y_0)$, and $\mathrm{col}_4(Y_0)$ is one of the eigenvectors of Q with respect to its eigenvalue of -1. Since the eigenspace of Q corresponding to its eigenvalue of -1 is

$$\mathrm{span}\Big\{\begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^T\Big\},$$

define

$$\mathrm{col}_4(Y_0) = \frac{1}{c}\begin{bmatrix} b & -b & a & a \end{bmatrix}^T$$

and

$$\mathrm{col}_2(Y_0) = J \cdot \mathrm{col}_4(Y_0) = \frac{1}{c}\begin{bmatrix} a & a & -b & b \end{bmatrix}^T.$$

The matrix

$$Y_0 = \frac{1}{c}\begin{bmatrix} b & a & -a & b \\ b & a & a & -b \\ a & -b & b & a \\ -a & b & b & a \end{bmatrix},$$

is both symplectic and orthogonal and it satisfies $Y_0^{-1} Q Y_0 = \Lambda$. □

Setting $D = B^{-1} S_G B$ and choosing Y_0 to be the matrix constructed in Lemma 5.8 gives $W = \Lambda D$. The matrices Λ and D are involutions (the latter because $S_G^2 = I$). As in Section 3.2, $W^{-1} = D\Lambda$, and there is a 2×2 matrix K such that

$$\frac{1}{2}\big(W + W^{-1}\big) = \begin{bmatrix} K^T & 0 \\ 0 & K \end{bmatrix}.$$

We show that the first column of K is $[1\ 0]^T$. Since $S_G^T = S_G$, $Y_0^{-1} = Y_0^T$, it follows by the third remark following Corollary 5.4 that

$$W = Y_0^{-1} S_F^T S_G Y_0 B^{-1} S_G B = Y_0^{-1} S_F^T Y(T/4) = Y_0^T S_F^T Y(T/4).$$

Set $v = Y_0^{-1}\gamma'(0)$. By the choice of the matrix Y_0,

$$v = Y_0^{-1}\gamma'(0) = Y_0^T\gamma'(0) = \begin{bmatrix} 0 \\ 0 \\ ||\gamma'(0)|| \\ 0 \end{bmatrix} = ||\gamma'(0)||e_3.$$

Because $\gamma'(s)$ is a solution to the linearized equation $\dot{\xi} = JD^2\Gamma(\gamma(s))\xi$ and because $\gamma'(0) = Y(0)Y_0^{-1}\gamma'(0)$, then $\gamma'(s) = Y(s)Y_0^{-1}\gamma'(0)$ for all s. Hence,

$$Wv = Y_0^T S_F^T Y(T/4)v = Y_0^T S_F^T\gamma'(T/4). \tag{5.16}$$

Since γ satisfies $\gamma(s+T/4) = S_F\gamma(s)$ for all s and $S_F^{-1} = S_F^T$, it then follows that

$$\gamma'(s) = S_F^{-1}\gamma'(s+T/4) = S_F^T\gamma'(s+T/4).$$

Setting $s = 0$ in this gives $\gamma'(0) = S_F^T\gamma'(T/4)$, and consequently that

$$Y_0^T S_F^T\gamma'(T/4) = Y_0^T\gamma'(0) = Y_0^{-1}\gamma'(0) = v. \tag{5.17}$$

Equations (5.16) and (5.17) now combine to show that $Wv = v$, i.e, that 1 is an eigenvalue of W and e_3 is an eigenvector for W corresponding to this eigenvalue. The first column of K is as claimed. The rest of K comes from the formula for the inverse of a symplectic matrix and the definition of D:

$$K = \begin{bmatrix} 1 & * \\ 0 & c_2^T(S_G J c_4) \end{bmatrix},$$

where c_i is the i^{th} column of $B = Y(T/8)$.

5.3.3 Numerical Calculations. Having not fixed E, we used an invariant scaling of the coordinates and time in equations (5.11) through (5.14) to preselect a period T before numerically computing the initial conditions for a periodic simultaneous binary collision orbit. For $\varepsilon > 0$, if $Q_1(s)$, $Q_2(s)$, $P_1(s)$, $P_2(s)$ is a periodic simultaneous collision orbit of equations (5.11) through (5.14), then replacing E with $\varepsilon^{-2}E$ shows that $\varepsilon Q_1(\varepsilon s)$, $\varepsilon Q_2(\varepsilon s)$, $P_1(\varepsilon s)$, $P_2(\varepsilon s)$ is also a periodic simultaneous binary collision orbit with energy $\varepsilon^{-2}E$ and period $\varepsilon^{-1}T$. Furthermore, it is straight forward to show that monodromy matrices for the periodic simultaneous binary collision orbits corresponding to values of $\varepsilon \neq 1$ are all similar to that for $\varepsilon = 1$. Thus the linear stability of a periodic simultaneous binary collision orbit for one $\varepsilon > 0$ implies the linear stability of the periodic simultaneous binary collision orbits for all $\varepsilon > 0$.

We rigorously computed the value of $c_2^T(S_G J c_4)$ for the periodic simultaneous binary collision orbit whose period is $T = 8$. This means that the first time of a simultaneous binary collision for this orbit is at $s = 1$. We set $Q_1(0) = Q_2(0) = \xi$ and $-P_1(0) = P_2(0) = \eta$, and defined a function $F(\xi,\eta)$ to be equal to the vector quantity $(Q_1(1), P_2(1))$. We used Newton's method and a good initial guess to find a root (ξ,η) of F. This involved computing the Jacobian of F which was done using the linearized equations. With an absolute error tolerance of 6×10^{-11}, this numerical method shows that the initial conditions

$$Q_1(0) = Q_2(0) = 1.62047369909693, \quad -P_1(0) = P_2(0) = 2.57486992651942,$$

lead to a periodic solution with a period of $T = 8$, and a value of $E \approx -1.142329388$. Using MATLAB and a Runge-Kutta-Fehlberg algorithm, we computed the columns of the matrix $Y(T/8)$ with an absolute error tolerance of 2.5×10^{-12}. From this we got

$$c_2^T(S_G J c_4) = -0.68024151010592.$$

Using the scaling of coordinates and time described above, the initial conditions for the periodic simultaneous binary collision orbit shown in Figures 3 and 4 are

$$Q_1(0) = Q_2(0) = 1, \quad -P_1(0) = P_2(0) = 2.57486992651942$$

with a period T satisfying $T/8 = 1.62047369909693$, and energy $E \approx -2.999682732$.

For the periodic simultaneous binary collision orbit, the rigorous estimate of the eigenvalue $c_2^T(S_G J c_4)$ of K and its distinctiveness from the eigenvalue 1 of K combine with Lemma 5.5 to give the following stability result.

Theorem 5.9. *The periodic simultaneous binary collision orbit in the 2D-symmetric equal mass four-body problem is linearly stable.*

When $c_2^T(S_G J c_4)$ is real and between -1 and 1, it is the real part of an eigenvalue with unit modulus for W (see [21]). For the periodic simultaneous collision orbit, the real part of $\exp(3\pi i/4)$, that is $-(1/2)\sqrt{2}$, is fairly close to the rigorously estimated value of $c_2^T(S_G J c_4)$. Raising $\exp(3\pi i/4)$ to the fourth power gives $\exp(3\pi i) = -1$, and so two of the eigenvalues of the monodromy matrix of the periodic simultaneous binary collision orbit are close to -1. The symmetry reductions used to compute the eigenvalues over just one-eighth of the period and the rigorous estimate of $c_2^T(S_G J c_4)$ showing that it is clearly between -1 and 1, assures the linear stability of the periodic simultaneous binary collision orbit.

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