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THE TROPICAL JACOBIAN OF AN ELLIPTIC CURVE IS THE GROUP $S^1(Q)$

by

Darryl G. Wade

A thesis submitted to the faculty of

Brigham Young University

in partial fulfillment of the requirements for the degree of

Master of Science

Department of Mathematics

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BRIGHAM YOUNG UNIVERSITY

GRADUATE COMMITTEE APPROVAL

of a thesis submitted by

Darryl G. Wade

This thesis has been read by each member of the following graduate committee and by majority vote has been found to be satisfactory.

Date	Tyler Jarvis, Chair
Date	William Lang
Date	Jasbir Chahal

BRIGHAM YOUNG UNIVERSITY

As chair of the candidate's graduate committee, I have read the thesis of Darryl G. Wade in its final form and have found that (1) its format, citations, and bibliographical style are consistent and acceptable and fulfill university and department style requirements; (2) its illustrative materials including figures, tables, and charts are in place; and (3) the final manuscript is satisfactory to the graduate committee and is ready for submission to the university library.

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ABSTRACT

THE TROPICAL JACOBIAN OF AN ELLIPTIC CURVE IS THE GROUP $S^1(Q)$

Darryl G. Wade

Department of Mathematics

Master of Science

We establish consistent definitions for divisors, principal divisors, and Jacobians of a tropical elliptic curve and show that for a tropical elliptic cubic $\mathcal{C}$, the associated Jacobian (or zero divisor class group) is the group $S^1(\mathbb{Q})$.

ACKNOWLEDGMENTS

All of the computations and figures for this research were done using Maple and the tropical Maple code package created by Nathan Grigg. We express great thanks to the members of the BYU Tropical Geometry research group, especially Gretchen Rimmasch, Nathan Grigg, and Dr. Tracy Hall for their help. I would also like to thank my advisor, Dr. Tyler Jarvis, the members of my committee, Dr. William Lang and Dr. Jasbir Chahal, for their help and direction. Finally, I would like to thank my family for their support, especially my wife, Stacey, for her abundant patience.

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1 Introduction

Tropical geometry is a field of math that has been drawing a lot of attention lately. There is a lack of uniformity in basic definitions, so this thesis aims to lay down clearly some definitions and results about tropical elliptic curves.

Tropical elliptic curves are the tropical analog of classical elliptic curves. In this thesis we will give a brief introduction to tropical mathematics, discuss some definitions in classical algebraic geometry, and define their tropical analogs. Finally we will prove that the group associated with a non-singular tropical elliptic curve is isomorphic to the group $S^1(\mathbb{Q})$. (Recall that $S^1(\mathbb{Q})$ is the additive group of $\mathbb{Q}$ modulo 1.)

2 Background

2.1 Introduction to tropical algebra

We begin by defining the tropical semifield $\mathcal{Q}$ to be the set

$$\mathbb{Q} \cup \{\infty\}$$

together with the binary operations

$$x \oplus y := \min\{x, y\};$$

$$x \odot y := x + y.$$

We will employ the common notation $a^n = a \odot a \odot \cdots a = na$. (We will omit the symbol $\odot$ usually, but it will be clear from context when tropical multiplication is implied.) This structure satisfies all of the axioms of a field, except for that of additive inverses (i.e., there is no subtraction), and thus is called a *semi-field*. [7]

Remark 2.1. Many other authors have used $\mathcal{R} = \mathbb{R} \cup \{\infty\}$ as the base set. This is acceptable for the results of this paper—every argument works exactly the same. However, we choose to use $\mathcal{Q} = \mathbb{Q} \cup \{\infty\}$, because the tropical semi-field $\mathcal{Q}$ is algebraically closed as shown by [4]. The reader will note that the graphs contained herein appear the same whether done using $\mathcal{Q}$ or $\mathcal{R}$ as our base semi-field. The only fundamental difference is in the topology.

We may define tropical polynomials similarly to classical polynomials.

Definition 2.2. The function $f(x)$ is a *tropical polynomial* if

$$f(x) = a_n x^n \oplus \cdots \oplus a_1 x \oplus a_0$$

for some non-negative $n \in \mathbb{Z}$ and $a_i \in \mathcal{Q}$. We call the integer n the *degree* of the polynomial.

This polynomial is equivalent to the function

$$f(x) = \min\{nx + a_n, (n-1)x + a_{n-1}, \ldots, x + a_1, a_0\}.$$

Definition 2.3. A tropical polynomial of more than one variable, $x_1, \ldots, x_n$ is a function with the form

$$f(x_1, \ldots, x_n) = a_{d_1} x^{d_1} \oplus \cdots \oplus a_{d_k} x^{d_k}$$

where $d_j \in \mathbb{N}_0^n$ and thus $x^{d_j} = x_1^{e_{j,1}} \cdots x_n^{e_{j,n}}$ for some positive integer values $e_{j,i}$, and a_{d_j} is in $\mathcal{Q}$. The *degree* of this polynomial is the integer

$$d = \max\left\{ \sum_{i=1}^{n} e_{j,i} \mid j = 1, \ldots, k \right\}.$$

The value $\displaystyle\sum_{i=1}^{n} e_{j,i}$ is the degree of the j^{th} monomial.

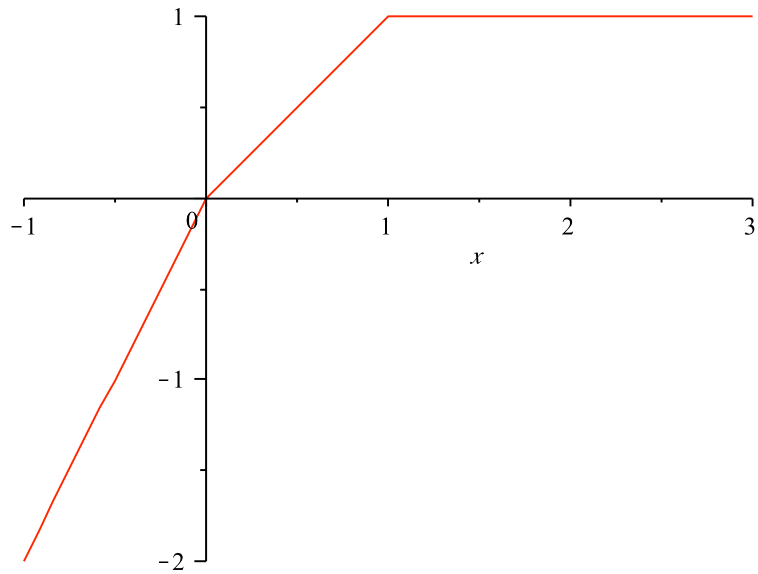


Figure 1: Graph of $f(x) = x^2 \oplus x \oplus 1$.

Definition 2.4. The *support of* f is defined to be the set $\text{supp}(f) = \{-d_j = (-e_{j,1}, \ldots, -e_{j,n}) \mid a_{d_j} \neq \infty\}$.

It is important to point out that f might not be the only polynomial that gives the corresponding function. For example, we may have $f(x) = x^2 \oplus 1$ and $g(x) = x^2 \oplus 1x \oplus 1$. These are certainly distinct polynomials, but they both give the same function, because the value of $\min\{2x, x+1, 1\}$ is attained by the $x+1$ term only when it is also attained by the $2x$ and 1 terms as well, as seen in Figure 2. [2]

In the classical setting, we define the *zero locus* of the function, or the set of points $Z(f) = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid f(x_1, \ldots, x_n) = 0\}$. In the tropical case, the analogous object is the *corner locus* of the polynomial. This is defined to be the set of points $(x_1, \ldots, x_n) \in \mathcal{Q}^n$ for which the minimum (tropical sum) is attained in at

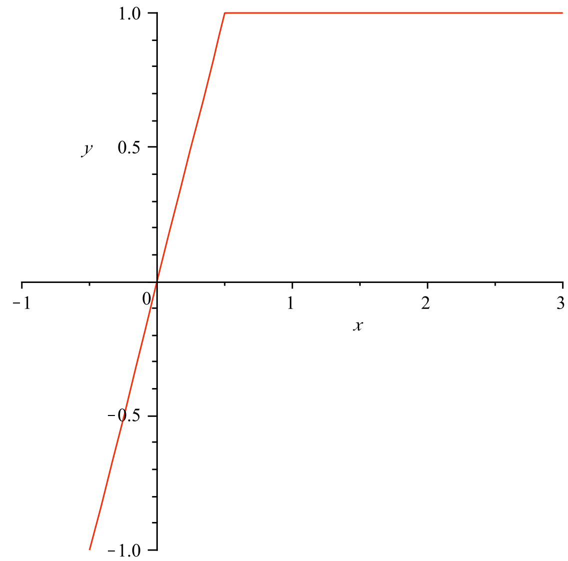


Figure 2: Graph of $f(x) = x^2 \oplus 1$ and $g(x) = x^2 \oplus 1x \oplus 1$.

least two of the tropical monomials [7]. For example, if $f(x) = x^2 \oplus x \oplus 1$, then the corner locus of f, denoted $K(f)$, is the set

$$\begin{aligned} K(f) &= \{x \mid 2x = x \leq 1\} \cup \{x \mid 2x = 1 \leq x\} \cup \{x \mid x = 1 \leq 2x\} \\ &= \{x = 0 \leq 1\} \cup \emptyset \cup \{x = 1 \leq 2\} \end{aligned}$$

Notice that $K(f) = \{0, 1\}$ is the set of corners of the graph of $y = f(x)$—hence the name "corner locus."

2.2 Tropical affine and projective spaces

In dealing with complex curves we often visualize the curves using the affine real or complex planes, but to be complete, we must actually handle the curves in the complex projective plane. This makes curves compact and allows us to count all intersections correctly. We will briefly discuss the tropical idea of projective space.

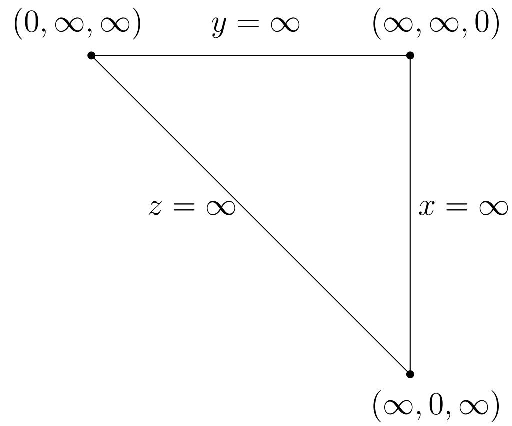


Figure 3: 2-simplex model of $\mathbb{TP}^2(\mathcal{Q})$.

Classical affine n-space is the set $\mathbb{A}^n(\mathbb{C}) = \mathbb{C}^n = \{(z_1,\ldots,z_n) \mid z_i \in \mathbb{C}\}$. Analogously, tropical affine n-space is the set $\mathbb{TA}^n(\mathcal{Q}) = \mathcal{Q}^n = \{(x_1,\ldots,x_n) \mid x_i \in \mathcal{Q}\}$. Classical projective n-space is defined as

$$\mathbb{P}^n(\mathbb{C}) = (\mathbb{A}^{n+1}(\mathbb{C}) \setminus \{(0,\ldots,0)\})/\sim$$

where the equivalence relation $\sim$ is given by $(z_0,\ldots,z_n) \sim (\lambda z_0,\ldots,\lambda z_n)$ if $\lambda \neq 0$. Since the tropical additive identity is ∞ instead of 0, tropical projective n-space is the set

$$\mathbb{TP}^n(\mathcal{Q}) = (\mathbb{TA}^{n+1}(\mathcal{Q}) \setminus \{(\infty,\ldots,\infty)\})/\sim$$

where the equivalence relation $\sim$ on the points of $\mathbb{TA}^{n+1}(\mathcal{Q}) \setminus \{\infty\}$ is given by $(x_0,\ldots,x_n) \sim (\lambda x_0,\ldots,\lambda x_n)$ if $\lambda \neq \infty$.

We can think of this graphically as an n-simplex. For example, $\mathbb{TA}^2(\mathcal{Q})$ looks topologically like the triangle, without the lower left edge, while $\mathbb{TP}^2(\mathcal{Q})$ is the closure of this (i.e., including the lower edge), seen in figure 3.

When dealing with polynomials in tropical projective space, as in classical projective space, we must use *homogeneous polynomials*. This means that if the polynomial has degree d, then each monomial must have degree d; i.e., if $d_j = (e_0, \ldots, e_n) \in \text{supp}(f)$, then $\sum_{i=0}^{n} e_i = d$. This ensures that $\lambda^d f(x) = f(\lambda x)$ for all points x and $\lambda \neq \infty$. Also, we may use the concept of support of f when it is homogeneous.

2.3 Tropical graphs and the tropical corner locus

For a tropical polynomial in two variables

$$f(x, y) = a_{n,n}x^n y^n \oplus a_{n,n-1}x^n y^{n-1} \oplus \cdots \oplus a_{1,1}xy \oplus a_{1,0}x \oplus a_{0,1}y \oplus a_{0,0},$$

the graph of $z = f(x, y)$ consists of portions of planes determined by each monomial (the portion of course determined by the minimum function.) If we then project the corners (i.e., the edges and vertices) of the graph down to the xy-plane, we have the graph of the corner locus $K(f)$. For example, below we have the graph [3] and the corner locus of the function $f(x, y) = 4 \oplus x \oplus y \oplus xy \oplus 5x^2 \oplus 3y^2$. Notice that each line segment of the corner locus corresponds to a set of values (x, y) for which two monomials of f are equal and less than the value of the other monomials. The vertices coincide with values where three or more monomials are equal and minimal. Note also that the convex regions bordered by the corner locus correspond to values where only one monomial attains the minimal value. For example, above $y = 4$ and to the right of $x = 4$ lie the values (x, y) for which 4 is strictly less than each of x, y, $x + y$, $5 + 2x$, and $3 + 2y$. [7]

2.4 Tropical dual graphs

Each tropical curve f in two affine (or three homogeneous) variables uniquely determines a dual graph, denoted $\Delta(f)$. The dual graph is useful to see what possible

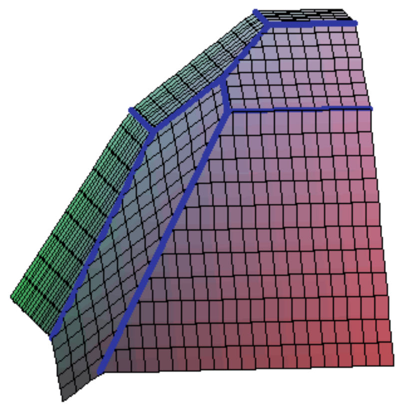


Figure 4: Graph of $z = 4 \oplus x \oplus y \oplus xy \oplus 5x^2 \oplus 3y^2$.

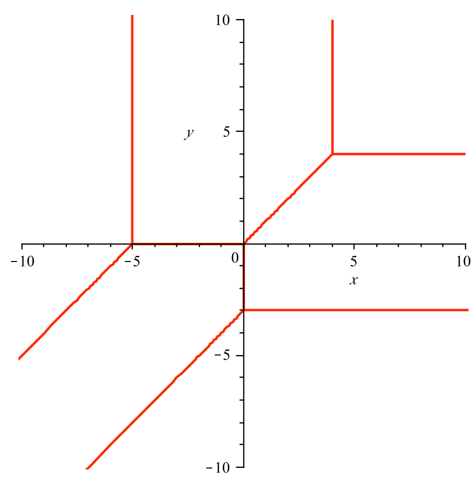


Figure 5: The corner locus of $f(x, y) = 4 \oplus x \oplus y \oplus xy \oplus 5x^2 \oplus 3y^2$.

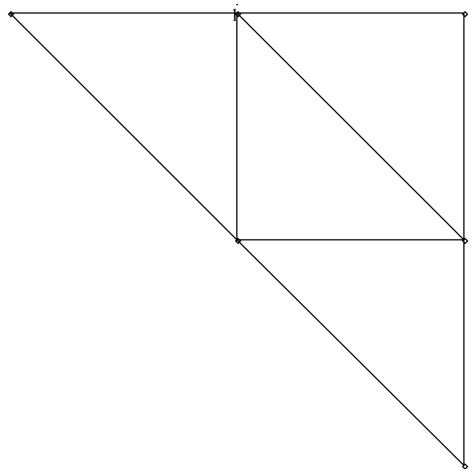


Figure 6: The dual graph of $f(x, y) = 4 \oplus x \oplus y \oplus xy \oplus 5x^2 \oplus 3y^2$.

shapes our curves can attain, so that we may deal with very specific examples in the proof of our main results. We construct it as described in [7] and [2].

Given a function $f(x_1, x_2) = a_{d_1}x^{d_1} \oplus \cdots \oplus a_{d_m}x^{d_m}$, we construct $\Delta(f)$ to be a graph with vertices taken from the points of $\text{supp}(f)$. For every pair of points $-d_i$ and $-d_j$ in $\text{supp}(f)$, we connect them with an edge if and only if there is some point $x = (x_1, x_2) \in \mathbb{Q}^n$ for which the monomials $a_{d_i}x^{d_i}$ and $a_{d_j}x^{d_j}$ attain the minimum at (x_1, x_1); that is to say

$$\begin{aligned} e_{i,1}x_1 + e_{i,2}x_2 + a_{d_i} &= e_{j,1}x_1 + e_{j,2}x_2 + a_{d_j} \\ &< e_{k,1}x_1 + e_{k,2}x_2 + a_{d_k} \end{aligned}$$

for all k not equal to i or j. Any point of $\text{supp}(f)$ not connected to another by an edge is disregarded. The example in Figure 6 shows the dual graph corresponding to the corner locus and graph that we show above in Figures 5 and 4.

Proposition 2.5. Each empty interior region of $\Delta(f)$ corresponds to a vertex of the corner locus $K(f)$, with the number of sides of the region corresponding to the

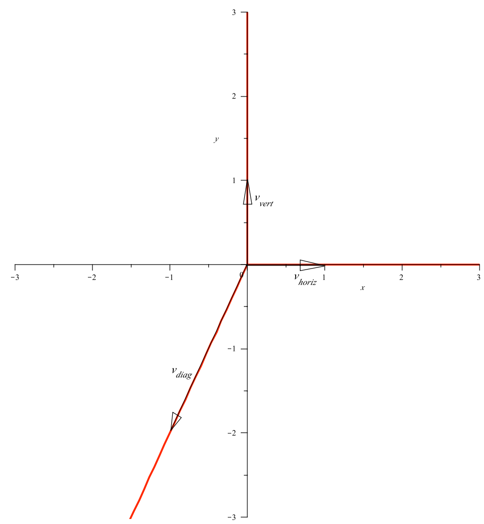


Figure 7: The corner locus of $f(x, y) = x^2 \oplus y \oplus 0$ with primitive direction vectors.

valence at the vertex. Each edge of $\Delta(f)$ corresponds to a ray or line segment of $K(f)$ with perpendicular slope.

Proof. Both [7] and [2] show this. $\square$

Thus the dual graph describes the combinatorial type of the curve by describing the slopes of segments and rays and relative positions of vertices, but provides no information regarding the lengths of segments or the location of vertices of $K(f)$.

Definition 2.6. The *weight* of a line segment or ray of the corner locus is the lattice length of the edge of $\Delta(f)$ that corresponds to it.

Definition 2.7. The *primitive direction vector* (or *primitive integral vector*) of a ray or segment of the graph or corner locus of a tropical function in two variables at the vertex V is the vector $v = (x_1, x_2)$ of shortest length such that x_i is an integer, and such that the ray or segment extends in the same direction from V as v does from the origin. For example, if $f(x, y) = x^2 \oplus y \oplus 0$ is the line with vertex V at

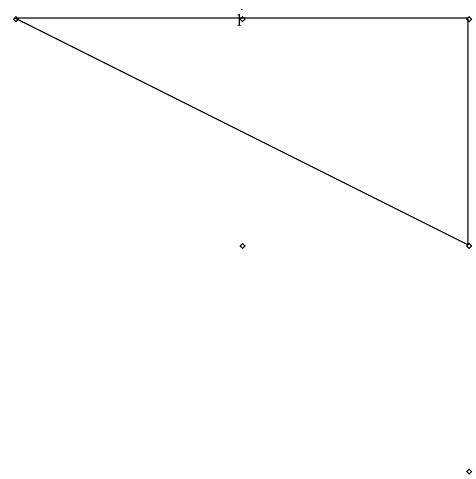


Figure 8: The dual graph of $f(x, y) = x^2 \oplus y \oplus 0$.

the origin, then the primitive direction vector of the vertical ray is $(0, 1)$; for the horizontal ray it is $(1, 0)$; and for the diagonal ray it is $(-1, -2)$. See Figure 7.

Proposition 2.8. Every vertex of the corner locus of a tropical function in two variables satisfies the *balancing condition*, i.e., if a vertex V has n rays or segments emanating from it, each with weight w_i and primitive direction vector v_i, then $\sum_{i=1}^{n} w_i v_i = 0$.

Proof. Both [2] and [6] show this. $\square$

For example, we see from the dual graph of f in Figure 8 that the vertical ray has weight 2, the horizontal has weight 1, and the diagonal has weight 1. So the sum $\sum_{i=1}^{3} w_i v_i = 2(0, 1) + 1(1, 0) + 1(-1, -2) = (0, 0)$, as expected.

Proposition 2.9. The outer edges of the dual graph $\Delta(f)$ always bound a convex region. Also, every connected interior region of $\Delta(f)$ is convex.

Proof. This is shown to be a consequence of the balancing condition in [2]. $\square$

3 Tropical elliptic curves

3.1 Cycles

Proposition 3.1. If f is a tropical cubic curve, then $\Delta(f)$ has an interior vertex if and only if $K(f)$ contains a unique cycle, which we will denote $\mathcal{C}^*(f)$ or simply $\mathcal{C}^*$.

Proof. For each interior region R touching the interior vertex v, there is a vertex of $K(f)$ that is joined by line segments to the vertexes of $K(f)$ corresponding to the interior regions adjoining R. Since there must be at least three edges touching v for it to be a vertex, there must be at least three line segments that join to make the cycle. The cycle is unique simply because there is only one point that can be an interior vertex of $\Delta(f)$. $\square$

Definition 3.2. If a tropical curve has a finite unique cycle, then it is called a *tropical elliptic curve.*

Each connected component of the closure of the complement of $\mathcal{C}^*$ is called a *tentacle*. We note that because of the constraints on the dual graphs of a tropical elliptic cubic, tentacles may only have certain slopes and relationships with each other.

Proposition 3.3. The segments and rays of a tentacle may only have slope $0, \infty, \frac{1}{3}$, $\pm\frac{1}{2}, \frac{2}{3}, \pm 1, \frac{3}{2}, \pm 2$, or 3.

Proof. The slope of each portion of a tentacle must be perpendicular to the slope of an edge of the dual graph that does not touch the interior vertex. Thus the slopes of tentacles are restricted to these. $\square$

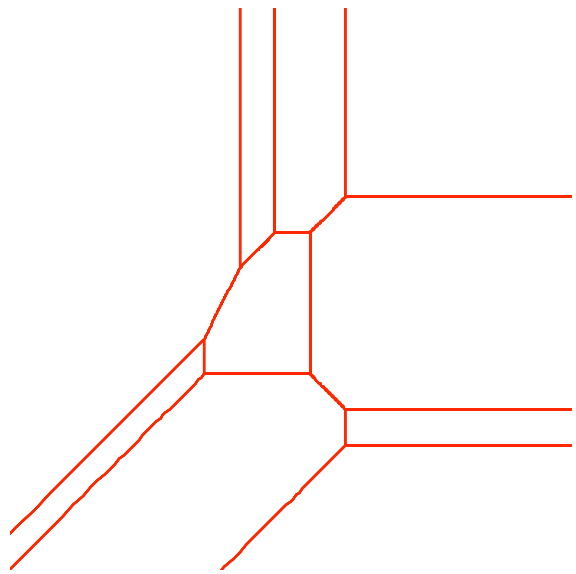


Figure 9: The corner locus of f showing two vertical rays and two diagonal rays protruding directly from the cycle (left); a tentacle composed of one vertical ray, one horizontal ray, and a line segment (upper right); and a tentacle composed of two horizontal rays, one diagonal ray, and two line segments (lower right).

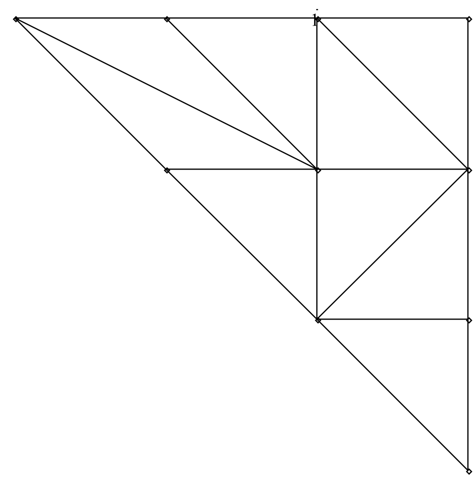


Figure 10: The dual graph of f above.

For an interesting example, we have the corner locus of $f(x,y) = 0 \oplus x \oplus y \oplus 2x^2 \oplus 5x^3 \oplus 6y^2 \oplus 13y^3 \oplus 1xy \oplus 5x^2y \oplus 6xy^2$, Figure 9, and it's dual graph, Figure 10. Notice that some tentacles emanate as single rays from the cycle. Some are composed of short segments and multiple rays heading in different directions. Some include different rays going in the same direction.

Proposition 3.4. The only possible slopes of cycle sides are $0, \infty, \frac{1}{2}, \pm 1$, and 2.

Proof. Each edge corresponding to a side of the cycle must originate with the interior point of $\Delta(f)$ and extend to a boundary vertex. By Proposition 2.5, the sides of the cycle must be perpendicular to the edges of $\Delta(f)$, so these slopes are the only possibilities. □

There is a special subset of tropical curves that have dual graphs with vertices lying at $(0,0,0), (-d,0,0)$, and $(0,-d,0)$, where d is the degree of the curve. Such curves are said to have *full support*.

3.2 Transversal intersections and intersection multiplicity

The intersection of two tropical curves $\mathcal{C}$ and $\mathcal{D}$ is simply the set of values in $\mathbb{TA}^2(\mathcal{Q})$ that lie on both $\mathcal{C}$ and $\mathcal{D}$. However, using this as in classical algebraic geometry would result in rays of $\mathcal{C}$ and $\mathcal{D}$ coinciding without coincident components and other problems, so we wish to define a more general form of intersection that allows for analogous tropical theorems, such as Bézout's theorem, the group law, etc. The definition will be composed of two types of intersections: *transverse intersections*, or those that occur when no vertex of $\mathcal{C}$ lies on $\mathcal{D}$, or vice versa; and *stable intersections* which take care of the rest.

Sometimes curves intersect transversally at a single point with multiplicity greater than 1. For example, in the classical setting, $f(x, y) = x^2 - y$ intersects with $g(x, y) = y$ at $(0, 0)$ with multiplicity 2. We will explain how this works in the tropical case.

Proposition 3.5. For any tropical functions f, g, $K(f) \cup K(g) = K(fg)$.

Proof. This was shown by Aaron Hill and is found in [1]. □

An important consequence of this is the following.

Proposition 3.6. If $\mathcal{C}$ is any curve in $\mathbb{TA}^2(\mathcal{Q})$ and f and g are tropical functions in two variables, then $\mathcal{C} \cap_{st} K(fg) = (\mathcal{C} \cap_{st} K(f)) \cup (\mathcal{C} \cap_{st} K(g))$.

Proof. This is an immediate consequence of Proposition 3.5. □

In order to count the multiplicity of the intersection of $K(f)$ and $K(g)$ at a given point, we can look at the dual graph $\Delta(fg)$.

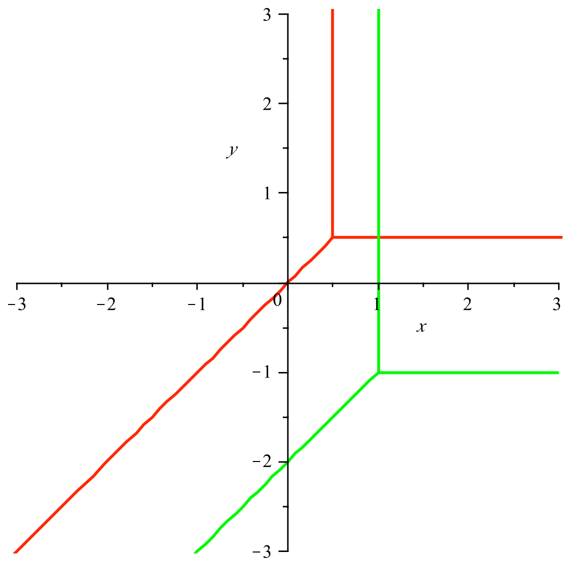


Figure 11: Corner loci of $f(x, y) = x \oplus y \oplus \dfrac{1}{2}$ and $g(x, y) = x \oplus 2y \oplus \dfrac{1}{2}$, or if taken together, of fg.

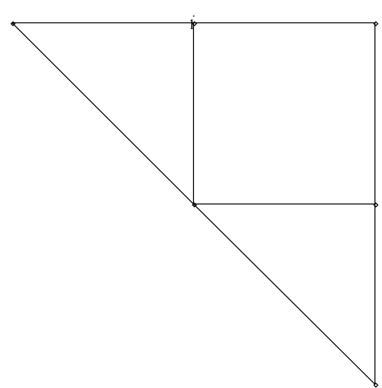


Figure 12: Dual graph $\Delta(fg)$.

Definition 3.7. Let P be an intersection point of $K(f)$ and $K(g)$, where the two edges meeting have weights m_1 and m_2, and primitive integer direction vectors (v_1, v_2) and (w_1, w_2), respectively. The *intersection multiplicity* μ_P [12] of C and D at P is the absolute value of

$$m_1 m_2 \begin{vmatrix} v_1 & v_2 & 0 \\ w_1 & w_2 & 0 \\ 1 & 1 & 1 \end{vmatrix}.$$

For example, set $f(x, y) = x \oplus y \oplus \frac{1}{2}$ and $g(x, y) = x \oplus 2y \oplus \frac{1}{2}$. The intersection lies on the horizontal ray of $K(f)$ and the vertical ray of $K(g)$, as in Figure 11. The dual graph $\Delta(fg)$ is Figure 12. The weights m_1 and m_2 are both 1, since the corresponding edges have lattice length 1. The primitive direction vectors are $(1, 0)$ and $(0, 1)$, so the intersection multiplicity at the intersection is

$$1 \cdot 1 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1.$$

3.3 Stable Intersection

Finding points of intersection of tropical curves is simple when the intersection is transverse, but because graphs of tropical functions may intersect along rays, even without sharing a common component (i.e., sharing a common factor in the function), we must use a more general idea of intersection called *stable intersection*.

There is a similar concept in classical intersection theory. For example, if we have the cubic $f(x, y) = (y + x^2)(y + 2)$, and $g_a(x, y) = y + a$, then as $a > 0$ varies continuously, the intersection $Z(f) \cap Z(g_a)$ also varies continuously, except at $a = 2$. When $a = 2$, the two share a component, and thus intersect at infinitely many points. But simply by taking the limit of the intersections as $a \to 2$, we can tell that there are two special points of intersection for when $a = 2$, at $(-\sqrt{2}, -2)$

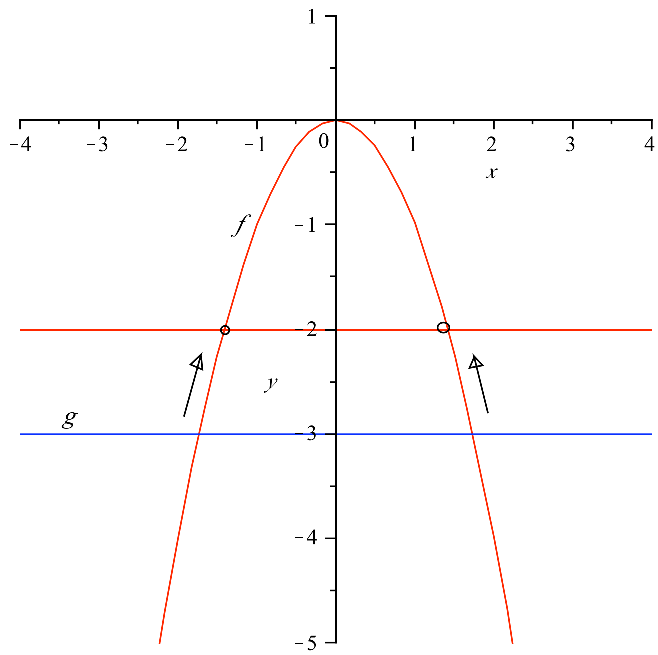


Figure 13: Example of classical stable intersection.

and $(\sqrt{2}, -2)$. (See the figure below.) These points are the stable intersection of f and g_2, denoted $Z(f) \cap_{st} Z(g_2)$.

Extending this idea to the tropical setting, for f, g tropical polynomials, we define the stable intersection of f and g, denoted $K(f) \cap_{st} K(g)$ to be the following:

Definition 3.8. Let f and g be tropical functions in two variables. Then for a generic $v \in \mathbb{Q}^2$, $K(f)$ intersects transversally with $K(g) + v$, as shown in [11]. So we define the *stable intersection* of $K(f)$ with $K(g)$ to be the set

$$K(f) \cap_{st} K(g) = \lim_{v \to 0} K(f) \cap (K(g) + v)\,,$$

counting multiplicities of each point in the set. This limit is shown to be well-defined in [11] as well.

For example, the corner loci of the lines $f(x, y) = x \oplus y \oplus 1$ and $g(x, y) = x \oplus y \oplus \frac{1}{2}$

coincide along the diagonal ray. But we see that perturbing $K(g)$ by $v = (0, 0.1)$ gives us a single transversal intersection. Taking smaller and smaller perturbations like this will give intersection points that get arbitrarily close to the point $\left(\frac{1}{2}, \frac{1}{2}\right)$, and so this point is the point of stable intersection.

As in the classical algebraic geometry, a form of Bézout's Theorem holds, which helps us know when we've found all of our stable intersection points.

Proposition 3.9 (Bézout's Theorem)**.** If $\mathcal{C}$ and $\mathcal{D}$ are tropical curves in $\mathbb{TP}^n(\mathcal{Q})$ of degrees k and m, respectively, then the number of stable intersection points of $\mathcal{C} \cap_{st} \mathcal{D}$, counting multiplicity, is km.

Proof. This is proved in [8]. □

3.4 Tropical singularities

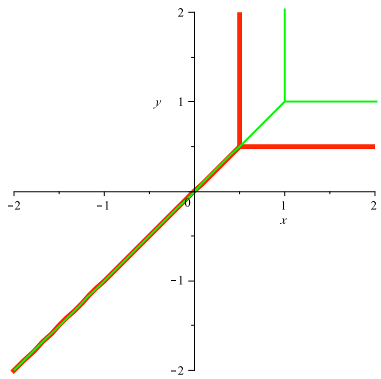
Tropical curves can contain points, which just like in the classical case, are called singularities, that cause problems when computing the group law. Such points must be identified and then thrown out.

Definition 3.10. A tropical *singularity* is a point p of the curve $\mathcal{C}$ at which at least one of the following two conditions is met:

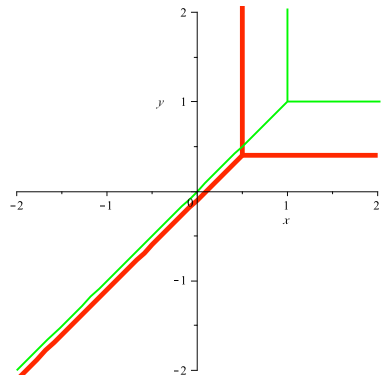
1. There is no tropical line which intersects the curve at p with multiplicity one;

2. The point has valence at least four (i.e., there are at least four rays or line segments emanating from the point p).

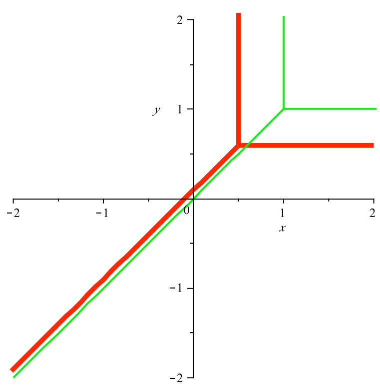
Remark 3.11. In the classical setting, there are a few equivalent conditions like the above ones that all can detect singularities of classical curves. Unfortunately, the



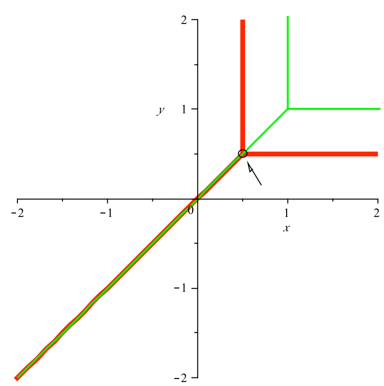
(a) $K(f)$ appears to intersect $K(g)$ along the diagonal ray.



(b) $K(f)$ is perturbed slightly downwards to show the single intersection near $(1/2, 1/2)$.



(c) Similarly perturbing $K(f)$ up shows the same on a different ray.



(d) The point of stable intersection is $(1/2, 1/2)$.

Figure 14: Example of perturbing a corner locus to find the stable intersection.

analogous tropical conditions behave badly in general. Luckily for our purposes, the above mentioned conditions work well for detecting singularities of tropical *cubics*, and so we are able to consistently apply them in this thesis.

We can often detect singularities with the help of the dual graph. If $\Delta(f)$ contains a region with more than three sides, as in Figure 20, then the corresponding vertex will satisfy condition 2., as in Figure 11. If the lattice length of any edge of the dual graph is more than one, as in Figure 15, then the corresponding ray or segment of the curve has weight $w > 1$. Thus when we compute the intersection multiplicity of any line with that segment it is also greater than 1. Therefore, the points of this segment or ray satisfy condition 1, as in Figure 11.

Another important example is the following. A cubic may have an infinite cycle which contains a point at infinity. For example, if $h(x, y) = 3 \oplus x \oplus y \oplus (-2)xy \oplus xy^2 \oplus y^2 \oplus (-1)y^3$, then $\Delta(h)$ is represented in Figure 15 and $K(h)$ is Figure 16.

Proposition 3.12. Any line that intersects an infinite cycle at the infinite point, must intersect with multiplicity two or more, and therefore, the infinite point of a cycle is singular.

Proof. The stable intersection of a line with the infinite cycle must be the limit of the intersections of the curve with a sequence of lines which approach the line at the infinite point. Since we can find such a sequence of lines that all have vertex within the cycle, it is clear that there are two intersections of each line on the cycle each approaching the infinite point. Thus the stable intersection of any line with this curve at the infinite point must have multiplicity 2 or more. $\square$

There is one other special type of point on some tropical elliptic curves that can cause problems—that is a point at infinity. There is some difficulty in discussing the

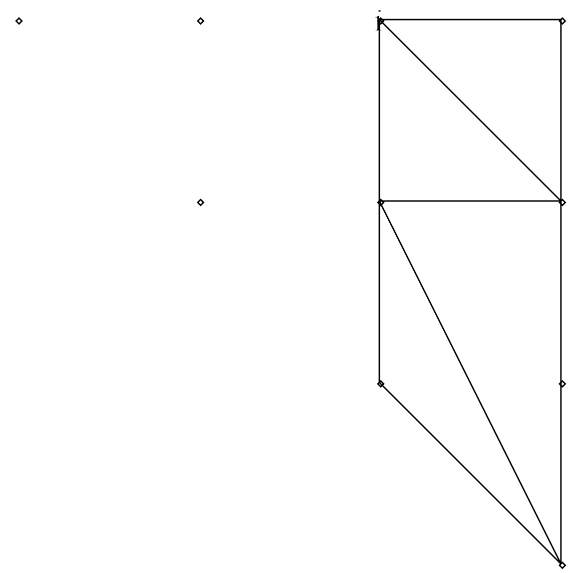


Figure 15: The dual graph of the curve h with an infinite cycle.

intersections of these points, and so we will not deal with them here. In the scope of this thesis, we will only consider tropical elliptic curves that are non-singular, (i.e., that don't contain any points of singularity), and we will only be considering the points of these curves that have no infinite coordinate, i.e. the *finite* points of nonsingular curves.

4 Divisors and the Group Law

4.1 From classical geometry

In classical algebraic geometry, the most general way to prove the group law on cubic curves is to use divisors and construct the Jacobian. Define a group structure on the curve in the following way (see [10], or [9].)

Definition 4.1. Set G equal to the free Abelian group on the points of the curve

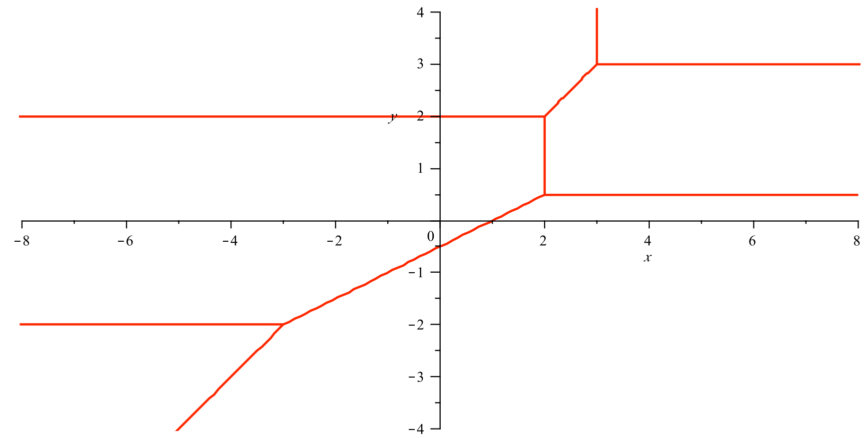


Figure 16: Corner locus of the curve h with an infinite cycle. Note that the two infinite rays extending left intersect at $(0, \infty, \infty) \in \mathbb{TP}^2(\mathcal{Q})$.

$\mathcal{C}$. The elements of this group are called *divisors*. So a divisor D is essentially a formal sum, $D = \sum_{P \in \mathcal{C}} \mu_P P$, of points on the curve, where μ_P is some integer. For obvious reasons, we will call the integer μ_P the *multiplicity* of D at P.

Definition 4.2. The divisor of a polynomial, *div* f, is the divisor $\sum \mu_P P$, where P ranges over the points of the stable intersection of f and $\mathcal{C}$ and μ_P is the multiplicity of the intersection at P.

Definition 4.3. The *degree* of a divisor D is the sum of the multiplicities $\sum_{P \in \mathcal{C}} \mu_P$.

Definition 4.4. The subgroup of degree zero divisors of $\mathcal{C}$ is denoted $Div^0(\mathcal{C})$. A divisor is called *principal* if it is the difference of the divisors of two homogeneous polynomials f and g of same degree; i.e., $D = div\ \dfrac{f}{g} = div\ f - div\ g$.

We define an equivalence relation on the elements of this group.

Definition 4.5. Two elements P and Q are *linearly equivalent*, written $P \sim Q$, if and only if there exists a principal divisor $D = div\ \frac{f}{g} = div\ f - div\ g$ such that $D = P - Q$.

We define the Jacobian (or divisor class group) as the following quotient.

Definition 4.6. The *Jacobian* of the curve $\mathcal{C}$ is $Jac(\mathcal{C}) = Div^0(\mathcal{C})/\sim$.

4.2 Tropical divisors and linear equivalence

As in the classical setting, we define the concept of tropical divisors, principal divisors, and the Jacobian.

Definition 4.7. Set $Div(\mathcal{C})$ to be the free Abelian group generated by the finite, non-singular points of the curve $\mathcal{C}$. A *divisor* is an element of $Div(\mathcal{C})$, i.e., $D = \sum_{P \in \mathcal{C}} \mu_P P$ where μ_P is an integer.

Definition 4.8. The sum $\sum_{P \in \mathcal{C}} \mu_P$ is the *degree* of the divisor D. The set of degree zero divisors of $\mathcal{C}$ is denoted $Div^0(\mathcal{C})$. A tropical divisor D is *principal* if there are tropical homogeneous polynomials f and g of the same degree and *each is the product of lines* such that $D = div\ f - div\ g$.

Definition 4.9. Two tropical divisors D and D' are *linearly equivalent*, written $D \sim D'$, if their difference is a principal divisor, i.e., if there are polynomials f and g, each of which is the product of n lines for some $n \geq 0$, such that $D - D' = div\ \frac{f}{g} = div\ f - div\ g$.

Remark 4.10. This definition of tropical principal divisors is different than those that other authors have suggested. Some authors have suggested including the restriction that $\Delta(f) = \Delta(g)$ [13] [14]. But this definition does not allow the relation $\sim$ to be transitive. For example, the transitivity of $\sim$ on the curve in Figure 21 will be discussed below. But defining principal divisors as we have, we get that $\sim$ is an equivalence relation.

There is another more obvious way to define the equivalence relation. That is to simply let f and g be homogeneous of the same degree without any extra conditions. We choose not to work with this definition since it seems that we are left with very few equivalence classes. Using this definition may in fact give the same result, but so far it seems that there are too many cases to consider, and most authors seem to restrict their definitions to be more workable like ours. We also suspect that there may be a way to show that for any function f of degree n and any curve $\mathcal{C}$ there is a product $g = \prod L_i$ of n lines such that $div\ f = div\ g$. However, without proving such a conjecture, we are not certain that the results will match ours.

Proposition 4.11. The relation $\sim$ is an equivalence relation.

Proof. Clearly, $D - D = div\ \frac{f}{f}$ for any polynomial f, so $\sim$ is reflexive. Also, if $D - D' = div\ f - div\ g$, then $D' - D = div\ g - div\ f$, so $D \sim D'$ if and only if $D' \sim D$; thus $\sim$ is symmetric.

Lemma 4.12. The divisor of the product of two functions is the sum of the divisors of each funciton, i.e., $div\ fg = div\ f + div\ g$.

Proof. This follows from the definition of $div\ f$ and Proposition 3.6. □

If f and g are products of m lines and h and k are products of n lines such that $D - D' = div\ f - div\ g$ and $D' - D'' = div\ h - div\ k$, then $D - D'' = (D - D') + (D' - D'') = div\ f - div\ g + div\ h - div\ k = div\ fh - div\ gk = div\ \frac{fh}{gk}$

is principal. Therefore $D \sim D''$, and $\sim$ is transitive. Thus $\sim$ is an equivalence relation. $\square$

Corollary 4.13. $R \sim S$ and $R + P \sim S + Q$ implies that $P \sim Q$.

Proof. If $div\ \dfrac{f}{g} = R - S$ and $div\ \dfrac{h}{k} = S + Q - R - P$, then

$$div\ \frac{fh}{gk} = R - S + S + Q - R - P = Q - P$$

so that $P \sim Q$. $\square$

Using this definition for $\sim$, we can define the *tropical Jacobian*.

Definition 4.14. The tropical Jacobian of a curve $\mathcal{C}$ is $Jac(\mathcal{C}) = Div^0(\mathcal{C})/\sim$.

The following theorem is the main result of this paper.

Theorem 4.15. *If $\mathcal{C}$ is a nonsingular tropical elliptic curve, then $Jac(\mathcal{C})$ is isomorphic to the group of rational points of S^1, which we will call $\mathcal{S}$.*

To prove this theorem, we must prove a few lemmas. We will show that the curve is reduced to the cycle by showing that all points on a tentacle are equivalent, and that distinct points on the cycle are not equivalent to one another. We will also define the group operation and the isomorphism to $\mathcal{S}$. [12] [13] [14]

5 Proof of the tropical group law

5.1 Tentacles

Lemma 5.1. If P, Q are points of the same tentacle $\mathcal{T}$ of $\mathcal{C}$, then $P \sim Q$.

First we will distinguish types of tentacles so that we may treat the proof methodically.

Definition 5.2. An *upper-right* tentacle is one that emanates from the cycle at an angle in $[0, 90)$ degrees. A *left* tentacle is one that emanates from the cycle at an angle in $[90, 225)$ degrees. A *lower* tentacle is one that emanates from the cycle at an angle in $[225, 360)$ degrees.

So every tentacle may be characterized as either an upper-right, left or lower tentacle.

Definition 5.3. A *side* tentacle consists of a ray emanating directly from the cycle, with no connecting line segments. A *corner* tentacle is one that is not a side tentacle.

First we must note that we may restrict our examples and proofs to only the upper-right tentacles. This is because when dealing with a left or lower tentacle, we may apply a transformation that permutes the projective variables x, y, and z, then find the appropriate divisors according to the method explained below, and reverse transform these divisors resulting in what we need to prove the result on the original curve. [1]

For example, if L is a vertical ray of a curve $K(f(x, y))$, we can switch the variables x and y, resulting in $K(f(y, x)$ having the portion L instead manifest as a horizontal ray. We may then apply the methods below on the horizontal ray, to find divisors D and D' that show linear equivalence between given the points on L. Then by switching the y and x variables back, in both the curve and the divisors, we get divisors D and D' that show linear equivalence on the vertical ray L.

Proof of Lemma 5.1. Suppose that P and Q are points on the tentacle $\mathcal{T}$ with the x coordinate of P less than that of Q. (The case where $P = Q$ is trivial.) We will consider a portion L (either a ray or line segment) of the tentacle, considering the cases of L being part of a corner tentacle, or a side tentacle, etc. We shall first assume that L has positive slope.

If $m \neq 1$, then we may set f to be the line with vertex at the point P, as in Figure 17. Set g to be the line that intersects L at Q and P. (Since P and Q are in general position when the slope $m \neq 1$, this line is unique.) Then $\mathcal{C} \cap_{st} K(f) = div\ f = 2P + R$ where R lies on the diagonal ray of $K(f)$. Also, $\mathcal{C} \cap_{st} K(g) = div\ g = P + Q + R$. Thus, $div\ \dfrac{f}{g} = P - Q$, and $P \sim Q$.

If $m = 1$, as in Figure 18, then choose f to be any line with an stable intersection of multiplicity 1 at P, and choose g to the line that intersects L at Q and has diagonal ray that overlaps that of $K(g)$. Then $div\ f = P + R_1 + R_2$ and $div\ g = Q + R_1 + R_2$. Thus $div \dfrac{f}{g} = P - Q$, and $P \sim Q$.

Now we consider the ray L with slope $m = 0$, as in Figure 19. Let us first assume that L is the topmost ray, i.e., that if the point $(x_0, y_0, 0) \in L$, then $(x_0, y, 0) \notin \mathcal{C}$ for all $y > y_0$. Then set δ to be some positive value less than the minimum distance between tentacles. Let $d(\cdot, \cdot)$ denote Euclidean distance. Then if $d(P, Q) < \delta$, we can choose a line f so that $K(f)$ intersects L at P and the vertex of $K(f)$ lies more than δ below L. Then we may choose the line g so that $K(g)$ intersects L at Q, and the downward rays of $K(f)$ and $K(g)$ overlap. Then $div\ f = P + R_1 + R_2$ and $div\ g = Q + R_1 + R_2$. Thus $div \dfrac{f}{g} = P - Q$, and $P \sim Q$.

If $d(P, Q) \geq \delta$, we may certainly find points $P = S_1, S_2, \ldots, S_r = Q \in L$ such that $d(S_i, S_{i+1}) < \delta$, so that $P \sim S_2 \sim \cdots \sim S_{r-1} \sim Q$. So since $\sim$ is transitive,

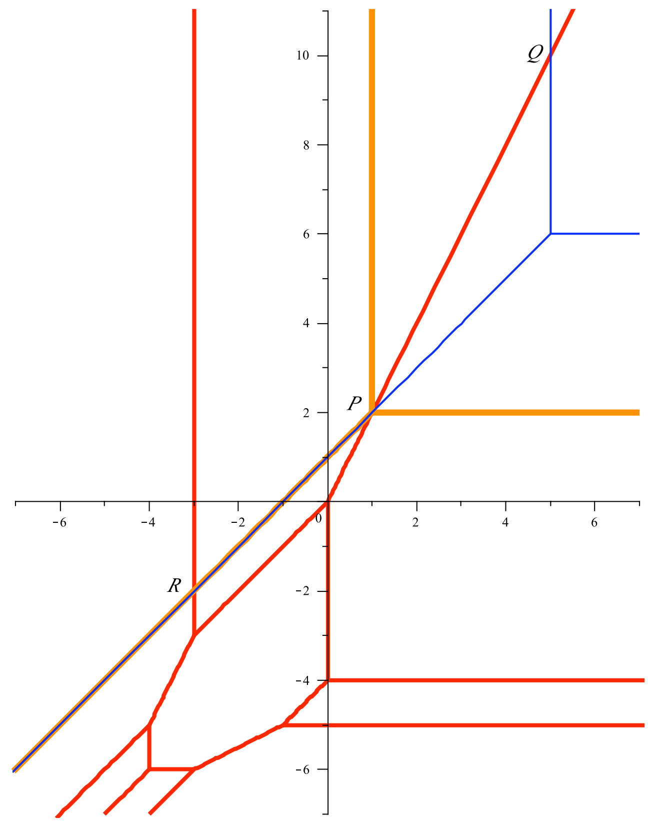


Figure 17: The curve $\mathcal{C}$, given by $x^2 \oplus y \oplus xy \oplus 3x^3 \oplus 4x^2y \oplus 6xy^2 \oplus 9y^3 \oplus 4y^2$, showing how to deal with a tentacle ray of slope $m \neq 1$.

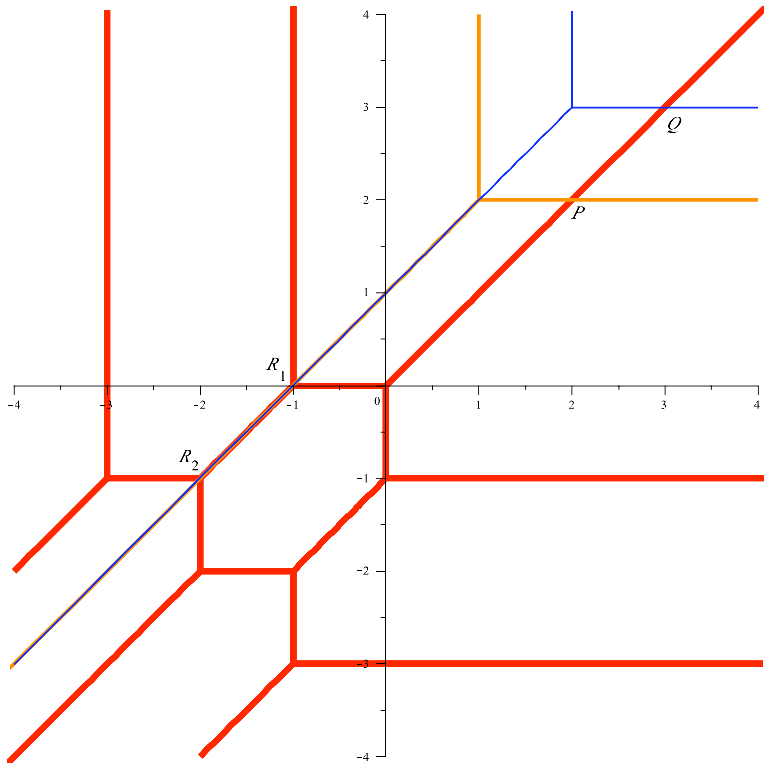


Figure 18: The curve $\mathcal{C}$ given by $x \oplus y \oplus xy \oplus 1x^2 \oplus 1y^2 \oplus 2x^2y \oplus 2xy^2 \oplus 4x^3 \oplus 4y^3$, showing how to deal with a tentacle ray of slope $m = 1$.

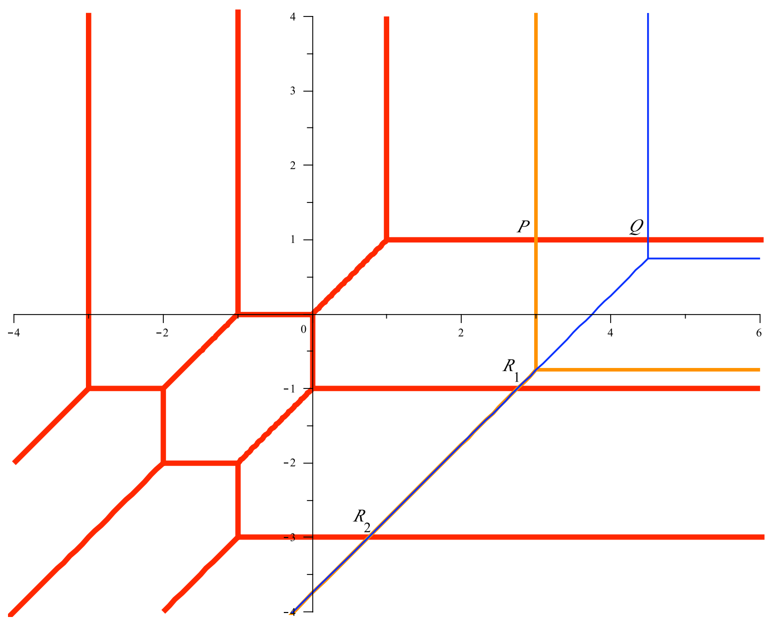
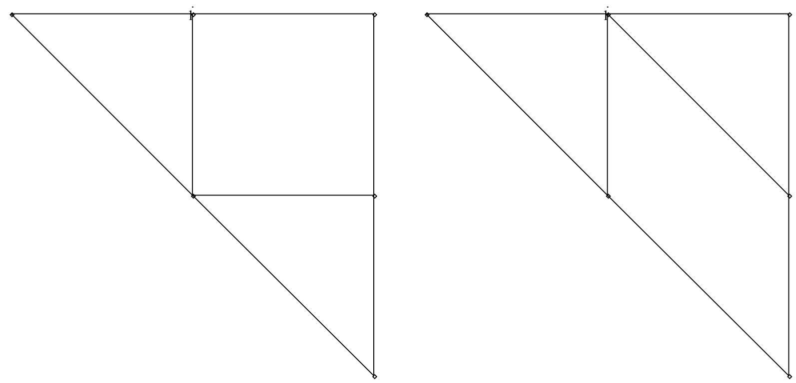


Figure 19: The curve $\mathcal{C}$, given by $1 \oplus x \oplus y \oplus xy \oplus 1x^2 \oplus 1y^2 \oplus 2x^2y \oplus 2xy^2 \oplus 4x^3 \oplus 4y^3$, showing how to deal with a tentacle ray of slope $m = 0$.



(a) The dual graph $\Delta(fh)$. (b) The dual graph $\Delta(gk)$.

Figure 20: The dual graphs for fh and gk are not the same.

$P \sim Q$.

Remark 5.4. We can now discuss why adding the restriction to the definition for linear equivalence that $\Delta(f) = \Delta(g)$ will result in a counter-example to transitivity. In Figure 21, we can see that $K(fh)$ is the union of the orange and pink lines, while $K(gk)$ is the union of the blue and green lines. The dual graphs of fh and gk are shown in Figure 20. We showed that $P \sim Q$ using $P - Q = div\ fh - div\ gk$. But since the duals are different, this could not work with the extra restriction on our definition.

We apply these methods to each ray or segment of the corner tentacle using Proposition 4.11 and Corollary 4.13. Thus we can reduce each ray to the point it emanates from on a segment, then reduce the segment to the point it emanates from. Thus we have reduced the topmost rays to a point. Then using transitivity again, we can show linear equivalence for points that lie on rays below these topmost rays and segments.

We can show that $P \sim Q$ when $\mathcal{T}$ is a side tentacle lying below a corner tentacle

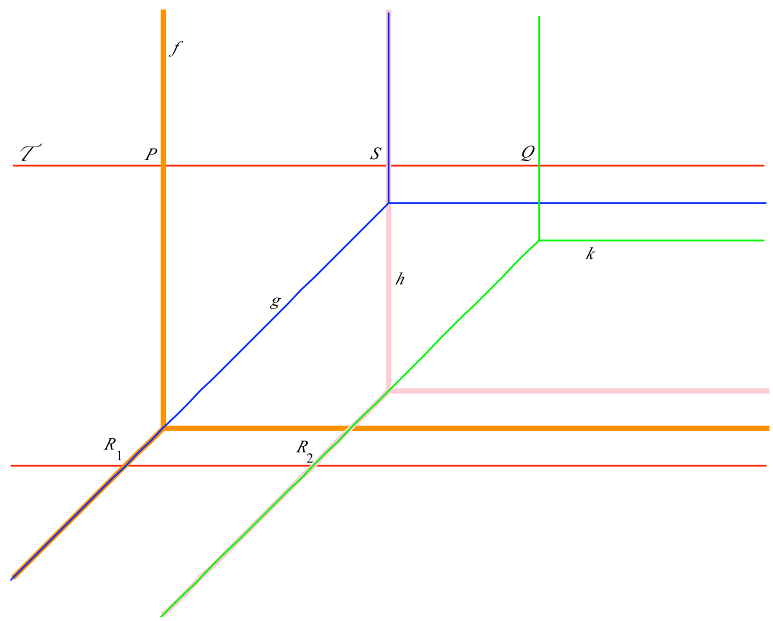


Figure 21: The curve $\mathcal{C}$, as in Figure 19, showing how to deal with points on the tentacle that are far away from each other.

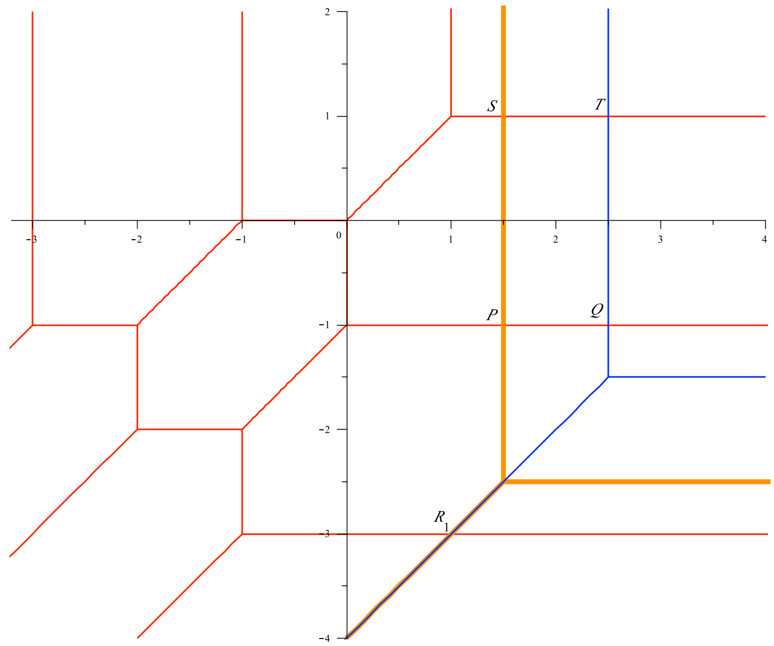


Figure 22: Showing how to deal with a side tentacle that lies below a corner tentacle.

or when L is a ray of the corner tentacle $\mathcal{T}$ lying below another ray or line segment of the corner tentacle $\mathcal{T}$. So suppose that L is the ray of a side tentacle below the corner tentacle, or that L is a ray of the corner tentacle $\mathcal{T}$ that lies below some other ray or segment of $\mathcal{T}$, and assume that $d(P, Q) < \delta$.

Then we may employ the same method as with the topmost ray to yield polynomials f and g with $div\ f = P+S+R_1$ and $div\ g = Q+T+R_1$. Thus $P+S \sim Q+T$. But $S \sim T$, so by the corollary we have $P \sim Q$. Then again by the transitivity, the points of this ray are all linearly equivalent to the point that it emanates from. We may thus inductively collapse each portion of the tentacle down to the point on the cycle from which it emanates, whether it be a side tentacle or a corner tentacle.

Thus we see that all points on a given tentacle are linearly equivalent. $\square$

Thus given a point P on a tentacle $\mathcal{T}_i$, we see that the divisor $P - \mathcal{O} \sim T_i - \mathcal{O}$, where T_i represents the point on the cycle where $\mathcal{T}_i$ meets $\mathcal{C}^*$. Therefore, we have the following corollary.

Corollary 5.5. The Jacobian $Jac(\mathcal{C}) = Div^0(\mathcal{C})/\sim$ is isomorphic to the Jacobian $Jac(\mathcal{C}^*) = Div^0(\mathcal{C}^*)/\sim$ of the curve restricted to the cycle.

5.2 Cycles

Now that we know the points of a given tentacle are all equivalent to the intersection of that tentacle with the cycle, we can treat all intersections $f \cap \mathcal{C}$ as intersections of f with the cycle $\mathcal{C}^*$. Thus we now turn our attention to the points on $\mathcal{C}^*$.

To begin, we define a special distance function for points on the cycle called the *directed distance*, denoted by ρ. Heuristically, we add up the distances along the counter-clockwise path, call it $\mathcal{P}$, from A to B. However, the distance along a given side will depend on the slope of that side.

Definition 5.6 (Directed cycle distance)**.** Recall that $d(P, Q)$ denotes the Euclidean distance between P and Q. Suppose that A and B are points on the cycle $\mathcal{C}^*$. Let $\mathcal{P}$ be the path from A to B along the cycle *going counter-clockwise.* Let $P_1, \ldots, P_n$ be the points on the cycle such that $P_1 = A$, $P_n = B$, and $P_2, \ldots, P_{n-1}$ are the vertices of the cycle that the path $\mathcal{P}$ touches between A and B, in order. That is, when traversing the path $\mathcal{P}$ from A to B, one starts at P_1, then one encounters P_2, then P_3, etc. until P_n is reached. For $i = 1, \ldots, n-1$, set $d_i = d(P_i, P_{i+1})$. If the absolute value of the slope of the side of the cycle from P_i to P_{i+1} is $\dfrac{1}{2}$ or 2, then set $q_1 = \sqrt{5}$; if it is 1, set $q_i = \sqrt{2}$; and if it is 0 or infinity, set $q_i = 1$. Then using these values for this specific pair of A and B we define the *directed cycle distance* $\rho(A, B)$ from A to B to be the sum

$$\rho(A, B) = \sum_{i=1}^{n} \frac{d_i}{q_i}.$$

For example, Figure 23 shows the distance from A to B to be the distance from A to the corner of s_1, plus the length of s_2, plus the length of s_3, plus the distance from the corner of s_4 to B. Thus

$$\rho(A, B) = \sum_{i=1}^{4} \frac{d_i}{q_i} = \frac{\sqrt{2}}{2\sqrt{2}} + \frac{2}{1} + \frac{\sqrt{2}}{\sqrt{2}} + \frac{3}{2} = 5.$$

Of course, we may also now define the circumference of the cycle.

Definition 5.7. The *circumference* c, of the cycle is $c = \rho(A, B) + \rho(B, A)$, for two distinct points $A, B \in \mathcal{C}^*$.

So the circumference of the cycle in Figure 23 is

$$c = \rho(A, B) + \rho(B, A) = 5 + 6 = 11.$$

Notice that for any two distinct points A and B, $\rho(A, B) = c - \rho(B, A)$.

Now we will construct a homomorphism from the Jacobian of the curve to $S^1(\mathbb{Q})$, and this will show that no two distinct points of the cycle are linearly equivalent along the way. We designate some point of $\mathcal{C}^*$ to be the origin and name it $\mathcal{O}$. (Of course, we could choose a point of a tentacle to be our origin, but after taking the quotient over the equivalence relation, this is the same as choosing the point where the tentacle meets the cycle for our origin.) Of the points on the cycle that are topmost, name the point farthest to the right P. Then define L to be the line with

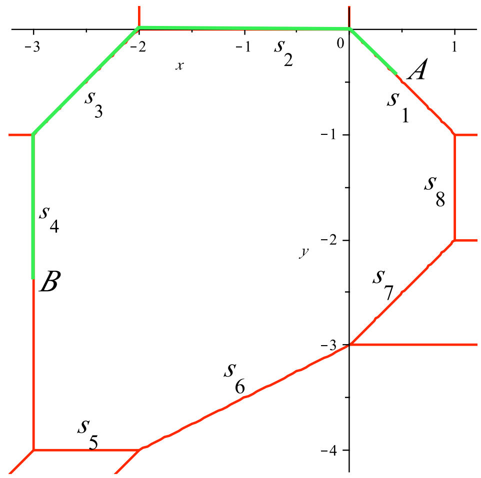


Figure 23: Finding $\rho(A, B)$.

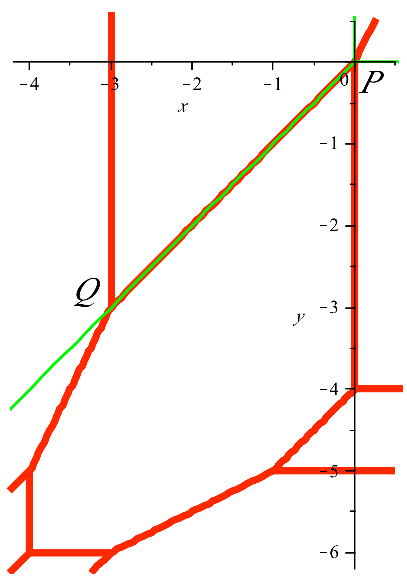


Figure 24: An example of how to choose P and Q on the cycle.

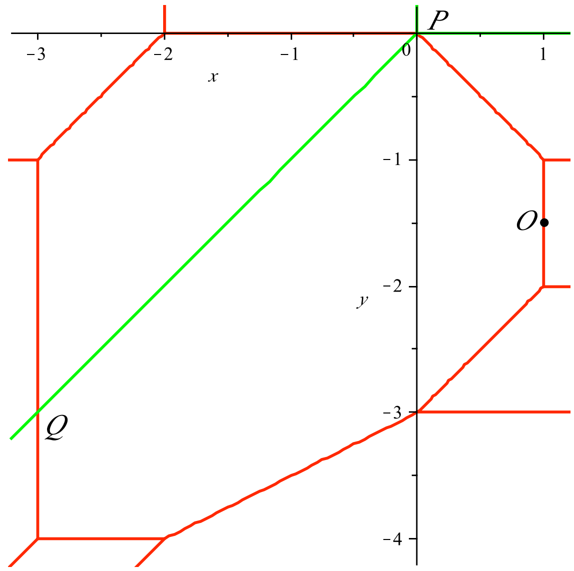


Figure 25: The cycle of $6y^3 \oplus 3y^2 \oplus 4xy^2 \oplus 3x^2y \oplus 2x^2 \oplus 1y \oplus xy \oplus x \oplus 0$ with $\mathcal{O}$, P, and Q labeled.

vertex at P and select Q to be the unique remaining point of the intersection of L with $\mathcal{C}^*$. (Notice that P chosen in this way must be an intersection of multiplicity 2, because shifting this line in the $x = y$ direction by $\varepsilon > 0$ results in two intersections– one on each of the vertical and horizontal rays. That is why there is exactly one other point of intersection with the cycle. This is also not necessarily the only way that could choose P We could actually choose for P any point on the cycle which has double intersection multiplicity, and then choose Q to be the remaining intersection point. Choosing P as we have done just simplifies our examples.) See Figures 24 and 25.

Define a function $\alpha : \mathcal{C}^* \to \mathcal{S}$, (where $\mathcal{S}$ is the usual additive group on $\mathbb{Q} \mod 1$, or the group S^1 restricted to rational points), by

$$\alpha(X) = \rho(\mathcal{O}, X) \mod c.$$

Clearly, the point $\mathcal{O}$ is mapped to the identity $\alpha(\mathcal{O}) = 0$.

Since $Div^0(\mathcal{C}^*)$ is a free Abelian group generated by the points of the cycle, and since $\mathcal{S}$ is Abelian, the map α extends to a homomorphism $\varphi : Div^0(\mathcal{C}^*) \to \mathcal{S}$ defined by

$$\varphi\left(\sum \mu_X X\right) = \sum \mu_X \alpha(X).$$

Proposition 5.8. The map φ is a homomorphism.

Proof. This is true almost by definition. If $D = \sum \mu_X X$ and $E = \sum \nu_X X$, then since $D + E = \sum(\mu_X + \nu_X)X$, we have

$$\begin{aligned}
\varphi(D+E) &= \varphi\left(\sum(\mu_X + \nu_X)X\right) \\
&= \sum(\mu_X + \nu_X)\alpha(X) \\
&= \sum \mu_X \alpha(X) + \sum \nu_X \alpha(X) \\
&= \varphi(D) + \varphi(E).
\end{aligned}$$

Therefore, φ is a homomorphism. $\square$

When the vertex lies on or within the cycle, then shifting it in any one of the vertical, horizontal, or $x = y$ directions, while keeping the vertex on or within the cycle, shifts exactly two points of intersection with the cycle. To be more specific, and to emphasize the importance of this concept, we will state it as a lemma.

Lemma 5.9. For g a tropical line with $div\ g = D + E + F$ and for which the vertex of $K(g)$ lies on or within the cycle, if g' is a line with vertex on or within the cycle and $div\ g' = D + E' + F'$, then $\rho(E, E') = \rho(F', F) = c - \rho(F, F')$.

Proof. Let g be a line with vertex V on or within the cycle, with $div\ g = D+E+F$, and suppose that g' is also a line with vertex V' on or within the cycle, and with

$div\ g' = D + E' + F'$. If D is on the vertical ray of $K(g)$, then it must be on the vertical ray of $K(g')$. Likewise, if D lies of the horizontal ray or the diagonal ray of $K(g)$, then it must be on the same ray of $K(g')$, respectively. Let's assume that the x coordinates of the vertices lie some distance $d(V_x, V_x') = \delta$ apart, and that the y-components lie a distance $d(V_y, V_y') = \varepsilon$ apart. (This is again Euclidean distance.) Of course, if D is on the horizontal ray, then $\varepsilon = 0$; if on the vertical ray, $\delta = 0$; and if on the diagonal ray, then $\delta = \varepsilon$.

Suppose that D is on the vertical ray (as in our example, Figure 26). Then the y-components of the intersections corresponding to the horizontal rays, say F and F', are separated by exactly ε. So if F and F' lie on the same side s_1 of the cycle, then since the only possible slopes of s are $\infty, \pm 1$, and $\frac{1}{2}$, $\rho(F', F) = \rho_{s_1}(F', F) = \varepsilon$. If E and E' also lie on one side s_2, then since the possible slopes are $\infty, 0, 1, \frac{1}{2}$, and 2, again we have $\rho(E, E') = \rho_{s_2}(E, E') = \varepsilon$. (See in our example, Figure 26, the shift from $K(g)$ to $K(g')$, or from $K(g')$ to $K(g'')$.)

If the E and E' or F and F' don't lie on the same side of the cycle, then we can break up the shift into finitely many smaller shifts for which the intersections lie on the same sides for each of the smaller shifts. So even if the shift slides intersection points around corners, our result regarding the opposing distances remains true.

If D lies on either of the other two rays, then the proof is nearly identical. We simply employ a permutation of variables x, y, and z to put D on the vertical ray, apply the above argument, and then use the reverse permutation to put the variables back. Therefore, our result is true in all cases. $\square$

We get the following lemma (due to Dr. Tracy Hall [5]).

Lemma 5.10. For any tropical line g such that $div\ g = D + E + F$, we get

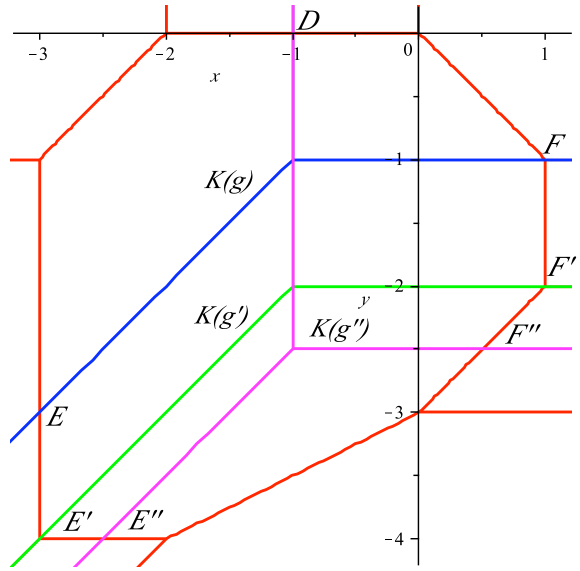


Figure 26: Demonstrating the change in intersection points when a vertical shift is made.

$\varphi(div\ g) = 2\varphi(P) + \varphi(Q)$.

Proof. Suppose that g is a line, which we may assume to have vertex lying on or within the cycle, with $div\ g = D+E+F$. Then we may shift g to a line g' with vertex that lies on or within the cycle and also on the line f which has $div\ f = 2P+Q$. Say $div\ g' = D+E'+F'$. We need only use at most two shifts from the three directions that we may shift as above. By Lemma 5.9, we know that when we shift g to g', because of the fact that $\rho(E,E') = \rho(F',F)$, we get $\varphi(E') = \varphi(E) + \rho(E,E')$, and

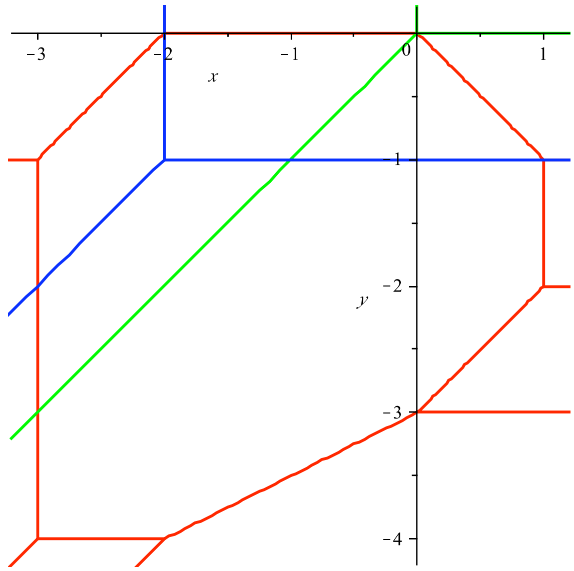


Figure 27: The corner loci of f and g.

$\varphi(F') = \varphi(F) + \rho(F, F')$. Thus

$$\begin{aligned}
\varphi(D+E+F) &= \varphi(D) + \varphi(E) + \varphi(F) \\
&= \varphi(D) + \varphi(E') - \rho(E, E') + \varphi(F') - \rho(F, F') \\
&= \varphi(D) + \varphi(E') + \varphi(F') - \rho(E, E') + \rho(F', F) \\
&= \varphi(D) + \varphi(E') + \varphi(F') + 0 \\
&= \varphi(D + E' + F').
\end{aligned}$$

For example, Figure 27 shows $f(x, y) = 0 \oplus x \oplus y$ and $g(x, y) = 0 \oplus 2x \oplus 1y$. When we shift g over one unit to the right, as shown in Figure 28, we get Figure 29.

Now that the vertex of $g'(x, y) = 0 \oplus 1x \oplus 1y$ lies on f, we can employ one more shift in the upward $x = y$ direction to slide the vertex so that the newly shifted g'' is equal to f (see Figure 30). Again from Lemma 5.9, $\varphi(D + E' + F') = \varphi(2P + Q)$, so that $\varphi(D + E + F) = 2\varphi(P) + \varphi(Q)$. $\square$

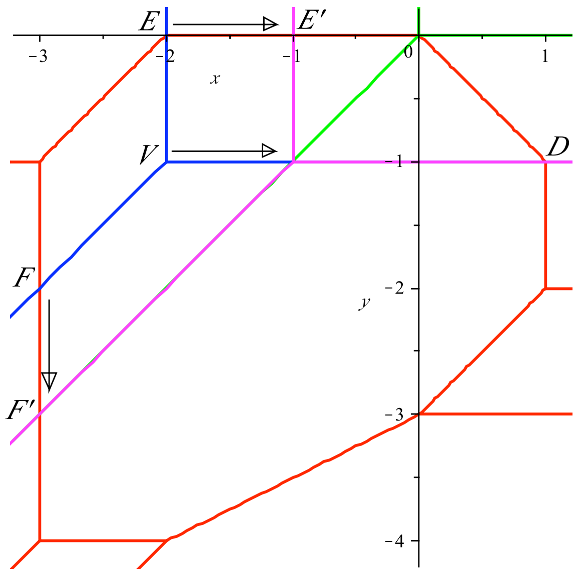


Figure 28: The graph of g is shifted (to g') so that its vertex V lies on the graph of f, resulting in Figure 29.

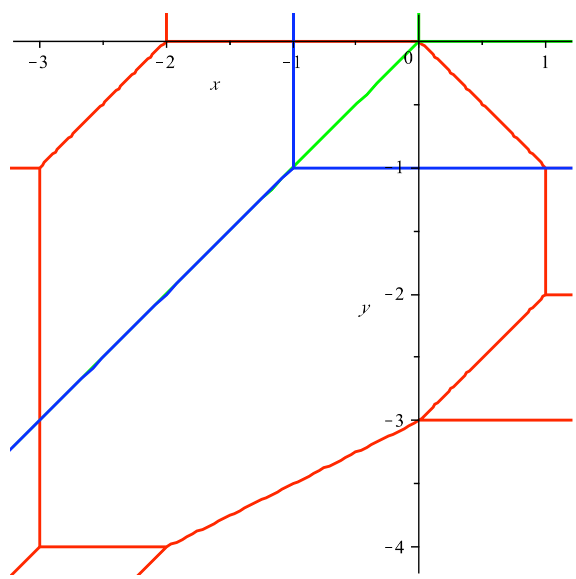


Figure 29: The corner locus of g' now lies on $K(f)$.

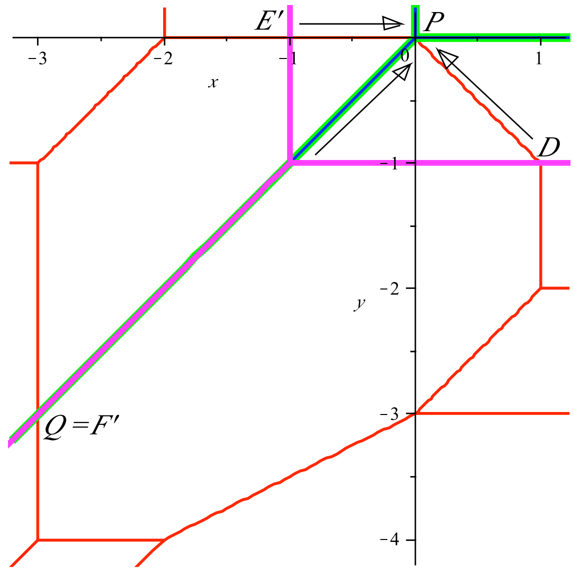


Figure 30: Now the graph of g' is shifted diagonally up to match up with f.

This gives the following corollaries.

Corollary 5.11. If D and D' are divisors on the cycle with $D \sim D'$, then $\varphi(D - D') = 0$.

Proof. Since $D - D' = div\ g - div\ g'$ for some g and g', both the products of n lines,

$$\varphi(D - D') = \varphi(D) - \varphi(D') = n(2P + Q) - n(2P + Q) = 0.$$

□

Corollary 5.12. If X and Y are points on the cycle and $D = X - \mathcal{O}$ and $D' = Y - \mathcal{O}$ are the corresponding divisors in $Div^0(\mathcal{C}^*)$, then $D \sim D'$ if and only if $X = Y$.

Proof. Obviously $X = Y$ implies $D \sim D'$. If $D \sim D'$ where $D = X - \mathcal{O}$ and $D' = Y - \mathcal{O}$, then

$$\varphi(D - D') = \varphi(X - Y) = \varphi(X) - \varphi(Y) = 0$$

by Corollary 5.11. But then $\varphi(X) = \varphi(Y)$. Thus X and Y are both the same distance from P. Therefore, $X = Y$. $\square$

Therefore, while every point on a given tentacle is linearly equivalent to every other point on that tentacle, no two points on the cycle are linearly equivalent. Therefore, since $\varphi : Div^0(\mathcal{C}^*) \to \mathcal{S}$ is a surjective homomorphism, we have the exact sequence

$$0 \to \ker \varphi \to Div^0(\mathcal{C}^*) \to \mathcal{S} \to 0.$$

Corollaries 5.11 and 5.12 shows us that $\ker \varphi$ is equal to the subgroup of principal divisors of $Div^0(\mathcal{C}^*)$. Therefore, $Jac(\mathcal{C}) \cong Jac(\mathcal{C}^*)$ by Corollary 5.5. Also,

$$Jac(\mathcal{C}^*) = Div^0(\mathcal{C}^*)/\sim \quad \cong Div^0(\mathcal{C}^*)/\ker \varphi,$$

and by the exactness of the sequence above, $Div^0(\mathcal{C}^*)/\sim \quad \cong \mathcal{S}$. Thus we have shown that

$$Jac(\mathcal{C}) \cong \mathcal{S}$$

as we stated in Theorem 4.15.

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