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# Fusion of the Parastrophic Matrix and Weak Cayley Table

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Fusion of the Parastrophic Matrix and Weak Cayley Table

by

Nathan Perry

A thesis submitted to the faculty of

Brigham Young University

in partial fulfillment of the requirements for the degree of

Master of Science

Department of Mathematics

Brigham Young University

August 2009

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BRIGHAM YOUNG UNIVERSITY

GRADUATE COMMITTEE APPROVAL

of a thesis submitted by

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This thesis has been read by each member of the following graduate committee and by majority vote has been found to be satisfactory.

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As chair of the candidate's graduate committee, I have read the thesis of Nathan Perry in its final form and have found that (1) its format, citations, and bibliographical style are consistent and acceptable and fulfill university and department style requirements; (2) its illustrative materials including figures, tables, and charts are in place; and (3) the final manuscript is satisfactory to the graduate committee and is ready for submission to the university library.

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## Fusion of the Parastrophic Matrix and Weak Cayley Table

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The parastrophic matrix and Weak Cayley Tables are matrices that have close ties to the character table. Work by Ken Johnson has shown that fusion of groups induces a relationship between the character tables of the groups. In this paper we will demonstrate a similar induced relationship between the parastrophic matrices and Weak Cayley Tables of the fused groups.

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# 1 Introduction

For a subset  $W$  of a finite group  $G$  let  $\overline{W}$  be the element in  $\mathbb{C}G$  which is the sum of all the elements of  $W$ . If  $W = \{g_1, \dots, g_s\}$ , then define  $W^{-1} = \{g_1^{-1}, \dots, g_s^{-1}\}$  [9]. An *S-ring* (sometimes referred to as a *Schur ring*) is a subalgebra of  $\mathbb{C}G$  which has as basis  $\{\overline{\omega}_1, \overline{\omega}_2, \dots, \overline{\omega}_m\}$ , for some partition  $G = \omega_1 \cup \omega_2 \cup \dots \cup \omega_n$ . Additionally we impose the following conditions:

1.  $\omega_1 = \{e\}$ .
2. for all  $1 \leq i \leq m$ ,  $\omega_i^{-1} = \omega_j$  for some  $1 \leq j \leq m$ .

The quantities  $\omega_1, \dots, \omega_m$  are the *basic sets* while  $\overline{\omega}_1, \dots, \overline{\omega}_m$  are the *basic elements* of the S-ring. An *S-ring isomorphism* is an algebra isomorphism which maps basic elements to basic elements. For early work on S-rings and basic definitions see [16], chapter IV.

The conjugacy classes,  $C_1, \dots, C_k$ , of  $G$  form a partition of  $G$  and the corresponding  $\overline{C}_1, \dots, \overline{C}_k$  of  $\mathbb{C}G$  form a basis for a particular S-ring over  $G$  which is called the *class algebra*. It is a well known fact that the class algebra is the center of  $\mathbb{C}G$ . It is also known as the *centralizer S-ring* of  $G$ . The class algebra is useful because it determines, and is determined by, the character table of  $G$ , see [10] chapter 30.

Let  $H$  and  $G$  be finite groups. We say that  $H$  *fuses* to  $G$ , if there exists a subalgebra of the class algebra of  $H$  which is S-ring isomorphic to the class algebra of  $G$ . In [11] Johnson and Smith showed how fusion relates the character tables of the groups, with a formulation known as the magic rectangle condition. This condition was later used in [9] to explore fusions from abelian groups. We will define this condition in chapter 10.

**Example 1.1.** Let  $H = C_6$ , with generator  $t$ , and  $G = S_3$ . The conjugacy classes of

$G$  are:

$$C_1 = \{1\},$$

$$C_2 = \{(123), (132)\},$$

$$C_3 = \{(12), (13), (23)\}.$$

Let

$$H_1 = 1 = t^6,$$

$$H_2 = t^2 + t^4,$$

$$H_3 = t + t^3 + t^5.$$

Since  $H$  is abelian we see that  $\mathbb{C}H = Z(\mathbb{C}H)$ . Thus  $R$ , the S-ring spanned by  $\{H_1, H_2, H_3\}$ , is a subalgebra of  $Z(\mathbb{C}H)$ . Also it is easy to check that

$$\phi : R \mapsto Z(\mathbb{C}G),$$

$$\phi(H_i) = \overline{C_i},$$

is an isomorphism of S-rings. Thus  $C_6$  fuses to  $S_3$ .

In [8], Frobenius introduced a concept known as the parastrophic matrix. This matrix was related to the character table but came from a different construction. In particular the parastrophic matrix intertwines the first and second regular representations of  $G$  ([2], p. 249). This gave rise to the notion of a Frobenius Algebras, where the first and second regular representations are equivalent ([3], [15]). In this thesis we shall construct a condition similar to the magic rectangle condition, but for parastrophic matrices instead of the character table. Additionally, we shall see that fusion also gives us a relationship between linear factors of the determinant of the parastrophic matrices of the two class algebras.

In [12], the authors formulate a construction known as the Weak Cayley Table. They show that if two groups have the same Weak Cayley Table, they will have the same character table. Additionally, the Weak Cayley Table gives even more information about the group. In particular, knowledge of the Weak Cayley Table is equivalent to knowledge of the one and two characters, one characters being characters under the standard definition, and two characters being a concept originally introduced by Frobenius [6]. In particular, in [13], the authors showed that the one and two characters determine the number of involutions in a group, thus the Weak Cayley Table does as well. In this work we shall explore what the Weak Cayley Table tells us about fusion. We will again get a magic rectangle type formulation, and a relationship between the determinants of the matrices.

Our investigation of the determinants of the parastrophic matrix and the Weak Cayley Table will demonstrate the relationship between the linear factors of these determinants and the central, orthogonal, idempotents of the S-ring. These results will demonstrate the duality that exists between the linear factors of the determinant of the parastrophic matrix and the Weak Cayley Table.

Finally, the last chapter will be a translation of Frobenius' work *Über vertauschbare Matrizen* (On commuting Matrices) [7]. We note that there is no English translation of this paper in the literature. This is one of Frobenius' five articles that were published in the 1896 volume of the *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*. These five papers are generally regarded as the birth of representation theory ([2], p 40). The primary result of the paper, according to Frobenius, has to do with the factorability of the determinant of a matrix into linear factors. It was this work that prompted our work in this paper on the linear factors of the determinants of the parastrophic matrix and Weak Cayley Table.

## 2 Irreducible Modules of Commutative Algebras

First let us recall some basic definitions from module theory:

**Definition 2.1.** A module is *irreducible* if its only submodules are  $\{0\}$  and itself.

**Definition 2.2.** A module is *completely reducible* if it is the direct sum of irreducible modules.

Now let us recall a basic result from module theory, known as Schur's lemma, which we shall state here without proof:

**Theorem 2.3** (Schur's Lemma, [10], 78). *Let  $A$  be a  $\mathbb{C}$ -algebra and  $V$  and  $W$  irreducible  $A$ -modules.*

(1) *If  $\phi : V \rightarrow W$  is an  $A$ -homomorphism, then either  $\phi$  is an  $A$ -isomorphism, or  $\phi(v) = 0$  for all  $v \in V$ .*

(2) *If  $\phi : V \rightarrow V$  is an  $A$ -homomorphism, then  $\phi$  is a scalar multiple of the identity linear transformation  $1_V$ .*

□

We shall use the second part of this result to prove the following theorem:

**Theorem 2.4.** *Let  $A$  be a commutative algebra over  $\mathbb{C}$ . All irreducible  $A$  modules have dimension 1 with respect to  $\mathbb{C}$ .*

*Proof.* Let  $V$  be an irreducible right  $A$ -module. Pick  $a \in A$ , since  $A$  is abelian:

$$vab = vba, \quad v \in V, \quad b \in A.$$

Thus the linear transformation  $v \mapsto va$  is an  $A$ -homomorphism. By Schur's lemma (2) this homomorphism is simply a scalar multiple of the identity linear transformation  $1_V$ , i.e.  $va = \lambda_a v$ ,  $\lambda_a \in \mathbb{C}$ . Thus the subspace generated by the element  $v$  is in fact an  $A$ -module. But  $V$  is irreducible, so  $vA = v\mathbb{C} = V$ , and  $V$  has dimension 1. □

**Definition 2.5.** If  $M$  is an ideal of the algebra  $A$ , then the *order* of  $M$  is the dimension of  $M$  with respect to the base field of  $A$ .

**Corollary 2.6.** *If  $A$  is a commutative algebra over  $\mathbb{C}$ , and  $M$  is a minimal (simple) ideal of  $A$ , then  $M$  has order 1.*

*Proof.* First note that  $M$  is an  $A$ -module with multiplication defined simply by the multiplication of  $A$ . If  $N$  is an  $A$ -submodule of  $M$ , then  $NA \subseteq N$ , thus  $N$  is an ideal of  $A$  contained in  $M$ . Thus  $N = 0$  or  $N = M$ , and  $M$  is an irreducible  $A$  module. So by the above theorem,  $M$  has order 1.  $\square$

It is worthy of note here that the algebra does not necessarily have to be over the field  $\mathbb{C}$ . In the proof of the second part of Schur's lemma it is only important to know that all the eigenvalues of  $\phi$  are contained within the field. Thus one generalization of the above theorem, which would still yield the same results, would be that  $A$  is a commutative algebra over the algebraically closed field  $F$ .

Other results that we shall rely heavily upon are Maschke's Theorem and Wedderburn's Theorem. These two theorems we shall state here without proof:

**Theorem 2.7** (Maschke's Theorem, [10], p. 70). *Let  $G$  be a finite group, and let  $V$  be a  $\mathbb{C}G$ -module. If  $U$  is an  $\mathbb{C}G$ -submodule of  $V$ , then there is an  $\mathbb{C}G$ -submodule  $W$  of  $V$  such that*

$$V = U \oplus W.$$

$\square$

**Theorem 2.8** (Wedderburn's Theorem, [5], p. 854). *Let  $R$  be a nonzero ring with an identity. Then the following are equivalent:*

1. every  $R$ -module is completely reducible.

2. the ring  $R$ , considered as a left  $R$ -module, is a direct sum:

$$R = L_1 \oplus L_2 \oplus \cdots \oplus L_n,$$

where each  $L_i$  is a simple module with  $L_i = Re_i$ , for some  $e_i \in R$ , with

(a)  $e_i e_j = 0$  if  $i \neq j$ ;

(b)  $e_i^2 = e_i$  for all  $i$ ;

(c)  $\sum_{i=1}^n e_i = 1$ .

□

Note that we can apply Maschke's theorem inductively to any finite dimensional  $\mathbb{C}G$ -module to show that it is completely reducible. We will now use this result to show that every  $R$ -module of our commutative S-ring,  $R$  is completely reducible:

**Proposition 2.9.** *Let  $R$  be a commutative S-ring over the finite group  $G$ , and let  $V$  be an  $R$ -module. Then  $V$  is a completely reducible  $R$ -module.*

*Proof.* Let  $m$  be the dimension of  $V$ . Let  $\phi$  be the left representation of the action of  $R$  on  $V$ . And let:

$$A_i = \phi(\overline{C}_i),$$

where  $G = C_1 \cup \cdots \cup C_k$  is a partition of  $G$  into its conjugacy classes. Since  $R$  is commutative, the  $A_i$  matrices commute. Additionally since  $C_i^{-1} = C_j$  for some  $1 \leq j \leq k$ , we have  $A_i^T = A_j$ . Thus the  $A_i$  matrices are normal. Thus we see that the  $A_i$  matrices are simultaneously diagonalizable and that there are vectors  $v_j \in V$ ,  $1 \leq j \leq m$ , such that the  $v_j$  vectors are eigenvectors of  $A_i$  for all  $1 \leq i \leq k$ :

$$A_i v_j = \lambda_{ij} v_j.$$

Let  $V_j$  be the subspace of  $V$  spanned by  $v_j$ . Then

$$\overline{C}_i V_j = \lambda_{ij} V_j = V_j.$$

Thus  $V_j$  is a  $S$ -submodule of  $V$  and  $V$  is completely reducible.  $\square$

Now that we have shown that every  $S$ -ring-module is completely reducible we can use Wedderburn's theorem to obtain a decomposition of  $S$ -rings into idempotents. We shall refer to such a decomposition as a *Wedderburn decomposition* and the set of idempotents as a *Wedderburn basis*.

### 3 First and Second Representations

The contents of this chapter are taken largely from Kiokemeister's work, [14], although they originally came from the work of Frobenius. Let  $A$  be a linear associative algebra with basis  $e_1, \dots, e_n$  over the field  $F$ . We define  $u$  to be the vector of basis elements:

$$u = (e_1, e_2, \dots, e_n), \quad u' = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}.$$

If  $a \in A$  then we define:

$$au = (ae_1, \dots, ae_n), \quad u'a = \begin{bmatrix} e_1 a \\ \vdots \\ e_n a \end{bmatrix}$$

For every element  $a \in A$ , there exist matrices  $R(a)$  and  $S(a)$ , with entries in

$F$ , satisfying ([4], p. 6):

$$au = uR(a), \quad u'a = S(a)u', \quad (3.1)$$

The matrices  $R(a)$  and  $S(a)$  are, respectively, the *first and second representations* of the element  $a$ . Let  $R_i = R(e_i)$ .

Given an element  $a$  in  $A$ ,  $a$  can be written as a unique linear sum of basis elements:  $a = \sum_{i=1}^n \alpha_i e_i$ . We define  $\widehat{a}$  to be the vector  $(\alpha_1, \alpha_2, \dots, \alpha_n)$ . Note that in some literature  $\bar{a}$  is used instead of  $\widehat{a}$ , but we will use  $\widehat{a}$  as we have already defined  $\bar{a}$  differently above. By the definition of  $\widehat{a}$  we have the following obvious identities:

$$a = \widehat{a}u' = u\widehat{a}'. \quad (3.2)$$

We also have the following properties:

**Proposition 3.3.** *For elements  $a, b \in A$ ,  $\alpha \in F$  we have:*

- (1)  $\widehat{\alpha a} = \alpha \widehat{a}$ .
- (2)  $\widehat{a + b} = \widehat{a} + \widehat{b}$ .
- (3)  $(\widehat{ab})' = R(a)\widehat{b}'$ ,  $\widehat{ab} = \widehat{a}S(b)$ .

*Proof.* (1) follows from the definition of  $\widehat{h}$  and the distributive property of algebras.

(2) is a consequence of the addition in the algebra.

For (3) consider the product  $ab$ . By 3.2 and associativity we know:

$$ab = a(u\widehat{b}') = (au)\widehat{b}'.$$

So by 3.1 we get:

$$ab = (uR(a))\widehat{b}' = u(R(a)\widehat{b}').$$

But we know that  $ab = u(\widehat{ab})'$ , thus  $(\widehat{ab})' = R(a)\widehat{b}'$  because the vector representation



is unique. The proof that  $\widehat{ab} = \widehat{aS}(b)$  is similar. □

## 4 Parastrophic Matrices and Parastrophic Forms

Retain the notation from the previous section. Let  $R = \mathbb{C}[x_1, \dots, x_n]$  be a polynomial ring and let  $\xi = (x_1, x_2, \dots, x_n) \in R^n$ . Recall that  $R_i = R(e_i)$ .

**Definition 4.1.** The *parastrophic matrix*,  $Q(\xi)$ , of the algebra  $A$  is:

$$Q(\xi) = \begin{pmatrix} \xi R_1 \\ \xi R_2 \\ \vdots \\ \xi R_n \end{pmatrix}.$$

Note that  $Q$  is a map from  $\mathbb{C}^n$  to the  $n \times n$  matrices over  $\mathbb{C}$ .

**Definition 4.2.** The *parastrophic form*,  $\pi(\xi)$ , of  $A$  is the determinant of the parastrophic matrix.

To prove some results about the parastrophic form we shall use some of the tools of Kioekemeister [14]. In particular the following construction will be very helpful:

**Definition 4.3.** If  $D$  is a subset of  $A$ , let  $U(D)$  be the set of all  $\eta \in F^n$  satisfying  $\widehat{dQ}(\eta) = 0$  for all  $d \in D$ .  $U(D)$  is called the *orthogonal module* of  $D$ .

Kioekemeister proved a number of results about this construction. In particular we shall make use of the following two theorems which we shall state here without proof:

**Theorem 4.4** ([14], p. 149). *If  $A$  contains a right ideal  $M$  of order 1, then*

$$\pi(\eta) = g(\eta)h(\eta),$$

where  $g$  is a linear form in  $x_1, x_2, \dots, x_n$ , and  $\eta$  is a variable in  $\mathbb{C}^n$ . The orthogonal module  $U(M)$  consists of all solutions of  $g(\eta) = 0$ .

**Theorem 4.5** ([14], p. 145). *If  $M$  is a right ideal of  $A$  of order  $r$ , then  $U(M)$  is an  $F$ -module of dimension  $n - r$ .*

These two results will help us to gain a greater understanding of the factors of the parastrophic form. In particular we will see that there is a very close connection between the idempotents of the algebra and the linear factors of the parastrophic form.

**Theorem 4.6.** *If  $A$  is a commutative S-ring of  $G$  over  $\mathbb{C}$ , with basic elements  $\{e_1, \dots, e_n\}$ , and the Wedderburn decomposition of  $A$  is given by*

$$A = A\epsilon_1 \oplus \dots \oplus A\epsilon_m,$$

with  $\epsilon_i = \sum_{j=1}^n \alpha_{ij} e_j$ , then  $n = m$  and there are  $n$  linear factors of the parastrophic form given by  $g_i = \sum_{j=1}^n \alpha_{ij} x_j$ .

*Proof.* Since  $A$  is an S-ring over a group it is semi-simple. Thus by Wedderburn's theorem, we can decompose  $A$ :

$$A = L_1 \oplus L_2 \oplus \dots \oplus L_m,$$

where  $L_i = A\epsilon_i$  for some set of orthogonal idempotents  $\epsilon_1, \dots, \epsilon_m$ , and each  $L_i$  is a minimal (simple) ideal. By Corollary 2.6 each  $L_i$  has order 1, thus  $L_i$  has dimension 1 over  $\mathbb{C}$  and is equal to the span of  $\epsilon_i$  over  $\mathbb{C}$ . So  $A = \text{Span}_{\mathbb{C}}\{\epsilon_1, \dots, \epsilon_m\}$ . Since  $A$  has dimension  $n$ , and the  $\epsilon_i$  are linearly independent, we get that  $m = n$ .

Now let us choose another basis for  $A$ , namely the basic elements of  $A$  as an S-ring of  $G$ :  $\{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n\}$ , where  $e_1 = \{1\}$ ,  $G = \bigcup_{i=1}^n e_n$ , and  $e_i^{-1} = e_j$  for some  $1 \leq$

$j \leq n$ . Recall that we define  $u$  to be the vector of basis elements:  $u = (\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n)$ . Recall that we defined  $R_i = R(\bar{e}_i)$ . Let us calculate the first column of all of the  $R_i$ 's:

$$\bar{e}_i u = \bar{e}_i(\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n) = (\bar{e}_i \bar{e}_1, \bar{e}_i \bar{e}_2, \dots, \bar{e}_i \bar{e}_n) = (\bar{e}_i, \bar{e}_i \bar{e}_2, \dots, \bar{e}_i \bar{e}_n).$$

Note in particular that the first entry of  $\bar{e}_i u$  is simply  $\bar{e}_i$ , since  $\bar{e}_1 = 1$ . But recall that  $\bar{e}_i u = u R_i$ . Thus the first column of  $R_i$  must have an one in the  $i$ -th row, and zeroes everywhere else.

Recall

$$U(L_i) = \{\eta \in \mathbb{C}^n \mid \widehat{d}Q(\eta) = 0, d \in L_i\}.$$

But for all  $d$  in  $L_i$ ,  $d = c\bar{e}_i$  for some  $c \in \mathbb{C}$ . Thus we can write

$$U(L_i) = \{\eta \in \mathbb{C}^n \mid \widehat{e}_i Q(\eta) = 0\}.$$

Let  $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ . Since the first column of  $R_i$  has a one in the  $i$ -th row, and zeroes everywhere else, the first entry of  $\eta R_i$  must be simply  $\eta_i$ . Then the first column of

$$Q(\eta) = \begin{pmatrix} \eta R_1 \\ \eta R_2 \\ \vdots \\ \eta R_n \end{pmatrix}$$

must be the column matrix  $\eta^T$ . But if  $\eta$  is in  $U(L_i)$ , then  $\bar{e}_i Q(\eta)$  must be the zero column vector. In particular  $\bar{e}_i$  times the first column of  $Q(\eta)$  must be a zero. But the first column vector of  $Q(\eta)$  is simply  $\eta^T$ , thus the dot product  $\widehat{e}_i \cdot \eta$  must equal zero.

We can write  $\epsilon$  as a sum of the  $\bar{e}_i$ 's:

$$\epsilon = \sum_{i=1}^n \alpha_i \bar{e}_i.$$

If  $f$  is the form

$$f(x_1, x_2, \dots, x_n) = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n,$$

then  $f(\eta) = \hat{\epsilon} \cdot \eta = 0$ . Thus all vectors in  $U(L_i)$  are zeroes of the form  $f$ , and thus  $U(L_i)$  is contained in the hyperplane,  $P$ , defined by  $f$ . Since  $\epsilon_i$  is non-zero,  $f$  is a non-zero linear form, so  $P$  has dimension  $n-1$ . By Theorem 4.5,  $U(L_i)$  has dimension  $n-1$ , thus  $P = U(L_i)$ .

By Theorem 4.4, for every  $L_i$  we get a  $g_i$ , a linear factor of the parastrophic form, such that the solutions of  $g_i$  are precisely  $U(L_i)$ . Thus  $g_i$  and  $f$  both define the same hyperplane, so  $g_i$  must be a scalar multiple of  $f$ . In particular, up to a common constant factor, the coefficients of  $g_i$  are completely determined by the coefficients of  $\epsilon_i$ . The parastrophic form,  $\pi(\xi)$ , is the determinant of an  $n$  by  $n$  matrix so it must have degree no greater than  $n$ . But for each  $L_i$  we get a non-zero linear factor  $g_i$ , thus  $\pi(\xi) = g_1 g_2 \dots g_n$ .  $\square$

## 5 Fusion of the Parastrophic Form

Suppose the group  $H$  fuses to the group  $G$ . Let  $Z(\mathbb{C}H) = \mathbb{C}\epsilon_1 \oplus \mathbb{C}\epsilon_2 \oplus \dots \oplus \mathbb{C}\epsilon_n$  be a Wedderburn decomposition of  $Z(\mathbb{C}H)$ . By the fusion, there exists an S-ring  $R$ , a sub-S-ring of  $Z(\mathbb{C}H)$ , that is isomorphic to  $Z(\mathbb{C}G)$ . Let  $R = \mathbb{C}\delta_1 \oplus \mathbb{C}\delta_2 \oplus \dots \oplus \mathbb{C}\delta_m$  be a Wedderburn decomposition of  $R$ . Since  $R$  is contained in  $Z(\mathbb{C}H)$  we can express each  $\delta_i$  as a unique sum of the  $\epsilon_j$ 's:  $\delta_i = \sum_{j=1}^n \alpha_{ij} \epsilon_j$ . Since the  $\epsilon_j$ 's are orthogonal idempotents we have

$$\delta_i^2 = \left( \sum_j \alpha_{ij} \epsilon_j \right)^2 = \sum_j \alpha_{ij}^2 \epsilon_j.$$

But since  $\delta_i$  is an idempotent we also know

$$\delta_i^2 = \delta_i = \sum_j \alpha_{ij} \epsilon_j.$$

By uniqueness of the vector representation, these equations show that  $\alpha_{ij} = \alpha_{ij}^2$ . Thus  $\alpha_{ij}$  must be one or zero. If  $\alpha_{ij} = 1$  we shall say that  $\epsilon_j$  is a *component* of  $\delta_i$ .

**Proposition 5.1.** *Let  $\Delta_i$  be the set of the indices of the components of  $\delta_i$  and let  $N = \{1, 2, \dots, n\}$ . Then the  $\Delta_i$ 's are pairwise disjoint, and  $\bigcup_{i=1}^m \Delta_i = N$ .*

*Proof.* We can rewrite each  $\delta_i$ :

$$\delta_i = \sum_{j \in \Delta_i} \epsilon_j.$$

Now suppose  $j \in N$  is in  $\Delta_i \cap \Delta_k$ . Then the product

$$\delta_i \delta_k = \left( \sum_{l \in \Delta_i} \epsilon_l \right) \left( \sum_{l \in \Delta_k} \epsilon_l \right) = \dots + \epsilon_j + \dots \neq 0,$$

which contradicts the orthogonality of the  $\delta_i$ 's unless  $i = k$ . Thus the  $\Delta_i$  are pairwise disjoint. Now, by contradiction, suppose that there exists  $j \in N$  such that  $j$  is not in  $\Delta = \bigcup_{i=1}^m \Delta_i$ . Then by Wedderburn's Theorem and disjointness of the  $\Delta_i$ 's we have:

$$1 = \sum_{i=1}^m \delta_i = \sum_{i=1}^m \sum_{j \in \Delta_i} \epsilon_j = \sum_{j \in \Delta} \epsilon_j.$$

But we also know that  $1 = \sum_{j=1}^n \epsilon_j$ , and thus see that  $\sum_{j \in N - \Delta} \epsilon_j = 0$ , contradicting the linear independence of the  $\epsilon_j$ 's. We conclude that each  $\epsilon_j$  occurs in one, and exactly one of the  $\Delta_i$ 's.  $\square$

Let  $E = \{e_1, \dots, e_n\}$  and  $D' = \{d'_1, \dots, d'_m\}$  be the S-ring bases of  $Z(\mathbb{C}H)$  and  $Z(\mathbb{C}G)$  respectively. Let  $\phi$  be the isomorphism from  $Z(\mathbb{C}G)$  to  $R$ . Since  $\phi$  is an S-ring isomorphism, the set  $D = \{\phi(d'_1), \dots, \phi(d'_m)\} = \{d_1, \dots, d_m\}$  forms an S-ring

basis for  $R$ . By the S-ring properties, we can define a  $D_i$  for each  $d_i = \sum_{j \in D_i} e_j$ , where each  $j$  occurs in precisely one  $D_i$ . Again we call  $e_j$  a component of  $d_i$  if  $j \in D_i$ .

Define:

$$\begin{aligned} \sigma : Z(\mathbb{C}H) &\mapsto R, \\ \sigma(e_j) &= \frac{d_i}{|D_i|}, \text{ if } j \in D_i. \end{aligned}$$

Note this is a module homomorphism but not an algebra homomorphism. Also we have

$$\sigma(d_i) = \sum_{j \in D_i} \sigma(e_j) = d_i$$

which shows that  $\sigma$  is the identity on  $R$ . We can use this homomorphism to define a homomorphism,

$$\begin{aligned} \theta : \mathbb{C}[x_1, \dots, x_n] &\mapsto \mathbb{C}[y_1, \dots, y_m], \\ \theta(x_j) &= \alpha y_i, \text{ if } \sigma(e_j) = \alpha d_i, \alpha \in \mathbb{C}. \end{aligned}$$

We are now in a position to understand the relationship between the parastrophic form of  $H$  and the parastrophic form of  $G$ . We can write each  $\epsilon_i = \sum_{j=1}^n \beta_{ij} e_j$  and  $\delta_i = \sum_{j=1}^m \gamma_{ij} d_j$ . We have seen above that the linear factors of the parastrophic forms of  $H$  and  $G$  are  $g_i = \sum_{j=1}^n \beta_{ij} x_j$  and  $f_i = \sum_{j=1}^m \gamma_{ij} y_j$  respectively. Thus:

$$\sum_{l=1}^m \gamma_{il} d_l = \delta_i = \sum_{j \in \Delta_i} \epsilon_j = \sum_{j \in \Delta_i} \sum_{k=1}^n \beta_{jk} e_k.$$

If we now apply  $\sigma$ , which is the identity on  $R$ , we see that

$$\sigma \left( \sum_{j \in \Delta_i} \sum_{k=1}^n \beta_{jk} e_k \right) = \sum_{l=1}^m \gamma_{il} d_l.$$

This equation in turn shows that  $\theta(\sum_{j \in \Delta_i} g_j) = f_i$ , which gives us the desired fusion of parastrophic forms. We state this result as a theorem:

**Theorem 5.2.** *Using the notation from this section, if the group  $H$  fuses to the group  $G$ , then for the linear factors of the parastrophic form we have:*

$$\theta\left(\sum_{j \in \Delta_i} g_j\right) = f_i,$$

where the  $g_j$ 's and  $f_i$ 's are the linear factors of the parastrophic forms of  $H$  and  $G$  respectively.

**Example 5.3.** Let  $H = C_6$  and  $G = S_3$ . We have previously shown that  $H$  fuses to  $G$ . Let  $t$  be the generator for  $H$ . Let  $e_i = t_i$  for  $1 \leq i \leq 6$ . Let  $J$  be a primitive third root of unity. Then the idempotents in the Wedderburn decomposition of  $Z(\mathbb{C}H)$  are:

$$\begin{aligned}\epsilon_1 &= \frac{e_6 + e_1 + e_3 + e_5 + e_2 + e_4}{6}, \\ \epsilon_2 &= \frac{e_6 - e_1 - e_3 - e_5 + e_2 + e_4}{6}, \\ \epsilon_3 &= \frac{e_6 - (1 + J)e_1 + e_3 + Je_5 + Je_2 - (1 + J)e_4}{6}, \\ \epsilon_4 &= \frac{e_6 + (1 + J)e_1 - e_3 - Je_5 + Je_2 - (1 + J)e_4}{6}, \\ \epsilon_5 &= \frac{e_6 + Je_1 + e_3 - (1 + J)e_5 - (1 + J)e_2 + Je_4}{6}, \\ \epsilon_6 &= \frac{e_6 - Je_1 - e_3 + (1 + J)e_5 - (1 + J)e_2 + Je_4}{6}.\end{aligned}$$

Let  $R$  be the sub-S-Ring of  $Z(\mathbb{C}H)$  that is isomorphic to  $Z(\mathbb{C}G)$ . Then the S-Ring basis for  $R$  is:

$$d_1 = e_6,$$

$$d_2 = e_1 + e_3 + e_5,$$

$$d_3 = e_2 + e_4.$$

The idempotents in the Wedderburn decomposition of  $R$  are:

$$\delta_1 = \frac{d_1 + d_2 + d_3}{6},$$

$$\delta_2 = \frac{d_1 - d_2 + d_3}{6},$$

$$\delta_3 = \frac{2d_1 - d_3}{3}.$$

The homomorphism  $\sigma : Z(\mathbb{C}H) \rightarrow R$ , from above has the following form  $\sigma(e_6) = d_1$ ,  $\sigma(e_i) = \frac{d_2}{3}$  for  $i = 1, 3, 5$ , and  $\sigma(e_i) = \frac{d_3}{2}$  for  $i = 2, 4$ . It is easy to check that  $\sigma(\epsilon_1) = \delta_1$ ,  $\sigma(\epsilon_2) = \delta_2$ , and  $\sigma(\epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6) = \delta_3$ . The fusion of the parastrophic forms of  $H$  and  $G$  follows immediately by the relationship between the idempotents and the factors of the parastrophic form.

This result is perhaps easiest to understand in terms of change of basis matrices. If we order the  $\epsilon_i$ 's such that all the components of  $\delta_j$  occur together contiguously, and the  $e_i$ 's such that all the the components of  $d_j$  occur contiguously, (see the example below). We showed above that  $\sigma(\sum_{j \in \Delta_i}) = \delta_i$ . But in our matrix all the components of  $\delta_i$  occur in a block. Thus if we sum across this block we will get  $\sum_{j \in \Delta_i}$ . And if  $e_k$  is a component of  $d_l$  then  $\sigma(e_k) = \frac{d_l}{|D_l|}$ . Thus in our matrix where all the components of  $d_l$  occur in a block, if we now sum across this block and divide by  $|D_l|$ , then the new matrix that we have created will give us the change of basis matrix from the  $d_i$  basis to the  $\delta_i$  basis.

**Example 5.4.** Continuing the example from above we see that the change of basis



matrix from the  $e_i$  basis to the  $\epsilon_i$  basis is:

$$Q = \frac{1}{6} \left( \begin{array}{c|ccc|cc} 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & -1 & -1 & -1 & 1 & 1 \\ \hline 1 & -(1+J) & 1 & J & J & -(1+J) \\ 1 & 1+J & -1 & -J & J & -(1+J) \\ 1 & J & 1 & -(1+J) & -(1+J) & J \\ 1 & -J & -1 & 1+J & -(1+J) & J \end{array} \right).$$

Note that lines have been added to show where the contiguous blocks of components occur. The change of basis matrix from the  $d_i$  basis to the  $\delta_i$  basis is:

$$S = \frac{1}{6} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 4 & 0 & -2 \end{pmatrix}.$$

It is now easy to check that if we sum the entries of the blocks of  $Q$  and divide by the cardinality of  $D_l$  we go from  $Q$  to  $S$ .

## 6 Fusion of the Parastrophic Matrix

**Definition 6.1.** Given a finite group  $G$  of order  $n$ , and an ordering of the elements of  $G$ ,  $\{g_1, g_2, \dots, g_n\}$ , the multiplication table matrix,  $M_G$ , is the matrix in which  $(M_G)_{ij} = x_k$  if  $g_i g_j^{-1} = g_k$ .

**Definition 6.2.** Given a finite group  $G$  of order  $n$ , let  $C_1, C_2, \dots, C_k$  be the conjugacy classes of  $G$ . Then  $\overline{C_1}, \overline{C_2}, \dots, \overline{C_k}$  form a basis for the center of the group ring,  $Z(\mathbb{Z}G)$ . Thus  $\overline{C_i} \overline{C_j} = \sum_{m=1}^k \lambda_{ijk} \overline{C_k}$  for some  $\lambda_{ijk} \in \mathbb{Z}$ . The  $\lambda_{ijk}$  are called the *structure constants*. The matrix  $P_G$ , where  $(P_G)_{ij} = \sum_{m=1}^k \lambda_{ijk} x_k$ , is called the parastrophic matrix of  $G$ .

**Definition 6.3.** Let  $A$  be an  $n \times n$  matrix over the indeterminates  $x_1, \dots, x_n$ , and  $B$  be an  $m \times m$  matrix over the determinates  $y_1, \dots, y_m$ , with  $n \geq m$ . Then a *parastrophic fusion* of  $A$  to  $B$  is determined by:

1. a partition  $R_1, \dots, R_a$  of  $N = \{1, 2, \dots, n\}$ , thought of as a partition of the rows (or columns) of  $A$ ,
2. a function  $\rho : \mathbb{Q}[x_1, \dots, x_n] \mapsto \mathbb{Q}[y_1, \dots, y_m]$ , where  $\rho(x_i) = \frac{y_j}{k_{ij}}$  for some  $k_{ij} \in \mathbb{Z}$ .

For each pair of  $R_i, R_j$ , we obtain a block of the matrix  $M_G$ , which we denote by  $R_{(i,j)}$ . Note that  $R_{(i,j)}$  will not necessarily be a contiguous block, but rather the set of all elements which occur in the  $i^*$  row and  $j^*$  column, with  $i^* \in R_i$  and  $j^* \in R_j$ . Let  $R_{(i,j)}^*$  be the sum of all the entries in  $R_{(i,j)}$ . This determines a matrix  $U$  where  $(U)_{i,j} = R_{(i,j)}^*$ . We say that  $A$  fuses parastrophically to  $B$  if  $\rho(U) = B$ .

**Example 6.4.** The parastrophic matrix for  $C_6 = \langle t \rangle$ , the cyclic group of order six, where  $C_1 = 1, C_2 = t^2, C_3 = t^4, C_4 = t, C_5 = t^3$ , and  $C_6 = t^5$ , is:

$$P_{C_6} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_2 & x_3 & x_1 & x_5 & x_6 & x_4 \\ x_3 & x_1 & x_2 & x_6 & x_4 & x_5 \\ x_4 & x_5 & x_6 & x_2 & x_3 & x_1 \\ x_5 & x_6 & x_4 & x_3 & x_1 & x_2 \\ x_6 & x_4 & x_5 & x_1 & x_2 & x_3 \end{pmatrix}.$$

The parastrophic matrix of  $S_3$ , where  $C_1 = 1, C_2 = \{(123), (132)\}$ , and  $C_3 = \{(12), (13), (23)\}$ , is:

$$P_{S_3} = \begin{pmatrix} w_1 & w_2 & w_3 \\ w_2 & 2w_1 + w_2 & 2w_3 \\ w_3 & 2w_2 & 3w_1 + 3w_2 \end{pmatrix}.$$

We can partition  $P_{C_6}$  in the following manner:

$$P_{C_6} = \left( \begin{array}{c|cc|ccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \hline x_2 & x_3 & x_1 & x_5 & x_6 & x_4 \\ x_3 & x_1 & x_2 & x_6 & x_4 & x_5 \\ \hline x_4 & x_5 & x_6 & x_2 & x_3 & x_1 \\ x_5 & x_6 & x_4 & x_3 & x_1 & x_2 \\ x_6 & x_4 & x_5 & x_1 & x_2 & x_3 \end{array} \right).$$

Now we can make the following identification:  $\rho(x_1) = w_1$ ,  $\rho(x_i) = \frac{w_2}{2}$  for  $i = 2$  or  $i = 3$  and  $\rho(x_i) = \frac{w_3}{3}$  for  $4 \leq i \leq 6$ . Now we may apply  $\rho$  to the block sums to show that there is a parastrophic fusion from  $P_{C_6}$  to  $P_{S_3}$ .

Note that through the proper ordering of the elements of  $G$  we can obtain the structure constants through a partitioning of the group matrix. In particular if  $C_1, C_2, \dots, C_k$  are the conjugacy classes of  $G$ , order the elements of  $G$ :

$$G = g_{11}, g_{21}, \dots, g_{2a_2}, \dots, g_{k1}, \dots, g_{ka_k},$$

such that  $C_i = \bigcup_{m=1}^{a_i} g_{im}$ . Now consider the group matrix,  $X_G$ , of  $G$  by this ordering. Partition  $X_G$  according to the sizes of the conjugacy classes. In other words all the elements of  $C_1$  will be in the first block, all the elements of  $C_2$  will be in the second block, etc. Now note that if we take the sum of all the elements in the  $ij$ -th block, we will simply get the quantity  $\overline{C_i C_j}$ . So now if we express this sum in terms of the  $C_k$ 's we will recover the structure constants.

**Example 6.5.** Put the elements of  $S_3$  in the following order:

$S_3 = \{1, (123), (132), (12), (13), (23)\}$ . Note that the conjugacy classes occur in contiguous blocks by this ordering, i.e.  $C_1 = x_1$ ,  $C_2 = \{x_2, x_3\}$ , and  $C_3 = \{x_4, x_5, x_6\}$ .

Now consider the group matrix, partitioned by conjugacy class sizes:

$$X_{S_3} = \left( \begin{array}{c|ccc|ccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \hline x_2 & x_3 & x_1 & x_6 & x_4 & x_5 \\ x_3 & x_1 & x_2 & x_5 & x_6 & x_4 \\ \hline x_4 & x_5 & x_6 & x_1 & x_2 & x_3 \\ x_5 & x_6 & x_4 & x_3 & x_1 & x_2 \\ x_6 & x_4 & x_5 & x_2 & x_3 & x_1 \end{array} \right).$$

Now sum over the blocks to get a new matrix,  $X'_{S_3}$ :

$$X'_{S_3} = \begin{pmatrix} x_1 & x_2 + x_3 & x_4 + x_5 + x_6 \\ x_2 + x_3 & 2x_1 + x_2 + x_3 & 2x_4 + 2x_5 + 2x_6 \\ x_4 + x_5 + x_6 & 2x_4 + 2x_5 + 2x_6 & 3x_1 + 3x_2 + 3x_3 \end{pmatrix}$$

Now simply substitute in  $\bar{C}_1$ ,  $\bar{C}_2$ , and  $\bar{C}_3$  to get:

$$X'_{S_3} = \begin{pmatrix} \bar{C}_1 & \bar{C}_2 & \bar{C}_3 \\ \bar{C}_2 & 2\bar{C}_1 + \bar{C}_2 & 2\bar{C}_3 \\ \bar{C}_3 & 2\bar{C}_3 & 3\bar{C}_1 + 3\bar{C}_2 \end{pmatrix}.$$

Which by a simple identification yields the parastrophic matrix of  $S_3$ , which we saw above.

**Theorem 6.6.** *There exists a parastrophic fusion from  $P_H$  to  $P_G$  if  $H$  fuses to  $G$ .*

*Proof.* Let  $C_1, \dots, C_k$  be the conjugacy classes of  $G$ . Order the elements of  $G$ , as above, according to conjugacy class:

$$G = g_{11}, g_{21}, \dots, g_{2a_2}, \dots, g_{k1}, \dots, g_{ka_k}.$$

Since there is a fusion from  $H$  to  $G$  there exists  $R$  an S-Ring over  $H$  that is isomorphic to  $Z(\mathbb{Z}G)$ . Thus there exist subsets  $H_1, H_2, \dots, H_k$  of  $H$  such that  $\langle \overline{H}_1, \dots, \overline{H}_k \rangle$  is isomorphic to  $\langle \overline{C}_1, \dots, \overline{C}_k \rangle$  with isomorphism  $f$ . Order the elements of  $H$ :

$$h_{11}, h_{21}, \dots, h_{2b_2}, \dots, h_{k1}, \dots, h_{kb_k},$$

such that  $H_i = \bigcup_{m=1}^{a_i} h_{im}$ . This gives us a map  $\sigma$  from the elements of  $H$  to the elements of  $G$  simply by mapping by order, i.e.  $\sigma(h_{11}) = g_{11}$ ,  $\sigma(h_{21}) = g_{21}$ , etc. So now if we partition  $X_H$ , created using the above ordering, according to the sizes of the  $H_i$ 's then in the  $ij$ -th block we will get the quantity  $\overline{H}_i \overline{H}_j$ , which has the same structure constants as  $\overline{C}_i \overline{C}_j$ . If we apply  $\sigma$  to the elements of  $\overline{H}_i \overline{H}_j$  in the natural way, we will in fact obtain the quantity  $\overline{C}_i \overline{C}_j$ . Thus giving us a partition and identification that gives us a parastrophic fusion from  $X_H$  to  $P_G$ .

Now we must slightly alter this fusion to get a parastrophic fusion from  $P_H$  to  $P_G$ . Let  $B_1, \dots, B_p$  be the conjugacy classes of  $H$ . Note that  $\langle \overline{H}_1, \dots, \overline{H}_k \rangle$  is in  $Z(\mathbb{Z}G)$ , thus each  $H_i$  is equal to a union of conjugacy classes of  $H$ . We can order the conjugacy classes, similarly to how we ordered the elements:

$$B_{11}, B_{21}, \dots, B_{2c_2}, \dots, B_{k1}, \dots, B_{kc_k},$$

such that  $H_i = \bigcup_{m=1}^{c_i} B_{im}$ . As we saw above we obtain the parastrophic matrix from the group matrix by taking sums over a partition determined by the sizes of the classes, and then a simple identification of variables. Now to get  $P_G$  we must simply partition  $P_H$  by the sizes  $c_1, c_2, \dots, c_k$ , so that two classes are in the same block if they occur in the same  $H_i$ . The identification,  $\tau$ , that we need is simply  $\tau(\sum_{m=1}^{c_i} B_{im}) = C_i$ . Thus  $P_H$  fuses parastrophically to  $P_G$ .

□

## 7 Fusion of the Weak Cayley Table

Let  $S$  be a commutative S-ring over the finite group  $G = \{g_1 = 1, g_2, \dots, g_n\}$  with the partition:

$$C_1 = \{1\}, C_2, \dots, C_r.$$

Define  $S_1, \dots, S_r \subset G \times G$ , where  $(g_j, g_k) \in S_i$  if  $g_k g_j^{-1} \in C_i$ . Since  $g_k g_j^{-1}$  is in  $C_i$  for some unique  $i$  we see that the  $S_i$ 's form a partition of  $G \times G$ . Define the  $n \times n$  matrices,  $A_1, \dots, A_r$  by

$$(A_i)_{j,k} = \begin{cases} 1 & \text{if } (g_j, g_k) \in S_i \\ 0 & \text{otherwise} \end{cases}.$$

Note that the  $A_i$ 's are the *adjacency matrices* of the *association scheme* determined by  $S$ . [1]

Define the *structure constants*,  $p_{i,j}^k$ , by

$$\overline{C}_i \overline{C}_j = \sum_k p_{i,j}^k \overline{C}_k.$$

Define the  $r \times r$  matrices  $B_1, \dots, B_r$  by

$$(B_i)_{j,k} = p_{i,k}^j.$$

The matrices  $B_i$  determine a faithful representation of the (left) regular representation of  $S$ . Then ([1], p. 57) shows that  $A_i \mapsto B_i$  determines an isomorphism of matrix algebras. In particular, for each  $i$  the matrices  $A_i, B_i$  have the same minimal polynomial. So they have the same eigenvalues.

Let  $R = \mathbb{Z}[x_1, \dots, x_r]$  be a polynomial ring, and let

$$W(S) = \sum_i x_i A_i.$$

If  $S = Z(\mathbb{Z}G)$  then we call  $W(S)$  the *weak Cayley table* of the  $G$ . Note that this definition agrees with the definition given by Johnson, Mattarai, and Sehgal. [12]

Now the  $B_i$  determine the left regular representation of the S-ring  $S$ . [14] Thus the  $A_i$  do also. Since  $S$  is commutative, the matrices  $A_1, \dots, A_r$  commute. Additionally, since  $S_i^{-1} = S_j$  for some  $j$  by the properties of S-rings,  $A_i^T = A_j$ , so the  $A_i$  matrices are normal. Thus we see that the  $A_i$  matrices are simultaneously diagonalizable. Therefore there is a  $n \times n$  matrix  $P$  such that  $D_i = PA_iP^{-1}$  is a diagonal matrix for  $1 \leq i \leq r$ . Thus

$$PW(S)P^{-1} = \sum_i x_i D_i$$

is also a diagonal matrix. This shows that  $\det(W(S)) = \det(\sum_i x_i D_i)$  is a product of linear factors. We now describe these linear factors: the fact that  $A_1, \dots, A_r$  are simultaneously diagonalizable means that there are subspaces  $E_1, \dots, E_r$  of  $\mathbb{C}^n$  such that  $E_1 \oplus \dots \oplus E_r = \mathbb{C}^n$  and each  $E_i$  is an eigenspace for each  $A_i$ . Let  $\lambda_{ij}$  denote the corresponding eigenvalue.

The above remarks about minimal polynomials show that the  $B_1, \dots, B_r$  are simultaneously diagonalizable where  $B_i$  is conjugated to the diagonal matrix  $\text{diag}(\lambda_{i1}, \dots, \lambda_{ir})$ . It follows that the factors of  $\det(W(S))$  are the same as the factors of  $\det(\sum_i x_i B_i)$  and that these factors are  $f_j = \sum_i x_i \lambda_{ji}$  for  $1 \leq j \leq r$ .

Now suppose that  $S_1$  and  $S_2$  are S-rings over  $G$  where  $S_2 \subset S_1$ . Let  $\dim S_i = n_i$ . Let  $\{C_1, \dots, C_{n_1}\}$  and  $\{C'_1, \dots, C'_{n_2}\}$  be the corresponding partitions of  $G$ . Let  $R_1 = \mathbb{Z}[x_1, \dots, x_{n_1}]$  and  $R_2 = \mathbb{Z}[x'_1, \dots, x'_{n_2}]$  be polynomial rings. Since  $S_2 \subset S_1$

every  $C_i$  is a subset of some  $C'_{j(i)}$ ; define the ring homomorphism  $\phi : R_1 \rightarrow R_2$  by  $\phi(x_i) = x'_{j(i)}$ . We say that  $\phi$  is the *ring homomorphism determined by the inclusion* of  $S_2$  in  $S_1$ .

Now let  $W(S_i)$ ,  $i = 1, 2$ , denote the weak Cayley table matrix of  $S_i$  using the variables from  $R_i$ . Then from the definition of  $\phi$  and  $W(S)$  we see that

$$\phi W(S_1) = W(S_2).$$

Now suppose that the group  $H$  fuses to the group  $G$ . Let  $C_1, \dots, C_r$  denote the classes in  $G$  and let  $D_1, \dots, D_r$  denote the corresponding partition of  $H$ , so that the correspondence  $\bar{C}_i \leftrightarrow \bar{D}_i$  determines the isomorphism of  $Z(\mathbb{Z}G)$  with the subring  $\langle \bar{D}_1, \dots, \bar{D}_r \rangle$  of  $Z(\mathbb{Z}H)$ . Thus if we have  $\bar{C}_i \bar{C}_j = \sum_k p_{i,j}^k \bar{C}_k$ , then we also have  $\bar{D}_i \bar{D}_j = \sum_k p_{i,j}^k \bar{D}_k$ .

Let  $B_1, \dots, B_r$  denote the faithful representation of the left regular representation of  $Z(\mathbb{Z}G) = \langle \bar{C}_1, \dots, \bar{C}_r \rangle$  and let  $B'_1, \dots, B'_r$  denote the corresponding matrices for  $\langle \bar{D}_1, \dots, \bar{D}_r \rangle \cong Z(\mathbb{Z}G)$ . Then the fusion shows that  $B_i = B'_i$ ,  $1 \leq i \leq r$ , and so determines an equality of algebras:

$$\langle B_1, \dots, B_r \rangle = \langle B'_1, \dots, B'_r \rangle.$$

Let  $A_1, \dots, A_r$  be the  $n \times n$  matrices for the  $\bar{D}_1 \dots \bar{D}_r$ 's and let  $A'_1, \dots, A'_r$  be the corresponding matrices for  $Z(\mathbb{Z}G)$ .

Now let  $S_1 = Z(\mathbb{Z}H)$  where the classes of  $H$  are  $F_1, \dots, F_s$ . Let  $R = \mathbb{Z}[x_1, \dots, x_s]$  be a polynomial ring and let  $S_2 = \langle \bar{D}_1, \dots, \bar{D}_r \rangle$  be the sub-algebra determined by the fusion. Let  $R' = \mathbb{Z}[y_1, \dots, y_r]$  be another polynomial ring and let  $\phi : R \rightarrow R'$  be the ring homomorphism determined by the inclusion of  $S_2$  in  $S_1$ .

Thus we have a fusion of  $W(S_1)$  to  $W(S_2)$ , where  $S_2 \cong Z(\mathbb{Z}G)$ . Further, the



matrices  $B_i, B'_i$  are equal and have eigenspaces which are one-dimensional. Thus they are simultaneously diagonalizable to the diagonal matrices  $D_1, \dots, D_r$ . It follows that  $\det(\sum_i y_i B_i)$  and  $\det(\sum_i y_i A_i)$  both factor as a product of linear factors and that these factors are the same. This gives a fusion of  $\det(W(S_1))$  to  $\det(W(S_2))$  as required.

## 8 Duality of the Parastrophic and Cayley Forms

Let  $G$  be a group with conjugacy classes  $C_1, C_2, \dots, C_k$ . Let  $T = \mathbb{C}G$ , and let  $\rho$  be the left regular representation of  $T$ . Then the  $A_i$ 's defined in the previous section can be calculated using  $\rho$ :  $A_i = \sum_{g \in C_i} \rho(g)$ . Or simply  $A_i = \rho(\overline{C_i})$ . If  $S = Z(\mathbb{C}G)$ , recall the definition of the weak Cayley table:  $W(S) = \sum_{i=1}^k x_i A_i$ . So we can now express the Weak Cayley Table in terms of  $\rho$ :

$$W(S) = \sum_{i=1}^k x_i \rho(\overline{C_i}).$$

Let  $\phi$  be the left regular representation of  $S$ . Then the  $B_i$ 's from the previous section can be calculated:

$$B_i = \phi(\overline{C_i}).$$

**Definition 8.1.** Using the notation from the preceding paragraph we define

$$V(S) = \sum_{i=1}^k x_i B_i$$

to be the *partial weak Cayley table*. We define the determinant of  $V(S)$  to be the *partial Cayley form*. The determinant of  $W(S)$  we call the *Cayley form*.

**Definition 8.2.** Recall that we defined the weak Cayley table of a group,  $G$ , to be the weak Cayley Table of the algebra  $Z(\mathbb{C}G)$ . Similarly we define the partial Cayley form of  $G$  to be the partial Cayley form of  $Z(\mathbb{C}G)$ .

In the previous section we showed that  $W(S)$  and  $V(S)$  have the same eigenvalues, but that in general these eigenvalues occur with different multiplicities. We shall now proceed to find the eigenvalues of  $V(S)$  and then use these to find the eigenvalues of  $W(S)$ .

**Proposition 8.3.** *If  $\{e_1, \dots, e_k\}$  is the  $S$ -ring basis for  $S$ , and  $\{\epsilon_1, \dots, \epsilon_k\}$  is the Wedderburn idempotent basis for  $S$ . We can write  $e_i = \sum_{j=1}^k \beta_{ij} \epsilon_j$ . Then the eigenvalues of  $V(S)$  are  $\sum_{i=1}^k x_i \beta_{ij}$  for all  $1 \leq j \leq k$ .*

*Proof.* First note that by (3.3), we know that  $\widehat{e_i \epsilon_k}' = B_i \widehat{\epsilon_k}'$ . Thus we see:

$$B_i \widehat{\epsilon_k}' = \widehat{e_i \epsilon_k}' = \left( \widehat{\left( \sum_{j=1}^n \beta_{ij} \epsilon_j \right) (\epsilon_k)} \right)' = \widehat{\beta_{ik} \epsilon_k}' = \beta_{ik} \widehat{\epsilon_k}'.$$

So the vectors  $\widehat{\epsilon_k}'$  are in fact eigenvectors of the  $B_i$  with corresponding eigenvalues  $\beta_{i1}, \dots, \beta_{ik}$ . Since  $V(S) = \sum_{i=1}^k x_i B_i$  we see:

$$V(S) \widehat{\epsilon_j}' = \left( \sum_{i=1}^k x_i B_i \right) \widehat{\epsilon_j}' = \sum_{i=1}^k x_i \beta_{ij} \widehat{\epsilon_j}'.$$

This gives  $\sum_{i=1}^k x_i \beta_{ij}$  as an eigenvalue of  $V(S)$ , and thus a linear factor of the partial Cayley form of  $G$ . □

**Theorem 8.4.** *Continuing the notation from the above proposition, the eigenvalues of  $W(S)$  are  $\sum_{i=1}^k x_i \beta_{ij}$ , where the multiplicity is equal to the dimension of  $\epsilon_j S$ .*

*Proof.* Let  $\delta_1, \dots, \delta_n$  be a Wedderburn basis for  $T$ . Then using an argument identical to the one in chapter 4, we can show that each  $\delta_i$  is a component of one, and exactly one,  $\epsilon_j$ . In other words  $\epsilon_j = \sum_{i \in E_j} \delta_i$ , for some properly chosen  $E_j$ . Note that now that we have moved from the sub-algebra  $S$  into the larger algebra  $T$ , that  $\widehat{\epsilon_i}$  can have multiple meanings. It can either be the vector representation relative to the  $S$  basis or the vector representation relative to the  $T$  basis. In particular  $B_i \widehat{\epsilon_j}'$  only makes

sense if we use the vector representation relative to the  $S$  basis, and similarly  $A_i\tilde{\epsilon}_j$  only makes sense if we use the vector representation relative to the  $T$  basis.

Now again by (3.3), we know

$$\widehat{e}_i\epsilon'_j = B_i\tilde{\epsilon}_j$$

and

$$\widehat{e}_i\epsilon'_j = A_i\tilde{\epsilon}'_j.$$

Where in each equation we choose the appropriate meaning of  $\widehat{e}_j$ . So if  $B_i\tilde{\epsilon}_j = \beta_{ij}\tilde{\epsilon}'_j$  then  $\widehat{e}_i\epsilon'_j = \beta_{ij}\tilde{\epsilon}'_j$  relative to the  $Z(\mathbb{C}G)$  basis. But of course the choice of basis is not going to affect the multiplication, thus  $\widehat{e}_i\epsilon'_j = \beta_{ij}\tilde{\epsilon}'_j$  relative to the  $\mathbb{C}G$  basis. As a result, we have  $A_i\tilde{\epsilon}'_j = \widehat{e}_i\epsilon'_j = \beta_{ij}\epsilon_j$ . Thus  $\beta_{ij}$  is an eigenvalue of  $A_i$ .

Now for  $l \in E_j$ , since  $\delta_l$  is an idempotent in  $T$ , by the same arguments used above we can show  $\widehat{\delta}'_l$  is an eigenvector of  $A_i$ . Additionally since  $\tilde{\epsilon}'_j = \sum_{l \in E_j} \widehat{\delta}'_l$  and  $\tilde{\epsilon}'_j$  is an eigenvector with eigenvalue  $\beta_{ij}$ , we see that  $\beta_{ij}$  must also be the eigenvalue of  $\widehat{\delta}'_l$ . So the multiplicity of  $\beta_{ij}$  as an eigenvalue of  $A_i$  must be the cardinality of  $E_j$ . Note that this cardinality can also be found by taking the dimension of  $\epsilon_j S$ .

Now we have the eigenvalues of the  $A_i$ 's, and can use them to find the eigenvalues of  $W(S) = \sum_{i=1}^k x_i A_i$ . Since  $\tilde{\epsilon}'_j$  is an eigenvector of  $A_i$  for all  $1 \leq i \leq k$  with corresponding eigenvalues  $\beta_{ij}$ , we see:

$$W(S)\tilde{\epsilon}'_j = \left( \sum_{i=1}^k x_i A_i \right) \tilde{\epsilon}'_j = \sum_{i=1}^k x_i \beta_{ij} \tilde{\epsilon}'_j.$$

Thus  $\sum_{i=1}^k x_i \beta_{ij}$  is an eigenvalue of  $W(S)$  and has multiplicity  $\epsilon_j S$ . □

**Example 8.5.** Let  $T = \mathbb{C}S_3$  and  $S = Z(\mathbb{C}S_3)$ . We have already seen that  $S =$

$\text{Span}_{\mathbb{C}}\{1, (123) + (132), (12) + (13) + (23)\}$ . Let

$$e_1 = 1, \quad e_2 = (123) + (132), \quad e_3 = (12) + (13) + (23).$$

Then the idempotents that yield the Wedderburn decomposition are

$$\epsilon_1 = \frac{e_1 + e_2 + e_3}{6}, \quad \epsilon_2 = \frac{e_1 + e_2 - e_3}{6}, \quad \epsilon_3 = \frac{2e_1 - e_2}{3}.$$

So the factors of the parastrophic form are simply

$$g_1 = \frac{x_1 + x_2 + x_3}{6}, \quad g_2 = \frac{x_1 + x_2 - x_3}{6}, \quad g_3 = \frac{2x_1 - x_2}{3}.$$

If we now solve for the  $e_i$ 's in terms of the  $\epsilon_i$ 's we get

$$e_1 = \epsilon_1 + \epsilon_2 + \epsilon_3, \quad e_2 = 2\epsilon_1 + 2\epsilon_2 - \epsilon_3, \quad e_3 = 3\epsilon_1 - 3\epsilon_2.$$

So the factors of the partial Cayley Form are

$$f_1 = x_1 + 2x_2 + 3x_3 \quad f_2 = x_1 + 2x_2 - 3x_3, \quad f_3 = x_1 - x_2.$$

If we let  $Q$  be the change of basis matrix from the  $e_i$  basis to the  $\epsilon_i$  basis, then the rows of  $Q$  are simply the coefficients of  $\epsilon_i$  expressed as a linear combination of the  $e_i$ 's. Thus we can see that the coefficients of the linear factors of the parastrophic form are just the entries in the rows of  $Q$ . The matrix  $Q^{-1}$  will give us a change of basis from the  $\epsilon_i$  basis to the  $e_i$  basis. Thus the rows of  $Q^{-1}$  are the coefficients of  $e_i$  expressed as a linear combination of the  $\epsilon_i$ 's. So the columns of  $Q^{-1}$  will give us the coefficients of the linear factors of the partial Cayley form.

**Example 8.6.** Continuing the example from above, we can use the equations for the

$\epsilon_i$ 's to find the change of basis matrix,  $Q$ , and then calculate its inverse  $Q^{-1}$ :

$$Q = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \\ \frac{2}{3} & -\frac{1}{3} & 0 \end{pmatrix}, \quad Q^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & -1 \\ 3 & -3 & 0 \end{pmatrix}.$$

Note how the coefficients on the columns of the  $Q^{-1}$  matrix give the linear factors of the partial Cayley form.

The definition of the  $A_i$ 's, and thus the definition of  $W(S)$ , depends on the underlying group structure. The  $B_i$ 's and  $V(T)$  can be defined for arbitrary algebras and not just for the group algebra. Thus although the terms weak Cayley table and Cayley form do not make sense in the general case of an algebra, the terms partial weak Cayley table and partial Cayley form do. Thus the duality of the parastrophic form is with the partial Cayley form, and not the Cayley form itself. We summarize these results in the following theorem:

**Theorem 8.7.** *Given an algebra  $S$  with basis  $E = \{e_1, \dots, e_k\}$ . Let  $\epsilon = \{\epsilon_1, \dots, \epsilon_k\}$  be the idempotents in the Wedderburn decomposition of  $S$ . If  $Q$  is the change of basis from the  $E$  basis to the  $\epsilon$  basis, then the coefficients of the rows of  $Q$  give the coefficients of the linear factors of the parastrophic form. The coefficients of the columns of  $Q^{-1}$  give the coefficients of the linear factors of the partial Cayley form.*

□

## 9 Fusion of the Partial Cayley Form

The duality of the parastrophic form and the partial Cayley form (8.7), along with our proof of the fusion of the parastrophic form (5.2), suggest a method for proving

the fusion of the partial Cayley form. The difficulty arises, however, because a linear factor of the parastrophic form corresponds to precisely one idempotent of the Wedderburn basis of the algebra, whereas a linear factor of the partial Cayley form does not correspond to a particular basis element but rather has coefficients that come from all the basis elements. As a result we must dig a bit deeper into the inner workings of the fusion map. Many of the tools that we use, however, are the same as in the parastrophic form case.

If the group  $H$  fuses to the group  $G$  let  $S = Z(\mathbb{C}H)$  and let  $R$  be the sub-S-ring of  $S$  that is isomorphic to  $Z(\mathbb{C}G)$ . Let  $e_1, \dots, e_n$  be the S-ring basis for  $S$  and  $d_1, \dots, d_m$  be the S-ring basis for  $R$ . Let  $\epsilon_1, \dots, \epsilon_n$  and  $\delta_1, \dots, \delta_m$  be the Wedderburn bases for  $S$  and  $R$  respectively. If  $N = \{1, \dots, n\}$ , recall that we can define  $D_1, \dots, D_m \subseteq N$  and  $\Delta_1, \dots, \Delta_m \subseteq N$  such that:

$$d_i = \sum_{j \in D_i} e_j, \quad \delta_i = \sum_{j \in \Delta_i} \epsilon_j, \quad N = \bigcup_{i=1}^m D_i = \bigcup_{i=1}^m \Delta_i.$$

This setup so far is exactly the same as in chapter 5, so refer to that chapter for the details.

We can express each  $e_i$  and  $d_i$  as a sum of the  $\epsilon_j$ 's and  $\delta_j$ 's:

$$e_i = \sum_{j=1}^n \alpha_{ij} \epsilon_j, \quad d_i = \sum_{j=1}^m \beta_{ij} \delta_j.$$

This gives

$$d_i = \sum_{j=1}^m \beta_{ij} \delta_j = \sum_{j=1}^m \beta_{ij} \left( \sum_{k \in \Delta_j} \epsilon_k \right). \tag{9.1}$$

But

$$d_i = \sum_{j \in D_i} e_j = \sum_{j \in D_i} \sum_{k=1}^n \alpha_{jk} \epsilon_k. \quad (9.2)$$

Thus using 9.1 and 9.2 we can equate coefficients to get:

$$\sum_{j \in D_i} \alpha_{jk} = \beta_{il}, \quad k \in \Delta_l.$$

We can renumber the  $\epsilon_k$  idempotents, so that for  $1 \leq k \leq m$ ,  $\epsilon_k$  is a component of  $\delta_k$ . This gives us that

$$\sum_{j \in D_i} \alpha_{jk} = \gamma_{ik} = \beta_{ik}, \quad 1 \leq k \leq m.$$

In chapter 5, we defined a module-homomorphism,  $\sigma : S \mapsto R$ , where if  $j \in D_i$ , then  $\sigma(e_j) = \frac{d_i}{|D_i|}$ . Here we must modify this map slightly to obtain our fusion, namely we define

$$\Omega : S \mapsto R,$$

$$\Omega(e_j) = d_i, \quad \text{if } j \in D_i.$$

We use  $\Omega$  to define a map  $\nu : \mathbb{C}[x_1, \dots, x_n] \mapsto \mathbb{C}[y_1, \dots, y_m]$  where:

$$\nu(x_j) = y_i, \quad \text{if } \Omega(e_j) = d_i.$$

We showed in chapter 8, that the linear factors of the partial Cayley forms of  $H$  and  $G$  are respectively:

$$g_i = \sum_{j=1}^n \alpha_{ji} x_j, \quad f_i = \sum_{j=1}^m \beta_{ji} y_j. \quad (9.3)$$

Thus we see that

$$f_i = \sum_{j=1}^m \left( \sum_{k \in D_j} \alpha_{ki} \right) y_j.$$

Now we can use  $\nu$ :

$$f_i = \sum_{j=1}^m \sum_{k \in D_j} \alpha_{ki} \nu(x_k).$$

But  $N$  is the disjoint union of the  $D_j$ 's:  $N = \bigcup_{j=1}^m D_j$ . Thus we see

$$f_i = \sum_{k=1}^n \nu(\alpha_{ki} x_k) = \nu(g_i).$$

Which gives us our desired fusion. We state this result as a theorem:

**Theorem 9.4.** *Using the notation from this section, if  $H$  fuses to  $G$  then using a proper ordering of the linear factors of the parastrophic forms of  $H$  we have:*

$$f_i = \nu(g_i), \quad 1 \leq i \leq m,$$

where the  $g_i$  and  $f_i$  are the linear factors of the parastrophic forms of  $H$  and  $G$  respectively. Additionally for  $m + 1 \leq i \leq n$  we have

$$\nu(g_i) = f_j$$

where  $i \in \Delta_j$ .

## 10 The Magic Rectangle Condition

In this chapter we shall present the definition of the magic rectangle condition given by Humphries and Johnson ([9], pp 1-2). Let the character table of  $H$  be  $X_{ij} = \{\chi_i(C_j)\}$ , i.e.  $\chi_1, \dots, \chi_r$  is the set of irreducible characters of the group  $H$ . Let  $d_i$  be the degree of  $\chi_i$ , and let  $k_j$  be the size of the  $j^{\text{th}}$  conjugacy class  $C_j$ . Then the magic rectangle



condition requires a partition  $\{B_i\}$ ,

$$B_1 = \{e\}, B_2 = \{C_{i_1}, \dots, C_{i_{r_2}}\}, \dots, B_f = \{C_{j_1}, \dots, C_{j_{r_f}}\},$$

of the conjugacy classes of  $H$ , and a partition  $\{\psi_i\}$ ,

$$\psi_1 = \{e\}, \psi_2 = \{\chi_{u_1}, \dots, \chi_{u_{s_2}}\}, \dots, \psi_f = \{\chi_{v_1}, \dots, \chi_{v_{s_f}}\}$$

of the irreducible characters. Consider the rectangle of the character table consisting of the columns corresponding to the elements of  $B_j = \{C_{t_1}, \dots, C_{t_{r_j}}\}$  and the rows corresponding to the elements of  $\psi_i = \{\chi_{w_1}, \dots, \chi_{w_{s_i}}\}$ . Abbreviate  $\chi_{w_i}$  by  $\chi_i$  and  $C_{t_i}$  by  $C_i$ . The magic rectangle condition is that for each  $i$  the number

$$\tau_{ij} = \frac{\sum_{m=1}^{r_j} k_m \chi_i(m)}{[\sum_{m=1}^{r_j} k_m] d_i}$$

is constant and equal to the the common value for each  $j$  of

$$\frac{\sum_{m=1}^{s_i} d_m \chi_m(j)}{\sum_{m=1}^{s_i} d_m^2}.$$

The fused table is an  $f \times f$  table which has rows corresponding to the  $\psi_i$  and classes corresponding to  $B_j$  such that the value of  $\psi_i$  on  $B_j$  (with slight abuse of notation) is  $\nu_i \tau_{ij}$  where

$$\nu_i = \sqrt{\sum_{m=1}^{s_i} d_m^2}$$

**Example 10.1.** Let  $H = C_6$  with generator  $t$ , then the character table of  $H$  is

	$e$	$t^2$	$t^4$	$t$	$t^3$	$t^5$
$\chi_0$	1	1	1	1	1	1
$\chi_3$	1	1	1	-1	-1	-1
$\chi_2$	1	$\omega$	$\omega^2$	1	$\omega$	$\omega^2$
$\chi_4$	1	$\omega^2$	$\omega$	1	$\omega^2$	$\omega$
$\chi_1$	1	$\omega$	$\omega^2$	-1	$\rho$	$\rho^{-1}$
$\chi_5$	1	$\omega^2$	$\omega$	-1	$\rho^{-1}$	$\rho$

Here  $\omega = e^{\frac{2\pi i}{3}}$  and  $\rho = e^{\frac{2\pi i}{6}}$ . The partitions of the classes and characters are given by the lines, i.e.  $B_1 = \{e\}$ ,  $B_2 = \{t^2, t^4\}$ , etc. Now when we fuse using the construction above we obtain the character table of  $S_3$ :

	$\{e\}$	$\{(1, 2, 3)\}$	$\{(1, 2)\}$
$\chi_0$	1	1	1
$\chi_1$	1	1	-1
$\chi_2$	2	-1	0

## 11 An Additional Example

All of the examples we have done so far in this thesis have had to do with the fusion of  $C_6$  to  $S_3$ . In this chapter we shall explore the fusion of  $C_{12}$  to  $A_4$ . First let  $H = C_{12}$  with generator  $t$ , and let  $G = A_4$ . Then the conjugacy classes of  $G$  are:

$$C_1 = \{1\},$$

$$C_2 = \{(12)(34), (13)(24), (14)(23)\},$$

$$C_3 = \{(134), (142), (243), (123)\},$$

$$C_4 = \{(132), (143), (234), (124)\}.$$

To obtain the fusion we let

$$D_1 = 1,$$

$$D_2 = \{t^3, t^6, t^9\},$$

$$D_3 = \{t, t^4, t^7, t^{10}\},$$

$$D_4 = \{t^2, t^5, t^8, t^{11}\}.$$

Then one can check that the algebra

$$R = \mathbb{C}\bar{D}_1 \oplus \mathbb{C}\bar{D}_2 \oplus \mathbb{C}\bar{D}_3 \oplus \mathbb{C}\bar{D}_4$$

is isomorphic to  $Z(\mathbb{C}G)$ , with isomorphism  $\phi : R \mapsto Z(\mathbb{C}G)$ , where:

$$\phi(\bar{D}_i) = \bar{C}_i, \quad 1 \leq i \leq 4.$$

The character table of  $C_{12}$  is:

	1	$t^3$	$t^6$	$t^9$	$t$	$t^4$	$t^7$	$t^{10}$	$t^2$	$t^5$	$t^8$	$t^{11}$
$\chi_0$	1	1	1	1	1	1	1	1	1	1	1	1
$\chi_3$	1	$\omega^9$	$\omega^6$	$\omega^3$	$\omega^3$	1	$\omega^9$	$\omega^6$	$\omega^6$	$\omega^3$	1	$\omega^9$
$\chi_6$	1	$\omega^6$	1	$\omega^6$	$\omega^6$	1	$\omega^6$	1	1	$\omega^6$	1	$\omega^6$
$\chi_9$	1	$\omega^3$	$\omega^6$	$\omega^9$	$\omega^9$	1	$\omega^3$	$\omega^6$	$\omega^6$	$\omega^9$	1	$\omega^3$
$\chi_1$	1	$\omega^3$	$\omega^6$	$\omega^9$	$\omega^1$	$\omega^4$	$\omega^7$	$\omega^{10}$	$\omega^2$	$\omega^5$	$\omega^8$	$\omega^{11}$
$\chi_4$	1	1	1	1	$\omega^4$	$\omega^4$	$\omega^4$	$\omega^4$	$\omega^8$	$\omega^8$	$\omega^8$	$\omega^8$
$\chi_7$	1	$\omega^9$	$\omega^6$	$\omega^3$	$\omega^7$	$\omega^4$	$\omega^1$	$\omega^{10}$	$\omega^2$	$\omega^{11}$	$\omega^8$	$\omega^5$
$\chi_{10}$	1	$\omega^6$	1	$\omega^6$	$\omega^{10}$	$\omega^4$	$\omega^{10}$	$\omega^4$	$\omega^8$	$\omega^2$	$\omega^8$	$\omega^2$
$\chi_2$	1	$\omega^6$	1	$\omega^6$	$\omega^2$	$\omega^8$	$\omega^2$	$\omega^8$	$\omega^4$	$\omega^{10}$	$\omega^4$	$\omega^{10}$
$\chi_5$	1	$\omega^3$	$\omega^6$	$\omega^9$	$\omega^5$	$\omega^8$	$\omega^{11}$	$\omega^2$	$\omega^{10}$	$\omega^1$	$\omega^4$	$\omega^7$
$\chi_8$	1	1	1	1	$\omega^8$	$\omega^8$	$\omega^8$	$\omega^8$	$\omega^4$	$\omega^4$	$\omega^4$	$\omega^4$
$\chi_{11}$	1	$\omega^9$	$\omega^6$	$\omega^3$	$\omega^{11}$	$\omega^8$	$\omega^5$	$\omega^2$	$\omega^{10}$	$\omega^7$	$\omega^4$	$\omega^1$

where  $\omega$  is a primitive  $12^{th}$  root of unity. Note that the ordering of the labels of the table is slightly unconventional, but will allow us to see the fusion more clearly. Using the the proposition from the last section, we see the coefficients of the idempotents of  $Z(CH)$  are simply the entries of the rows of the character table, normalized by a constant multiplier of  $\frac{1}{12}$ , e.g. the first couple of idempotents are:

$$\epsilon_0 = \frac{1}{12}(1 + t^3 + t^6 + t^9 + t^1 + t^4 + t^7 + t^{10} + t^2 + t^5 + t^8 + t^{11}),$$

$$\epsilon_3 = \frac{1}{12}(1 + \omega^9 t^3 + \omega^6 t^6 + \omega^3 t^9 + \omega^3 t + t^4 + \omega^9 t^7 + \omega^6 t^{10} + \omega^6 t^2 + \omega^3 t^5 + t^8 + \omega^9 t^{11}),$$

$\vdots$

The character table of  $A_4$  is:

	1	(12)(34)	(123)	(132)
$\tau_1$	1	1	1	1
$\tau_2$	1	1	$J$	$J^2$
$\tau_3$	1	1	$J^2$	$J$
$\tau_4$	3	-1	0	0

where  $J$  is a primitive third root of unity. Then the idempotents of  $Z(\mathbb{C}G)$  are:

$$\delta'_1 = \frac{1}{12}(\bar{C}_1 + \bar{C}_2 + \bar{C}_3 + \bar{C}_4),$$

$$\delta'_2 = \frac{1}{12}(\bar{C}_1 + \bar{C}_2 + J\bar{C}_3 + J^2\bar{C}_4),$$

$$\delta'_3 = \frac{1}{12}(\bar{C}_1 + \bar{C}_2 + J^2\bar{C}_3 + J\bar{C}_4),$$

$$\delta'_4 = \frac{1}{4}(3\bar{C}_1 - \bar{C}_2).$$

Now let  $S$  be the sub-S-ring of  $Z(\mathbb{C}H)$  that is isomorphic to  $Z(\mathbb{C}G)$ . The S-ring basis for  $S$  is:

$$d_1 = 1,$$

$$d_2 = t^3 + t^6 + t^9,$$

$$d_3 = t + t^4 + t^7 + t^{10},$$

$$d_4 = t^2 + t^5 + t^8 + t^{11}.$$

The Wedderburn basis is

$$\delta_1 = \frac{1}{12}(d_1 + d_2 + d_3 + d_4),$$

$$\delta_2 = \frac{1}{12}(d_1 + d_2 + Jd_3 + J^2d_4),$$

$$\delta_3 = \frac{1}{12}(d_1 + d_2 + J^2d_3 + Jd_4),$$

$$\delta_4 = \frac{1}{4}(3d_1 - d_2).$$

Let  $e_i = t^i$ . Then we define  $\sigma : Z(\mathbb{C}H) \mapsto S$  by

$$\sigma(e_1) = d_1,$$

$$\sigma(e_i) = \frac{d_2}{3} \text{ for } i = 3, 6, 9,$$

$$\sigma(e_i) = \frac{d_3}{4} \text{ for } i = 1, 4, 7, 10,$$

$$\sigma(e_i) = \frac{d_4}{4} \text{ for } i = 2, 5, 8, 11.$$

Then we can check that  $\sigma(\epsilon_0) = \delta_1$ ,  $\sigma(\epsilon_4) = \delta_2$ ,  $\sigma(\epsilon_8) = \delta_3$ , and  $\sigma(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_5 + \epsilon_6 + \epsilon_7 + \epsilon_9 + \epsilon_{10} + \epsilon_{11}) = \delta_4$ . Thus this map gives us the fusion of the parastrophic forms as well as the simplified version of the magic rectangle condition.

## 12 Translation

As mentioned in the introduction many of the techniques used in this paper were inspired by work done by Frobenius in his seminal work *On Commuting Matrices*. As there is no translation of the work in the literature, we have provided a translation here. In the translation we have attempted to retain as much of Frobenius' original notation and numbering as possible to provide consistency with the original work. Note that not only does this work provide some of the basis for the development of character theory, but also gives many standard results in linear algebra. Topics such as the characteristic and minimal polynomials are covered in detail. In fact this paper provides one of the first proofs of the Cayley-Hamilton theorem.

## On Commuting Matrices

In 1884 Weierstrass published his work *Zur Theorie der aus  $n$  Haupteinheiten gebildeten complexen Groessen* in the Goettinger Nachrichten, the results of which he had already given in his lectures. In 1885 Dedekind presented his own investigations concerning the subject in a paper of the same name, results which he had already to some extent shared in the second publication of Dirichlet's lectures on Number Theory. As Dedekind points out, however, these findings can also be viewed as consequences of the following algebraic theorem. This theorem, however, is more general, than the form it takes in the previous papers and can be expressed in purely algebraic form as follows:

I. Suppose  $a_{\alpha\beta\gamma}$  ( $\alpha, \beta = 1, 2, \dots, n; \gamma = 1, 2, \dots, m$ ) are any  $mn^2$  quantities that satisfy the following condition:

$$\sum_{\lambda} a_{\alpha\lambda\gamma} a_{\lambda\beta\delta} = \sum_{\lambda} a_{\alpha\lambda\delta} a_{\lambda\beta\gamma},$$

and let

$$a_{\alpha\beta} = \sum_{\gamma} a_{\alpha\beta\gamma} x_{\gamma}.$$

Then the determinant of  $n$ -th degree  $|a_{\alpha\beta}|$  is a product of  $n$  linear functions of  $m$  independent variables  $x_1, x_2, \dots, x_m$ .

The results of this theorem can be made even more general. In fact Study pointed out in his paper *Ueber Systeme von complexen Zahlen* (Goettinger Nachrichten, 1889) that many of these earlier results were different from known theorems regarding linear transformations only in their presentation. I will therefore use the tools for matrices that I developed for my work *Ueber lineare Substitutionen und bilineare Formen* (Crelle's Journal, Bd. 84), (which I will henceforth cite as  $L$ ). Using these

constructions in  $L$ , the theorem can be formulated in the following manner:

II. *Let  $f(x,y,z,\dots)$  be a function of  $m$  variables, let  $A, B, C, \dots$  be  $m$  matrices that commute with each other, and let  $a_1, a_2, a_3, \dots$  ( $b_1, b_2, b_3, \dots$ ;  $c_1, c_2, c_3, \dots$  respectively) be roots of the characteristic polynomial of  $A$  ( $B, C$  respectively). These roots can be ordered so that the determinant of  $f(A,B,C,\dots)$  is equal to the product*

$$f(a_1, b_1, c_1, \dots)f(a_2, b_2, c_2, \dots)f(a_3, b_3, c_3, \dots).$$

*This ordering of the roots is in fact independent of the choice of  $f$ .*

The application of this theorem to the function  $r - f(x, y, z, \dots)$ , where  $r$  is a constant, gives the following seemingly more general theorem.

III. *The quantities  $f(a_1, b_1, c_1, \dots), f(a_2, b_2, c_2, \dots), f(a_3, b_3, c_3, \dots) \dots$  are the roots of the characteristic polynomial of the matrix  $f(A, B, C, \dots)$ .*

Since the ordering of the roots is the same for every function  $f$ , this theorem gives the method for defining this ordering. Namely, if  $x, y, z, \dots$  are independent variables, then the application of the theorem to the matrix  $Ax + By + Cz + \dots$ , shows that

$$a_1x + b_1y + c_1z + \dots, \quad a_2x + b_2y + c_2z + \dots, \quad a_3x + b_3y + c_3z + \dots, \dots$$

are the the roots of the characteristic polynomial of, thus establishing the desired ordering. This theorem is a generalization of the following theorem developed in  $L$  §3, III:

IV. *Let  $r_1, r_2, \dots, r_n$  be the roots of the characteristic polynomial of the matrix  $A$ , then  $f(r_1), f(r_2), \dots, f(r_n)$  are the roots of the characteristic polynomial of the matrix  $f(A)$ .*

If one accepts this simple theorem as proven, then to obtain Theorem III one



needs only the following special case:

V. *Let  $A$  and  $B$  be commuting matrices, and let  $x$  and  $y$  be two variables, then the roots of the characteristic polynomial of the matrix  $Ax+By$  are linear functions of  $x$  and  $y$ .*

If we accept this theorem as proven, then the determinant of the matrix  $Ir + Ax + By$  is equal to  $(r + a_1x + b_1y)(r + a_2x + b_2y) \dots$ . Let  $x = 1$  and  $y = 0$ , then  $a_1, a_2, \dots$  are the roots of the characteristic polynomial of  $A$ , or for the sake of brevity, the roots of  $A$ . Similarly for  $b_1, b_2, \dots$  and  $B$ . If the matrix  $C$  commutes with  $A$  and  $B$ , then  $C$  commutes with  $Ax + By$  as well. Therefore the roots  $a_1x + b_1y, a_2x + b_2y, \dots$  of  $Ax + By$  and the roots  $c_1, c_2, \dots$  of  $C$  can be ordered so that the roots of  $(Ax + By) + Cz$  are equal to  $(a_1x + b_1y) + c_1z, (a_2x + b_2y) + c_2z, \dots$ . It is clear that this theorem can be extended to an arbitrary number of matrices, which all commute with each other. Also by theorem L §1, II the theorem remains valid if one adds in arbitrary functions of  $A, B, C, \dots$  to the system. In particular if one adds  $A^2$  to the system, and if  $h_1, h_2, \dots$  are the roots of  $A^2$  in the proper ordering, then  $a_\nu x + b_\nu y + c_\nu z + \dots + h_\nu u$  ( $\nu = 1, 2, \dots, n$ ) are the roots of  $Ax + By + Cz + \dots + A^2u$ . Let  $y = z = \dots = 0$ , then  $a_\nu x + h_\nu u$  are the roots of  $Ax + A^2u$ . By theorem IV, since  $a_\nu x + h_\nu u = a_\nu x + a_\nu^2 u$ ,  $h_\nu = a_\nu^2$ . Similarly the roots of the matrix

$$Ax + By + Cz + \dots + A^2u + B^2v + ABw$$

are equal to

$$a_\nu x + b_\nu y + c_\nu z + \dots + a_\nu^2 u + b_\nu^2 v + h_\nu w$$

where  $h_1, h_2, \dots, h_n$  are the roots of  $AB = BA$  in appropriate order. On the other hand, the roots of the matrix  $pA + qB$  are equal to  $pa_\nu + qb_\nu$ . Thus the roots of the matrix  $Ax + By + Cz + \dots + (pA + qB)^2$  are equal to  $a_\nu x + b_\nu y + c_\nu z + \dots + (pa_\nu + qb_\nu)^2$ . These two results show that  $h_\nu = a_\nu b_\nu$ . By the decomposition of the determinant of

$Ax + By$  into linear factors, one sees that the ordering  $a_1, a_2, \dots, a_n$  corresponds to the ordering  $b_1, b_2, \dots, b_n$ . Thus  $a_1b_1, a_2b_2, \dots, a_nb_n$  is the corresponding ordering of the roots of  $AB$ . Similarly  $(a_1b_1)c_1, (a_2b_2)c_2, \dots, (a_nb_n)c_n$  is the corresponding ordering of the roots of  $(AB)C$ . Thus we obtain theorem III for an arbitrary product of matrices, and similarly for an arbitrary linear combination of such products.

Thus, as a result of these simple comments, we simply have to worry about the proof of theorem V. In so doing I will repeat the work of  $L$  as little as possible and simultaneously take the opportunity to prove some of the results of that paper, that I proved there using infinite series, in a simpler manner.

The above theorems I was already aware of at the time of the publication of  $L$ , as can be seen in some of the implications found therein. A portion of my results on commuting matrices have been put together in  $L$  §7, Theorems XII through XV. I have not based this paper off of those results for the following reason: If the matrices  $A, B, C, \dots$  are functions of the matrix  $R$ , then any two of them commute with each other. For this case, all the presented theorems are consequences of theorem IV. They would be trivial assuming the following theorem were true: If the matrices  $A, B, C \dots$  commute with each other, then they can all be represented as functions of a single matrix  $R$ . This theorem would be analogous to the well known theorem of Abel out of the theory of algebraic equations. In the algebraic version, one can choose an  $R$  that is a function of  $A, B, C \dots$ . But this is not true for the general case. Because if:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then  $A^2 = B^2 = AB = BA = 0$ . Thus every function of  $A$  and  $B$  has the form  $F = aA + bB + cI$ , and every function of  $F$  has the form  $pI + qF$ .

Whether the addition of this requirement would make theorem true, I have

not been able to determine. Together with this question is another question regarding commuting matrices, which I have so far not solved, namely the question regarding the composition of  $m$  linear independent matrices (of degree  $n$ ), which commute with each other, and the greatest value, that  $m$  can have.

### §1.

If the determinant  $a = |A|$  is nonzero, then there is an inverse matrix  $A^{-1}$ , which is uniquely determined by the conditions:

$$AA^{-1} = A^{-1}A = I. \quad (12.1)$$

Multiply by  $a$ , to get the adjoint form  $aA^{-1}$ , that will be represented by  $\bar{A}$ . It is made up of the elements  $b_{\alpha\beta}$ , where  $b_{\alpha\beta}$  corresponds to the cofactor of  $a_{\beta\alpha}$ , and therefore is an integer function of the elements of  $A$ , and thus can be constructed when  $a \neq 0$ . The adjoint satisfies the following equations

$$A\bar{A} = \bar{A}A = aI. \quad (12.2)$$

The determinant

$$|rE - A| = \phi(r) \quad (12.3)$$

is called the *characteristic function*. The equation  $\phi(r) = 0$  is called the characteristic equation of  $A$ . The adjoint form of  $rE - A$  is a matrix  $F$ , whose elements are integer functions of  $r$  of the  $(n - 1)^{th}$  degree, and that thus can be written as  $F(r)$ . Then:

$$(rE - A)F(r) = F(r)(rE - A) \quad (12.4)$$

and

$$(rE - A)F(r) = \phi(r)E. \tag{12.5}$$

Write

$$F(r) = F_0 + F_1r + F_2r^2 + \dots$$

according to the powers of  $r$ . Then from equation (4.) we get that the matrices  $F_0, F_1, F_2, \dots$  commute with  $A$ . Let

$$\phi(r) = a_0 + a_1r + a_2r^2 + \dots + a_nr^n.$$

Then from equation (5.) we get the equations:

$$-AF_0 = a_0I,$$

$$-AF_1 + F_0 = a_1I,$$

$$-AF_2 + F_1 = a_2I,$$

...

$$-AF_{n-1} + F_{n-2} = a_{n-1}I,$$

$$F_{n-1} = a_nI.$$

If  $B$  is another matrix, then multiply these equations on the right by  $B^0, B^1, \dots, B^n$ , and add them together. Let

$$F(B) = F_0 + F_1B + \dots + F_{n-1}B^{n-1}.$$

Then we obtain

$$-AF(B) + F(B)B = \phi(B) \tag{12.6}$$

Because of the particular difficulty, which forces one in the construction of  $F(B)$  to pay attention to the place of  $B$ , this result is only useful when  $B$  commutes with  $F_0, F_1, F_2 \dots$ . Thus:

$$(B - A)F(B) = \phi(B). \tag{12.7}$$

Out of equation (5.) we get another equation, simply by replacing  $r$  with a matrix  $B$  that commutes with  $A$  and  $F(r)$ . I made ample use of this principle in my work  $L$ . Let  $B = A$ , then one obtains the equation

$$\phi(A) = 0. \tag{12.8}$$

This fundamental theorem was first found by Cayley, and I believe, was first published in *A Memoir on the Theory of Matrices*, Phil. Trans. vol. 148., albeit without a general proof. The above form was proven by Pasch in, *Ueber bilineare Formen und deren geometrische Anwendung*, Math. Ann. Bd. 38, S. 48. Using the same methods one can obtain the second fundamental theory of matrix theory:

VI. Let  $\zeta(r)$  be the greatest common divisor of all the co-factors of the  $(n-1)$  degree of the matrix  $rI-A$ , and let  $\psi(r) = \phi(r)/\zeta(r)$ , then

$$\psi(A) = 0 \tag{12.9}$$

is the equation of least degree, for which  $A$  is a root. And if  $\chi(A) = 0$ , then  $\psi(r)$  divides  $\chi(r)$ .

Define the entire function  $F$  of the variables  $r$  and  $s$ :

$$(\phi(r) - \phi(s))/(r - s) = F(r, s) = F(s, r).$$

From the equation

$$\phi(r) - \phi(s) = (r - s)F(s, r),$$

we get

$$\phi(r)I - \phi(A) = (rI - A)F(A, r),$$

so by (8.) we see that

$$(rI - A)F(A, r) = \phi(r)I \tag{12.10}$$

and so

$$(rI - A)^{-1} = F(A, r)/\phi(r). \tag{12.11}$$

The adjoint matrix of  $rI - A$  is thus  $F(A, r)$ , and is thus an entire function of  $A$ , whose elements are entire functions of  $r$ . It follows that  $F_0, F_1, F_2, \dots$  are entire functions of  $A$ , so  $B$  commutes with all of these matrices as long as  $B$  commutes with  $A$ . With this condition we get the following equation (7.):

$$(B - A)F(A, B) = \phi(B).$$

The elements of the matrix  $F(A, r)$  are the cofactors of  $(n - 1)^{th}$  degree from  $rE - A$ , and are all divisible by  $\psi(r)$ . If one orders the determinant (3.) according to the elements of a row, one realizes, that  $\phi(r)$  is also divisible by  $\psi(r)$ . Accordingly

the elements of the matrix

$$F(A, r)/\zeta(r) = G(A, r)$$

that is a entire function of  $A$ , are entire functions of  $r$ . And thus by (10.)

$$(rI - A)G(A, r) = \psi(r)I.$$

Using the principle developed in detail above one obtains a proper equation, by replacing  $r$  with a matrix  $B$  that commutes with  $A$ . If  $B = A$ , then one obtains equation (9.).

On the other hand if  $\chi(r)$  is a polynomial satisfying  $\chi(A) = 0$ . Define  $H(r, s)$ :

$$(\chi(r) - \chi(s))/(r - s) = H(r, s) = H(s, r),$$

then

$$\chi(r)I - \chi(A) = (rI - A)H(A, r),$$

thus

$$(rI - A)H(A, r) = \chi(r)I,$$

and consequently

$$\chi(r)G(A, r) = \psi(r)H(A, r).$$

The matrix  $G(A, r)$  is made up of  $n^2$  elements  $g_{\alpha\beta}(r)$ , that are entire functions of  $r$  and by hypothesis have no common divisors. The elements of  $H(A, r)$  are also entire functions of  $r$ . From the  $n^2$  equations

$$\chi(r)g_{\alpha\beta}(r) = \psi(r)h_{\alpha\beta}(r),$$

it follows that  $\chi(r)$  is divisible by  $\psi(r)$ . Consequently  $\psi(A) = 0$  the equation of least degree that has  $A$  as a root, and every other equation that has  $A$  as a root has the form  $\psi(A)g(A) = 0$ , where  $g(r)$  is an entire function on  $r$ . In particular,  $\psi(r)$  divides  $\phi(r)$ . By differentiating the determinant by  $r$ , one notices that every root of  $\psi(r) = 0$  is also a root of  $\phi(r) = 0$ , and thus a power of  $\psi(r)$  is divisible by  $\phi(r)$ .

I first published Theorem VI in *L.* §3, and proved using infinite series. I alluded to the preceding proof, however, in *L.* §3 and particularly in §13. This consequential theorem has so far received little attention. The special case where  $\psi(r)$  is a divisor of  $r^m - 1$ , which I paid special attention to in *L.* §3, VIII, was proven by Lipschitz in *Beweis eines Satzes aus der Theorie der Substitution*, Acta Math. BD. X. His proof was similar to the above proof in the important aspects. Kronecker also handled this theorem in detail in *Ueber die Composition der Systeme von  $n^2$  Groessen mit sich selbst*, Sitzungber. 1890. These authors failed to notice, however, that I had already proven this theorem as a special case of theorem VI. Also the English and American authors, with whom I have worked on matrix theory, have largely ignored my work, along with the great work by Laguerre, *Sur le calcul des systemes lineaires*, Journ. de l'ecole polyt. tom. 25 cah. 42 p. 215. Another, less simple, proof was given by E. Weyr, *Zur Theorie der bilinearen Formen*, Monatshefte fuer Math. und Physik, Bd. 1 S. 187.

## §2.

If a matrix satisfies the equation  $A^k = 0$ , then  $\psi(r)$  is a divisor of  $r^k$ . So  $\psi(r) = r^m$  is a power of  $r$ , and consequently  $\phi(r) = r^n$ . Conversely, if zero is the only root of the characteristic polynomial, then  $A^n = 0$ . The following theorem applies to such matrices. The theorem is a special case of Theorem V, and the same proof from *L.* §3, VII is repeated here:



VII. Let  $A$  and  $B$  be commuting matrices, and  $A$  nilpotent, then the determinant of  $A+B$  equals the determinant of  $B$ .

From the equation  $r^n = \phi(r) = |rI - A|$  it follows that if  $r = -1$ , then  $|A + E| = 1$ . Let  $s$  be an arbitrary quantity, then  $(B + sI)^{-1}$  commutes with  $A$ . Let  $C = (B+sI)^{-1}A$ , then because of this commutating property  $C^n = (B+sI)^{-n}A^n = 0$ , and it follows that  $|C + E| = 1$ . Since  $(B+sI)(C+E) = A+B+sI$ ,  $|B+sI||C+E| = |A+B+E|$ . Let  $s = 0$ , then  $|B| = |A+B|$ .

The meaningful contributions that Weierstrass made to the work of Cauchy and Jacobi is that he taught how to decompose even nilpotent matrices, or more generally matrices whose characteristic equation only has one root, unless the least power that vanishes is the  $n$ -th. Theorem VII makes it possible to continue the following development without appeal to this decomposition.

### §3.

Let  $a, b, c, \dots$  be the roots of the characteristic function of  $A$ :

$$\phi(r) = (r - a)^\alpha (r - b)^\beta (r - c)^\gamma \dots \quad .$$

Then according to a generalization of Lagrange's Interpolation Formula there is a unique entire function of degree  $n - 1$ , which is divisible by  $(r - b)^\beta (r - c)^\gamma \dots$ , and for which  $f(r) - 1$  is divisible by  $(r - a)^\alpha$ . Define similarly the functions  $g(r), h(r), \dots$  for the roots  $b, c, \dots$ . Then

$$f(r) + g(r) + h(r) + \dots = 1 \tag{12.1}$$

because the difference between the left and side is a entire function of degree  $n - 1$ ,

that is divisible by  $(r - a)^\alpha, (r - b)^\beta, (r - c)^\gamma, \dots$  and thus by the entire function of degree  $n$ ,  $\phi(r)$ .

The function  $f(r)$  can also be defined as the coefficient of  $(s - a)^{-1}$  in

$$\frac{\phi(r) - \phi(s)}{r - s} \frac{1}{\phi(s)}. \quad (12.2)$$

The coefficient of  $s^{-1}$  is equal to 1. Because the function is infinite for  $s = a, b, c, \dots$  but not for  $s = r$ , then the residue theorem gives equation (1.). Because (2.) is an entire function of  $r$ , then the residues of  $f(r), g(r), h(r), \dots$  are also entire functions of degree at most  $n - 1$  of  $r$ . The expansion of the second term of the difference

$$\frac{\phi(r)}{(r - s)\phi(s)} - \frac{1}{r - s}$$

by powers of  $s - a$  includes no negative powers of  $s - a$ . The expansion of the first term is,

$$\phi(r) \left( \frac{1}{r - a} + \frac{s - a}{(r - a)^2} + \frac{(s - a)^2}{(r - a)^3} + \dots \right) \left( \frac{a_0}{(s - a)^\alpha} + \frac{a_1}{(s - a)^{\alpha-1}} + \dots \right),$$

if the last series is the expansion of  $\frac{1}{\phi(s)}$ . Thus  $f(r) = \frac{\phi(r)\zeta(r)}{(r-a)^\alpha}$ , where  $\zeta(r)$  is an entire function of  $r$  of degree  $\alpha - 1$ . Consequently  $f(r)$  is divisible by  $\phi(r)(r - a)^{-\alpha}$ . Similarly  $g(r)$  is divisible by  $\phi(r)(r - b)^{-\beta}, \dots$ . And since every function  $g(r), h(r), \dots$  is divisible by  $(r - a)^\alpha$ , by (1.)  $f(r) - 1$  is divisible by  $(r - a)^\alpha$ .

I will now change the notation and denote the  $n$  roots of  $\phi(r)$  by  $a_1, a_2, \dots, a_n$ , the distinct roots by  $a_1, a_2, \dots, a_m$ , and their corresponding functions of degree  $n - 1$  by  $\phi_1(r), \phi_2(r), \dots, \phi_m(r)$ . By equation (1.):

$$\sum \phi_\lambda(r) = 1 \quad (12.3)$$

and consequently

$$\sum \phi_\lambda(A) = E. \quad (12.4)$$

Further we know that  $\phi_\lambda(r)(\phi_\lambda(r) - 1)$  is divisible by  $\phi(r)$ , and if  $\chi$  does not equal  $\lambda$ , by  $\phi_\chi(r)\phi_\lambda(r)$  as well. Therefore:

$$(\phi_\lambda(A))^2 = \phi_\lambda(A), \quad \phi_\chi(A)\phi_\lambda(A) = 0. \quad (12.5)$$

If

$$\sum a_\lambda \phi_\lambda(A) - A = A_0, \quad (12.6)$$

then by Theorem IV the roots of the characteristic equation of  $A_0$  are all zero, and thus  $A_0^n = 0$ . The characteristics expressed by equations (4.) and (5.) of the matrices  $\phi_\lambda(A)$  were also handled by Study, *Recurrirende Reihen und bilineare Formen*, Monatshefte fuer Math. und Physik, Bd. II. I also obtained these results in a different manner in my work *Ueber die schiefe Invariante einer bilinearen oder quadratischen Form*, Crelle's Journ. Bd. 86, §6.

If  $B$  is a matrix that commutes with  $A$  and  $x_1, x_2, \dots, x_m$  and  $y$  are variables, then

$$\begin{aligned} & (x_1 I + y\phi_1(A)B)(x_2 I + y\phi_2(A)B) \dots (x_m I + y\phi_m(A)B) \\ &= x_1 x_2 \dots x_m \left( I + \frac{y\phi_1(A)B}{x_1} + \frac{y\phi_2(A)B}{x_2} + \dots + \frac{y\phi_m(A)B}{x_m} \right). \end{aligned}$$

All other elements elements in the development of the product vanish. For example,  $\phi_1(A)B\phi_2(A)B = \phi_1(A)\phi_2(A)B^2 = 0$ . If one multiplies the matrix on the right hand

side with  $\sum x_\lambda \phi_\lambda(A)$ , then by (5.) we get:

$$\sum x_\lambda \phi_\lambda(A) + y(\phi_1(A)B + \phi_2(A)B + \cdots + \phi_m(A)B).$$

By (4.) then we get

$$\left(\sum x_\lambda \phi_\lambda(A)\right) \prod (x_\lambda I + y\phi_\lambda(A)B) = \left(yB + \sum x_\lambda \phi_\lambda(A)\right) \prod (x_\lambda), \quad (12.7)$$

and therefore the determinants of these two matrices are equal. The determinant of  $x_\lambda I + y\phi_\lambda(A)B$  is the homogeneous characteristic function  $-\phi_\lambda(A)B$ , and is thus an entire homogeneous function of  $n$ -th degree of  $x_\lambda$  and  $y$ , where the coefficient of  $x_\lambda^n$  is equal to 1. The determinant of  $\sum x_\lambda \phi_\lambda(A)$  is by theorem IV a product of  $n$  factors  $\sum x_\lambda \phi_\lambda(a_x)$ . The determinant of the identity does not vanish, because by (4.) if  $x_1 = x_2 = \cdots = x_m = 1$ , then the determinant is also 1. Consequently the determinant of  $yB + \sum x_\lambda \phi_\lambda(A)$  is nonzero and a product of  $n$  linear functions of the variables  $x_1, x_2, \dots, x_m$  and  $y$ . Let  $x_\lambda = xa_\lambda - r$ , then by (4.) and (6.)

$$\sum x_\lambda \phi_\lambda(A) = x \sum a_\lambda \phi_\lambda(A) - rI = xA - rI + xA_0.$$

Because  $A_0$  commutes with  $xA + yB - rI$  and  $A_0^n = 0$ , then by §2 the determinant of  $xA + yB - rI + xA_0$  is equal to the determinant of  $xA + yB - rI$ . Therefore this determinant is a product of  $n$  linear functions of  $x, y$ , and  $r$ . Thus we get Theorem V, from which the more general Theorem III follows.

#### §4.

Let  $A_1, A_2, \dots, A_m$  be  $m$  matrices, which commute with each other. For  $\gamma =$

1, 2, \dots m let

$$a_{\alpha\beta\gamma} \quad (\alpha, \beta = 1, 2, \dots n)$$

be the elements of the matrix  $A_\gamma$ , then since  $A_\gamma A_\delta = A_\delta A_\gamma$

$$\sum_{\lambda} a_{\alpha\lambda\gamma} a_{\lambda\beta\delta} = \sum_{\lambda} a_{\alpha\lambda\delta} a_{\lambda\beta\gamma}. \quad (12.1)$$

Then if  $x_1, x_2, \dots x_m$  and  $r$  are variables:

$$\left| \sum_{\gamma} A_{\gamma} x_{\gamma} - rI \right| = \prod_{\chi} (r_1^{(\chi)} x_1 + \dots + r_m^{(\chi)} x_m - r) \quad (12.2)$$

where  $r_{\gamma}^{(1)}, r_{\gamma}^{(2)}, \dots r_{\gamma}^{(n)}$  are the roots of the characteristic equation of  $A_{\gamma}$ . Let

$$A = \sum_{\gamma} A_{\gamma} x_{\gamma}, \quad a_{\alpha\beta} = \sum_{\gamma} a_{\alpha\beta\gamma} x_{\gamma},$$

then

$$r^{(\chi)} = \sum_{\gamma} r_{\gamma}^{(\chi)} x_{\gamma} \quad (12.3)$$

are the roots of  $A$ . By formula (2.) the roots of the matrices  $A_1, A_2, \dots A_m$  and  $A$  are ordered in a particular manner, and for notational comfort let  $r_{\gamma}^{(\chi)}$  ( $r^{(\chi)}$  respectively) be the  $\chi$ -th root of  $A_{\gamma}$  ( $A$  respectively). Let  $f(u_1, u_2, \dots u_m)$  be a function of  $u_1, u_2, \dots u_m$ , then  $f(r_1^{(\chi)}, r_2^{(\chi)}, \dots r_m^{(\chi)})$  is the  $\chi$ -th root of the matrix  $f(A_1, A_2, \dots A_m)$ .

By setting coefficients equal to each other we get from (2.)

$$\sum_{\chi} r^{(\chi)} = \sum_{\alpha} a_{\alpha\alpha} = \sum_{\alpha, \lambda} a_{\alpha\alpha\lambda} x_{\lambda}$$

and

$$\sum_{\chi, \lambda} r^{(\chi)} r^{(\lambda)} = \sum_{\alpha, \beta} (a_{\alpha\alpha} a_{\beta\beta} - a_{\alpha\beta} a_{\beta\alpha}) = \sum_{\alpha, \beta} \left( \sum_{\chi} a_{\alpha\alpha\chi} x_{\chi} \right) \left( \sum_{\lambda} a_{\beta\beta\lambda} x_{\lambda} \right) - \left( \sum_{\chi} a_{\alpha\beta\chi} x_{\chi} \right) \left( \sum_{\lambda} a_{\beta\alpha\lambda} x_{\lambda} \right)$$

consequently we get

$$\sum_{\chi} (r^{(\chi)})^2 = \sum_{\alpha, \beta, \chi, \lambda} a_{\alpha\beta\chi} a_{\beta\alpha\lambda} x_{\chi} x_{\lambda}. \quad (12.4)$$

Let

$$\sum_{\alpha, \beta} a_{\alpha\beta\chi} a_{\beta\alpha\lambda} = c_{\chi\lambda} = c_{\lambda\chi}, \quad (12.5)$$

then

$$\sum_{\chi} (r^{(\chi)})^2 = \sum_{\alpha, \beta} c_{\alpha\beta} x_{\alpha} x_{\beta}, \quad (12.6)$$

thus

$$c_{\alpha\beta} = \sum_{\chi} r_{\alpha}^{(\chi)} r_{\beta}^{(\chi)}. \quad (12.7)$$

Now add to hypotheses (1.) the additional conditions that  $m = n$ , and that the quantities

$$a_{\alpha\beta\gamma} = a_{\alpha\gamma\beta}. \quad (12.8)$$

Then the elements  $g_{\rho\sigma}$  of the matrix  $A_{\beta} A_{\gamma} = A_{\gamma} A_{\beta}$  equal

$$g_{\rho\sigma} = \sum_{\alpha} a_{\rho\alpha\beta} a_{\alpha\sigma\gamma} = \sum_{\alpha} a_{\rho\alpha\gamma} a_{\alpha\sigma\beta}.$$

Since the first expression remains unchanged when  $\sigma$  and  $\gamma$  are switched, the same is true for the second expression. Consequently

$$g_{\varrho\sigma} = \sum_{\varrho\alpha\sigma} a_{\alpha\gamma\beta} = \sum_{\alpha} a_{\alpha\beta\gamma} a_{\varrho\sigma\alpha},$$

thus

$$A_{\beta}A_{\gamma} = A_{\gamma}A_{\beta} = \sum_{\alpha} a_{\alpha\beta\gamma}A_{\alpha}. \quad (12.9)$$

By Theorem III the  $\chi$ -th root of  $\sum_{\alpha} a_{\alpha\beta\gamma}A_{\alpha}$  equal to  $\sum_{\alpha} a_{\alpha\beta\gamma}r_{\alpha}^{(\chi)}$  and the roots of  $A_{\beta}A_{\gamma}$  equal to  $r_{\beta}^{(\chi)}r_{\gamma}^{(\chi)}$  and consequently from (9.):

$$r_{\beta}^{(\chi)}r_{\gamma}^{(\chi)} = \sum_{\alpha} a_{\alpha\beta\gamma}r_{\alpha}^{(\chi)}. \quad (12.10)$$

According to Theorem III the  $\chi$ -th root of  $\sum_{\alpha} a_{\alpha\beta\gamma}A_{\alpha}$  equals  $\sum_{\alpha} a_{\alpha\beta\gamma}r_{\alpha}^{(\chi)}$  and the  $\chi$ -th root of  $A_{\beta}A_{\gamma}$  equals  $r_{\beta}^{(\chi)}r_{\gamma}^{(\chi)}$ , and consequently from (9.):

$$r_{\beta}^{(\chi)}r_{\gamma}^{(\chi)} = \sum_{\alpha} a_{\alpha\beta\gamma}r_{\alpha}^{(\chi)}. \quad (12.11)$$

The equations

$$r_{\beta}r_{\gamma} = \sum_{\alpha} a_{\alpha\beta\gamma}r_{\alpha} \quad (12.12)$$

between the unknowns  $r_1, r_2, \dots, r_n$  have thus  $n$  systems of solutions

$$r_1 = r_1^{(\chi)}, r_2 = r_2^{(\chi)}, \dots, r_n = r_n^{(\chi)} \quad (\chi = 1, 2, \dots, n) \quad (12.13)$$

and no other if one ignores the solution  $r_1 = r_2 = \dots = r_n = 0$ , in case this solution

is not included in (12.). Let  $\sum r_\gamma x_\gamma = r$ , then

$$r_\beta r = \sum_\alpha a_{\alpha\beta} r_\alpha. \quad (12.14)$$

Consequently the determinant

$$|A - rI| = \prod_x (R_1^{(x)} x_1 + \cdots + r_n^{(x)} x_n - r)$$

vanishes. Thus  $r = \sum r_\gamma x_\gamma$  is equal to one of the  $n$  functions  $\sum_\gamma r_\gamma^{(x)} x_\gamma$ , or is made up of one of the equations in (12.). These comments lead to the following theorem, that differs from our previous results, in that no inequality is present in the hypotheses:

VIII. *If the  $n^3$  quantities  $a_{\alpha\beta\gamma}$  satisfy the equations*

$$a_{\alpha\beta\gamma} = a_{\alpha\gamma\beta}, \quad \sum_\lambda a_{\alpha\lambda\gamma} a_{\lambda\beta\delta} = \sum_\lambda a_{\alpha\lambda\delta} a_{\lambda\beta\gamma},$$

*then the coefficients of the linear factors, into which the determinant*

$$\left| \sum_\gamma a_{\alpha\beta\gamma} x_\gamma - r I_{\alpha\beta} \right| = \prod_x (r_1^{(x)} x_1 + \cdots + r_n^{(x)} x_n - r)$$

*decomposes, satisfy the equations*

$$r_\beta r_\gamma = \sum_\alpha a_{\alpha\beta\gamma} r_\alpha$$

*and are the only solution of the equations.*

By formula (7.) we get

$$|c_{\alpha\beta}| = |r_\alpha^{(x)}|^2 \quad (12.15)$$

from which we get the wonderful theorem of Dedekind:



IX. If the  $n^3$  quantities  $a_{\alpha\beta\gamma}$  satisfy the equations

$$a_{\alpha\beta\gamma} = a_{\alpha\gamma\beta}, \quad \sum_{\lambda} a_{\alpha\lambda\gamma} a_{\lambda\beta\delta} = \sum_{\lambda} a_{\alpha\lambda\delta} a_{\lambda\beta\gamma},$$

and if the determinant made up of the quantities

$$c_{\chi\lambda} = \sum_{\alpha,\beta} a_{\alpha\beta\chi} a_{\beta\alpha\lambda} = \sum_{\alpha\beta} a_{\alpha\alpha\beta} a_{\beta\chi\lambda}$$

is nonzero, then the equations

$$r_{\beta} r_{\gamma} = \sum_{\alpha} a_{\alpha\beta\gamma} r_{\alpha}$$

have exactly  $n$  distinct solutions  $r_{\alpha} = r_{\alpha}^{(x)}$ , and the determinant made up of these solutions is nonzero.

If  $|r_{\alpha}^{(x)}| = 0$ , then one can order the quantities  $x_1, x_2, \dots, x_n$ , that are not all zero, so that the  $n$  quantities (3.), that are the roots of the characteristic equation of  $A$ , simultaneously vanish, and consequently a power of  $A$  vanishes. The necessary and sufficient conditions for the determinant to vanish is that there is a matrix in the representation  $\sum A_{\gamma} x_{\gamma}$  which is nilpotent but nonzero (compare Weierstrass, a. a. O. S. 402).

If  $|r_{\alpha}^{(x)}| = 0$ , then  $(s_{\alpha}^{(x)})$  is the complementary system to  $(r_{\alpha}^{(x)})$ , where the complement comes from inversion. So if  $e_{\alpha\beta}$  are the elements of the identity matrix  $I$ , then:

$$\sum_{\chi} r_{\alpha}^{(\chi)} s_{\beta}^{(\chi)} = e_{\alpha\beta}, \quad \sum_{\alpha} r_{\alpha}^{(\chi)} s_{\alpha}^{(\lambda)} = e_{\chi\lambda}. \quad (12.16)$$

Then by (10.) we get

$$a_{\alpha\beta\gamma} = \sum_{\chi} s_{\alpha}^{(\chi)} r_{\beta}^{(\chi)} r_{\gamma}^{(\chi)} \quad (12.17)$$

and

$$s_{\alpha}^{(\chi)} r_{\gamma}^{(\chi)} = \sum_{\beta} a_{\alpha\beta\gamma} s_{\beta}^{(\chi)}, \quad (12.18)$$

and thus

$$s_{\alpha}^{(\chi)} r^{(\chi)} = \sum_{\beta} a_{\alpha\beta} s_{\beta}^{(\chi)}. \quad (12.19)$$

By these equations we can completely describe the behavior of the  $n$  quantities

$$s_1^{(\chi)}, s_2^{(\chi)}, \dots, s_n^{(\chi)}.$$

Every root,  $r$ , of the equation  $|A - rI| = 0$  corresponds to  $n$  quantities  $s_1, s_2, \dots, s_n$ , whose behavior is described by the equations.

$$s_{\alpha} r = \sum_{\beta} a_{\alpha\beta} s_{\beta}. \quad (12.20)$$

Let

$$|re_{\alpha\beta} - a_{\alpha\beta}| = \phi(r), \quad (12.21)$$

and let this determinant the cofactors corresponding to the elements  $re_{\alpha\beta} - a_{\alpha\beta}$  equal  $\phi_{\alpha\beta}(r)$ . Then if  $r$  is a root of the equation  $\phi(r) = 0$ , then by equations (13.) and (19.),  $\phi_{\alpha\beta} = \rho r_{\alpha} s_{\beta}$ , where  $\rho$  is independent of  $\alpha$  and  $\beta$ . However if  $\phi'(r) = \sum \phi_{\alpha\alpha}$

then by (15.)  $\sum r_\alpha s_\alpha = 1$ . Consequently if  $\rho = \phi'(r)$ , then

$$\phi_{\alpha\beta}(r) = \phi'(r)r_\alpha s_\beta. \quad (12.22)$$

If  $r$  is a variable, then by decomposition into partial fractions

$$\frac{\phi_{\alpha\beta}(r)}{\phi(r)} = \sum_x \frac{\phi_{\alpha\beta}(r^{(x)})}{\phi'(r^{(x)})} \frac{1}{r - r^{(x)}}.$$

Consequently the determinant (20.) contains

$$\phi_{\alpha\beta}(r) = \phi(r) \sum_x \frac{r_\alpha^{(x)} s_\beta^{(x)}}{r - r^{(x)}} \quad (12.23)$$

Let  $r_\alpha^{(x)}$  be a system of  $n^2$  quantities, whose determinant is nonzero, then the  $a_{\alpha\beta\gamma}$  defined in equation 16 are the most general system that satisfy the conditions of theorem IX.

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