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Abstract: A multilinear routing method using diffusion analogy model as a sub-model is proposed in this study. The diffusion analogy model can be considered as the next level of approximation to the full Saint-Venant equations. The applicability of the method is first tested by simulating the outflow hydrographs generated by routing the hypothetical inflow hydrographs in hypothetical channel reaches using the Saint-Venant equations and, subsequently, by studying the flood wave propagation in a 23 km reach of Tiber River in Central Italy. The results demonstrate the suitability of the diffusion analogy model as a sub-model for its use in the multilinear routing method for real-time applications.

Keywords: Diffusion; Multilinear; Routing; Streamflow; Modelling.

1. INTRODUCTION

Flood routing problems are solved using the hydraulic and hydrological methods. However, the hydraulic based methods, which generally use the full Saint-Venant equations, are of limited use primarily due to non-availability of topographical and hydrological inputs required at smaller spatial scales. Further, the computational limitations of the numerical schemes adopted in the solution procedure also restrict the use of the full Saint-Venant equations for many practical problems such as flood forecasting. An alternative way of overcoming these data and computational problems is by using the simplified hydraulic routing methods, which are derived from the Saint-Venant equations, but at the same time, they are not data intensive. While many researchers [Cunge et al., 1980; Lai, 1986] favour the use of the full Saint-Venant equations in flood routing studies, various other researchers [NERC, 1975; Ferrick, 1985] argued for using the simplified routing methods. Two categories of the simplified routing methods are used in practice: 1) the linear simplified routing methods which use constant parameters for routing a given flood wave, and 2) the variable parameter simplified routing methods which use the model structure of the simplified routing methods, but the parameters varying at every routing time step. The variable parameter Muskingum-Cunge (VPMC) method and its variants [Ponce and Yevjevich, 1978; Ponce and Chaganti, 1994], the multilinear Muskingum method [Perumal, 1992] and the multilinear discrete cascade model [Perumal, 1994] are some of the available variable parameter simplified methods. While assessing the future developments of flood routing methods Fread [1981] and Lai [1986] opined that the simplified methods would continue to be used in practice, especially as components of precipitation-runoff basin models for routing overland and channel flows associated with the network of headwater streams.

The current study focuses on the development of an improved multilinear routing method using diffusion analogy model [Hayami, 1951] as the submodel. The diffusion analogy model can be considered as the next level of approximation to full Saint-Venant equations [Fread, 1981]. This method is more appropriate for its use in real-time flood forecasting schemes. The applicability of the method is first tested by simulating the outflow hydrographs generated by routing the hypothetical inflow hydrographs in hypothetical channel reaches using the Saint-Venant equations and, subsequently, testing the field applicability of the method by studying the flood wave propagation in a 23 km reach of Tiber River in Central Italy.

2. MULTILINEAR ROUTING MODELS

The linear flood routing methods such as the Muskingum and Kalinin-Milyukov methods are widely used as the basic models in real-time flood forecasting schemes. However, flood waves are inherently nonlinear in nature and, therefore, it is generally desirable to use nonlinear models for studying flood wave movement in channels. But the nonlinear models are more difficult to apply in the field than the linear models. The convenience of linear analysis can still be used for modelling the nonlinear hydrological processes by working within the limitation imposed by its assumption. One simple method by which the nonlinearity of the flood routing process could be taken care off is by using the multilinear modelling approach advocated by Keefer and McQuivey [1974] and followed by Becker [1976], and Becker and Kundzewicz [1987]. This approach was further extended by employing the Muskingum method [Perumal, 1992] and the discrete cascade model [Perumal, 1994] as the sub-models of the multilinear routing methods. All these multilinear routing methods amply demonstrate the appropriateness of this approach by closely reproducing the solutions of the full Saint-Venant equations in off-line mode, which were considered as the benchmark solutions.

Moramarco and Singh [1999] investigated the use of diffusion analogy model as a sub-model of two multilinear modelling approaches. The principle employed in these approaches is to distinguish on the input hydrograph, different input components, each of which is subsequently routed through the diffusion analogy sub-model. These two modelling approaches differ from each other by inflow portioning approach, depending on whether the inflow is divided up either horizontally or vertically, i.e., the horizontal distinctions represent the different zones of discharge, whereas the vertical distinctions are introduced at fixed times. The former approach has been used by the early investigators of the multilinear routing methods [Keefer and McQuivey, 1974; Becker, 1976; Becker and Kundzewicz, 1987]. Becker and Kundzewicz [1987] introduced the vertical portioning of the inflow hydrograph into different time zones with each zone characterized by a unique instantaneous response function defined by the average flow characteristics of the respective zones. This scheme of portioning is alternatively known as the time distribution scheme. Perumal [1992, 1994] and Moramarco and Singh [1999] restricted the time distribution scheme of the inflow hydrograph to that of the routing time interval, thus, accounting for the nonlinear feature of the flood wave characteristics in a more efficient manner. Moramarco and Singh [1999] directly employed the response function of the diffusion analogy submodel for convoluting with the inflow hydrograph. However, the theoretically correct way of obtaining the Δt-hour response function to be used for convolution with the given inflow hydrograph ordinate is the one obtained by convoluting a unit inflow ordinate with the instantaneous unit hydrograph (IUH) of the diffusion analogy model. However, when the routing interval Δt is small, say an hour, the difference between the IUH and the corresponding Δt-hour response function may not be significant, implying that one may directly employ the IUH for convolution with the given inflow hydrograph ordinates.

3. MULTILINEAR DIFFUSION ANALOGY ROUTING MODEL

The routed outflow at any time in response to the given inflow hydrograph observed till the same time may be expressed by the discrete convolution approach as:

$$Q(i\Delta t) = I_b + \sum_{i=1}^{j} h(i\Delta t)\left(I\left((j-i-1)\Delta t\right) - I_b\right) \quad (1)$$

where I_b is the initial flow in the considered river reach; $j\Delta t$ denotes the time t in discrete intervals with j=1 corresponding to the first ordinate; and $h(i\Delta t)$ denotes the ordinate of the Δt-hour response function of the diffusion analogy sub-model at time $i\Delta t$. The Δt-hour response function of the diffusion analogy sub-model corresponding to the unit ordinate input is expressed using the convolution approach as:

$$h(t) = \int_{t-\Delta t}^{t} u(\tau) d\tau \tag{2}$$

where $u(t)$ is the IUH of the diffusion analogy model [Hayami, 1951] expressed as:

$$u(t) = \frac{x}{\sqrt{4\pi D t^3}} \exp\left[\frac{-(x-ct)^2}{4Dt}\right] \tag{3}$$

where x is the routing reach length, c is the wave celerity in m/s, and D is the hydraulic diffusivity in m^2/s expressed as:

$$D = \frac{Q_0}{2S_0 B} \tag{4}$$

where Q_0 is the reference discharge at time level t, S_0 is the channel bed slope, and B is the water surface width corresponding to Q_0.

The wave celerity c is expressed as:

$$c = \left.\frac{dQ}{dA}\right|_{Q=Q_0} = \frac{1}{B}\left.\frac{dQ}{dy}\right|_{Q=Q_0} \tag{5}$$

The reference discharge used in the estimation of the parameters c and D of the diffusion analogy model is estimated as [Perumal, 1992, 1994]:

$$Q_O = Q_b + a(I(t) - Q_b) \tag{6}$$

where a is an empirical coefficient with $0 < a < 1$.

Though the diffusion analogy sub-model used in the proposed method has a larger range of applicability limit than the Muskingum method used as the sub-model in the multilinear Muskingum (MM) method [Perumal, 1992] and the discrete cascade sub-model [Perumal, 1994] used in the multilinear discrete cascade routing method, its application to nonlinear modelling on the same lines as these multilinear models is difficult to achieve. The difficulty lies with the establishment of the pulse response, required for convolution with the inflow hydrograph, as the closed form integration of the IUH given by equation (2) cannot be obtained. Although there are a number of built-in functions available in the modern day personal computers for a close estimation of the pulse response function, the use of such a solution approach is not mathematically elegant.

To overcome this problem, the use of the Adam-Moulton multi-step numerical integration method [Atkinson, 2003] is used for the numerical integration of equation (2) and the expression of pulse response for $t \geq 2\Delta t$ is expressed as:

$$h(i\Delta t) = \frac{\Delta t}{12}\left(8u_{t-\Delta t} + 5u_t - u_{t-2\Delta t}\right) \tag{7}$$

where Δt is the duration of the pulse response or routing time interval; and $h(t)$ is the ordinate of the IUH at time t.

The pulse response at time Δt is expressed as:

$$h(\Delta t) = \frac{u(0) + u(\Delta t)}{2} = \frac{u(\Delta t)}{2} \tag{8}$$

where $u(0) = 0$.

4. APPLICATION

The proposed multilinear diffusion analogy routing method was applied first for simulating a number of hypothetical flood hydrographs obtained by routing hypothetical inflow hydrographs in rectangular and trapezoidal channel reaches, with no lateral inflow within the routing reach. The inflow hydrograph, defined by a mathematical function, is routed in a given channel reach for a specified distance using the proposed method and is compared with the corresponding solutions obtained using the Saint-Venant equations. The form of the inflow hydrograph used in the numerical experiments of the present study is expressed as:

$$I(t) = I_b + \left(I_p - I_b\right)\left(\frac{t}{t_p}\right)^{\frac{1}{(\gamma-1)}} \exp\left[\frac{\left(1-\frac{t}{t_p}\right)}{(\gamma-1)}\right] \quad (9)$$

where I_b is the initial steady flow in the reach (100 m^3/s), I_p is the peak inflow, t_p is the time to peak, and γ is the skewness factor.

Different inflow hydrographs were routed in the considered rectangular channel with a width of 100 m and in the symmetrical trapezoidal channel with a side slope of *z*: horizontal to 1: vertical and with a bottom width of 100 m. The details of channel configurations and inflow hydrographs used in the routing experiments of the proposed study are given in Table 1. A total of 3360 numerical experiments each with unique combination of inflow hydrograph and channel configuration formed by the parameters given in Table 1 were made for each of the considered rectangular and trapezoidal channels.

For all the routing experiments, the value of coefficient a used in the equation (6) for estimating the reference discharge was taken as 0.30 for both the rectangular and trapezoidal channels. This best value of a was arrived at by a trial and error approach using the values between zero and unity for simulating few cases of benchmark solutions, i.e., the Saint-Venant solutions. The Benchmark solutions were obtained using explicit numerical methods.

The method was also tested for field applications by studying the recorded flood events in a 23 km stretch of the reach of Tiber River in central Italy between Ponte Felcino and Torgiano sections. The cross-section of the reach is approximated as a rectangular channel section with a width of 45 km, and the reach is characterized by a Manning's roughness coefficient $n = 0.04$ and the bed slope, S_0 of the reach is estimated as 0.0014. Out of the five events simulated, three events (1985, 1986 and 1992) may be considered as having negligible lateral flow $(< 1\%)$ and the other two events (1991 and 1982) have a small loss and gain of volume as -5.15%- and 12.28%, respectively. The best value of the coefficient a for simulating all these five events was arrived at by trial and error approach starting with a value of $a = 0.30$, and it was found that all the five events could be closely reproduced for a value of $a = 0.24$.

5. PERFORMANCE CRITERIA

The performance of the method in reproducing the benchmark solutions is evaluated by comparing its solution with the corresponding solution of the Saint-Venant equations based on the following criteria:

1) Accuracy of reproduction of the hydrograph shape and size given by the measure of variance explained [Nash and Sutcliffe, 1970] expressed as:

$$VAREX \text{ (in \%)} = \left(1 - \frac{\sum_{i=1}^{N} (Q_{oi} - Q_{ci})^2}{\sum_{i=1}^{N} (Q_{oi} - \overline{Q}_{oi})^2} \right) \times 100 \tag{10}$$

where Q_{oi} is the ith ordinate of the benchmark discharge hydrograph ordinate at the outlet of the reach, Q_{ci} is the ith ordinate of the routed or computed discharge hydrograph by the proposed model and N is the total number of discharge hydrograph ordinates to be simulated.

2) The accuracy of reproduction of the peak discharge of the benchmark solution is estimated by the following measure expressed as:

$$q_{per}(in\%) = \left(\frac{q_{pc}}{q_{po}} - 1 \right) \times 100 \tag{11}$$

where q_{pc} is the routed or computed peak of the discharge hydrograph at the outlet by the proposed method, and q_{po} is the corresponding benchmark hydrograph. The positive value of q_{per} indicates overestimation of the benchmark peak and the negative value indicates its underestimation.

Table 1. Combination of channel and flow characteristics used in the routing experiments by the multilinear diffusion analogy routing method.

Characteristics		Values
Skewness factor, γ	:	1.05, 1.15, 1.25, 1.50
Channel bed slope, S_o	:	0.0002, 0.0005, 0.0008, 0.001, 0.002, 0.003, 0.005, 0.01
Manning's roughness, n	:	0.01, 0.02, 0.03, 0.04, 0.05
Initial discharge, I_b (m^3/s)	:	100.0
Peak discharge, I_p (m^3/s)	:	1000; 2500; 5000; 7500; 10,000; 12,500; 15,000
Time to peak discharge, t_p (h)	:	10.0, 15.0, 20.0
Channel bottom width, b_m (m)	:	100.0
z		0.0; 1.0

3) The accuracy of reproduction of the time to peak of the benchmark solution is estimated by the following measure expressed as:

$$t_{per}(in\%) = \left(\frac{t_{pc}}{t_{po}} - 1 \right) \times 100 \tag{12}$$

where t_{pc} is the time corresponding to routed or computed peak of the discharge hydrograph at the outlet, and t_{po} is the time corresponding to peak of the benchmark discharge hydrograph at the outlet.

As the solution of the method is obtained by convolution approach, the volume of the routed hydrograph is always conserved provided the duration of convolution exceeds the memory of the system.

6. DISCUSSION OF RESULTS

A reach length of 40 km was used in all the experimental runs to arrive at the outflow hydrographs using the proposed multilinear diffusion analogy routing method and these solutions are compared with the corresponding benchmark solutions of the Saint-Venant equations. Figures 1 and 2, respectively, show the typical routing results obtained using the proposed method when a given inflow hydrograph was routed in the considered rectangular and trapezoidal channels for a reach length of 40 km and the corresponding benchmarks solutions are also shown therein. Figures 3 and 4 show the variation of various measures of the accuracy criteria used in this study with reference to the variable of maximum of $(1/S_0)\partial y/\partial x$ estimated for each run. The variable $[(1/S_0)\partial y/\partial x]_{\max}$ has been considered to represent the order of diffusivity associated with each run, and it is expected that higher its magnitude the lower would be the accuracy of the proposed method in reproducing the respective benchmark solution closely.

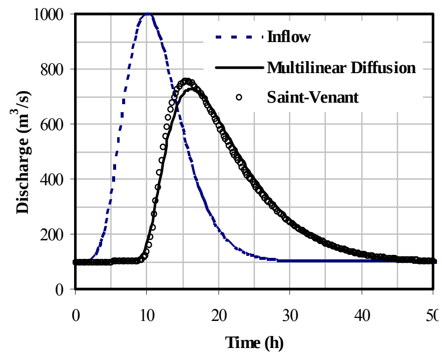

Figure 1. Hydrograph reproductions by the proposed multilinear diffusion and the Saint-Venant methods in rectangular channel (S_o=0.0002, n=0.04, I_b=100 m^3/s, I_p=1000 m^3/s, γ = 1.15)

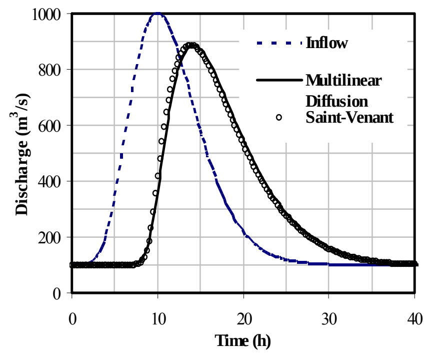

Figure 2. Hydrograph reproductions by the proposed multilinear diffusion and the Saint-Venant methods in a trapezoidal channel (S_o=0.0002, n=0.02, z=1.0, I_b=100 m^3/s, I_p=1000 m^3/s, γ=1.15)

It is inferred from the typical routing results presented in Figures 1 and 2 that the proposed method has the capability in reproducing the benchmark solutions closely. This inference is further confirmed from the results shown in Figures 3 and 4 depicting the variability of the three measures of the performance criteria, *viz.*, the *VAREX*, $q_{per}(in\%)$, and $t_{per}(in\%)$ estimated for the two sets of 3360 runs, each made for the considered rectangular and trapezoidal channel reaches, against the diffusivity measure $[(1/S_0)\partial y/\partial x]_{\max}$ estimated for each run. It is seen from Figures 3a and 4a that the variance explained for almost all the runs made in these channels is greater than 98.5 % with few runs deviating away from this general trend when $[(1/S_0)\partial y/\partial x]_{\max}$ is nearer to unity, i.e., when $[(1/S_0)\partial y/\partial x]_{\max}$ is maximum. There are about ten runs amongst the 3360 runs made for each of the considered rectangular and trapezoidal channels that could not successfully compute the benchmark solutions due to numerical problems, and they have not been considered in Figures 3 and 4. Figures 3b and 4b bring out the variability of the measures of the errors of peak discharge estimates within the range of ±5% for almost all the 3360 runs in each of these channels. It is seen that there exists a trend in the variability of this error with positive values associated with lower order diffusivity and the negative values associated with higher order diffusivity. Figures 3c and 4c depicts the measure of errors in the time to peak which almost lie within the band of ±2.5% and these errors are almost evenly distributed about the zero error line. Based on the inferences made from this study, one may consider that the proposed multilinear diffusion analogy routing method may be used as the improved basic model in flood forecasting models. The proposed model structure maintains the simplicity of the linear routing scheme which is desirable for adopting in a flood forecasting scheme and, yet, is capable of accounting the nonlinearity of the routing process in an efficient manner.

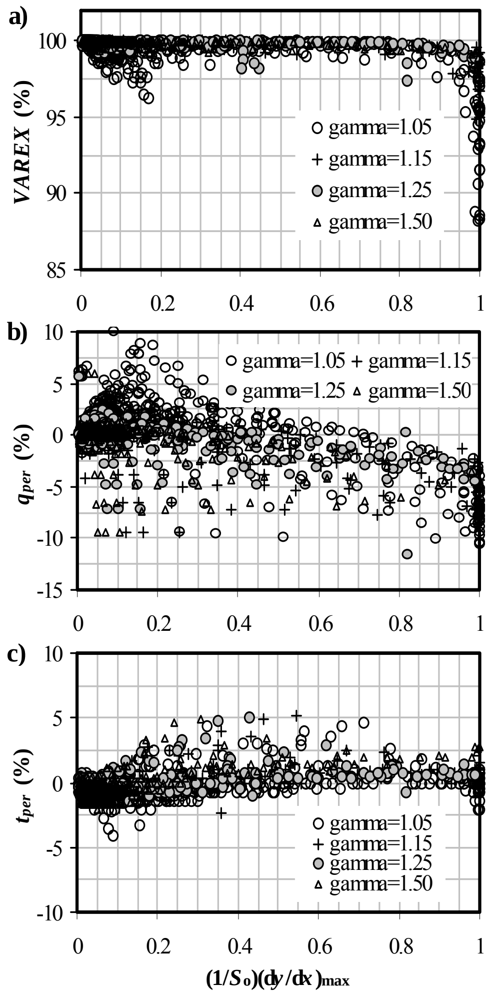

Figure 3. Error estimates of the multilinear diffusion model in comparison with the Saint-Venant's solutions for rectangular channel.

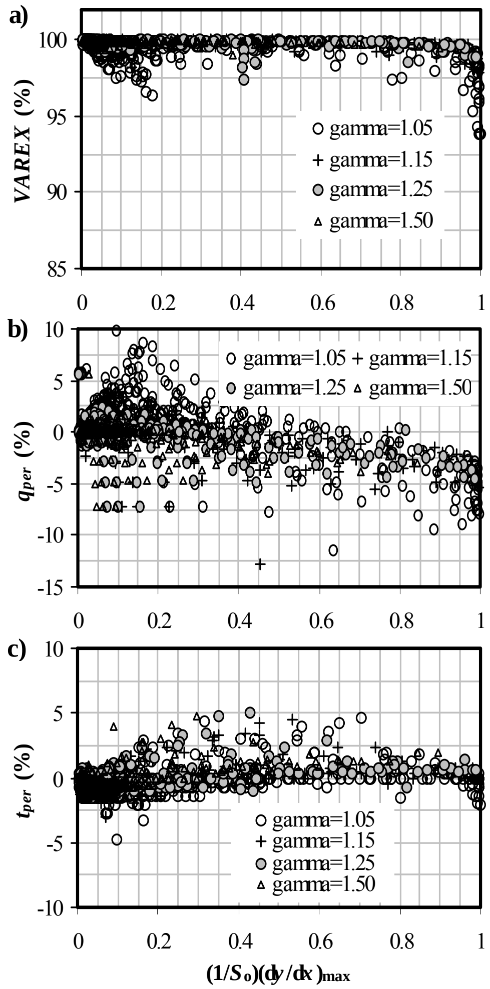

Figure 4. Error estimates of the multilinear diffusion model in comparison with the Saint-Venant's solutions for trapezoidal channel.

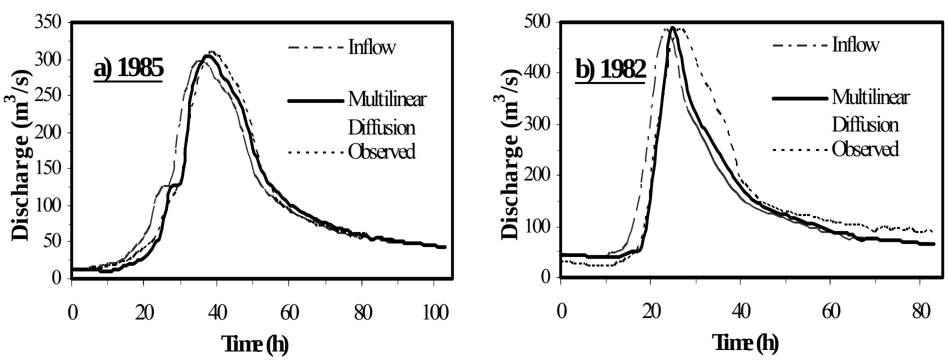

Figure 5. Typical flood routing simulations using the proposed method (Between Ponte Felcino and Torgiano sections of the Tiber River in Central Italy).

It is inferred from the simulation of the five flood events in the reach between the Ponte Felcino and Torgiano sections of the Tiber River in Central Italy that three events (1992, 1985 and 1986) with negligible lateral flow could be reproduced very closely with variance explained criteria $> 97.8\%$. The other two events (1991 and 1982) could be reproduced in best possible manner with variance explained being 94.0% and 92.0%, respectively and these events were subjected to -5.0% and 12.0% lateral flow. Two of the typical reproductions of the field events are shown in Figure 5. It may be seen that the method was able to reproduce the rising limb of the observed hydrograph of the 1982 event quite closely, although there was significant lateral flow contribution within the reach.

CONCLUSIONS

The study demonstrates that the proposed multilinear routing method using the diffusion analogy response function as the sub-model is capable of simulating the benchmark solutions (i.e., the Saint-Venant solutions) closely for a wide range of diffusivity exhibited in the routing process. The method was also tested for simulating few field events of the Tiber River in the reach between Ponte Felcino and Torgiano, and the results obtained indicates the suitability of the method for field application. This method maintains the simplicity of the linear routing scheme which is desirable for adoption in a flood forecasting algorithm and, yet is capable of accounting the nonlinearity of the routing process in an efficient manner. A further investigation of this method is necessary for studying a number of field events for assessing the suitability of this method before recommending it for field application.

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