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GENERALIZED RANDOM WALKS, THEIR TREES, AND THE TRANSFORMATION METHOD OF OPTION PRICING

by

Thomas G. Stewart

A thesis submitted to the faculty of

Brigham Young University

in partial fulfillment of the requirements for the degree of

Master of Science

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GRADUATE COMMITTEE APPROVAL

of a thesis submitted by

Thomas G. Stewart

This thesis has been read by each member of the following graduate committee and by majority vote has been found to be satisfactory.

Date	H. Dennis Tolley, Chair
Date	Jeffrey Humpherys
Date	Scott D. Grimshaw

BRIGHAM YOUNG UNIVERSITY

As chair of the candidate's graduate committee, I have read the thesis of Thomas G. Stewart in its final form and have found that (1) its format, citations, and bibliographical style are consistent and acceptable and fulfill university and department style requirements; (2) its illustrative materials including figures, tables, and charts are in place; and (3) the final manuscript is satisfactory to the graduate committee and is ready for submission to the university library.

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ABSTRACT

GENERALIZED RANDOM WALKS, THEIR TREES, AND THE TRANSFORMATION METHOD OF OPTION PRICING

Thomas G. Stewart

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Master of Science

The random walk is a powerful model. Chemistry, Physics, and Finance are just a few of the disciplines that model with the random walk. It is clear from its varied uses that despite its simplicity, the simple random walk it very flexible. There is one major drawback, however, to the simple random walk and the geometric random walk. The limiting distribution is either normal, lognormal, or a levy process with infinite variance. This thesis introduces an new random walk aimed at overcoming this drawback. Because the simple random walk and the geometric random walk are special cases of the proposed walk, it is called a generalized random walk.

Several properties of the generalized random walk are considered. First, the limiting distribution of the generalized random walk is shown to include a large class of distributions. Second and in conjunction with the first, the generalized random walk is compared to the geometric random walk. It is shown that when parametrized properly, the generalized random walk does converge to the lognormal distribution. Third, and perhaps most interesting, is one of the limiting properties of the generalized random walk. In the limit, generalized random walks are closely connected with a u

function. The u function is the key link between generalized random walks and its difference equation. Last, we apply the generalized random walk to option pricing.

ACKNOWLEDGEMENTS

It is the early afternoon of 13 August 2008, and tomorrow I will presumably start a new phase of my life. I am headed to North Carolina to continue my graduate education. I am excited for this new chapter in my life, and I am grateful for those individuals that have guided me, prayed for me, and laughed with me during the chapter that is ending.

I thank my parents, Monte and Anne Stewart. I thank them for their deep love for me and for the gospel. I thank them for their fearless determination to seek goodness and righteousness, even when it requires personal sacrifice.

I thank Dr. Tolley. This thesis topic is his idea, a lot of his work, and a lot more of his patience. But Dr. Tolley deserves more than just thanks for being my adviser, he deserves thanks for being my mentor in the fullest sense of that word. Permit me to recount my very first encounter with Dr. Tolley and one of my most recent. I met Dr. Tolley went I was sixteen. My family moved into a home a few doors down from his. He and his wife visited, and our first conversation went something like this: "What grade are you in?" Tenth. "What math class are you in?" Calculus. "Do you know what the derivative of x^2 is?". The second and more recent encounter was at BYU in my graduate office. We sat together reading from the Book of Mormon, and we discussed *the happiest thought of the gospel.* I retell these two stories because they exemplify three of Dr. Tolley's greatest traits: a love of learning, a love of the gospel of Jesus Christ, and a love of Zion's students. Indeed, they are traits that I have come to admire and seek for myself.

CONTENTS

TABLES

Table

FIGURES

Figure

1. INTRODUCTION TO THE TRANSFORMATION METHOD OF OPTION PRICING

The Black-Scholes-Merton model of option pricing is plagued with one major drawback: the assumption that stock returns are normally distributed is too restrictive. While this assumption may be approximately true, it is poor for an industry investing over a trillion dollars annually. Indeed, the development of alternate methods—jump-diffusion models, stochastic volatility rate models, Monte Carlo simulation models, and finite difference models—indirectly suggests that practitioners are grateful for the pioneering work of Black, Scholes, and Merton, but not completely satisfied. Practitioners have stated their dissatisfaction in more direct terms. Consider these charges:

> There is considerable time series evidence against the hypothesis that log-differenced asset prices are normally distributed at short time scales. Consequently, the main assumption of geometric Brownian motion fails for short timescales, so that the Black and Scholes equation loses its validity (Schulz 2003).

> Despite its success, the Black-Scholes formula has become increasingly unreliable over time in the very markets where one would expect it to be most accurate (Rubinstein 1994).

Despite the model's shortcomings, it is valuable to remember that the Black-Scholes-Merton model does exactly what it claims to do: it *fairly* prices options where the underlying asset price follows the prescribed behavior. (We discuss what *fair* means in Chapter 3.)

The Transformation method of option pricing is simply what it claims to be; it is a method of option pricing. It is based on the pricing methods of Ross, Cox, and Rubenstien's Binomial asset pricing model (Cox et al. 1979). The significant advantage of the Transformation method is that the distribution of the underlying asset price is generated from what will be defined as a *generalized random walk*. Where the limiting distribution of asset prices in the binomial model is fixed at lognormal, the distribution of asset prices generated from a generalized random walk is flexible. So long as the user specifies a quantile function, the generalized random walk coupled with standard binomial tree pricing methods will price an option. The specifics of this process are described in Chapter 8.

Before explaining the details of the Transformation method in Chapter 8, Chapters 2 and 3 deal with the preliminaries of options and options valuation. Chapter 5 is an overview of the Binomial and Black-Scholes-Merton models. Chapter 6 outlines how these models have been expanded. The focus of the chapter is to outline the specific extensions that address the distributional assumption of stock price returns. Chapter 11 explores the possibility that the Binomial model is a special case of the Transformation method. The last chapter offers examples of options priced with the Transformation method.

2. BACKGROUND ON OPTIONS AND OTHER DERIVATIVE SECURITIES

The key to understanding options and other derivative securities is to understand their purpose. Options are a means of managing risks. Consider this example of a cattle rancher.

Calf production consists of grazing lands, mama cows, a few bulls, horses, a skilled cowboy or cowgirl, a white pickup truck, and other equipment. The calves born in spring and summer are sold in the fall.

Consider what the cattle rancher does not know each winter when beginning the calf production cycle anew: the future price of calves, the amount of rain on the grazing lands, the prevalence of forest fires in the summer, etc. There are a whole host of unknowns that affect the cattle rancher's success.

Now suppose a speculator offers to buy the calves for \$M each in the fall. If the fall price of calves is greater than \$M, then the speculator makes a profit. However, if the price of calves is lower than \$M, then the speculator loses money. The rancher, on the other hand, can lock in the price. The rancher has transferred one production risk to a willing recipient, the speculator.

The particular contract between the rancher and speculator is called a forward contract. Forward contracts are part of a larger class of financial instruments (contracts) called derivative securities. The value of these financial instruments is based on the value of an underlying asset. In the cattle ranching example, the value of the forward contract is based on the price of calves.

Options are a specific type of derivative security. Unlike the forward contract described above, an option gives the holder the opportunity but not the obligation to buy or sell the underlying asset. In the cattle example, the investor could have said: "Give me \$Z now, and if you want, you can sell your calves to me for \$M each in the fall." Or the investor may have proposed: "I'll give you \$Z now, and if I want, you will sell your calves to me for \$M each." The first proposal is an example of a put option, and the second one is an example of a call option. Call options and put options are the two basic option types. Note that the contract holder has the option to buy or sell the underlying asset—hence the name *option*.

Not only are options derived from commodities (like cattle), but they are also built on stocks, bonds, indexes, and even futures contracts.

As a risk management tool, options are increasingly prevalent. The largest U.S. options exchange is the Chicago Board Options Exchange (CBOE) (Hull 2006) which reported over \$600 billion volume for 2007 (CBOE 2008a). The 2007 volume is a marked increase over the \$312 billion volume in 2006 (CBOE 2007). Astoundingly, CBOE's volume only represents a third of all options traded. The entire industry reports 2.8 billion contracts in 2007 (CBOE 2008b).

Given the number of options contracts and their value, it is easy to understand why considerable effort is dedicated to properly pricing an options contract. It is also understandable why this effort reveals itself in the form of option pricing models; a usable and generalizable pricing method must—in light of reality's complexity—simplify reality with easy-to-use approximations. Indeed, a pricing method built on easy-to-use assumptions is a pricing model.

Option pricing models are built on three assumptions, and these three assumptions will provide (a) framework to organize a discussion of current option pricing models and (b) context to understand the Transformation method, the proposed model. In the next chapter, each of these assumptions is discussed. Chapter 5 contains an explanation of the Black-Scholes-Merton model and the Binomial model. Chapter 6 continues with a brief discussion of other option pricing models. Lastly, Chapter 8 describes the proposed model.

3. BACKGROUND ON OPTION PRICING

There are considerable financial incentives to properly pricing options contracts. The starting point, therefore, of any pricing model is a clear definition of *proper* pricing. If considered in the context of the cattle rancher's example, a proper price may be any price the rancher is willing to pay and the speculator is willing to accept for the contract. Or a proper price may be considered in terms of the speculator's risk. The point is, linked to any definition of *proper* pricing is an assumption of how the market should value risk.

The value of risky assets is not only tied to the market's risk premium, it is also tied to the market's valuation of riskless assets. In one sense, a risk premium is only the relative value of a risky asset compared to a nonrisky asset. Because of this, changes of the interest rate over time will also affect the value of risky assets.

The value of options contracts, as noted before, rests on the value of some underlying asset. The character of the underlying asset's price naturally comes into play when pricing options.

A model of option pricing must address these three ideas, which we can summarize with three questions.

A1: How does the market value risk? (3.1)

A2: How does the interest rate change with time? (3.2)

A3: How does the price of the underlying asset change with time? (3.3)

Models of option price are distinguished on the basis of Assumptions 2 and 3. For example, the Black-Scholes-Merton model assumes a constant interest rate with a geometric Brownian motion price path. Another option price model, the stochastic interest rate model, calls for a nonconstant interest rate. It is natural that these two assumptions dominate much of option pricing's discussion.

Assumption 1, on the other hand, generally enjoys some level of consensus among option pricing models. How does the market value risk? In light of the cattle ranching example, it may be tempting to consider risk and its value in terms of the risk preferences of the buyer and seller. Considering these preferences has been successful in other asset pricing models, most notably the Capital asset pricing model. However, option pricing models attack risk valuation indirectly. Assumption 1 is often discussed in terms of efficient markets, expectation, or arbitrage. There are differences between the three ideas; the efficient market idea assumes more than the other two. The efficient market assumption goes something like this:

$$\textit{All information is contained within the market's prices.} \tag{3.4}$$

At first blush, it may not appear that the efficient market hypothesis even addresses the question of risk's value. Yet the efficient market hypothesis says something about arbitrage, and arbitrage says something about the price of risk. Because markets are efficient, the items in those markets are properly priced. If by chance an item were overpriced or underpriced, traders would immediately detect the misprice and trade to profit from it. The opportunities to profit from mispriced items are referred to as arbitrage opportunities.

The concept of arbitrage is not new. It is closely related to the concept of equilibrium and has been discussed in that context since the early era of formal economics. The earliest use of the concept grows from currency exchanges. If the exchange rate between country A and country B is known and if the exchange rate between country A and country C is known, then arbitration of exchange—in its 1811 sense (see OED)—is to determine the exchange rate between country B and country C. While he never uses the words arbitrage or arbitration of exchange, arbitrage is exactly what the early economist Augustin Cournot explains in Cournot (1838), where he derives what he calls the "equations of exchange." That is, he finds the exchange rates between an arbitrary number of countries. Central to his solution is the idea that ignoring transaction costs, the rate from A to B must be equal to the essential rate from A to C to B. If such conditions do not hold, an individual could without risk earn profit by exchanging currency. Which is why, perhaps understating the concept's importance, Cournot wrote: "[I]f this relation temporarily ceases to be satisfied, banking transactions continually tend to reestablish it" (p 32). Further, Cournot emphasizes the equilibrium concept by suggesting there is a "state of equilibrium" when the condition is met.

While the concept obviously existed in Cournot's time, the word arbitrage apparently was not used until 37 years after he published his book. In its earliest sense, the word refers to moments when the exchange rates are not in equilibrium. As noted above, there is an opportunity to earn a riskless profit at these moments. Such an opportunity is called arbitrage. The present-day meaning is not far from its original usage. Today arbitrage is any trading opportunity—not just of currency—that may earn a profit with zero risk.

A closely related concept to arbitrage is the law of one price or the single-price law. It states: Identical products must be priced identically in perfectly competitive markets. In a perfect market, suppose two identical products are priced unequally. There is sure profit to be made by buying the less expensive product and immediately selling it at the higher price. From this example, one may conclude that in perfect markets

A violation of the single-price law implies that arbitrage opportunities exist. (3.5)

It turns out that the connection between the two concepts is even stronger than this statement. We discuss this idea later.

The academic development of the single-price law dates back beyond the word arbitrage. In fact, it seems to appear first in 1657 in the context of gambling (Rubinstein 2005). Yet, given the single-price law's close connection to arbitrage, it is safe to assume that the idea of arbitrage existed earlier than the seventeenth-century foreign currency markets.

The ideas of arbitrage and single-price play a central role in modern finance. In fact, they are components of the fundamental theorem of financial economics (see Rubinstein (2005)).

In practice, these ideas are applied to find equilibrium prices. Recall that the absence of arbitrage implies a state of equilibrium. Also note that (3.5) can be restated as follows:

If arbitrage opportunities do not exist then the single-price law holds.

For finance in general and option pricing in specific, this means that invoking the no-arbitrage condition also invokes the law of one price. Thus, two portfolios that return identical profits and losses in every possible outcome must also have identical values. Note that the portfolios need not be the same in composition. For example, one portfolio may be stock of company A and the other may be stock of company B. So long as the possible future returns are matched, the single-price law holds. A concrete example of pricing with the law of one price is given in chapter 5.

To restate, Assumption 1 deals with the market's value of risk. If Assumption 1 takes on the specific form of the efficient market hypothesis, then markets are also assumed to be arbitrage free. Arbitrage free markets obey the single-price law. With regards to Assumption 1 and risk, this means that identical risks are priced identically. The efficient market hypothesis does not state the risk's value in absolute terms, but rather in relative terms. This may not seem like a very strong statement, and in truth, it is not. However, it is possible to assume even less and arrive at the same conclusion. For example, an option pricing model may simply start with the arbitrage assumption and the law of one price.

There are some consequences for choosing the efficient market hypothesis over the no-arbitrage hypothesis, but they are minor and outside the scope of this paper.

The single-price law—despite its widespread and uncontroversial use—is an integral part of each pricing model. The Chapter 5 demonstrates how the single-price law motivates the pricing model's algorithm.

Chapter 6 offers detailed lists of several proffered assumptions. However, it is the random walk that is the foundation of nearly every model's assumption of stock price behavior. The next chapter discusses the simple random walk and the geometric random walk. In Chapter 5, the specifics of the Binomial model's random walk are discussed. While the models listed in Chapter 6 represent four decades of intense interest and improvement, it is the Binomial model and the Black-Scholes-Merton model that have shaped option pricing's discussion and elucidated the cleverness of the no-arbitrage method. Indeed, no-arbitrage provides a mechanism to avoid the complexity of buyer and seller preferences.

4.1 The Simple Random Walk

The simple random walk is easily illustrated with a coin and a character. Let the character stand on the equator at the prime meridian. In each time period, the individual flips the coin. If it is tails, he takes one step west towards Brazil in South America. If it is heads, he takes one step east towards Gabon of Africa. For example, the chap may flip the series H, H, T, H, H; in which case the character's path is two steps to Gabon, one to Brazil, and two more towards Gabon. In this case, at the end

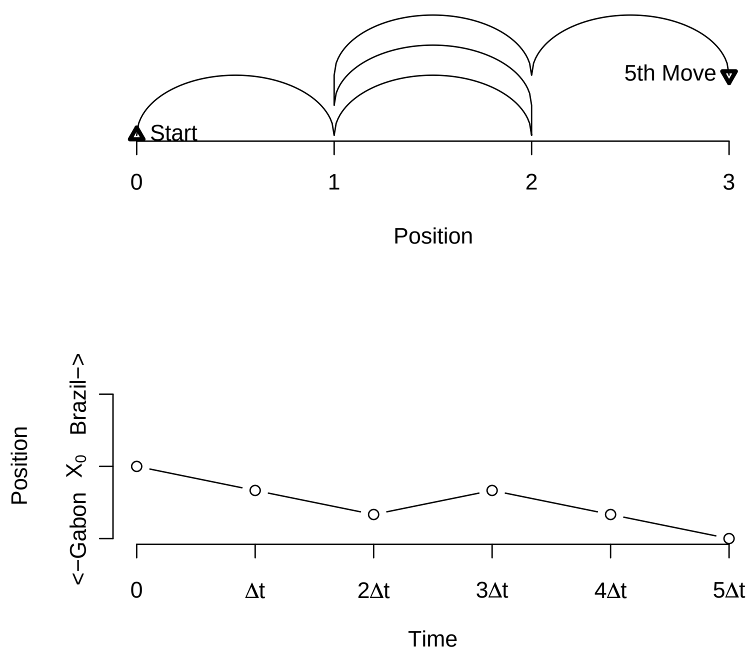


Figure 4.1: A Random Walk on the Equator.

of the five tosses, the individual is three steps closer to Gabon.

The notation for the simple binomial random walk is straight forward, and it is

Z_i	The random variable that dictates a left or right movement. Specifically, this is a Bernoulli random variable with $P(Z_i = 1) = p$. In the example, the coin flip outcome was used to determine Z_i on the i^{th} toss.
Δt	The length of each time interval. This is the time between successive coin tosses.
Δx	The size of each change of position.
n	The index of the time interval.
X_0	The starting position of the random walk.
X_n	The position of the random walk at the end of interval n.

Table 4.1: Simple Random Walk Notation.

the notation used through out the remainder of this document. See Table 4.1. The last variable, X_n, is the variable of interest. With the notation established above, consider the following definition and mathematical expression.

Definition: A simple random walk is a process that evolves over time, and its value is expressed as

$$X_n = X_0 + \Delta x \sum_{i=1}^{n} (2Z_i - 1).$$

Figure 4.1 plots the character's path over time. A binomial tree is a similar plot. Rather than plotting only the position of one path, a binomial tree plots all possible positions at each time point. In the simple binomial random walk, a binomial tree looks like Figure 4.2.

The application areas of the simple random walk are numerous. In the simple example above, the character may have been a person, or it may have been a particle of dust. With some modification, the random walk can also be used to describe how asset prices change over time. This modification of the simple random walk is the geometric random walk.

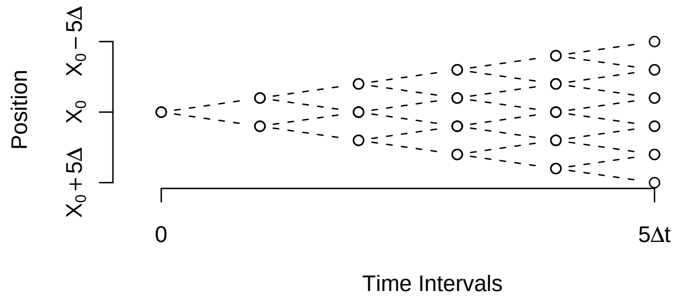


Figure 4.2: The Binomial Tree of a Simple Random Walk.

4.2 The Geometric Random Walk

Let X_n^G be the position of a character (a stock price) after n time intervals each of length Δt. The superscript G denotes a geometric random walk. Suppose that in each time period that the position of the character changes by a percentage amount—not an additive amount. Let $u - 1 > 0$ be the percentage change if the character moves up. And let u^{-1} be the proportional decreasee if the character moves down. Note that $u^{-1} < 1$ and this indicates a move down.

> **Definition:** A geometric random walk is a process that evolves over time, and its value is expressed as
>
> $$X_n^G = X_0^G u^{\sum_{i=1}^{n}(2Z_i - 1)}$$

There is a connection between a simple random walk and a geometric random walk. Consider the log of a geometric random walk compared to the simple random walk:

$$\log(X_n^G) = \log(X_0^G) + \log(u)\sum_{i=1}^{n}(2Z_i - 1)$$

$$X_n = X_0 + \Delta x \sum_{i=1}^{n}(2Z_i - 1).$$

It is clear that if X_n^G is a geometric random walk, then $\log(X_n^G)$ is a simple random walk with $\Delta x = \log(u)$ and $\log(X_0^G) = X_0$. This relationship is also demonstrated in a geometric binomial tree. See Figure 4.3. Note how the shape of the tree is different

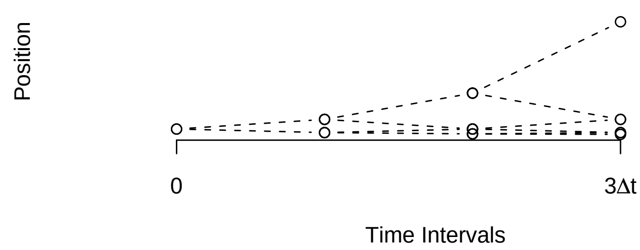


Figure 4.3: The Binomial Tree of a Geometric Random Walk.

from the simple binomial tree.

4.3 The Simple and Geometric Random Walk as a Difference Equation

The definitions for the simple and geometric random walks express the value of the process in absolute terms, i.e., the value of the process in this time interval is X_n. There are other frameworks for expressing a random walk. One such framework is the difference equation. The difference equation and differential equation framework approach random walks by considering the evolution of the process in one time interval. Given the nature of random walks, difference equations are a natural way of exploring the process's properties.

The difference equation that describes a simple random walk is

$$X_n = \Delta x(2Z_n - 2) + X_{n-1}.$$

Likewise, for geometric random walks, the equations is

$$X_n^G = u^{2Z_n - 1} X_{n-1}^G.$$

4.4 The Limit of the Simple and Geometric Random Walk

The applications of the simple and geometric random walk often times calls for the limit of the random walk. The limit of the random walk involves considering X_n as

$$\Delta t \to 0, \text{ or } \Delta x \to 0, \text{ or } n \to \infty$$

or considering some combination of the three. More importantly, limits can refer to the entire process or simply at one time point. That is, a process limit refers to the joint distribution of the random walk at a set of k time points,

$$\{t_1, t_2, \ldots, t_k\},$$

whereas, a point limit only considers the distribution of the random walk at one time point, say t_j. Lastly, a terminal limit considers the distribution of X_n as n approaches infinity without regard for any particular time point. Note that a process limit encapsulates both point limits and terminal limits.

The best known process limit, Brownian Motion, considers all three variables Δt, Δx, and n. With constants T and c, Brownian motion can be thought of as the limit of the simple random walk when

$$\frac{T}{\Delta t} = n \text{ and } \Delta x = c\sqrt{\Delta t}$$

as Δt approaches zero. Figure 4.4 is an example of a simple random walk under these conditions when Δt is small.

The limit of a geometric random walk is closely related to the limit of the simple random walk. Because $\log(X_n^G)$ is a simple random walk, its limit under the same conditions as above is Brownian motion. Specifically, if B_T denotes Brownian motion, then

$$X_T^G = e^{B_T}.$$

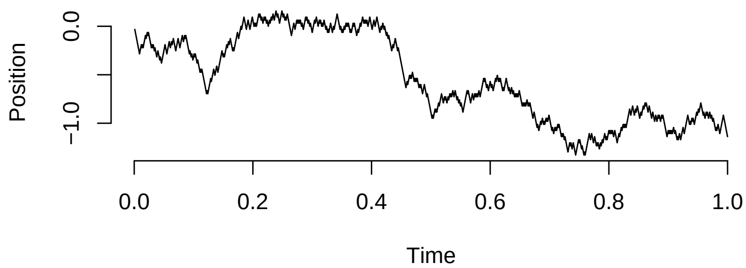


Figure 4.4: A Simple Random Walk *Close* to Brownian Motion.

Chapters 5 and 6 discuss Brownian motion in models of stock price behavior. Section 5.1 provides a proof, in the context of option, that the terminal limit of a geometric random walk is the lognormal distribution.

The basics of the Binomial asset pricing model are accessible in several sources; Shreve (2005) and Hull (2006) are excellent examples. The original source, which is also easily accessible, is Cox et al. (1979). Our framework for this model and every other model is founded on the three assumptions.

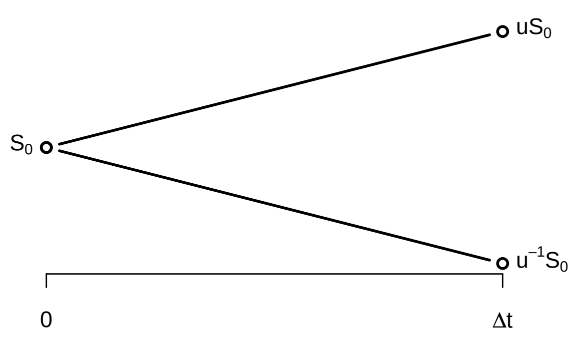
The Binomial Model

A1: No-Arbitrage
A2: Constant Interest Rate, r
A3: If S_t is the price of the underlying asset at time t and if Δt is a time interval, then

$$S_{t+\Delta t} = u^{2Z-1} S_t$$

where $Z \sim BINOMIAL(1, p)$, $p \in (0, 1)$, and $u \in (1, \infty)$.

A brief explanation of Assumption 3 is in order. Assumption 3 indicates that stock prices follow a geometric random walk. The notation simply reflects the specific application to asset price modeling. Suppose S_0 is the current asset price. After a time interval the asset price moves to uS_0 or $u^{-1}S_0$. The first will occur with probability p, and the latter will occur with probability $1 - p$.



In each interval the price changes by the same process. It moves up by a factor u or down by a factor u^{-1}. Consider a price path of five intervals. Figure 5.1 is an example. The dotted lines indicate all the possible price paths of the asset. The solid line is an example of one price path. As noted before, the graph of all possible price

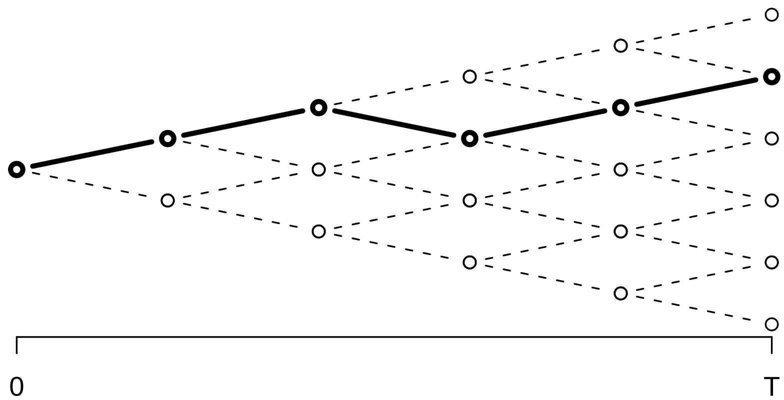


Figure 5.1: A Five Period Binomial Tree

paths is known as a binomial tree. The possible prices at the end of each interval are often called nodes.

Asset price modeling demostrates several different ways one can parametrize a geometric random walk. Consider Δt. As we did in the previous chapter, one can work directly with different time interval lengths. Or one can specify a specific time horizon, say T. In the second case, one also specifies the total number of time intervals as N. It follows, then, that $\Delta t = T/N$. In asset price modeling, this convention is popular, and it is what is used in this paper when discussing random walks applied to asset prices.

It is worth noting that the size of the interval (choosen with T and N), along with u and p, are all parameters chosen by the model user. A large p indicates a price path that generally grows up. A small p has the opposite effect. A large u increases the spread of the tree. The interval size changes the number of price changes. The next page contains examples of price paths from different parameter values. Note that the three parameters offer a large variety of asset price paths.

The purpose of the Binomial model is to properly price an option. The three assumptions come together in a rather simple way to find the option's price. Let V_t

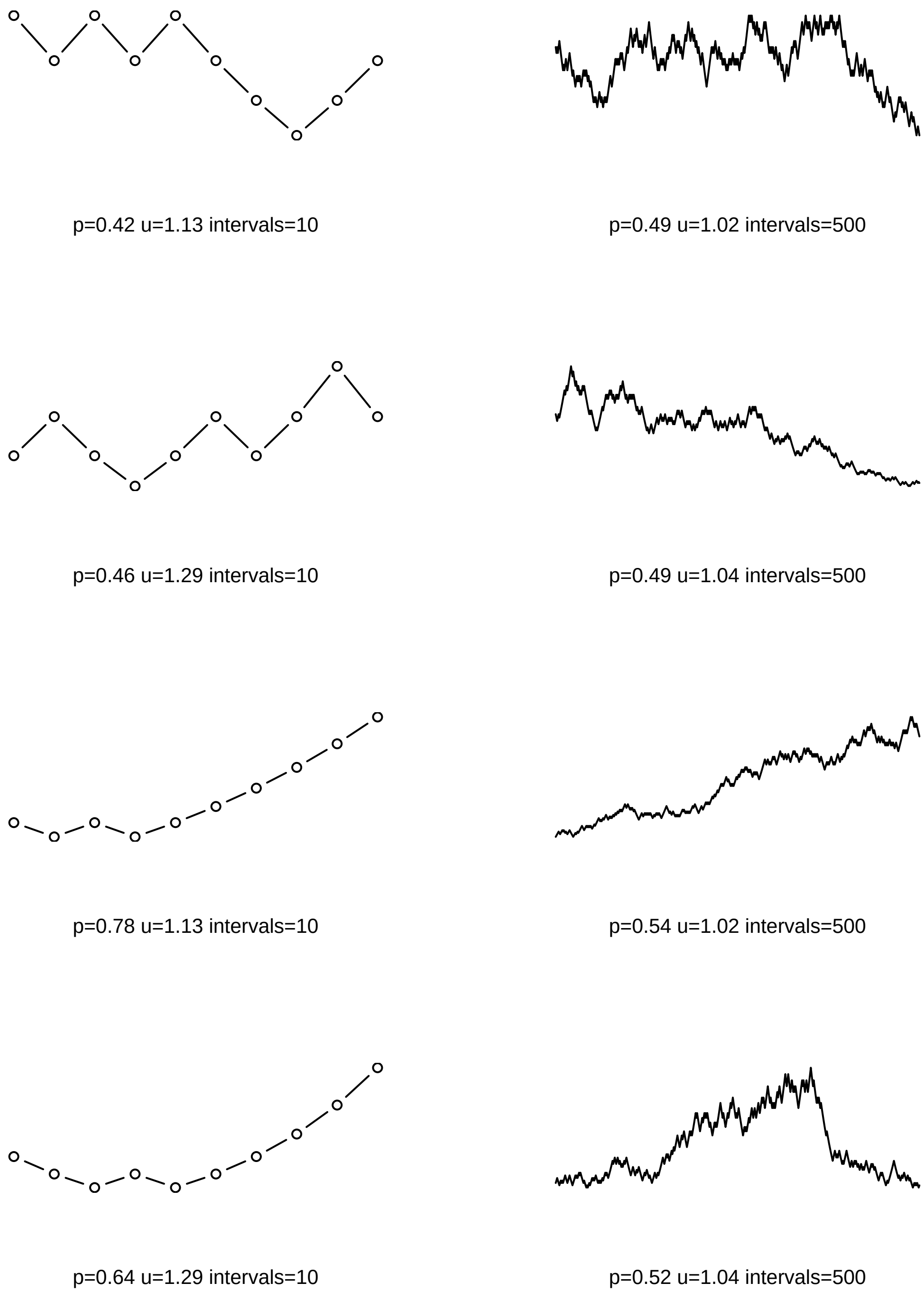


Figure 5.2: Examples of price paths from different parameter values.

be the value of the option at time t. Let K be the strike price and $t = T$ be the maturity date. Recall that the value of the option at maturity, V_T, is either \$$X_T - K$ or \$0. That is,

$$V_T = \max\{X_T - K, 0\}.$$

If $\Delta t = T$—which is the one step case—then

$$S_T = uS_0 \text{ or } S_T = u^{-1}S_0, \text{ and}$$

$$V_T = \max\{uS_0 - K, 0\} \text{ or } V_T = \max\{u^{-1}S_0 - K, 0\}.$$

Both of these outcomes are graphically displayed in Figure 5.4. The (u) and the (u^{-1}) in $V(u)$ and $V(u^{-1})$ simply indicate which outcome occurred.

Recall that the no-arbitrage principle is the motivation for the binomial model's pricing. Suppose there exists a portfolio of assets and cash that mimics the outcomes of the option. That is, in every possible outcome, the portfolio and the option are worth the same amount. If it exists, then no-arbitrage requires that the portfolio and the option are priced identically.

The idea of creating an identical portfolio of cash and assets is called replication. The figures on page 20 illustrate the three markets involved in replication: options, assets, and cash. The options and assets graphs have already been discussed. The cash market reflects Assumption 2, a constant interest rate. The following example illustrates how replication leverages the law of one price. It is important to note that this example also proves the proposition that in the one step binomial model there always exists a portfolio of cash and assets that replicates the option.

Let X_t be the value of the portfolio at time t. Let Δ be the number of assets in the portfolio. The portfolio can be expressed:

$$X_0 = \underbrace{\Delta S_0}_{Asset\ Position} + \underbrace{(X_0 - \Delta S_0)}_{Cash\ Position}.$$

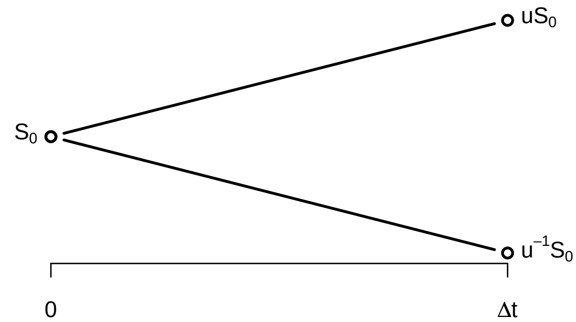


Figure 5.3: One Period Binomial Tree: Possible Asset Prices.

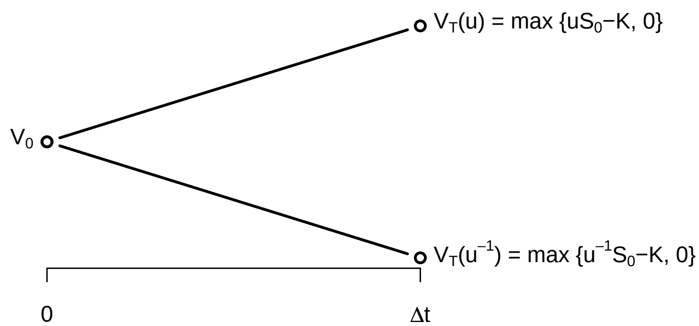


Figure 5.4: Possible Option Values.

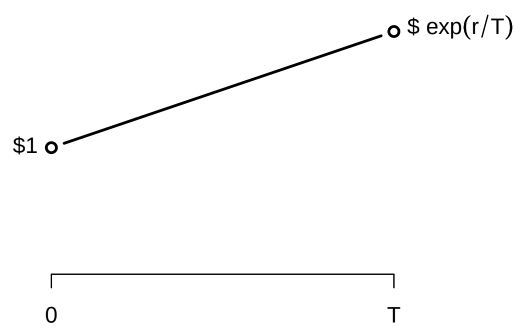


Figure 5.5: Cash Holdings.

At time T, the portfolio has two possible values, either

$$X_T(u) = \Delta u S_0 + (X_0 - \Delta S_0)e^{r/T} \text{ or}$$
$$X_T(u^{-1}) = \Delta u^{-1} S_0 + (X_0 - \Delta S_0)e^{r/T}.$$

The choice of Δ and X_0 is the solution to a simple set of linear equations:

$$\begin{bmatrix} S_0(u - e^{r/T}) & e^{r/T} \\ S_0(d - e^{r/T}) & e^{r/T} \end{bmatrix} \begin{bmatrix} \Delta \\ X_0 \end{bmatrix} = \begin{bmatrix} V_T(u) \\ V_T(u^{-1}) \end{bmatrix}.$$

So long as $u \neq 1$, then a solution exists. It is

$$\Delta = \frac{V_T(u) - V_T(u^{-1})}{S_0(u - u^{-1})} \text{ and}$$
$$X_0 = e^{r/T}\left[\tilde{p}V_T(u) + (1-\tilde{p})V_T(u^{-1})\right] \quad \text{where} \quad \tilde{p} = \frac{e^{r/T} - u^{-1}}{u - u^{-1}}.$$

Because the outcomes of the portfolio are identical to the outcomes of the option, the no-arbitrage criterion requires that the value of the option be the value of the portfolio, $X_0 = V_0$.

The one step model is an obvious oversimplification because asset prices can assume a host of values. However, replication can be applied to multistep models; and as figure 5.2 illustrates, multistep binomial price paths can look remarkably similar to asset price paths seen in stock exchanges.

The Multistep Binomial Model

(1) Choose values for r, u, p, and Δt. These values determine a binomial tree. Figure 5.6 is an example with five intervals.
(2) Note the last interval (with bold solid lines) is five one step trees. Using the same argument and equations as the one-step model, one can replicate each one step tree and solve for the nodes of Interval 4.
(3) With the nodes solved at Interval 4, it is possible to solve for the nodes at Interval 3. This is the layer with bold dotted lines.

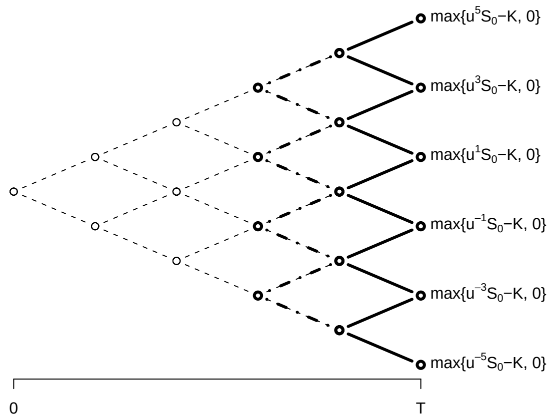


Figure 5.6: Five Step Tree.

(4) The replication process continues (backwards) from interval to interval until the last interval is solved.

The great flexibility and simplicity of the Multiperiod Binomial model is one of the reasons it is widely used. It turns out the binomial tree allows for alterations that are not tractable in the Black-Scholes-Merton model.

Because later chapters will build on this idea, it is worth emphazing the process of the Binomial model. First, a specific type of random walk is assumed. The type of random walk choosen implies a specific binomial tree. And with the binomial tree, the replication process prices the option. As will be shown, the Transformation method is simply the same three steps but with a different type of random walk.

Step (1) of the Multistep Binomial model requires the user to choose u, p, r and Δt. In Figure 5.2, the effect of these parameters is illustrated, and a suitable method of choosing the parameters is possibly guess-and-check. Another method for choosing the parameters is to consider the limit of the Binomial model as Δt approaches 0. The next section shows that if u and p are chosen in tandem, the limiting distribution can be the well-known lognormal. Thus, it is possible to choose u and p that correspond to a specific mean and variance with a limiting lognormal distribution.

5.1 The Limit of the Binomial Model

Let $T/\Delta t = R$, the number of intervals, and let Z_R be the number of intervals in which the stock price moves up. Then $R - Z_R$ is the number of intervals in which the stock price moves down. Because movements in each interval are independent of movements in the others, and because the probability of an upward shift remains constant, Z_R is a binomial random variable. That is, $Z_R \sim BINOMIAL(R, p)$. It follows then, that

$$S_T = S_0 u^Z u^{-(R-Z_R)} = S_0 u^{2Z_R - R}$$
$$\log(S_T) = \log(S_0) + (2Z - R)\log(u).$$

It is for this reason that the Binomial model gets its name; the logged asset price is a linear function of a binomial random variable. Because $\log(S_T)$ is a linear function of the binomial random variable, Z_R, we can directly apply the central limit theorem to find the asymptotic distribution. Note that

$$\begin{aligned} E\left[\log(S_T)\right] &= \log(S_0) + (2E[Z_R] - R)\log(u) \\ &= \log(S_0) + (2p-1)R\log(u) \text{ and} \end{aligned}$$

$$\begin{aligned} V\left[\log(S_T)\right] &= 4\log^2(u)V[Z] \\ &= 4\log^2(u)Rp(1-p). \end{aligned}$$

The central limit theorem applied to $\log(S_T)$ justifies the following statement:

$$\frac{\log(S_T) - \log(S_0) - (2p-1)\log(u)R}{2\log(u)\sqrt{p(1-p)R}} \xrightarrow{d} W \sim N(0,1) \text{ as } R \to \infty. \qquad (5.1)$$

The relationships between the expected value, the variance, u, p, and R are important.

Note that if the limit of $E[S_T]$ and $V[S_T]$ are going to be finite, then u and p must be chosen—together—to make this happen. The choice of u and p suggested by Cox et al. (1979) does produce a finite mean and variance, and it is parametrized so that $E[S_T] \to \log(S_0) + \mu T$ and $V[S_T] \to \sigma^2 T$. Their choice is

$$u = \exp\{\sigma\sqrt{T/R}\} \text{ and} \tag{5.2}$$

$$p = \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{T/R}. \tag{5.3}$$

Equation (5.1) parametrized with this choice of u and p,

$$\frac{\log(S_T) - \log(S_0) - (2p-1)\log(u)R}{2\log(u)\sqrt{p(1-p)R}} = \frac{\log(S_T) - \log(S_0) - \mu T}{\sqrt{\sigma^2 T - \frac{\mu^2 T^2}{\sigma^2 R}}},$$

and evaluated as R approaches infinity,

$$\lim_{R\to\infty} \frac{\log(S_T) - \log(S_0) - \mu T}{\sqrt{\sigma^2 T - \frac{\mu^2 T^2}{\sigma^2 R}}} = \frac{\log(S_T) - \log(S_0) - \mu T}{\sqrt{\sigma^2 T}},$$

justifies the statement that $\log(S_T)$ is asymptotically normal with

$$E[\log(S_T)] = \log(S_0) + \mu T \text{ and } V[\log(S_T)] = \sigma^2 T.$$

There is a strong tie between the Binomial model and the Black-Scholes-Merton model. Consider the price process of the binomial model,

$$S_{t+\Delta t} = u_{\Delta t}^{2Z_{\Delta t}-1} S_t,$$

where p and u are the functions of Δt suggested in (5.2). Consider the limit of the logged price process as Δt approaches 0. The strong connection between the models is this: the limit of the binomial model as Δt approaches 0 when p and u satisfy (5.2) is the Black-Scholes-Merton model.

5.2 The Black-Scholes-Merton model

The Black-Scholes-Merton model is the fundamental model of option pricing. It is the patriarch of a family of option pricing formulas that numbers in the hundreds.

The Black-Scholes-Merton Model

A1: No-Arbitrage
A2: Constant Interest Rate, r
A3: If S is the asset price, then the instantaneous change in S is

$$dS = \mu S\,dt + \sigma S\,dz,$$

where dz is a Wiener process, μ and σ are constants.

Not only has it motivated the creation of several other models, it enjoys widespread use in everyday options markets.

The appeal of the model is its simplicity. The pricing solution is closed-form, and is easily accessible on most computers. As is expected, the inputs are the same as the Binomial model: r, μ, σ, T, K, and S_0. The arbitrage free price when asset prices follow Assumption 3 of the Black-Scholes-Merton model is

$$\text{call price } = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

$$\text{put price } = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

$$\text{where } d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \text{ and}$$

$$d_2 = d_1 - \sigma\sqrt{T}.$$

The solution requires use of stochastic calculus, namely Ito's Lemma. However, the same reasoning that motivates the Binomial model's solution also motivates the Black-Scholes-Merton solution: replication and the law of one price. In essence, the Black-Scholes-Merton solution shows (under Assumption 3) that if instantaneous trading is allowed, then there is a portfolio of asset and cash that mimics the option at every moment.

The effectiveness of the Black-Scholes-Merton model and the Binomial model depend on two broad points:

(1) the approximate truth of the assumptions, especially Assumptions 2 and 3;

(2) the approximate truth of the input parameters: μ, σ, and r.

The next chapter explores the evidence of Assumption 2 and 3's truthfulness. There is evidence that both assumptions should be improved.

6. SINCE THE PIONEERS

Black, Scholes, and Merton were not the first to model asset price behavior with Brownian motion. Bachelier's 1900 dissertation is probably the first paper to do so. It was, however, the work of Osborne in his 1959 paper that marks the start of sustained research of stock price behavior. In the 1959 paper, Osborne suggests geometric Brownian motion as a suitable description of a stock price path. Several papers that followed generally agreed; for example, see Boness (1964). It is not surprising that after a decade of research on the topic, Black, Scholes, and Merton would incorporate geometric Brownian motion as a central assumption of their option pricing model.

Despite what appears to have been a strong consensus in the 1960s, there were some notable exceptions. Mendlebrot in his 1962 paper pointed out that empirical stock price distributions generally have fatter tails and more skew than a Wiener price process should have. Mendlebrot's observations were confirmed by Osborne himself in his 1962 paper. Recent authors have also found the same discrepancy (see Katz and McCormick (2005)).

The fact that geometric Brownian motion is not a perfect description of stock price behavior is not surprising. Model assumptions are only approximately true, and despite their flaws may still be *good enough*. Even before Black and Scholes published their celebrated 1973 paper, the duo published an empirical study to test their model's fit. They found that the model overpriced high variance stocks and underpriced low variance stocks. Further they found "non-stationarity in the variance" (Black and Scholes 1972).

The systematic discrepancy of the Black-Scholes-Merton model has been extensively studied. It dominates current research in option pricing, and it is now widely known as the volatility smile. The name comes from a well replicated graph of implied volatility. Suppose one has model inputs r, μ, T, S_0, K, but not σ. In addition, suppose one also has the option's true value at maturity. It is possible to use the Black-Scholes-Merton formula and answer this question: Given the true value and the above inputs, what does σ need to be so that the Black-Scholes-Merton formula accurately prices the option? The answer to that question is called implied volatility.

Now consider a set of options contracts that are identical in every way except for different strike prices. If Assumption 3 were exactly correct, then the implied volatility of each contract would be the same. In practice, when implied volatility is calculated and plotted against strike price, a smile appears. This is the graph mentioned earlier.

The existence of a volatility smile may not be sufficient evidence to conclude that Assumption 3 of Black-Scholes-Merton is not *true enough*. Yet, recent work by Rubinstein (1994) has shown that over time, the volatility smile has become more pronounced and more reminiscent of a smirk than a soft smile. Rubinstien's concludes: "Despite its success, the Black-Scholes formula has become increasingly unreliable over time in the very markets where one would expect it to be most accurate " (Rubinstein 1994). Indeed, the development of option pricing formulas intended to overcome this problem suggests that traders are less and less convinced that Assumption 3 is *true enough*.

Haug's book entitled *Option Pricing Formulas* lists 129 different entries. Of course, several formulas deal with options outside of the simple vanilla option discussed in this paper. But, there are several formulas that do apply in the simple case. Six broad categories of options models are discussed next.

Similar to the Black-Scholes-Merton model is the Constant Elasticity of Variance model (CEV).

Note the α in the $\sigma S^{\alpha} dz$ expression. When $\alpha = 1$ the CEV model reduces

Constant Elasticity of Variance Model

A3: If S is the asset price, then the instantaneous change in S is

$$dS = (r - q)S\,dt + \sigma S^{\alpha}\,dz,$$

where dz is a Wiener process and r, q, α are constants.

to the Black-Scholes-Merton model. When $\alpha \neq 1$, the volatility component, σS^{α} is affected by the size of S. For $\alpha > 1$ and small S, volatility is greater. For $\alpha > 1$ and large S, the volatility is also larger.

The Binomial, Black-Scholes-Merton, and CEV models assume a stock price that is continuous. Continuity of price, in one sense, removes the possibility of large dramatic price changes. Jump models incorporate the idea of large price shocks.

Jump-Diffusion Model

A3: If S is the asset price, then the instantaneous change in S is

$$dS = (r - q - \lambda k)S\,dt + \sigma S^{\alpha}\,dz + S\,dp,$$

where dz is a Wiener process, dp is a Poisson process, and r, q, λ, k are constants.

The expression of interest is $S\,dp$, where dp is a Poisson process and λ is the expected number of jumps per year. If $\lambda = 3$, then one might expect 3 dramatic price jumps in the years.

The 4 models presented up to this point had one random variable. The Stochastic Volatility model has two: the stock price and the volatility, V.

Stochastic Volatility Model

A3: If S is the asset price, then the instantaneous change in S is

$$dS = (r - q)S\,dt + \sqrt{V}S dz_s$$

$$dV = a(V_l - V)\,dt + \xi V^{\alpha} dz_v,$$

where dz_s and dz_v are a Wiener processes, and r, q, α, a, V_l are constants.

The key point for the Stochastic Volatility Model is that volatility is not constant. Rather volatility fluctuates with time. The $a(V_l - V)$ expression represents the volatility's drift and ξV^{α} represents the impact of randomness. The randomness is encapsulated in Brownian motion, dz_s and dz_v, as it is in the previous 3 models.

Also, notice that the 3 previous models all modify the Black-Scholes-Merton model by tinkering with $\sigma S\,dz$. The Implied Volatility model does the same, but rather than considering the underlying asset, it considers the options themselves.

Implied Volatility Model

A3: If S is the asset price, then the instantaneous change in S is

$$dS = [r(t) - q(t)]S\,dt + \sigma(S,t)S\,dz,$$

where dz is a Wiener process and r, q, and σ are deterministic functions.

The function $\sigma(S,t)$ is based on a training set of matured options. Generally smooth and simple, $\sigma(S,t)$ is chosen so that the pricing method correctly priced the training set.

Other models use a training set to calibrate the model. The Implied Binomial tree uses a training set to generate a binomial tree.

Implied Binomial Tree

A3: Price changes according to a binomial tree generated by a training set of mature options.

A 200 step binomial tree has 60,301 parameters and the algorithm to generate an implied binomial tree involves quadratic minimization and the rather strong "independent path" assumption.

The last model type is the stochastic interest rate model. The heart of this model is not Assumption 3, but it is Assumption 2. This is a general class of models and can be parametrized in several ways.

The key is that the interest rate is not constant. This particular model can be incorporated into the previous models as well. For example, a stochastic interest rate

Stochastic Interest Rate Model

A2: r is generated from a stochastic process. For example,

$$dr = \mu_r + r\,dz,$$

where μ_r is constant and dz is Brownian motion.

jump diffusion model is a possibility.

7. GENERALIZED BINOMIAL RANDOM WALKS AND GENERALIZED BINOMIAL TREES

The simple random walk and the geometric random walk are models with a wide range of applications. Their simplicity is partially responsible for their value and varied use. Not only is their core concept simple, but their use in the limit is also powerfully simple. The fact that both random walks are simple and discrete descriptions of micro-level behavior coupled with the fact that limit of these models converges to simple and continuous models is reason for the random walk's popularity.

But the limiting properties of the simple and geometric random walk is a blessing and a curse. The blessing is this: the simple random walk converges to a Wiener process (Brownian motion) under the right conditions. The curse is this: if it converges, the simple random walk only converges to a Wiener process or a Levy process with infinite variance.

The generalized random walk is motivated by overcoming the curse of the simple random walk. That is, it is a process that converges to some other terminal distribution than the normal.

Some notation in addition to Table 4.1 is found in Table 7.1.

Definition: A generalized random walk is a process with value

$$X_n = F^{-1}_{\mu_n,\sigma_n}\left[\Phi\left(\frac{\sum_{i=1}^n Z_i - np}{\sqrt{np(1-p)}}\right)\right].$$

$F_{\mu,\sigma}$	A cumulative distribution function parametrized by its mean, μ, and standard deviation σ.
Φ	The cumulative distribution function of the standard normal distribution.
μ_n, σ_n	Sequences of means and variances indexed by time interval, n.

Table 7.1: Additional Notation for the Generalized Random Walk.

7.1 Examples of a Generalized Binomial Tree

The shape of the simple and geometric binomial trees were determined by the inputs: Δt, Δx or u. The shape of a generalized binomial tree is determined by F and Δt. Below are two examples of generalized binomial trees. Note the role the choice of F plays. The first plot is the generalized binomial tree where F is the cumulative distribution of a Gamma random variable with mean equal to four and variance equal to eight. The second plot is the generalized binomial tree when F is Pareto with the same mean and variance.

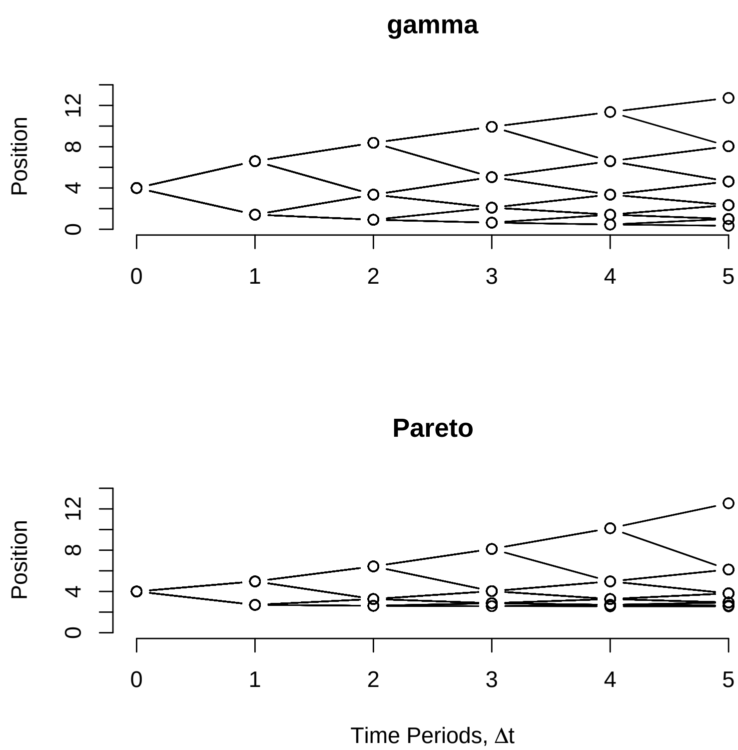


Figure 7.1: Two Examples of a Generalized Binomial Tree

7.2 The Mean and Variance Sequences

In the examples of Figure 7.1, μ_n and σ_n are constant sequences. Constant sequences are the simplest case of the generalized random walk. However, μ_n and σ_n are parameters that allow the user greater flexibility. As long as μ_n and σ_n converge to μ and σ respectively, the user can use μ_n and σ_n to incorporate his or her ideas about the process's changing mean and variance. For example, the sequence

$$\mu_n = \mu^{1-\frac{1}{n}}$$

introduces into the random walk an upward drift. Or, the sequence

$$\sigma_n = \sigma(1 - \frac{1}{n})$$

incorporates an increasing degree of spread.

The point is that the mean and variance sequences provide an added measure of flexibility that the simple random walk and the geometric random walk do not directly offer. The simple and geometric walks are confined to a constant variance, and both only allow indirect mean drift by fiddling with the parameter p of the Bernoulli random variable, Z_i.

7.3 The Limiting Distribution of the Generalized Random Walk

The generalized random walk can converge to a distribution other than the normal. As noted before, there are several ways to think about convergence. In this case, convergence is considered without regard for the length of a time interval, Δt. It is considered simply as n approaches infinity.

Lemma 1: Let X have cumulative distribution function $F_{\mu,\sigma}(x)$, and let F be continuous in x, μ, and σ. Let μ_n and σ_n be sequences that converge to μ and σ respectively as $n \to \infty$. Then, the terminal distribution of the generalized random walk, X_n, converges in distribution to X, $X_n \xrightarrow{d} X$.

Proof. Let F_{μ_n,σ_n} be the cumulative distribution function of X_n. It suffices to show

$$\lim_{n\to\infty} F_{\mu_n,\sigma_n}(x) = F_{\mu,\sigma}(x)$$

for all x. Choose an arbitrary x.
First, note that

$$\begin{aligned}
F_{X_n}(x) &= P(X_n \leq x) \\
&= P\left(F^{-1}_{\mu_n,\sigma_n}\left[\Phi\left(\frac{\sum_{i=1}^n Z_i - np}{\sqrt{np(1-p)}}\right)\right] \leq x\right) \\
&= P\left(\Phi\left(\frac{\sum_{i=1}^n Z_i - np}{\sqrt{np(1-p)}}\right) \leq F_{\mu_n,\sigma_n}(x)\right).
\end{aligned}$$

Denote the centered and scaled binomial random variable,

$$\frac{\sum_{i=1}^n Z_i - np}{\sqrt{np(1-p)}}$$

as W_n. Second, because W_n is a centered and scaled binomial random variable, the Central Limit Theorem implies that W_n converges in distribution to a standard normal random variable, call it W. Because Φ is continous, the sequence $\Phi(W_n)$ converges in distribution to $\Phi(W)$. The function Φ is the standard normal probability function and W is a standard normal random variable, so $\Phi(W)$ is a standard uniform random variable, call it U. The probability function of U is

$$P(U < u) = u \text{ for } 0 < u < 1.$$

To recap, because

$$\Phi\left(W_n\right) \xrightarrow{d} U$$

if follows that

$$\begin{aligned}
\lim_{n\to\infty} F_{\mu_n,\sigma_n}(x) &= \lim_{n\to\infty} P\left(\Phi\left(W_n\right) \leq F_{\mu_n,\sigma_n}(x)\right) \\
&= P\left(U \leq F_{\mu,\sigma}(x)\right) \\
&= F_{\mu,\sigma}(x).
\end{aligned}$$

And $\lim_{n\to\infty} F_{\mu_n,\sigma_n}(x) = F_{\mu,\sigma}(x)$ is the desired result. □

8. THE TRANSFORMATION METHOD OF OPTIONS PRICING

This chapter introduces pricing with the generalized binomial tree. As mentioned before, this will consist of assuming a specific form of a generalized random walk and generating the resulting tree. Let n be the size of the tree. That is, n is the number of steps. Let F be the target distribution—the distribution approximated by the final step of the binomial tree. Let S_0 be the central or starting node of the tree.

8.1 An Example of Creating a Generalized Binomial Tree

In the definition of the generalized random walk, there are sequences μ_n and σ_n to specify the drift and variance of the process. In this example, we use a technique that automatically incorporates μ_n and σ_n.

A tree of size n will have $2n+1$ nodes. However, the last step of the tree only has $n+1$ nodes. The second to last step of the tree has the remaining n nodes. As a matter of notation, let x_i be the nodes of the final step, and let y_i be the nodes of the penultimate step.

The generalized binomial tree is governed by a constant probability of an upward shift. Let this parameter be p. The parameters S_0, F, and n determine p. Specifically, p is chosen so that

$$S_0 = F^{-1}\left[\Phi\left(\frac{n/2 - np}{\sqrt{np(1-p)}}\right)\right]$$

holds when n is even.

The values of the nodes are found in two steps. In step 1, the nodes x_i are found, and in step 2 the nodes y_i are found. That is,

$$x_i = F^{-1}\left[\Phi\left(\frac{i - np}{\sqrt{np(1-p)}}\right)\right] \qquad i = 0, 1, \ldots n$$

$$y_i = \sqrt{x_{i+1}x_i} \qquad i = 0, 1, \ldots, n-1.$$

Let F be the CDF of the exponential distribution with unit mean, $\mu = 1$. Let $T_0 = .25$ and $n = 6$.

We first find p. Note that $F(t) = 1 - e^t$. We need to find p so that

$$\begin{aligned} \Phi^{-1}\left(F(.25)\right) &= \frac{3-6p}{\sqrt{6p(1-p)}} \\ -0.7681 &= \frac{3-6p}{\sqrt{6p(1-p)}}. \end{aligned}$$

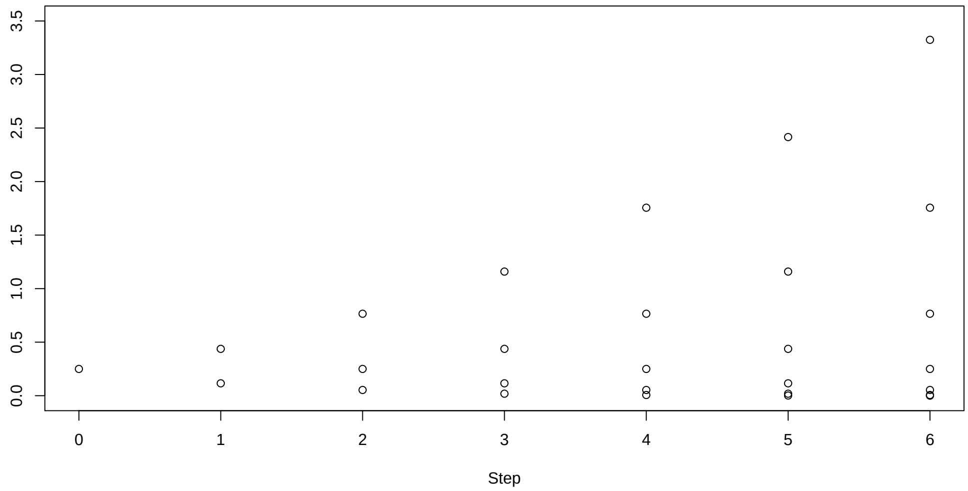
There are several ways of solving for p. We use a newton root finder. In this example, $p = 0.6496$.

Now we find x_0 through x_6. Note $F^{-1}(t) = -\log(1-t)$. Solving for each node gives

$$\begin{aligned} x_0 &= -\log\left(1 - \Phi\left(\frac{0 - 6 * 0.6496}{\sqrt{6(0.6496)(1-0.6496)}}\right)\right) \\ &= -\log\left(1 - 0.000426292\right) = 0.0004263829 \\ x_1 &= 0.0066011832 \\ x_2 &= 0.0536275731 \\ x_3 &= 0.2500368917 \\ x_4 &= 0.7655436808 \\ x_5 &= 1.7558816045 \\ x_6 &= 3.3240391670, \text{ and} \end{aligned}$$

$$\begin{aligned} y_0 &= \sqrt{x_0 x_1} = \sqrt{0.0004263829 \times 0.0066011832} = 0.001677686 \\ y_1 &= 0.018815032 \\ y_2 &= 0.115796683 \\ y_3 &= 0.437509043 \\ y_4 &= 1.159398148 \\ y_5 &= 2.415909606. \end{aligned}$$

Below is the generalized binomial tree.



The transformation method of option pricing is heuristically simple. It offers considerable flexibility to the user, and it has application outside of option pricing. Indeed, option pricing is a simple motivating example to demonstrate the variety of stochastic processes that can be generated by the generalized random walk. From a differential equation standpoint—and outside option pricing—the transformation method creates a process where the random variable evolves but maintains a common distribution indexed by time.

9. THE BINOMIAL MODEL AS A SPECIAL CASE OF THE TRANSFORMATION METHOD

The motivation behind developing the Transformation method is flexibility. Specifically, the Transformation method allows the user to specify the limiting distribution of the terminal layer of the binomial tree.

Because the Transformation method is built on the ideas of the standard Binomial model, it is natural to ask what type of relationship exists between the Binomial model and the Transformation method. Recall that both methods price options in two phases: Phase one is to create a binomial tree, and phase two is to use replication to price the tree. The Binomial model and the Transformation method differ in phase one and are identical in phase two. Therefore, the similarities and differences between the binomial model and the Transformation method stem from phase one, the creation of the binomial tree.

In the Binomial model, the process that generates the binomial tree,

$$S_{t+\Delta t} = u^{2Z-1} S_t$$
$$\text{where } Z \sim BINOMIAL(1,p),\ p \in (0,1),\ \text{and } u \in (1,\infty),$$

is parametrized by three parameters: Δt, p, and u. The Transformation method,

$$S_{t+\Delta t} = u(Z, \Delta t, F, S_t) S_t$$
$$\text{where } Z \sim BINOMIAL(1,p),\ p \in (0,1),\ \text{and } F \text{ is a CDF},$$

is parametrized by the parameters Δt, p and F.

The practical application of the Binomial model and the Transformation method generally involves generating a binomial tree with very small values of Δt. And any meaningful comparison of the two models is within the scope of practical application.

Recall from Section 5.1 that in the limiting case of the binomial model, p and u are chosen as functions of Δt so that the limiting distribution of S_t has a specific mean and variance along with the well-known lognormal distribution.

To compare the Binomial model and the Transformation model, consider the limit of p and consider the limit as Δt approaches 0 of

$$u(Z = 1, \Delta t, F, S_t)$$
$$\text{when } F \sim LOGNORMAL(\log(S_0) + \mu T,\, \sigma^2 T).$$

Note that the distribution of F is the limiting distribution of S_t when u and p are chosen according to (5.2). For clarity, in the remainder of this section the parameters of the Binomial model are subscripted with B, and the parameters of the Transformation Method are subscripted with T.

9.1 The Limit of p_T

Allow A and B to be placeholders for $\log(S_0) + \mu T$ and $\sigma^2 T$ respectively. If $log(X) \sim N(A, B^2)$ then the inverse probability function (quantile function) is

$$F^{-1}(p) = \exp\left[A + B\Phi^{-1}(p)\right].$$

Again, we choose p_T so that

$$\begin{aligned} S_0 &= \exp\left[A + B\Phi^{-1}\left(\Phi\left(\frac{R/2 - Rp_T}{\sqrt{Rp_T(1-p_T)}}\right)\right)\right] \\ &= \exp\left[A + B\frac{R(1/2 - p_T)}{\sqrt{Rp_T(1-p_T)}}\right]. \end{aligned}$$

With a little algebra, p_T is

$$p_T = \frac{1}{2} - \frac{t}{2\sqrt{1+t^2}} \quad \text{where} \quad t = \frac{\log(S_0) - A}{B\sqrt{R}}.$$

Now replace A and B with $\log(S_0) + \mu T$ and $\sigma^2 T$:

$$p_T = \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{T/(R + \frac{\mu^2 T}{\sigma^2})}.$$

Note the similarity to the p_B that Ross, Cox, and Rubinstein propose (5.3):

$$p_B = \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{T/R}.$$

The parameters p_T and p_B are asymptotically equivalent. Futhermore, the values of p_T and p_B are similar for relatively small values of R.

9.2 The Limit of u_T

Now consider u under choice one and two. The placeholders A and B are the same as above.

9.2.1 u_i Choice 1

Under the first choice,

$$u_i = \frac{x_{i,R}}{x_{i-1,R-1}}$$

where

$$\begin{aligned} x_{i,R} &= F^{-1}\left[\Phi\left(\frac{i - Rp_T}{\sqrt{Rp_T(1-p_T)}}\right)\right] \\ &= \exp\left[A_R + B_R \frac{i - Rp_T}{\sqrt{Rp_T(1-p_T)}}\right]. \end{aligned}$$

The value of u_i is

$$u_i = \frac{\exp\left[A_R + B_R \frac{i - Rp_T}{\sqrt{Rp_T(1-p_T)}}\right]}{\exp\left[A_{R-1} + B_{R-1} \frac{i-1-(R-1)p_T}{\sqrt{(R-1)p_T(1-p_T)}}\right]}$$
$$= \exp\left[(A_R - A_{R-1}) + B_R \frac{i - Rp_T - i + 1 + Rp_T}{\sqrt{Rp_T(1-p_T)}}\right]$$
$$= \exp\left[(A_R - A_{R-1}) + B_R \frac{1}{\sqrt{Rp_T(1-p_T)}}\right].$$

9.2.2 u_i Choice 2

Under the second choice,

$$u_i = \sqrt{\frac{x_i}{x_{i-1}}}$$
$$= \sqrt{\frac{\exp\left[A + B \frac{i-Rp_T}{\sqrt{Rp_T(1-p_T)}}\right]}{\exp\left[A + B \frac{i-1-Rp_T}{\sqrt{Rp_T(1-p_T)}}\right]}}$$
$$= \sqrt{\exp\left[B \frac{i - Rp_T - i + 1 + Rp_T}{\sqrt{Rp_T(1-p_T)}}\right]}$$
$$= \exp\left[B \frac{1}{2\sqrt{Rp_T(1-p_T)}}\right].$$

9.2.3 The Binomial Model as a Special Case

The key point to recognize from both choices is that u_i is independent of i. This means that u_i is independent of S_t and is a constant. This is an important result because u independent of S_t is an essential step to show that the Binomial model is a special case of the Transformation method.

The function u_T and the parameter u_B are asymptotically equivalent. Note that $u_B = e^{\sigma\sqrt{T/R}}$ approaches 1 as R approaches infinity. Further, $4p_T(1-p_T)$ approaches 1 and $A_R - A_{R-1}$ approaches 0 as R gets large. So, under either choice one or choice two, u_T approaches 1.

The combination of p_T and u_T's asymptotic equivalence to each parameter's counterpart in the Binomial model suggests that the binomial tree of the Transformation method is also asymptotically equivalent to the binomial tree of the Binomial model. However, considering the limiting properties of p_T and u_T is not sufficient to conclude that the Transformation method's binomial tree is asymptotically equivalent to the Binomial model's. In this paper, we leave the specifics of a more complete proof for future research, and we conclude with two observations. First, because p_T and u_T are asymptotically equivalent to the Binomial model counterparts, there is very good reason to believe that the resulting binomial trees are also equivalent. The second observation is that practical application of the binomial trees suggests that the trees are asymptotically equivalent.

The practical application of a binomial tree is option pricing, and a practical test of the limiting properties of the Transformation method is to price an option under the lognormal assumption. As noted before, the Binomial model converges to the Black-Scholes-Merton model as the number of steps increases. Therefore, if the Transformation method does asymptotically reduce to the Binomial model, then it should also achieve the Black-Scholes-Merton solution in the limit.

A complete simulation study of the asymptotic properties of the Transformation method is possible. One could sample several values of μ, σ, r, T, K, and S_0 and find the option price for increasingly more steps. That is left for future research. The goal of this section is to present two examples of option pricing that suggest that the Transformation method does reduce to the Binomial model. Figure 9.1 displays these two examples. Both examples suggest that in practical terms—option pricing—the Transformation method does reduce to the Binomial model when a lognormal distribution is assumed.

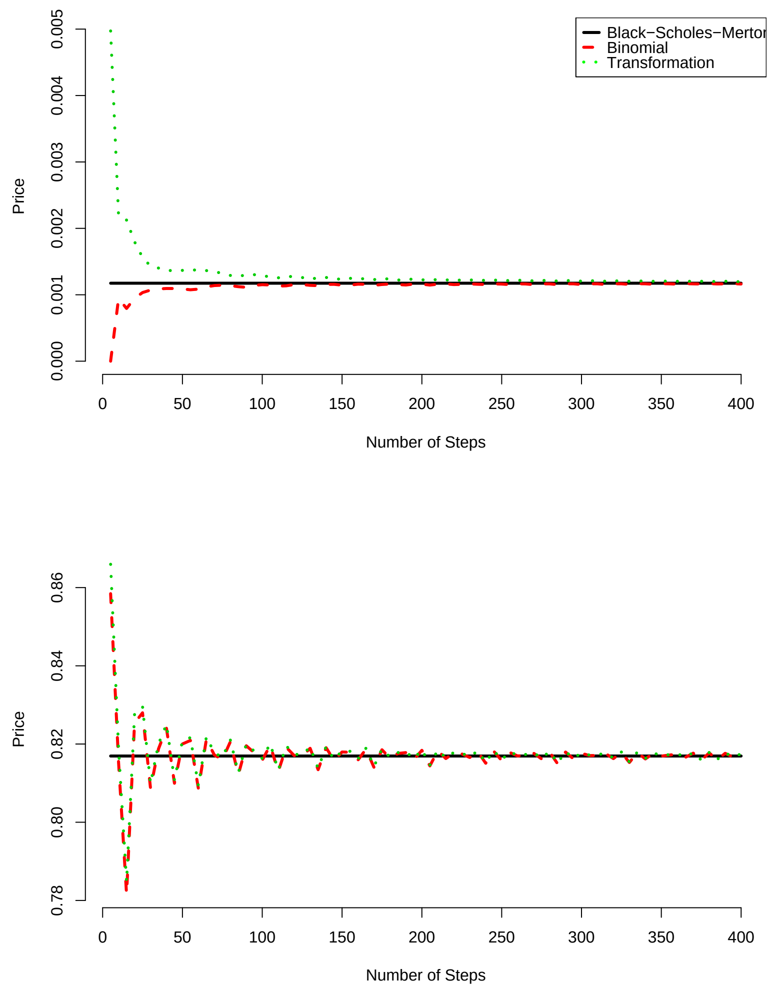


Figure 9.1: The Transformation Method and the Binomial Model

These plots are two examples of option pricing with the Transformation method. The green dashed line is the price under Transformation Pricing. The red dashed line is the price under the Binomial model. The solid black line is the Black-Scholes-Merton solution. The parameters for each option are:

	T	K	S_0	μ	σ	r
Top	1	1	.5	.6	.3	.05
Bottom	1	10	5	.6	.9	.05

10. THE U FUNCTION

The difference equation that describes a simple random walk is

$$X_n = \Delta x(2Z_n - 2) + X_{n-1}.$$

Likewise, for geometric random walks, the equation is

$$X_n^G = u^{2Z_n - 1} X_{n-1}^G.$$

The generalized random walk does not have an obvious counterpart to these difference equations. In spite of the unclear connection, consider the following fact. The X_n^G difference equation suggests exactly what we already know about the geometric random walk: in each time interval, the process goes up or down by a multiplicative constant. The constant is the same for the entire process, regardless of the process's position. The next lemma shows that the generalized random walk is connected with a difference equation,

$$X_n = u(Z, X_{n-1})X_{n-1},$$

where the multiplicative change factor depends on the position of the process. That is to say that the generalized random walk allows for different step sizes. For example, the generalized random walk may step up twenty-five percent when the process is close to the mean and step up forty percent when the process is a standard deviation below the mean.

As will be seen, the general idea of the proof consists of taking sequential time intervals and calculating the ratio

$$\frac{X_n}{X_{n-1}}$$

when $X_n > X_{n-1}$. The ratio will be

$$\frac{F^{-1}\left[\Phi\left(\frac{i-.5n}{.5\sqrt{n}}\right)\right]}{F^{-1}\left[\Phi\left(\frac{i-1-.5(n-1)}{.5\sqrt{n-1}}\right)\right]}$$

for some value of $i \in \mathbb{R}$ and some $n \in \mathbb{N}$.

For an arbitrary $z \in \mathbb{R}$, there is an i so that $z = (i - .5n)/(.5\sqrt{n})$. This means that the ratio can be rewritten as

$$\frac{F^{-1}\left[\Phi\left(z\right)\right]}{F^{-1}\left[\Phi\left(\frac{z\sqrt{n}-1}{\sqrt{n-1}}\right)\right]}.$$

This function will play a key role in the upcoming lemma.

Lemma 2: If $F^{-1}[\Phi(z)] \in C^1$, then

$$\lim_{n\to\infty} \sqrt{n-1}\left(\frac{F^{-1}[\Phi(z)]}{F^{-1}[\Phi\left(z\frac{z\sqrt{n}-1}{\sqrt{n-1}}\right)]} - 1\right) = \frac{d}{dz}\log\left(F^{-1}[\Phi(z)]\right).$$

We prove the lemma in three parts.

Part 1: For an arbitrary $z \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{z - \frac{z\sqrt{n}-1}{\sqrt{n-1}}}{\frac{1}{\sqrt{n-1}}} = 1.$$

Proof. The expression can be simplified, as follows:

$$\begin{aligned}
\lim_{n\to\infty} \frac{z - \frac{z\sqrt{n}-1}{\sqrt{n-1}}}{\frac{1}{\sqrt{n-1}}} &= \lim_{n\to\infty} z\sqrt{n-1} - z\sqrt{n} + 1 \\
&= \lim_{n\to\infty} z\left(\frac{\sqrt{n-1}}{\sqrt{n-1}+\sqrt{n}} - \frac{\sqrt{n}}{\sqrt{n-1}+\sqrt{n}}\right)\sqrt{n-1}+\sqrt{n}+1 \\
&= \lim_{n\to\infty} z\left(\frac{(n-1)+\sqrt{n-1}\sqrt{n}-\sqrt{n-1}\sqrt{n}-n}{\sqrt{n-1}+\sqrt{n}}\right)+1 \\
&= \lim_{n\to\infty} z\left(\frac{-1}{\sqrt{n-1}+\sqrt{n}}\right)+1 \\
&= 1. \qquad \square
\end{aligned}$$

Part 2: If the function $y(z)$ is differentiable, then

$$\frac{dy}{dz} = \lim_{n\to\infty} \frac{y(z) - y\left(\frac{z\sqrt{n}-1}{\sqrt{n-1}}\right)}{\frac{1}{\sqrt{n-1}}}.$$

Proof. By definition,

$$\frac{dy}{dz} = \lim_{a\to z} \frac{y(z)-y(a)}{z-a}.$$

Let $a = \frac{z\sqrt{n}-1}{\sqrt{n-1}}$. Note that $a \to z$ as $n \to \infty$. The ratio

$$\frac{\frac{y(z)-y(a)}{\frac{1}{\sqrt{n-1}}}}{\frac{y(z)-y(a)}{z-a}}$$

simplifies to

$$\frac{z - \frac{z\sqrt{n}-1}{\sqrt{n-1}}}{\frac{1}{\sqrt{n-1}}}.$$

Part 1 indicates that the ratio converges to one as n goes to infinity. Because

$$\frac{y(z) - y\left(\frac{z\sqrt{n}-1}{\sqrt{n-1}}\right)}{\frac{1}{\sqrt{n-1}}}$$

is asymptotically equivalent to the definition of the derivative, its limit is the derivative as well. □

Part 3: If $y \in C^1$ and $y(z) \neq 0$ for all $z \in \mathbb{R}$, then

$$\lim_{n\to\infty} \sqrt{n-1}\left(\frac{y(z)}{y\left(\frac{z\sqrt{n}}{\sqrt{n-1}}\right)} - 1\right) = \frac{d}{dz}\log[y(z)].$$

Proof. Part 2 shows that

$$\frac{dy}{dz} = \lim_{n\to\infty} \frac{y(z) - y\left(\frac{z\sqrt{n}-1}{\sqrt{n-1}}\right)}{\frac{1}{\sqrt{n-1}}}.$$

Because $y(z) \neq 0$ for all $z \in \mathbb{R}$ and because limit of products is the product of limits, if follows that

$$\begin{aligned}
\frac{dy}{dz} &= \lim_{n\to\infty} \frac{y(z) - y\left(\frac{z\sqrt{n-1}}{\sqrt{n-1}}\right)}{\frac{1}{\sqrt{n-1}}} \frac{y\left(\frac{z\sqrt{n-1}}{\sqrt{n-1}}\right)}{y\left(\frac{z\sqrt{n-1}}{\sqrt{n-1}}\right)} \\
&= \lim_{n\to\infty} \sqrt{n-1}\left(\frac{y(z)}{y\left(\frac{z\sqrt{n}}{\sqrt{n-1}}\right)} - 1\right) y\left(\frac{z\sqrt{n}-1}{\sqrt{n-1}}\right) \\
&= \lim_{n\to\infty} \sqrt{n-1}\left(\frac{y(z)}{y\left(\frac{z\sqrt{n}}{\sqrt{n-1}}\right)} - 1\right) \lim_{n\to\infty} y\left(\frac{z\sqrt{n}-1}{\sqrt{n-1}}\right) \\
&= y(z) \lim_{n\to\infty} \sqrt{n-1}\left(\frac{y(z)}{y\left(\frac{z\sqrt{n}}{\sqrt{n-1}}\right)} - 1\right) \\
\frac{1}{y(z)}\frac{dy}{dz} &= \lim_{n\to\infty} \sqrt{n-1}\left(\frac{y(z)}{y\left(\frac{z\sqrt{n}}{\sqrt{n-1}}\right)} - 1\right) \\
\frac{d}{dz}\log[y(z)] &= \lim_{n\to\infty} \sqrt{n-1}\left(\frac{y(z)}{y\left(\frac{z\sqrt{n}}{\sqrt{n-1}}\right)} - 1\right). \qquad \square
\end{aligned}$$

Because $F^{-1}[\Phi(z)] \in C^1$ and $F^{-1}[\Phi(z)] \neq 0$ for all $z \in \mathbb{R}$, then the results of part 3 apply to $F^{-1}[\Phi(z)]$. $\square$

There are several take home points from Lemma 2. First, because the expression

$$\sqrt{n-1}\left(\frac{F^{-1}[\Phi(z)]}{F^{-1}\left[\Phi\left(\frac{z\sqrt{n-1}}{\sqrt{n+1}}\right)\right]} - 1\right)$$

is not free of n, a function $u(Z, X_{n-1})$ does not exist for every F. However, there is a function $u(Z, X_{n-1}, \Delta t)$ that does satisfy the expression

$$X_n = u(Z, X_{n-1}, \Delta t)X_{n-1}.$$

The function is

$$u(Z = 1, X_{n-1}, \Delta t) = \sqrt{\frac{c}{\Delta t}\frac{\phi(z)}{f[\Phi(z)]F^{-1}[\Phi(z)]}} + 1$$

where z is the value of X_{n-1} in standard deviations. Call this the u function.

10.1 Starting with the U Function

It is important to point out that the u function was derived starting from the generalized random walk with F known. It is possible to work in the opposite direction. One can start with a u function and with the relationship described in the lemma, find the resulting F.

> **Corollary to Lemma 2:** Let $u^*(z)$ be a u function without known generalized random walk. That is,
>
> $$u^*(z) = \lim_{n\to\infty} \sqrt{n-1}(u(Z=1,z,n)-1)$$
>
> for an unknown generalized random walk. The probability function of the distribution of the limit of the generalized random walk must satisfy
>
> $$\Phi(z) = F\left[\exp\left\{\int_0^z u^*(\tau)\,d\tau\right\}\right]. \tag{10.1}$$

10.1.1 Example: When the U Function is Constant

Consider the simplest example of the corollary,

$$u(z)^* = m$$

where m is a constant.

Working through the algebra,

$$\begin{aligned}
m &= \frac{d}{dz}\log\left(F^{-1}_{\mu,\sigma}[\Phi(z)]\right) \\
\int_0^z \tau\,d\tau &= log\left(F^{-1}_{\mu,\sigma}[\Phi(z)]\right) \\
\exp\tau z + k &= F^{-1}_{\mu,\sigma}[\Phi(z)] \\
F_{\mu,\sigma}\left(\exp\tau z + k\right) &= \Phi(z) \\
P\left(X \le \exp\tau z + k\right) &= \Phi(z) \\
P\left((\log(X) - k)/\tau \le z\right) &= \Phi(z),
\end{aligned}$$

shows that $(\log(X) - k)/\tau$ is normally distributed which means that X is lognormal. This is the expected outcome, and suggests that the generalized random walk does revert to the geometric random walk as a special case. A more complete discussion of that topic is found in Chapter 9.

10.2 Examples of U Functions

The u function is central to the generalized random walk because it provides a meaningful comparison of generalized random walks of different distributions. Figure 10.1 is a plot of u functions from six different distributions. Note that this particular plot is on the X_n scale and not the standard deviation scale. Plots on the standard deviation scale are found in Figure 10.2.

Figure 10.1 is the plot of six different u functions. More precisely, each line is

$$u(Z = 1, \Delta t, X)$$

plotted against centered values of the asset price, $X - E(X)$. Each line represents a different choice of distribution, F. The distributions where chosen so that all six share the same mean and all but one share the same variance. The chi square is the exception. The other five distributions include the log normal, gamma, Weibull, Pareto, and generalized beta of the second type (GB2). In the particular case depicted in Figure 10.1, the mean is 0.911.

There are three key points to be gleaned from the figure. First, the u function for the log normally distributed asset is constant. As noted in chapter 9, this is to be expected. The solid horizontal line provides a basis to consider the other five u functions.

The second point to be gleaned from Figure 10.2 is interesting. Figure 10.2 summarizes how asset prices change relative to distance from the mean. The u functions of the gamma and Weibull distributions are higher before the mean and lower

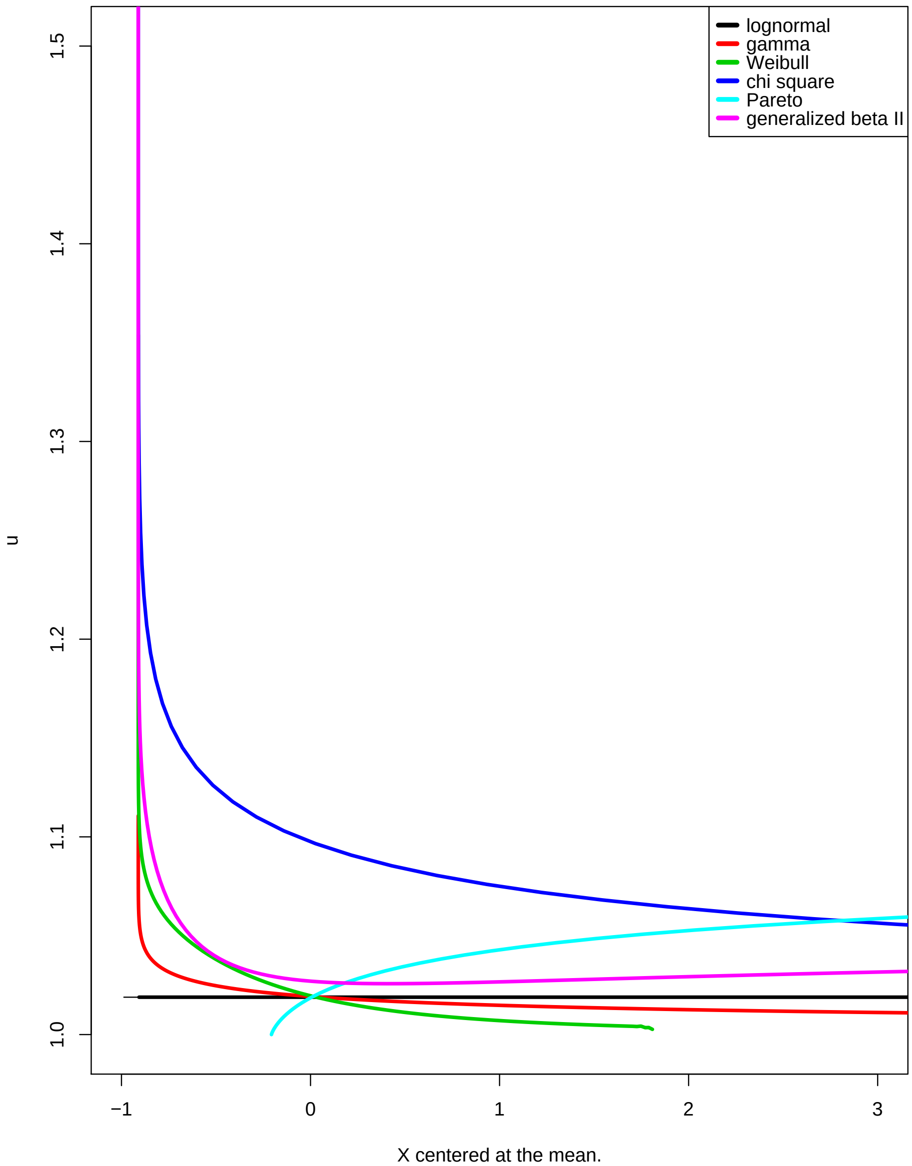


Figure 10.1: An example of six u functions on the X scale.

The distributions of all the u functions share the same mean and variance, expect for the chi square. The chi square u function only shares the same mean.

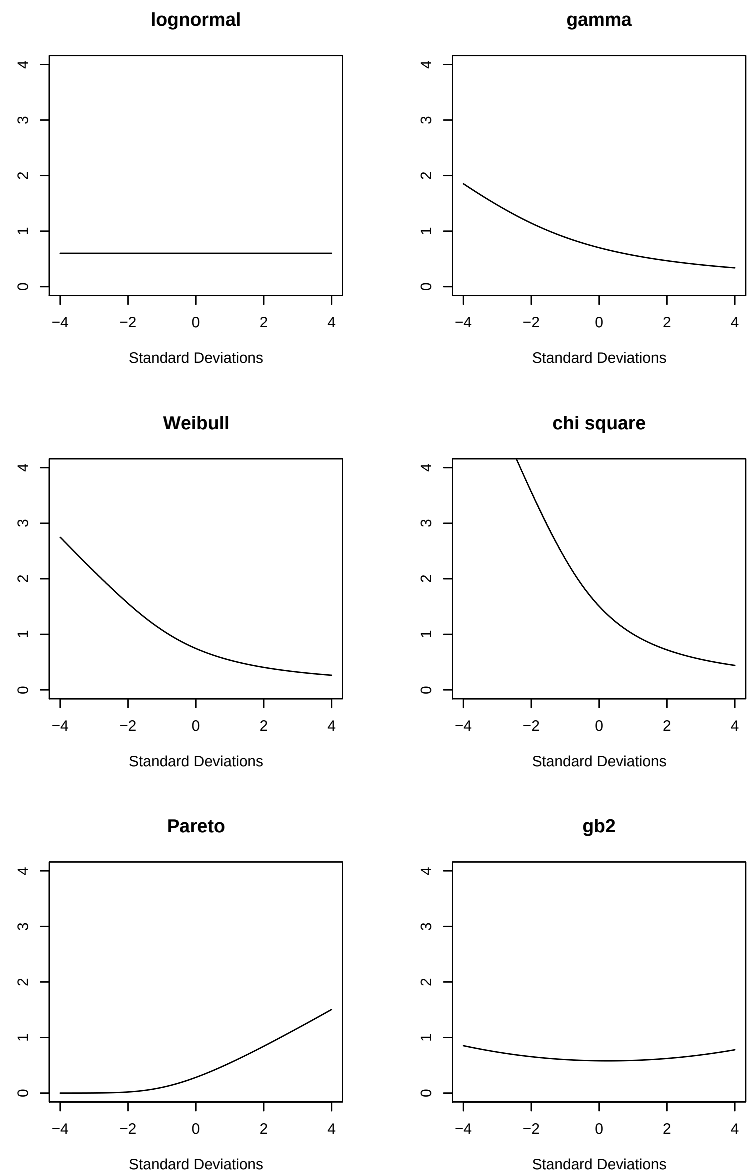


Figure 10.2: An example of six u functions on the X scale.

after the mean relative to the log normal u function. In terms of asset prices, the gamma and Weibull u functions suggest that lower and higher than expected prices will have a stronger shift toward the mean price. Not that the random walk will shift with greater frequency, but when it does shift up, it will move with greater impact.

The third point, and perhaps the most important, is that Figure 10.2 depicts the relative volatility of the random walk. Note that

$$\begin{aligned} u(Z=0, X_{n-1}, \Delta t) &= \frac{X_n\left(\frac{z\sqrt{n-1}}{\sqrt{n-1}}\right)}{X_{n-1}(z)} \\ &\approx u(Z=1, X_{n-1}, \Delta t)^{-1}. \end{aligned}$$

Thus, regions in the plot with u function values relatively larger than one will also have relatively drastic values when $Z = 0$. Using the word volatility to mean the variance of a small interval, volatility is

$$\frac{1}{2}\left[u(Z=1, X_{n-1}, \Delta t)^{-1} + u(Z=1, X_{n-1}, \Delta t)\right] X_{n+1}$$

which is minimized as $u(Z = 1, X_{n-1}, \Delta t)$ approaches one. Therefore, the plots in Figure 10.2 indicate where (in standard deviation units) a generalized random walk is going to experience higher and lower levels of volatility. Note that the gamma, Weibull, and chi-square u functions all indicated higher volatility below the mean, and the Pareto u function indicates higher volatility above the mean. Perhaps, most striking is the generalized beta u function. It indicates that volatility is minimized at the mean and gradually gets larger the farther the random walk strays from the mean.

Specific applications of the generalized random walk will determine which u function is best, but with regards to asset price behavior, a volatility structure like the one displayed with the generalized beta seems like a natural fit. Indeed, given a set of probability functions, examining the volatility structure suggested by the u functions provides a simple diagnostic to evaluate the generalized random walk's fit. In contrast, if one is modeling a random walk from the micro-level, the u functions appear to give a good indication of the resulting macro-level distribution.

11. USING THE TRANSFORMATION METHOD TO PRICE OPTIONS WHEN ASSET PRICES ARE NOT LOGNORMAL

If asset prices followed the behavior assumed by the Binomial or Black-Scholes-Merton models, there would be no need to consider an alternate model of option pricing. The value of the Transformation method is its application to option pricing when the asset price distribution is not lognormal. This chapter consists of three such examples. Specifically, the following distributions are featured:

(1) gamma

(2) Weibull

(3) generalized beta of the second type.

In each example, the price of the option is considered under two scenarios. The first scenario is under the assumption that the asset price distribution is lognormal; the second scenario is under the assumption that the asset price distribution is the example distribution. In each scenario, the means of both distributions are identical.

Each example also includes the out-of-the-money probability. This is the probability that the option will mature worthless, i.e.,

$$P(\text{ Asset Price at } T < K).$$

This quantity is a useful benchmark to judge the reasonableness of the results. If an option has a higher probability of earning profit, we may assume that the option is worth more. Each example lists this probability in the row named O.O.M. (Out Of the Money). Some of the examples meet this bench mark; other do not.

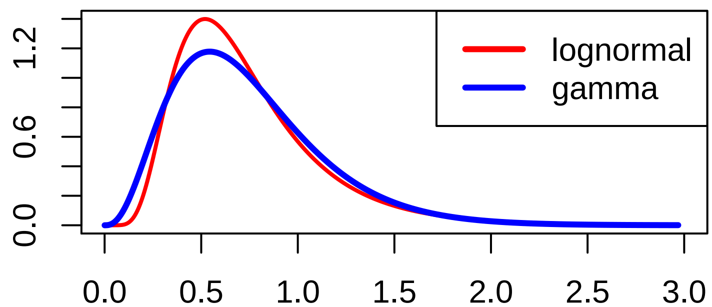


Figure 11.1: Pricing with the Gamma vs. Lognormal

Strike Price	.5	1	1.5	2	2.5
Lognormal Price	0.534	0.213	0.078	0.029	0.011
Gamma Price	0.538	0.207	0.063	0.015	0.003
O.O.M. Lognormal	0.285	0.804	0.954	0.989	0.997
O.O.M. Gamma	0.295	0.781	0.956	0.993	0.999
Contract Parameters	T	S_0	μ	σ	r
	1	1	-0.298	0.487	.05

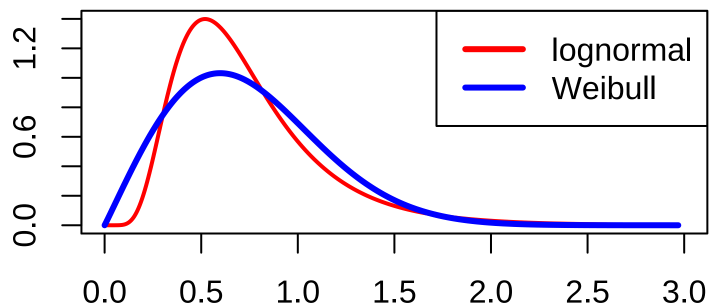


Figure 11.2: Pricing with the Weibull vs. Lognormal

Strike Price	.5	1	1.5	2	2.5
Lognormal Price	0.534	0.213	0.078	0.029	0.011
Weibull Price	0.541	0.198	0.044	0.005	0.000
O.O.M. Lognormal	0.285	0.804	0.954	0.989	0.997
O.O.M. Weibull	0.297	0.761	0.961	0.997	1.000
Contract Parameters	T	S_0	μ	σ	r
	1	1	-0.298	0.487	.05

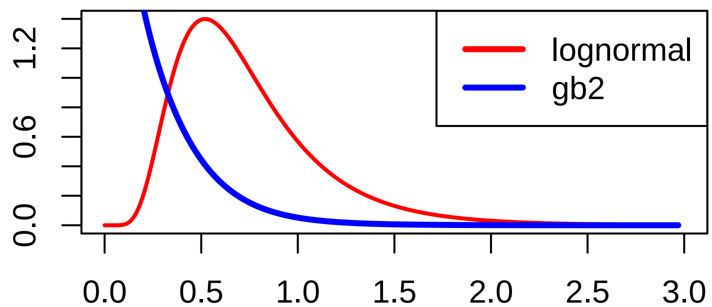


Figure 11.3: Pricing with the Beta vs. Lognormal

Strike Price	.5	1	1.5	2	2.5
Lognormal Price	0.534	0.213	0.078	0.029	0.011
GB2 Price	0.570	0.283	0.139	0.067	0.035
O.O.M. Lognormal	0.285	0.804	0.954	0.989	0.997
O.O.M. GB2	0.896	0.988	0.998	0.999	0.999
Contract Parameters	T	S_0	μ	σ	r
	1	1	-0.298	0.487	.05

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