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Non-Dimensional Approach for Static Balancing of Rotational Flexures

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Abstract

This work presents a nondimensional method for statically balancing flexural hinges, including those with stiffness that varies with load. Using a set of non-dimensional parameters, it is shown that one can quickly design a balancing mechanism for an idealized hinge/torsion spring system. This method is then extended to load-dependent systems, and is demonstrated with the design of a balanced cross-axis-flexural pivot with stiffness that varies as a function of compressive preload. A physical prototype is built and tested to verify the design method. The prototype demonstrates an average stiffness reduction of 87% over an 80 degree deflection range. The method enables improved static balancing for systems where the balancing pre-load influences the systems force-deflection behavior.

Keywords: compliant, flexure, cross-axis-flexural pivot, static balancing

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1. Background

A compliant mechanism obtains its motion from the deflection of its constituent members. Because this eliminates sliding contact of surfaces, friction and subsequent wear can be avoided, leading to higher performance [1]. Because of the strain energy associated with bending the flexible members, compli-

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ant mechanisms generally have higher actuation effort compared to traditional mechanisms [2]. Static balancing is one strategy for reducing the actuation effort of compliant mechanisms. However, the pre-load applied in statically balanced systems is often large enough to affect the stiffness of the system to
10 be balanced. This deviation from the expected stiffness reduces the balancing effect. The objective of this research is to develop a non-dimensional approach for static balancing of compliant hinges that is generalized to systems with load-dependent stiffness.

Static balancing is often accomplished by adding auxiliary springs that provide energy storage [3]. As the mechanism is actuated, energy stored in the
15 balancing elements is transferred to the deflected mechanism [4]. This means that less energy must be added during actuation, thus reducing actuation effort [5]. This strategy has been effectively incorporated into applications such as the design of surgical instruments and prosthetics [4, 6, 7].

20 Balancing elements commonly incorporate a negative stiffness mechanism, such as buckled beams in linear systems, or preloaded linear springs in rotational systems [8]. Other approaches use gravity balancing or systems of ideal springs [4, 9, 10].

Design of statically balanced systems generally requires the use of optimization routines [8, 11, 12]. Usually, the optimization problem minimizes the change
25 in a mechanism's stored energy or searches for an appropriate negative-stiffness mechanism [13]. Depending on the system under consideration, this optimization may incorporate finite element analysis (FEA) and topology optimization. This means that to design a statically balanced compliant mechanism, significant
30 resources must be available to develop and validate the model being used. Additionally, optimization routines utilizing FEA can quickly become cumbersome due to the relatively long solution time of non-linear FEA and the many function calls of most optimization routines.

Finally, building practical statically balanced mechanisms is difficult because
35 the balancing element is often bulky, making the system much larger than is convenient [10].

This paper describes the development of a non-dimensional approach for static balancing for idealized systems and load-dependent systems. The next section introduces nomenclature used in the method development that follows.

40 A non-dimensional approach for idealized systems is discussed first, followed by a generalization to load-dependent systems. An example load-dependent system is designed, and prototype hardware is built and tested.

2. Nomenclature

In this work, “load-independent (LI) joint” is a joint with a rotational stiffness that is not a function of applied lateral loads. This is modeled as a pin joint with a torsional spring. A “load-dependent (LD) joint” is a joint whose stiffness changes when a lateral load is applied. An LD joint can be modeled as an LI joint if a relationship can be found between the applied lateral loads and joint stiffness.

50 In this work the statically balanced system consists of an LI joint of finite, constant stiffness that is balanced by the addition of a pre-loaded constant-stiffness linear spring. The spring connects at points equidistant from the pivot, as shown in Figure 1. This simplified system can represent load-dependent systems with proper application of the pseudo-rigid-body model [1].

55 Variables and their relationships are included in the following lists. The first list is for variables directly related to balancing of LI compliant hinges, illustrated in Figure 1

k_θ = Torsional stiffness of LI joint, or corrected stiffness of LD joint (with applied loads)

60 k_l = Stiffness of balancing spring

k = Stiffness of balanced system

d = Distance from pivot center to balancing spring attachment points

x_0 = Free length of balancing spring

Figure 1: A LI system with balancing spring and associated variables.

Figure 2: A cross-axis-flexural pivot with associated variables.

$$P = k_l(2d - x_0) = \text{Preload applied to balancing spring}$$

$$\theta = \text{Angle of deflection of the LI or LD joint}$$

$$T = k\theta = \text{Torque required to deflect hinge through angle } \theta$$

$$\Pi_1 = k_\theta/(Pd) = \text{Pi group governing torsional stiffness}$$

$$\Pi_2 = k_l d/P = \text{Pi group governing stiffness of balancing spring}$$

The following list contains variables related to the design of a cross-axis-
 70 flexural pivot (CAFP) that has a stiffness that is load-dependent. See Figure 2
 for a depiction of geometric variables.

$$E = \text{Young's modulus of the flexure material}$$

$$b = \text{Width of CAFP flexure strip}$$

t = Thickness of CAFP flexure strip

75 $I = bt^3/12$ = Moment of inertia of CAFP flexure strip

L = Length of CAFP flexure strips

k'_θ = Uncorrected torsional stiffness of LD joint (no applied loads)

The following list contains variables used to correct the stiffness of a CAFP to account for the effects of applied loads, as adapted from Wittrick [14]. See
80 Figure 2 for a depiction of geometric variables. The loads V and H are applied to the moving block of the CAFP at the center of the pivot.

V = Vertical load applied to hinge

H = Horizontal load applied to hinge

α = Half the intersection angle of the CAFP flexures

85 $v = VL^2 \sec(\alpha)/(EI)$ = Non-dimensionalized applied vertical load

$h = HL^2 \csc(\alpha)/(EI)$ = Non-dimensionalized applied horizontal load

β_i = Dimensionless parameter describing the forces in CAFP flexures

ϕ_i = Dimensionless parameter describing the stiffness of the individual CAFP flexures

90 **3. Balancing of Load-Independent Hinge-Spring System**

Because a general solution to static balancing is sought, it is desirable to use dimensional analysis techniques to analyze the balanced systems.

Recall the Buckingham-Pi theorem:

If an equation involving k variables is dimensionally homogeneous,
95 it can be reduced to a relationship among $k - r$ independent dimensionless products, where r is the minimum number of reference dimensions required to describe the variables [15].

The energy of a load-independent system, E , can be written as the sum of the potential energies of the torsional and linear springs, as follows:

$$E = \frac{k_\theta}{2}\theta^2 + \frac{k_l}{2}(\sqrt{2d^2(1+\cos(\theta))} - x_0)^2 \quad (1)$$

100 Since the objective is to minimize torque (T), and $T = \frac{dE}{d\theta}$, we can take the derivative with respect to θ as

$$T = k_\theta\theta - \frac{k_l(\sqrt{2d^2(1+\cos(\theta))} - x_0)d^2 \sin(\theta)}{\sqrt{2d^2(1+\cos(\theta))}}. \quad (2)$$

Dividing by θ gives the mechanism stiffness ($k = \frac{T}{\theta}$) as

$$k = \frac{k_\theta\theta - \frac{k_l(\sqrt{2d^2(1+\cos(\theta))} - x_0)d^2 \sin(\theta)}{\sqrt{2d^2(1+\cos(\theta))}}}{\theta}. \quad (3)$$

This result will be used later. Setting the torque from Equation (2) equal to zero and using $x_0 = 2d - P/k_l$ gives:

$$k_\theta\theta = \frac{k_l(\sqrt{2d^2(1+\cos(\theta))} - (2d - P/k_l))d^2 \sin(\theta)}{\sqrt{2d^2(1+\cos(\theta))}} \quad (4)$$

105 Equation (4) is a homogeneous equation with four dimensioned variables - k_θ , k_l , P , and d and two dimensions (force and length). Thus, $k = 4$ and $r = 2$, and the system can be described by two non-dimensional parameters, designated Π_1 and Π_2 . Since it is desirable that these parameters have a physical, intuitive meaning it is convenient to select force P and distance d as repeating variables so that each pi-group deals with a stiffness term independently.
110 Through application of dimensional analysis, we have:

$$\Pi_1 = \frac{k_\theta}{Pd} \quad (5)$$

and

$$\Pi_2 = \frac{k_l d}{P}. \quad (6)$$

We now assume that a relationship between our pi-groups exists, written as $\Pi_1 = \phi(\Pi_2)$. As a result of the Buckingham-Pi theorem, this relationship will
 115 govern the stiffness of the system. If a relationship for Π_1 and $\phi(\Pi_2)$ can be found that results in a balanced system, a combination of system parameters k_θ , k_l , P , and d that follows this relationship will yield a statically balanced system.

A program was written to find the relationship between Π_1 and Π_2 for $0.2 \leq$
 120 $\Pi_1 \leq 1$. This script used a particle swarm optimization [16] routine to minimize $|\frac{k}{k_\theta}|$ (see Equation (3)) calculated over a range of $0 < \theta \leq 20^\circ$ for a given value of Π_1 . Optimization variables were k_l , P , and d . k_θ was found from Equation (5). After minimizing the normalized stiffness $|\frac{k}{k_\theta}|$, Π_2 was calculated from Equation (6) using the final variable values. A particle swarm algorithm was employed
 125 for this optimization because of its ability to find a global optimum.

This approach was repeated for other values of Π_1 in the range of $0.2 \leq$
 $\Pi_1 \leq 1$ to find Π_2 as a function of Π_1 . Figure 4 plots $|\frac{k}{k_\theta}|$ against Π_1 . The plot shows that in the region of $\Pi_1 = 0.5$ a close approximation of perfect balancing is achieved. By plotting Π_2 as a function of Π_1 , as in Figure 3, we can see
 130 that using $\Pi_1 \geq 0.5$ gives $\Pi_2 < 0$, which is inconvenient for design purposes (it could require k_l to be negative). Table 1 lists convenient Π terms along with the expected reduction in stiffness. A curve fit for the data in Figure 3 is

$$\Pi_2 = -102.54\Pi_1 + 51.104 : 0.2 \leq \Pi_1 \leq 0.81 \quad (7)$$

with an R^2 value of 1.

Rearranging the data in Figure 4 and applying a curve fit gives the % re-
 135 duction in stiffness as

$$\frac{k_\theta - k}{k_\theta} \times 100 = \begin{cases} 2414\Pi_1^3 - 3336\Pi_1^2 + 1679\Pi_1 - 206.7 & : 0.2 \leq \Pi_1 \leq 0.5 \\ 111.85\Pi_1^2 - 215.42\Pi_1 + 179.53 & : 0.5 \leq \Pi_1 \leq 0.8 \end{cases} \quad (8)$$

with an R^2 value of 0.9998.

Figure 3: Π_2 plotted as a function of Π_1 . The flat line occurring at $\Pi_1 = 0.82$ is an artifact of the optimization constraints.

This method only works when the flexure is load-independent; that is, when k_θ is not a function of applied lateral loads. Also, because the joint design has only four parameters (k_θ , k_l , P , and d), choosing more than two parameters
140 results in an over-constrained design.

4. Extending to Load-Dependent Joints

In load-dependent (LD) systems, torsional stiffness varies with applied load-
ing. Because the balancing spring exerts a pre-load on the flexure to be balanced,
its stiffness (k_θ) is no longer the same as its stiffness without any applied loads
145 (k'_θ). To use the Π groups with a LD joint, a prediction of joint stiffness under
load is required. Once the corrected stiffness k_θ is found for a specified pre-load
 P , the other joint parameters can be found from Π_1 and Π_2 .

In this work, we will consider the static balancing of a cross-axis-flexural
pivot, sometimes called a cross-spring pivot. This is a type of flexure formed
150 by crossing two flexible strips and has been used extensively to allow motion

Figure 4: Normalized balanced stiffness as a function of Π_1

in many applications [14, 17, 18, 19, 20]. Additionally, it has been the subject of other investigations into static balancing strategies [13]. Morsch and Herder were able to balance a CAFP using a pair of zero-free-length springs with average stiffness reduction of 70% in the physical prototype [13]. A final motivation for the use of a CAFP is the availability of published load-dependent behavior [14, 18]. In this work we use a different balancer topology and employ the methods

Table 1: Tabulated values of Π_1 and Π_2 , with the expected reduction in joint stiffness.

Π_1	Π_2	$\frac{k_\theta - k}{k_\theta} \times 100$
0.5	-0.1673	100.0
0.49	0.8581	98.8
0.48	1.884	97.6
0.47	2.909	96.3
0.46	3.934	95.0
0.45	4.96	93.6

described here to take into account the change in CAFP stiffness when subjected to a compressive load.

Wittrick established that the stiffness of cross-axis-flexural pivots is dependent on applied lateral loads [14, 18]. He discussed how applied loads change the moments and loads applied to the constituent flexures, which affects their deflections. This same principle applies to many flexure systems commonly in use. A balancing method that accounts for the change in stiffness due to applied loads can provide a more balanced system. In this case, the applied load is due to the compressive pre-load of the balancing spring.

Wittrick's results are summarized here for convenience. He gives the stiffness of a CAFP as [14]:

$$k_\theta = \frac{EI}{L}(\phi_1 + \phi_2) \quad (9)$$

where

$$\begin{aligned} \phi_i &= \beta_i(\cot \beta_i - \beta_i) \\ \beta_1^2 &= \frac{1}{8}(v + h) \\ \beta_2^2 &= \frac{1}{8}(v - h) \end{aligned} \quad (10)$$

Recall that v and h are non-dimensionalized horizontal and vertical loads. The balancing spring exerts a vertical load on the hinge because of its pre-load, P . Choosing an acceptable value of P and letting $V = -P$ and $H = 0$, allows the computation of k_θ for a given geometry. Choosing a value for Π_1 and its associated Π_2 for the desired stiffness reduction enables the calculation of the required d and k_l from Equations (5) and (6). Thus the Π groups reduce the balancing problem to a system of two equations and four unknowns. Choosing two unknowns as design parameters allows the equations to be solved.

Alternatively, if it is desirable to select a value of P and k_l with flexures of a given moment of inertia, the associated k_θ can be found to satisfy Equation (5), and L can be found with an optimization loop. Because Equations (9) and (4)

180 contain trigonometric terms, a non-gradient based optimization routine such as
particle swarm optimization is effective.

5. Example Design

This approach was followed to design a statically balanced CAFP. Because
the spring preload changes the behavior of the CAFP, implementing the load-
185 dependent method introduced here can result in a system with lower stiffness
than would otherwise be possible. Convenient values of k_l and P were chosen to
match those of a commercially available tension spring, and a flexure moment
of inertia was selected so that the CAFP could be built from available spring
steel. A flexure length was found along with torsional stiffness k_θ and d . The
190 resulting design variables are listed in Table 2.

Table 2: Design parameters for balanced CAFP.

Parameter	Value	Units
Π_1	0.49	
Π_2	0.8581	
E	30.0e6	lbs/in ²
L	2.5961	in
b	0.015	in
w	0.501	in
I	1.409e-7	in ⁴
k_θ	5.7054	lb-in/rad
k_l	1.3	lb/in
P	4.2	lb
d	2.7723	in
x_0	2.3139	in

This CAFP was fabricated and tested, as described in the following section.

Figure 5: A prototype joint designed using the method detailed in this paper.

6. Experimental Results and Discussion

The prototype balanced joint was designed and fabricated according to the design parameters of Table 2. The final hardware is shown in Figure 5. Rigid
 195 sections were machined from 6061 aluminum bar stock while the flexures were cut from spring steel. Torque was measured using a torque transducer while the joint was displaced with a worm-wheel gear-set. The experimental setup is shown in Figure 6.

Figure 7 shows the predicted and measured stiffness in both the unbalanced
 200 and balanced configurations. The balanced stiffness is not as low as predicted; this is most likely due to variance in the linear spring stiffness from nominal, the difficulty in ensuring precise application of the design pre-load P , as well as manufacturing error. The finite element results shown were obtained from an ANSYS simulation that used BEAM23 elements for the flexures and COMBIN14
 205 elements for the linear spring. Simulations showed a stress-limited deflection of about 40° , so the prototype was designed with this physical limit in mind.

Figure 8 shows the percent stiffness reduction calculated as $\frac{k'_\theta - k}{k'_\theta} \times 100$. An average stiffness reduction of 87% was achieved over 80° of deflection. Note that

Figure 6: Experimental setup used to measure unbalanced and balanced stiffness.

stiffness reduction in Table 1 is calculated as $|(k_\theta - k)/k_\theta|$, while the stiffness
 210 reduction shown in Figure 8 is $|(k'_\theta - k)/k'_\theta|$, and the compressive load P makes
 $k'_\theta < k_\theta$.

This prototype demonstrates the validity of the balancing method presented
 herein. Using non-dimensional parameters as a balancing criterion simplifies
 the design process, making the rapid design of balanced joints practical in many
 215 applications. By taking into account the change in stiffness of a flexure due to
 joint pre-load, a better balancing solution can be achieved than if the flexure
 stiffness is assumed to be independent of load.

7. Conclusion

It has been shown that the use of the Π groups presented herein can simplify
 220 the design of balancing mechanisms for compliant hinges that exhibit load-
 independent behavior. It has also been shown that the Π groups are equally
 valid when used in conjunction with load-dependent joints whose stiffness under
 load can be predicted.

A prototype CAFP was built and tested. Results show that the stiffness-
 225 correction method results in highly balanced joints with an 87% average stiffness
 reduction over an 80° deflection range. The method enables balancing results

Figure 7: The stiffness of the balanced and unbalanced joints plotted for comparison.

that would not be possible without considering the effect that the balancing pre-load has on the system stiffness. The balancing method and stiffness correction were demonstrated with a cross-axis-flexural pivot, but the result is general and can be applied to joints having load-dependent stiffness.

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Figure 8: The percent reduction in stiffness of the balanced joint.

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