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Zeng Lian
Brigham Young University - Provo

LYAPUNOV EXPONENTS AND INVARIANT MANIFOLD FOR RANDOM DYNAMICAL SYSTEMS IN A BANACH SPACE

by

Zeng Lian

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of a dissertation submitted by

Zeng Lian

This dissertation has been read by each member of the following graduate committee and by majority vote has been found to be satisfactory.

Date	Kening Lu, Chair
Date	Tiancheng Ouyang
Date	Christopher P. Grant
Date	Kenneth L. Kuttler
Date	Jeffrey Humpherys

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Date	Kening Lu Chair, Graduate Committee
Accepted for the Department	William E. Lang Graduate Coordinator
Accepted for the College	Thomas Sederberg, Associate Dean College of Physical and Mathematical Sciences

ABSTRACT

LYAPUNOV EXPONENTS AND INVARIANT MANIFOLD
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Zeng Lian

Department of Mathematics

Ph.D of Mathematics

Abstract

We study the Lyapunov exponents and their associated invariant subspaces for infinite dimensional random dynamical systems in a Banach space, which are generated by, for example, stochastic or random partial differential equations. We prove a multiplicative ergodic theorem. Then, we use this theorem to establish the stable and unstable manifold theorem for nonuniformly hyperbolic random invariant sets.

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Contents

1 Introduction

Lyapunov exponents play an important role in the study of the behavior of dynamical systems. They measure the average rate of separation of orbits starting from nearby initial points. They are used to describe the local stability of orbits and chaotic behavior of systems.

In this thesis, we study Lyapunov exponents and their associated random invariant subspaces for infinite dimensional random dynamical systems in a Banach space and establish the existence of stable and unstable manifolds of Pesin type for random invariant sets.

Infinite dimensional random dynamical systems arise in applications when randomness or noise is taken into account. They may be generated, for example, by stochastic partial differential equations and random partial differential equations.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\theta^n)_{n\in\mathbb{Z}}$ be a measurable P-measure preserving dynamical system on Ω. A discrete time linear random dynamical system (or a cocycle) on a Banach space X over the dynamical system θ^n is a measurable map

$$\Phi(n,\cdot,\cdot) : \Omega \times X \to X, \quad (\omega, x) \mapsto \Phi(n,\omega,x), \quad \text{for } n \in \mathbb{Z}^+$$

such that the map $\Phi(n,\omega) := \Phi(n,\omega,\cdot) \in L(X,X)$, the usual space of all bounded linear operators, and forms a cocycle over θ^n:

$$\Phi(0,\omega) = Id, \quad \text{for all } \omega \in \Omega,$$

$$\Phi(n+m,\omega) = \Phi(n,\theta^m\omega)\Phi(m,\omega), \quad \text{for all } m,n \in \mathbb{Z}^+, \quad \omega \in \Omega.$$

A typical example is the solution operator at time $t = n$ for a stochastic linear differential equation. The measurable dynamical system $(\theta^n)_{n\in\mathbb{Z}}$ is also called a metric dynamical system. The metric dynamical system models the noise of the system.

We write the time-one map $\Phi(1,\omega)$ as $S(\omega) := \Phi(1,\omega)$. Then $S(\omega)$ is the so-called random bounded linear operator in the sense that $S(\omega)$ is strongly measurable, i.e., $\omega \to S(\omega)x$ is measurable from Ω to X for each $x \in X$. We note that a random bounded linear operator $S(\omega)$ generates the linear random dynamical system $\Phi(n,\omega)$:

$$\Phi(n,\omega) = \begin{cases} S(\theta^{n-1}\omega)\cdots S(\omega), & n > 0, \\ I, & n = 0. \end{cases}$$

1.1 Lyapunov Exponents

In the following, we will see that the long term behavior of orbits of $\Phi(n,\omega)$ can be described by Lyapunov exponents and their associated random invariant linear subspaces. In order to state our main results, we need to introduce the following two quantities:

$$\kappa(\Phi)(\omega) = \lim_{n\to+\infty} \frac{1}{n} \log \|\Phi(n,\omega)\|$$

and

$$l_\alpha(\omega) = \lim_{n\to+\infty} \frac{1}{n} \log \|\Phi(n,\omega)\|_\alpha.$$

The quantity $\kappa(\Phi)(\omega)$ is the largest Lyapunov exponent and it describes the exponential growth rate of the norm of $\Phi(n,\omega)$ along forward orbits. $||\Phi(n,\omega)||_\alpha$ is the Kuratowski measure of operator $\Phi(n,\omega)$, and $l_\alpha(\omega)$ is the essential exponent. which is the accumulation point of Lyapunov exponents when there are infinite many Lyapunov exponents. $l_\alpha(\omega) = -\infty$ when $S(\omega)$ is compact. In the deterministic case that $\Phi(n,\omega) = S^n$ where S is a bounded linear operator, e^{l_α} is the radius of the essential spectrum of S. We will see that both limits exist almost surely under the assumption that the positive part of logarithms of the norm of $S(\cdot)$ is integrable. We will also see that $\kappa(\Phi)(\omega) \geq l_\alpha(\omega)$ and they are θ-invariant.

Our main results may be summarized as

Theorem A. *Assume that $S(\cdot) : \Omega \to L(X, X)$ is strongly measurable, where X is a separable Banach space, $S(\omega)$ is injective almost everywhere and*

$$\log^+ ||S(\cdot)|| \in L^1(\Omega, \mathcal{F}, P).$$

Then, there exists a θ-invariant subset $\tilde{\Omega} \subset \Omega$ of full measure such that for each $\omega \in \tilde{\Omega}$ only one of the following conditions holds

(I) $\kappa(\Phi)(\omega) = l_\alpha(\omega)$.

(II) There exist $k(\omega)$ numbers $\lambda_1(\omega) > \ldots > \lambda_{k(\omega)}(\omega) > l_\alpha(\omega)$ and a splitting

$$X = E_1(\omega) \oplus \cdots \oplus E_{k(\omega)}(\omega) \oplus F(\omega)$$

of finite dimensional linear subspaces $E_j(\omega)$ and infinite dimensional linear subspace $F(\omega)$ such that

1) Invariance: $k(\theta\omega) = k(\omega)$, $\lambda_i(\theta\omega) = \lambda_i(\omega)$, $S(\omega)E_j(\omega) = E_j(\theta\omega)$ and $S(\omega)F(\omega) \subset F(\theta\omega)$. $k(\omega)$ and $\lambda_i(\omega)$ are constant when θ is ergodic;

2) Lyapunov Exponents:

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|\Phi(n, \omega)v\| = \lambda_j(\omega) \text{ for all } v(\neq 0) \in E_j(\omega), 1 \leq j \leq k;$$

3) Exponential Decay Rate on $F(\omega)$:

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|\Phi(n, \omega)|_{F(\omega)}\| \leq l_\alpha(\omega)$$

and if $v(\neq 0) \in F(\omega)$ *and* $(\Phi(n, \theta^{-n}\omega))^{-1}v$ *exists for all* $n \geq 0$*, which is denoted by* $\Phi(-n, \omega)v$*, then*

$$\liminf_{n \to +\infty} \frac{1}{n} \log \|\Phi(-n, \omega)v\| \geq -l_\alpha(\omega);$$

4) Measurability and Temperedness: $k(\omega), \lambda_i(\omega)$*, and* $E_j(\omega)$ *are measurable and the projection operators associated with the decompositions*

$$X = \Big(\bigoplus_{i=1}^{j} E_i(\omega)\Big) \oplus \Big(\big(\bigoplus_{i=j+1}^{k(\omega)} E_i(\omega)\big) \oplus F(\omega)\Big) = \Big(\bigoplus_{i=1}^{k(\omega)} E_i(\omega)\Big) \oplus F(\omega)$$

are strongly measurable and tempered.

(III) There exist infinitely many finite dimensional subspaces $E_j(\omega)$*, infinitely many infinite dimensional subspaces* $F_j(\omega)$*, and infinitely many numbers*

$$\lambda_1(\omega) > \lambda_2(\omega) > \ldots > l_\alpha(\omega) \text{ with } \lim_{j \to +\infty} \lambda_j(\omega) = l_\alpha(\omega)$$

such that

1) Invariance: $\lambda_i(\theta\omega) = \lambda_i(\omega)$*,* $S(\omega)E_j(\omega) = E_j(\theta\omega)$*,* $S(\omega)F_j(\omega) \subset F_j(\theta\omega)$*.* $\lambda_i(\omega)$ *are constants when* θ *is ergodic;*

2) Invariant Splitting:

$$X = E_1(\omega) \oplus \cdots \oplus E_j(\omega) \oplus F_j(\omega) \text{ and } F_j(\omega) = E_{j+1}(\omega) \oplus F_{j+1}(\omega);$$

3) Lyapunov Exponents:

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|\Phi(n, \omega)v\| = \lambda_j(\omega), \text{for all } v(\neq 0) \in E_j(\omega);$$

4) Exponential Decay Rate on $F_j(\omega)$:

$$\lim_{n\to+\infty} \frac{1}{n} \log \|\Phi(n,\omega)\big|_{F_j(\omega)}\| = \lambda_{j+1}(\omega)$$

and if for $v(\neq 0) \in F_j(\omega)$ such that $\Phi(-n,\omega)v$ exists for all $n \geq 0$, then

$$\liminf_{n\to+\infty} \frac{1}{n} \log \|\Phi(-n,\omega)v\| \geq -\lambda_{j+1}(\omega);$$

5) Measurability and Temperedness: $\lambda_j(\omega)$ and $E_j(\omega)$ are measurable and the projection operators associated with the decomposition

$$X = \left(\bigoplus_{i=1}^{j} E_i(\omega) \right) \oplus F_j(\omega)$$

are strongly measurable and tempered.

Furthermore, the above statements hold for any continuous time random dynamical system $\Phi(t,\omega), t \in \mathbb{R}^+$, which satisfies

$$\sup_{0\leq s\leq 1} \log^+ ||\Phi(s,\cdot)|| \text{ and } \sup_{0\leq s\leq 1} \log^+ ||\Phi(1-s,\theta^s\cdot)|| \in L^1(\Omega,\mathcal{F},P).$$

Remark: (1) $\kappa(\Phi) = l_\alpha(\omega)$ means that the largest Lyapunov exponent is the same as the essential exponent. When $\Phi(n,\omega) = S^n$ is a deterministic dynamical system, it means that there are no eigenvalues outside of the circle with the radius, e^{l_α}, of the essential spectrum. (2) When case (II) holds, there are only finitely many Lyapunov exponents larger than $l_\alpha(\omega)$. This is because the largest Lyapunov exponent of the restriction $\Phi\big|_F$ equals to $l_\alpha(\omega)$. We note that the essential exponent of the restriction $\Phi\big|_F$ is the same as $l_\alpha(\omega)$ since the codimension of F is finite. (3) In the case (III), we have that $\kappa(\Phi\big|_{F_j}) > l_\alpha(\omega)$ for all j, thus there are infinitely many Lyapunov exponents and $l_\alpha(\omega)$ is the limit point of these Lyapunov exponents. (4) When $S(\omega)$ is compact, $l_\alpha(\omega) = -\infty$. In this case, as long as there is no trajectory decaying faster than any exponential function, this theorem implies that there are infinitely many Lyapunov exponents. We notice that the random dynamical system generated by a stochastic parabolic partial differential equation is compact while the random dynamical system generated by a stochastic wave equation is not compact.

The results we obtain here are the extension of fundamental results by Oseledets, Ruelle, Mané, Thieullen, and Schaumlöffel to a general infinite dimensional random dynamical system in a separable Banach space.

The study of Lyapunov exponents goes back to Lyapunov [19] and has a long history. The fundamental results on the theory of Lyapunov exponents for finite dimensional systems were first obtained by Oseledets [34] in 1968, which is now called the Oseledets multiplicative ergodic theorem. Alternative proofs of this truly remarkable theorem can be found in [30], [35], [11], [24], [37], [38], [22], [6], [1], [16], [15], [4], [9], and [2]. These proofs are based on either the triangularization of a linear cocycle and the classical ergodic theorem for the triangular cocycle or the singular value decomposition of matrices and Kingman's subadditive ergodic theorem. This theorem plays a crucial role in the celebrated Pesin's theory for describing the dynamics of nonuniformly hyperbolic diffeomorphisms on compact manifolds. It is also an important and indispensable tool for studying the dynamics of random dynamical systems on compact manifolds, see [20], [21], [17], [13], and a recent survey [14].

In his remarkable paper [39], Ruelle extended the multiplicative ergodic theorem to compact random linear operators in a separable Hilbert space with a base measurable metric dynamical system in a probability space. A typical example of these maps is the time-one map of the solution operator of a stochastic or random parabolic partial differential equation. In this case, one has to face the difficulties arising from that the phase space is not locally compact and the dynamical system may not be invertible over the phase space. Ruelle's results have been applied to study other stochastic partial differential equations and delay differential equations. See, for example, [31]and [32].

Later, Mané [23], extended the multiplicative ergodic theorem to compact operators $S(\omega)$ in a Banach space, where the base metric dynamical system is a homeomorphism over a compact topological space and $S(\omega)$ is continuous in ω. A drawback of Mané's results is that they cannot be applied to random dynamical systems generated by stochastic partial differential equations. Besides the obstacles Ruelle encountered in a Hilbert space, one also needs to overcome the problem that there is no inner product. Mané took a different approach from Ruelle's.

Thieullen [42] further extended Mané's results on Lyapunov exponents to bounded linear operators $S(\omega)$ in a Banach space, where the base metric dynamical system θ is a homeomorphism over a topological space Ω homeomorphic to a Borel subset of a separable metric space and $S(\omega)$ is continuous in ω.

In [41], Flandoli and Schaumlöffel obtained a multiplicative ergodic theorem for random isomorphisms on a separable Hilbert space with a measurable metric dynamical system over a probability space. This result is used to study hyperbolic stochastic partial differential equations. Schaumlöffel, in his thesis [40], extended the multiplicative ergodic theorem to a class of bounded random linear operators which map a closed linear subspace onto a closed linear subspace in a Banach space with certain convexity.

Recently, Mierczynski and Shen, in [25, 26, 27, 28, 29], have extensively studied the principal Lyapunov exponents for random parabolic equations and obtained results of Krein-Rutman type.

1.2 Random Stable and Unstable Manifolds.

We consider a nonlinear random dynamical system $\phi(n, \omega, x)$ in a Banach space X. We assume that $\phi(n, \omega, x)$ has a random invariant set $\mathcal{A}(\omega)$. By invariance we mean that

$$\phi(n, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta^n \omega).$$

We will see in section 9 that if the conditions of Theorem A holds for the linearized system along orbits in $\mathcal{A}(\omega)$ and there is no zero Lyapunov exponent and $l_\alpha < 0$, then the nonuniform hyperbolicity holds. Our results on the existence of stable and unstable manifolds may be stated as

Theorem B. *(Existence of Stable and Unstable Manifold) Let $\phi(\omega, \cdot)$ be a $C^k, k \geq 2$, random map on Banach space X. Let $\mathcal{A}(\omega)$ be invariant subset of X. Assume that $\phi(\omega, \cdot)$ is non-uniformly hyperbolic on $\mathcal{A}$. Then there exist stable and unstable manifolds of $\mathcal{A}$.*

The theorem we have here is an extension of Pesin's stable manifold theorem to infinite dimensional random dynamical systems in a Banach space. The stable manifold theorem for nonuniformly hyperbolic diffeomorphisms on a compact manifold was first established by Pesin in his celebrated paper [36]. The Pesin's theorem was later extended to Hilbert space by Ruelle [39] for not only deterministic dynamical systems but also random dynamical systems. Mané extended it to deterministic dynamical systems with a compact invariant set in a Banach space. The proof of Pesin's stable manifold theorem for random dynamical systems on a compact manifolds can be founded in [17]. The local theory of invariant manifolds for finite dimensional random dynamical systems are given in [43] and Arnold [1].

1.3 Nontechnical Overview.

Our proof of Theorem A is based on Mane's approach and consists of seven main steps. The most significant difference we have here is that the metric dynamical system θ is only a measurable function on a probability space, the base space Ω has no topology and the base flow θ is not continuous, and $S(\omega)$ is only strongly measurable. Since the phase space X is a Banach space, there is no inner product, thus the approach used by Ruelle is not applicable. However, some of ideas from Ruelle will be used. We will also use ideas from Thieullen. In order to overcome these difficulties, new techniques are needed to establish the existence of Lyapunov exponents and their associated random invariant subspaces.

Step 1. Volume Functions.

We first introduce a volume function in a Banach space and give its basic properties. Then, we study the exponential growth rate of the volume function under a linear transformation and its relationships with the Kuratowski measure of the transformation. These details are given in Section 4.

Step 2. Gap and Distance between Closed Subspaces.

In Section 5, we first review the properties of the gap and distance between two closed subspaces of a Banach space X, which are taken from Kato [12]. Under this distance, the set of closed linear subspaces of X is a complete metric space. We show that the direct sum is a continuous function. As a consequence, we have that a measurable base gives the measurability of the space.

Step 3. Exponential Growth Rates.

We study five exponential growth rates: (1) the exponential growth rate of the volume function along pullback orbits of linear random dynamical systems; (2) the largest Lyapunov exponent; (3) the essential exponent; (4) the essential growth rate along pullback orbits; and (5) the exponential growth rate of the norm of the system along the pullback orbits. Kingman's subadditive ergodic theorem is used to establish the existence of these rates. A key estimate is that the essential exponent is the upper bound of the asymptotic limit of the exponential growth rate of the volume function along pullback orbits. These details are given in Section 6.1.

Step 4. Random Invariant Unstable Subspace.

In Section 6.2, we consider a random invariant unstable subspace E^{λ} with the exponential growth rate λ between the largest Lyapunov exponent and the essential exponent. In Section 6.1, we first show by using the volume function that it is a finite dimensional space. Then, we prove that its dimension is bounded below by an integer m which is completely determined by the exponential growth rate of the volume function along pullback orbits and the dimension of E^{λ} is exact m when λ equals to the largest Lyapunov exponent $\kappa(\Phi)$. Again, the main tool is the volume function. We also use a lemma due to Pliss on the estimation of a partial sum and the Birkhoff ergodic theorem.

Step 5. Measurability of $E^{\kappa(\Phi)}(\omega)$.

We modify the theorem of measurable selection taken from [3]. By using this modified theorem and the volume function, we construct a sequence of measurable m-dimensional spaces and show that this sequence converges to $E^{\kappa(\Phi)}(\omega)$.

Step 6. Oseledets Spaces.

In Subsection 6.3, we show that $E^{\kappa(\Phi)}(\omega)$ is the Oseledets space associated with the largest Lyapunov exponent.

Step 7. Measurable Invariant Complementary Subspace.

It is well-known that any finite dimensional subspace of a Banach space has a complementary subspace. Here, we show that the finite dimensional random invariant subspace $E^{\kappa(\Phi)}(\omega)$ has a *measurable* and *invariant* complementary subspace. We first show that every finite dimensional measurable subspace has a measurable basis by using the modified measurable selection theorem. Then we prove a measurable version of the Hahn-Banach theorem: every strongly measurable bounded linear functional on a finite dimensional subspace of a separable Banach space X can be extended to a measurable bounded linear functional on X with the same norm. To construct the measurable and invariant complementary subspace, we first construct a strongly measurable projection which gives an approximation of the complementary subspace. Using this approximation as a coordinate axis and the construction of the stable manifold, we construct the measurable and invariant complementary subspace. These details are given in Section 7.

The proof of Theorem A is given in Section 8 based on induction. In Section 9, we prove Theorem B by using the Lyapunov-Perron's approach.

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2 Notations and Preliminaries

The results and proofs represented in this thesis require a certain amount of technical notations and lemmas which we collect in this section for future reference.

2.1 Random Dynamical Systems

In this section, we review some of the basic concepts on random dynamical systems in a Banach Space that are taken from Arnold [1].

Let $(\Omega, \mathcal{F}, P)$ be a probability space and X be a Banach space. Let $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$ and $\mathbb{T}^+ = \mathbb{R}^+$ or $\mathbb{Z}^+$. $\mathbb{T}$ and $\mathbb{T}^+$ are endowed with their Borel $\sigma-$ algebra.

Definition 1. A family $(\theta^t)_{t\in\mathbb{T}}$ of mappings from Ω into itself is called a metric dynamical system if

(1) $(\omega, t) \to \theta^t\omega$ is $\mathcal{F} \otimes \mathcal{B}(\mathbb{T})$ measurable;

(2) $\theta^0 = id_\Omega$, the identity on Ω , $\theta^{t+s} = \theta^t \circ \theta^s$ for all $t, s \in \mathbb{T}$;

(3) θ^t preserves the probability measure P.

Definition 2. A map

$$\phi : \mathbb{T}^+ \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \phi(t, \omega, x),$$

is called a random dynamical system (or a cocycle) on the Banach space X over a metric dynamical system $(\Omega, \mathcal{F}, P, \theta^t)_{t\in\mathbb{T}}$ if

(1) ϕ is $\mathcal{B}(\mathbb{T}^+) \otimes \mathcal{F} \otimes \mathcal{B}(X)$-measurable;

(2) The mappings $\phi(t,\omega) := \phi(t,\omega,\cdot) : X \to X$ form a cocycle over θ^t:

$$\phi(0,\omega) = Id, \quad \text{for all } \omega \in \Omega,$$
$$\phi(t+s,\omega) = \phi(t,\theta^s\omega) \circ \phi(s,\omega), \quad \text{for all } t,s \in \mathbb{T}^+, \quad \omega \in \Omega.$$

A typical example of ϕ is the solution operator of a random or stochastic partial differential equation. See, for example,[8], [7], [18], and [32].

Here we borrow an example from [1]

Example 1. (Linear and Affine RDS) (i) Linear RDS: Let the measurable function $A : \Omega \to \mathbb{R}^{d\times d}$ satisfy $A \in L^1(\Omega,\mathcal{F},\mathbb{P})$. Then

$$\dot{x}_t = A(\theta_t\omega)x_t$$

generates a unique C^∞ RDS Φ satisfying

$$\Phi(t,\omega) = I + \int_0^t A(\theta_s\omega)\Phi(s,\omega)ds$$

and

$$\det \Phi(t,\omega) = \exp \int_0^t \text{trace} A(\theta_s\omega)ds.$$

Also, differentiating $\Phi(t,\omega)\Phi(t,\omega)^{-1} = I$ yields

$$\Phi(t,\omega)^{-1} = I - \int_0^t \Phi(s,\omega)^{-1}A(\theta_s\omega)ds.$$

Affine RDS: The equation

$$\dot{x}_t = A(\theta_t\omega)x_t + b(\theta_t\omega), \quad A,b \in L^1(\Omega,\mathcal{F},\mathbb{P}),$$

generates a unique C^∞ RDS. The variation of constants formula yields

$$\begin{aligned}\varphi(t,\omega)x &= \Phi(t,\omega)x + \int_0^t \Phi(t,\omega)\Phi(u,\omega)^{-1}du \\ &= \Phi(t,\omega) + \int_0^t \Phi(t-u,\theta_u\omega)b(\theta_u\omega)du,\end{aligned}$$

where Φ is the matrix cocycle generated by $\dot{x}_t = A(\theta_t\omega)x_t$.

$\phi(t,\omega,x)$ is said to be a C^N random dynamical systems if ϕ is C^N in x.

A map $S : \Omega \to L(X,X)$ is called a linear random map in a separable Banach space X if it is strongly measurable, i.e., $S(\cdot)x : \Omega \to X$ is measurable for each $x \in X$, where $L(X,X)$ is the usual Banach space of bounded linear operators from X to X. A linear random map $S(\omega)$ generates a linear random dynamical system with one-sided time over the metric dynamical system $(\Omega, \mathcal{F}, P, (\theta^n)_{n\in\mathbb{Z}})$.

$$\Phi(n,\omega) = \begin{cases} S(\theta^{n-1}\omega)\cdots S(\omega), & n > 0 \\ I, & n = 0. \end{cases}$$

If $S^{-1}(\omega)$ exists and is also a linear random map, then $S(\omega)$ generates a linear random dynamical system with two-sided time over the metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_n)_{n\in\mathbb{Z}})$.

$$\Phi(n,\omega) = \begin{cases} S(\theta^{n-1}\omega)\cdots S(\omega), & n > 0 \\ I, & n = 0 \\ S^{-1}(\theta^{n}\omega)\cdots S^{-1}(\theta^{-1}\omega), & n < 0. \end{cases}$$

The next concept is of fundamental importance in the study of random dynamical systems.

Definition 3. (i) A random variable $R : \Omega \to (0, \infty)$ is called tempered with respect to a metric dynamical system θ^n if

$$\lim_{t \to \pm\infty} \frac{1}{n} \log R(\theta^n \omega) = 0 \quad P - a.s.$$

(ii) $R : \Omega \to [0, \infty)$ is called tempered from above if

$$\lim_{n \to \pm\infty} \frac{1}{n} \log^+ R(\theta^n \omega) = 0 \quad P - a.s.$$

(iii) $R : \Omega \to (0, \infty)$ is called tempered from below if $1/R$ is tempered from above.

2.2 Ergodic Theory

In this section we will state some fundamental results of ergodic theory which will be a language in the proof.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. $T : \Omega \to \Omega$ is called measure-preserving transform if T satisfies:

(i) T is surjective,

(ii) T is measurable,

(iii) $\mathbb{P}(T^{-1}A) = \mathbb{P}(A)$ for any $A \in \mathcal{F}$.

A set $A \in \mathbb{F}$ is called invariant if $A = T^{-1}A$. T is called ergodic if Ω and $\emptyset$ are the only invariant sets.

Theorem 1. *(Birkhoff's Ergodic Theorem) Let $T : \Omega \to \Omega$ be a measure-preserving transform, f be a measurable function. Then there is a full measure set $\tilde{\Omega} \subset \Omega$ and a measurable function $\bar{f}$ such that for any $\omega \in \tilde{\Omega}$*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f \circ T^i(\omega) = \bar{f}(\omega).$$

If T is ergodic, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f\circ T^i(\omega) = \int_\Omega f d\mathbb{P}.$$

By choosing special f such as characteristic functions, one have

Corollary 2. *For any measurable set A and ergodic transform T*

$$\lim_{N\to\infty}\frac{1}{N}\sharp\{T^n(\omega)\in A, 1\le n\le N\} = \mathbb{P}(A) \quad a.e..$$

The following theorem is taken from [1]

Theorem 3. (Subadditive Ergodic Theorem). *(M,Σ,ρ) denotes a probability space, and $f: M\to M$ a measurable map preserving ρ. Let $\{F_n\}_{n>0}$ be a sequence of measurable functions from M to $\mathbb{R}\bigcup\{-\infty\}$ satisfying the conditions:*

$$(a)\ integrability:\ F_1^+\in L^1(M,\Sigma,\rho),$$

$$(b)\ subadditivity:\ F_{m+n}\le F_m + F_n\circ f^m almost\ everywhere.$$

Then there exists an f-invariant measurable function $F: M\to\mathbb{R}\bigcup\{-\infty\}$ such that $F^+\in L^1(M,\Sigma,\rho)$,

$$\lim_{n\to\infty}\frac{1}{n}F_n = F\ a.s.$$

and

$$\lim_{n\to\infty}\frac{1}{n}\int F_n(x)\rho(dx) = \inf_{n\in\mathbb{N}}\frac{1}{n}\int F_n(x)\rho(dx) = \int F(x)\rho(dx).$$

2.3 Measures of noncompactness

We now review the measures of noncompactness and their properties. Let B be a subset of X. The Kuratowski measure of noncompactness, α, defined by

$$\alpha(B) = \inf\{d\ :\ B \text{ has a finite cover of diameter } < d\}.$$

The α-measure of the noncompactness satisfies

(i) $\alpha(B) = 0$ for $B \subset X$ if and only if B is precompact.

(ii) $\alpha(B \cup C) = \max\{\alpha(B), \alpha(C)\}$ for $B, C \subset X$.

(iii) $\alpha(B + C) \leq \alpha(B) + \alpha(C)$.

(iv) α(closed convex hull of B) $= \alpha(B)$.

(v) If $B_1 \supset B_2 \supset \cdots$ are nonempty closed sets of X such that $\alpha(B_i) \to 0$ as $i \to \infty$, then $\bigcap_{i\geq 1} B_i$ is nonempty and compact.

Let L be a map from X to X. We define

$$||L||_\alpha = \inf\big\{k > 0 \ : \ \alpha\big(L(B)\big) \leq k\alpha(B) \text{ for all bounded sets } B \subset X\big\}. \tag{1}$$

If L is a bounded linear operator, then

$$||L||_\alpha = \alpha\big(L(B(0,1))\big),$$

where $B(0,1)$ is the unit ball of X with center at 0. Furthermore, $||\cdot||_\alpha$ is multiplicative norm:

$$\|L_1 + L_2\|_\alpha \leq \|L_1\|_\alpha + \|L_2\|_\alpha, \quad \|L_1 \circ L_2\|_\alpha \leq \|L_1\|_\alpha \|L_2\|_\alpha$$

and $\|\cdot\|_\alpha$ is a continuous function on $L(X,X)$. The number $||L||_\alpha$ is related to the radius of the essential spectrum of L. The limit $\lim_{n\to\infty} ||L^n||_\alpha^{1/n}$ is equal to the radius of the essential spectrum of L, see Nussbaum [33].

3 Main Results

In this section, we state our main results. We consider a linear random map $S(\omega)$ in a separable Banach space X. We assume that $S(\omega)$ is injective almost surely. As we have seen in Section 2, $S(\omega)$ generates a one-side time random dynamical system $\Phi(n,\omega)$ on X over the metric dynamical system $(\theta^n)_{n\in\mathbb{Z}}$. We note that $(\Phi(n,\omega))^{-1}$ may not be a bounded linear operator since the range of $S(\omega)$ may not be the whole space.

Before we state our main results, introduce two important quantities

$$l_\alpha(\omega) = \lim_{n\to+\infty} \frac{1}{n} \log \|\Phi(n,\omega)\|_\alpha$$

and

$$\kappa(\Phi)(\omega) = \lim_{n\to+\infty} \frac{1}{n} \log \|\Phi(n,\omega)\|,$$

where $||\cdot||_\alpha$ is defined in (1), which is related to the essential spectrum and $\kappa(\Phi)(\omega)$ is the largest Lyapunov exponent. In the deterministic case when $\Phi(n,\omega) = S^n$ where S is a bounded linear operator, e^{l_α} is the radium of the essential spectrum of S. We will see that both limits exist almost surely under the assumptions of our main theorem. We will also see that $\kappa(\Phi)(\omega) \geq l_\alpha(\omega)$ and they are θ invariant.

Theorem 4. *(Multiplicative Ergodic Theorem) Assume that $S(\cdot) : \Omega \to L(X,X)$ is strongly measurable, $S(\omega)$ is injective almost everywhere and*

$$\log^+ ||S(\cdot)|| \in L^1(\Omega, \mathcal{F}, P).$$

Then there exists a θ-invariant subset $\tilde{\Omega} \subset \Omega$ of full measure such that for each $\omega \in \tilde{\Omega}$ only one of the following conditions holds

(I) $\kappa(\Phi)(\omega) = l_\alpha(\omega)$.

(II) There exist $k(\omega)$ *numbers* $\lambda_1(\omega) > \ldots > \lambda_{k(\omega)}(\omega) > l_\alpha(\omega)$ *and a splitting*

$$X = E_1(\omega) \oplus \cdots \oplus E_{k(\omega)}(\omega) \oplus F(\omega)$$

of finite dimensional linear subspaces $E_j(\omega)$ *and infinite dimensional linear subspace* $F(\omega)$ *such that*

1) *Invariance:* $k(\theta\omega) = k(\omega)$, $\lambda_i(\theta\omega) = \lambda_i(\omega)$, $S(\omega)E_j(\omega) = E_j(\theta\omega)$ *and* $S(\omega)F(\omega) \subset F(\theta\omega)$;

2) *Lyapunov Exponents:*

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|\Phi(n,\omega)v\| = \lambda_j(\omega) \text{ for all } v(\neq 0) \in E_j(\omega), 1 \leq j \leq k;$$

3) *Exponential Decay Rate on* $F(\omega)$*:*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|\Phi(n,\omega)|_{F(\omega)}\| \leq l_\alpha(\omega)$$

and if $v(\neq 0) \in F(\omega)$ *and* $(\Phi(n, \theta^{-n}\omega))^{-1}v$ *exists for all* $n \geq 0$, *which is denoted by* $\Phi(-n,\omega)v$, *then*

$$\liminf_{n\to+\infty} \frac{1}{n} \log \|\Phi(-n,\omega)v\| \geq -l_\alpha(\omega);$$

4) *Measurability and Temperedness:* $k(\omega), \lambda_i(\omega)$, *and* $E_j(\omega)$ *are measurable and the projection operators associated with the decompositions*

$$X = \Big(\bigoplus_{i=1}^{j} E_i(\omega)\Big) \oplus \Big(\big(\bigoplus_{i=j+1}^{k(\omega)} E_i(\omega)\big) \oplus F(\omega)\Big) = \Big(\bigoplus_{i=1}^{k(\omega)} E_i(\omega)\Big) \oplus F(\omega)$$

are strongly measurable and tempered.

(III) There exist infinitely many finite dimensional subspaces $E_j(\omega)$, infinitely many infinite dimensional subspaces $F_j(\omega)$, and infinitely many numbers

$$\lambda_1(\omega) > \lambda_2(\omega) > \ldots > l_\alpha(\omega) \text{ with } \lim_{j\to+\infty} \lambda_j(\omega) = l_\alpha(\omega)$$

such that

1) Invariance: $\lambda_i(\theta\omega) = \lambda_i(\omega)$, $S(\omega)E_j(\omega) = E_j(\theta\omega)$, $S(\omega)F_j(\omega) \subset F_j(\theta\omega)$;

2) Invariant Splitting:

$$E_1(\omega) \oplus \cdots \oplus E_j(\omega) \oplus F_j(\omega) = X \quad \text{and} \quad F_j(\omega) = E_{j+1}(\omega) \oplus F_{j+1}(\omega);$$

3) Lyapunov Exponents:

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|\Phi(n,\omega)v\| = \lambda_j(\omega), \text{for all } v(\neq 0) \in E_j(\omega);$$

4) Exponential Decay Rate on $F_j(\omega)$:

$$\lim_{n\to+\infty} \frac{1}{n} \log \|\Phi(n,\omega)\big|_{F_j(\omega)}\| = \lambda_{j+1}(\omega)$$

and if for $v(\neq 0) \in F_j(\omega)$ such that $\Phi(-n,\omega)v$ exists for all $n \geq 0$, then

$$\liminf_{n\to+\infty} \frac{1}{n} \log \|\Phi(-n,\omega)v\| \geq -\lambda_{j+1}(\omega);$$

5) Measurability and Temperedness: $\lambda_j(\omega)$ and $E_j(\omega)$ are measurable and the *projection operators associated with the decomposition*

$$X = \left(\bigoplus_{i=1}^{j} E_i(\omega) \right) \oplus F_j(\omega)$$

are strongly measurable and tempered.

Here $\lambda_i(\omega)$ are the so-called Lyapunov exponents and $E_i(\omega)$, $F_i(\omega)$ and $F(\omega)$ are the so-called Oseledets spaces.

As a corollary of Theorem 4, we have the following for compact linear random dynamical systems.

Theorem 5. *(Multiplicative Ergodic Theorem for Compact Linear Random Dynamical Systems) Assume that $S(\cdot) : \Omega \to L(X, X)$ is strongly measurable, $S(\omega)$ is injective and compact almost everywhere and*

$$\log^+ ||S(\cdot)|| \in L^1(\Omega, \mathcal{F}, P).$$

Then there exists a θ-invariant subset $\tilde{\Omega} \subset \Omega$ of full measure such that for each $\omega \in \tilde{\Omega}$ only one of the following conditions holds

(I) $\kappa(\Phi)(\omega) = -\infty$;

(II) There exist $k(\omega)$ numbers $\lambda_1(\omega) > \ldots > \lambda_{k(\omega)}(\omega) > -\infty$ and a splitting

$$X = E_1(\omega) \oplus \cdots \oplus E_{k(\omega)}(\omega) \oplus F(\omega)$$

of finite dimensional linear subspaces $E_j(\omega)$ and infinite dimensional linear subspace $F(\omega)$ such that

1) *Invariance:* $k(\theta\omega) = k(\omega)$, $\lambda_i(\theta\omega) = \lambda_i(\omega)$, $S(\omega)E_j(\omega) = E_j(\theta\omega)$ *and* $S(\omega)F(\omega) \subset F(\theta\omega)$;

2) *Lyapunov Exponents:*

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|\Phi(n,\omega)v\| = \lambda_j(\omega) \text{ for all } v(\neq 0) \in E_j(\omega), 1 \leq j \leq k;$$

3) *Exponential Decay Rate on* $F(\omega)$*:*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|\Phi(n,\omega)|_{F(\omega)}\| = -\infty$$

and if $v(\neq 0) \in F(\omega)$ *and* $(\Phi(n, \theta^{-n}\omega))^{-1}v$ *exists for all* $n \geq 0$, *which is denoted by* $\Phi(-n,\omega)v$, *then*

$$\liminf_{n\to+\infty} \frac{1}{n} \log \|\Phi(-n,\omega)v\| = +\infty;$$

4) *Measurability and Temperedness:* $k(\omega), \lambda_i(\omega)$, *and* $E_j(\omega)$ *are measurable and the projection operators associated with the decompositions*

$$X = \Big(\bigoplus_{i=1}^{j} E_i(\omega)\Big) \oplus \Big(\big(\bigoplus_{i=j+1}^{k(\omega)} E_i(\omega)\big) \oplus F(\omega)\Big) = \Big(\bigoplus_{i=1}^{k(\omega)} E_i(\omega)\Big) \oplus F(\omega)$$

are strongly measurable and tempered.

(III) *There exist infinitely many finite dimensional subspaces* $E_j(\omega)$, *infinitely many infinite dimensional subspaces* $F_j(\omega)$, *and infinitely many numbers*

$$\lambda_1(\omega) > \lambda_2(\omega) > \ldots > -\infty \text{ with } \lim_{j\to+\infty} \lambda_j(\omega) = -\infty$$

such that

1) Invariance: $\lambda_i(\theta\omega) = \lambda_i(\omega)$, $S(\omega)E_j(\omega) = E_j(\theta\omega)$, $S(\omega)F_j(\omega) \subset F_j(\theta\omega)$;

2) Invariant Splitting:

$$E_1(\omega) \oplus \cdots \oplus E_j(\omega) \oplus F_j(\omega) = X \quad \textit{and} \quad F_j(\omega) = E_{j+1}(\omega) \oplus F_{j+1}(\omega);$$

3) Lyapunov Exponents:

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|\Phi(n,\omega)v\| = \lambda_j(\omega), \textit{for all } v(\neq 0) \in E_j(\omega);$$

4) Exponential Decay Rate on $F_j(\omega)$*:*

$$\lim_{n\to+\infty} \frac{1}{n} \log \|\Phi(n,\omega)\big|_{F_j(\omega)}\| = \lambda_{j+1}(\omega)$$

and if for $v(\neq 0) \in F_j(\omega)$ *such that* $\Phi(-n,\omega)v$ *exists for all* $n \geq 0$*, then*

$$\liminf_{n\to+\infty} \frac{1}{n} \log \|\Phi(-n,\omega)v\| \geq -\lambda_{j+1}(\omega);$$

5) Measurability and Temperedness: $\lambda_j(\omega)$ *and* $E_j(\omega)$ *are measurable and the projection operators associated with the decomposition*

$$X = \left(\bigoplus_{i=1}^{j} E_i(\omega) \right) \oplus F_j(\omega)$$

are strongly measurable and tempered.

The next theorem is the multiplicative ergodic theorem for continuous time linear random dynamical systems.

Theorem 6. *(Multiplicative Ergodic Theorem for Continuous Time Linear Random Dynamical Systems) Let* $\Phi(t,\omega) : \mathbb{R}^+ \times \Omega \to L(X,X)$ *be a continuous time cocycle* *and X be a separable Banach space. Assume $\Phi(1,\cdot):\Omega\to L(X,X)$ is strongly measurable, $\Phi(1,\omega)$ is injective almost everywhere, and*

$$\sup_{0\le s\le 1}\log^+||\Phi(s,\cdot)|| \text{ and } \sup_{0\le s\le 1}\log^+||\Phi(1-s,\theta^s\cdot)||\in L^1(\Omega,\mathcal{F},P).$$

Let

$$l_\alpha(\omega)=\lim_{s\to+\infty}\frac{1}{s}\log\|\Phi(s,\omega)\|_\alpha$$

and

$$\kappa(\Phi)(\omega)=\lim_{s\to+\infty}\frac{1}{s}\log\|\Phi(s,\omega)\|.$$

Then there exists a θ^t-invariant subset $\tilde{\Omega}\subset\Omega$ of full measure such that for each $\omega\in\tilde{\Omega}$ only one of the following conditions holds

(I) $\kappa(\Phi)(\omega)=l_\alpha(\omega)$.

(II) There exist $k(\omega)$ numbers $\lambda_1(\omega)>\ldots>\lambda_{k(\omega)}(\omega)>l_\alpha(\omega)$ and a splitting

$$X=E_1(\omega)\oplus\cdots\oplus E_{k(\omega)}(\omega)\oplus F(\omega)$$

of finite dimensional linear subspaces $E_j(\omega)$ and infinite dimensional linear subspace $F(\omega)$ such that

1) Invariance: $k(\theta^t\omega)=k(\omega)$, $\lambda_i(\theta^t\omega)=\lambda_i(\omega)$, $\Phi(t,\omega)E_j(\omega)=E_j(\theta^t\omega)$ and $\Phi(t,\omega)F(\omega)\subset F(\theta^t\omega)$;

2) Lyapunov Exponents:

$$\lim_{t\to\pm\infty}\frac{1}{t}\log\|\Phi(t,\omega)v\|=\lambda_j(\omega) \text{ for all } v(\neq 0)\in E_j(\omega), 1\le j\le k;$$

3) Exponential Decay Rate on $F(\omega)$:

$$\limsup_{t\to+\infty}\frac{1}{t}\log\|\Phi(t,\omega)|_{F(\omega)}\|\le l_\alpha(\omega)$$

and if $v(\neq 0)\in F(\omega)$ and $(\Phi(t,\theta^{-t}\omega))^{-1}v$ exists for all $t\geq 0$, which is denoted by $\Phi(-t,\omega)v$, then

$$\liminf_{t\to+\infty}\frac{1}{t}\log\|\Phi(-t,\omega)v\|\geq -l_\alpha(\omega);$$

4) Measurability and Temperedness: $k(\omega),\lambda_i(\omega)$, and $E_j(\omega)$ are measurable and the projection operators associated with the decompositions

$$X=\Big(\bigoplus_{i=1}^{j}E_i(\omega)\Big)\oplus\Big(\big(\bigoplus_{i=j+1}^{k(\omega)}E_i(\omega)\big)\oplus F(\omega)\Big)=\Big(\bigoplus_{i=1}^{k(\omega)}E_i(\omega)\Big)\oplus F(\omega)$$

are strongly measurable and tempered.

(III) There exist infinitely many finite dimensional subspaces $E_j(\omega)$, infinitely many infinite dimensional subspaces $F_j(\omega)$, and infinitely many numbers

$$\lambda_1(\omega)>\lambda_2(\omega)>\ldots>l_\alpha(\omega)\ \text{with}\ \lim_{j\to+\infty}\lambda_j(\omega)=l_\alpha(\omega)$$

such that

1) Invariance: $\lambda_i(\theta^t\omega)=\lambda_i(\omega)$, $\Phi(t,\omega)E_j(\omega)=E_j(\theta^t\omega)$, $\Phi(t,\omega)F_j(\omega)\subset F_j(\theta^t\omega)$;

2) Invariant Splitting:

$$E_1(\omega)\oplus\cdots\oplus E_j(\omega)\oplus F_j(\omega)=X\ \text{ and }\ F_j(\omega)=E_{j+1}(\omega)\oplus F_{j+1}(\omega);$$

3) Lyapunov Exponents:

$$\lim_{t\to\pm\infty}\frac{1}{t}\log\|\Phi(t,\omega)v\| = \lambda_j(\omega), \textit{for all } v(\neq 0)\in E_j(\omega);$$

4) Exponential Decay Rate on $F_j(\omega)$:

$$\lim_{t\to+\infty}\frac{1}{t}\log\|\Phi(t,\omega)\big|_{F_j(\omega)}\| = \lambda_{j+1}(\omega)$$

and if for $v(\neq 0)\in F_j(\omega)$ such that $(\Phi(t,\theta^{-t}\omega))^{-1}v$ exists for all $t\geq 0$, which is denoted by $\Phi(-t,\omega)v$, then

$$\liminf_{t\to+\infty}\frac{1}{t}\log\|\Phi(-t,\omega)v\| \geq -\lambda_{j+1}(\omega);$$

5) Measurability and Temperedness: $\lambda_j(\omega)$ and $E_j(\omega)$ are measurable and the projection operators associated with the decomposition

$$X = \left(\bigoplus_{i=1}^{j} E_i(\omega)\right)\oplus F_j(\omega)$$

are strongly measurable and tempered.

The proof of Theorem 6 follows from Theorem 4 and the following facts.

Lemma 7. *Under the assumptions of Theorem 6, it holds almost everywhere that for every $v\in X$*

$$\liminf_{s\to+\infty}\frac{1}{s}\log\|\Phi(s,\omega)v\| \geq \liminf_{n\to+\infty}\frac{1}{n}\log\|\Phi(n,\omega)v\|, \tag{2}$$

$$\limsup_{s\to+\infty}\frac{1}{s}\log\|\Phi(s,\omega)v\| \leq \limsup_{n\to+\infty}\frac{1}{n}\log\|\Phi(n,\omega)v\|. \tag{3}$$

The proof of this lemma follows from a standard argument. For completeness, we provide the proof here.

Proof. For $s \in \mathbb{R}^+$, let $[s] := \sup\{n \in \mathbb{N} \cup \{0\} : n \leq s\}$ and $\{s\} := s - [s]$. By using the cocycle property, we obtain that

$$\|\Phi(s,\omega)v\| \leq \|\Phi(\{s\}, \theta_{[s]}\omega)\| \|\Phi([s],\omega)v\|$$

and

$$\|\Phi([s],\omega)v\| \leq \|\Phi(1 - \{s\}, \theta_{s-1}\omega)\| \|\Phi(s-1,\omega)v\|.$$

This implies

$$\limsup_{s\to+\infty} \frac{1}{s} \log \|\Phi(s,\omega)v\| \leq \limsup_{s\to+\infty} \frac{1}{[s]} \log \|\Phi([s],\omega)v\| + \limsup_{s\to+\infty} \frac{1}{[s]} \log \sup_{0\leq t\leq 1} \|\Phi(t, \theta_{[s]}\omega)\|.$$

Since $\log \sup_{0\leq t\leq 1} \|\Phi(t,\omega)\| \leq \sup_{0\leq t\leq 1} \log^+ \|\Phi(t,\omega)\| \in L^1(\Omega, \mathcal{F}, P)$, we have that

$$\limsup_{s\to+\infty} \frac{1}{[s]} \log \sup_{0\leq t\leq 1} \|\Phi(t, \theta_{[s]}\omega)\| \leq 0.$$

Similarly, we have

$$\begin{aligned}
&\liminf_{s\to+\infty} \frac{1}{[s]} \log \|\Phi([s],\omega)v\| \\
&\leq \liminf_{s\to+\infty} \frac{1}{s-1} \log \|\Phi(s-1,\omega)v\| + \limsup_{s\to+\infty} \frac{1}{[s]-1} \log \sup_{0\leq t\leq 1} \|\Phi(1-t, \theta_{[s]-1}\theta_t\omega)\|.
\end{aligned}$$

We also have that $\log \sup_{0\leq t\leq 1} \|\Phi(1-t, \theta_t\omega)\| \leq \sup_{0\leq t\leq 1} \log^+ \|\Phi(1-t, \theta_t\omega)\| \in L^1(\Omega, \mathcal{F}, P)$. Thus,

$$\limsup_{s\to+\infty} \frac{1}{[s]-1} \log \sup_{0\leq t\leq 1} \|\Phi(1-t, \theta_{[s]-1}\theta_t\omega)\| \leq 0 \text{ a.e..}$$

The proof is complete. $\square$

From Theorem 4, we have the following. The details are given in Section 7.

Theorem 8. *(Nonuniformly Exponential Dichotomy) If $\lambda_i(\omega) \neq 0$ and $l_\alpha(\omega) < 0$ for all $1 \leq i$, let $E^u(\omega) = \bigoplus_{\lambda_i > 0} E_i(\omega)$ and $E^s(\omega)$ to be the invariant complementary subspace of $E^u(\omega)$. Also denote $\Pi_s(\omega)$, $\Pi_u(\omega)$ to be projections onto $E^s(\omega)$, $E^u(\omega)$ respectively associate to the splitting $E(\omega) = E^s(\omega) \oplus E^u(\omega)$. Then there is a random variable $\beta : \Omega \to (0, \infty)$ and a tempered random variable $K(\omega) : \Omega \to [1, \infty)$ such that $\beta(\theta\omega) = \beta(\omega)$ for all $\omega \in \Omega$, which are constant when θ^n is ergodic, and*

$$||\Phi(n,\omega)\Pi_s(\omega)|| \leq K(\omega)e^{-\beta(\omega)n} \quad \textit{for } n \geq 0 \tag{4}$$

$$||\Phi(n,\omega)\Pi_u(\omega)|| \leq K(\omega)e^{\beta(\omega)n} \quad \textit{for } n \leq 0. \tag{5}$$

Here $\beta(\omega)$ is chosen to be smaller than the absolute values of all Lyapunov exponents. For example, one may choose

$$\beta(\omega) = \frac{1}{2}\min\{\|\lambda_i(\omega)\|\}.$$

However, along each orbit $\theta^n\omega$, $\beta(\omega)$ is a constant and $K(\omega)$ can increase only at a subexponential rate, which together with conditions (101) and (102) imply that the linear system $\Phi(n,\omega)$ is nonuniformly hyperbolic in the sense of Pesin.

4 Volume Function in Banach Spaces

In this section, we study a volume function defined in a Banach space. This function plays key roles in establishing Lyapunov exponents and measurable Oseledets invariant subspaces.

4.1 Volume Function $V_n(w_1, w_2, \ldots, w_n)$.

Let X be a Banach space with norm $\|\cdot\|$. For each positive integer n, we define a function V_n from the product space X^n to $\mathbb{R}^+$ by

$$V_n(w_1, w_2, \ldots, w_n) = \Big(\prod_{i=1}^{n-1} \mathrm{dist}(w_i, \mathrm{span}\{w_{i+1}, \ldots, w_n\})\Big)\|w_n\|, \tag{6}$$

where $w_i \in X$ for $i = 1, \cdots, n-1$, and

$$\mathrm{dist}(w_i, \mathrm{span}\{w_{i+1}, \ldots, w_n\}) = \inf\{\|w_i - v\| \mid v \in \ \mathrm{span}\{w_{i+1}, \ldots, w_n\}\}$$

for $i = 1, \ldots, n-1$.

For the sake of convenience, we define

$$\overline{\mathrm{dist}}(w_i, \mathrm{span}\{w_{i+1}, \ldots, w_n\}) = \begin{cases} \mathrm{dist}(w_i, \mathrm{span}\{w_{i+1}, \ldots, w_n\}), & 1 \le i < n, \\ ||w_n||, & i = n. \end{cases} \tag{7}$$

Remark 1. In Euclidian Space $\mathbb{R}^n$, with a given base (not necessary orthogonal), each vector v can be represented by a $n \times 1$ matrix, denoted by $\tilde{v}$, and the inner product satisfies that $< u, v >= \tilde{u}^T \ G \ \tilde{v}$, where G is a positive symmetric matrix. For vectors $v_1, v_2, \cdots, v_n$, by using Gram-Schmidt method, there is a lower triangular matrix T having 1s on its diagonal such that $(\tilde{v}_1, \cdots, \tilde{v}_n)T$ is a G-orthogonal matrix. Note that $\det T = 1$ and

$$[(\tilde{v}_1, \cdots, \tilde{v}_n)T]^T \; G \; [(\tilde{v}_1, \cdots, \tilde{v}_n)T] = diag(\overline{\mathrm{dist}}(v_i, \mathrm{span}\{v_{i+1}, \ldots, v_n\})^2)_{i=1,\ldots,n}.$$

Thus we have

$$V(v_1, \cdots, v_n) = \sqrt{\det G} |\det(\tilde{v}_1, \cdots, \tilde{v}_n)|.$$

So this volume function we defined in Banach Spaces becomes an usual volume (without direction) when goes back to Euclidian Spaces.

We first note that

$$0 \leq V_n(w_1, w_2, \ldots, w_n) \leq \prod_{i=1}^{n} \|w_i\|.$$

It is also not hard to see that this volume function satisfies the usual property

$$V_n(k_1 w_1, k_2 w_2, \ldots, k_n w_n) = \left(\prod_{i=1}^{n} |k_i| \right) V_n(w_1, w_2, \ldots, w_n) \tag{8}$$

for $k_i \in \mathbb{R},\ i = 1, \cdots, n.$

The next lemma gives a uniform lower bound for the distance between a vector and the span of other vectors.

Lemma 9. *Assume that vectors $v_i \in X, 1 \leq i \leq n$, satisfy that for some $C > 0$*

$$\overline{\mathrm{dist}}(v_j, \mathrm{span}\{v_i\}_{j<i\leq n}) > C,\ 1 \leq j \leq n.$$

Then

$$\mathrm{dist}(v_j, \mathrm{span}\{v_i\}_{1\leq i\leq n,\ i\neq j}) > \left(\frac{C}{r+C}\right)^{n-1} C,\ 1 \leq j \leq n, \tag{9}$$

where $r = \max_{1\le i\le n} \|v_i\|$.

Proof. Let $1 \le m \le n$. Then, there exist n real numbers $a_i^m, 1 \le i \le n$, such that $a_m^m = 1$ and

$$\operatorname{dist}(v_m, \operatorname{span}\{v_i\}_{1\le i\le n,\ i\ne m}) = \Big\| \sum_{i=1}^{n} a_i^m v_i \Big\|. \tag{10}$$

Let

$$m' = \min\Big\{ i \Big| \ |a_i^m| \ge \Big(\frac{C}{r+C}\Big)^{n-i} \Big\}.$$

Then $m' \le m$ since $a_m^m = 1$. Therefore, by (10), we have

$$\begin{aligned}
&\operatorname{dist}(v_m, \operatorname{span}\{v_i\}_{1\le i\le n,\ i\ne m})\\
\ge&\Big\| \sum_{i=m'}^{n} a_i^m v_i \Big\| - \sum_{i=1}^{m'-1} |a_i^m| \|v_i\|\\
\ge&|a_{m'}^m| \overline{\operatorname{dist}}(v_{m'}, \operatorname{span}\{v_i\}_{m'+1\le i\le n}) - r \sum_{i=1}^{m'-1} |a_i^m|\\
\ge&\Big(\frac{C}{r+C}\Big)^{n-m'} C - r\left(\Big(\frac{C}{r+C}\Big)^{n-m'} \frac{C}{r} - \frac{C}{r}\Big(\frac{C}{r+C}\Big)^{n-1} \right)\\
\ge&\Big(\frac{C}{r+C}\Big)^{n-1} C.
\end{aligned}$$

This completes the proof. □

The norm we used in the product space is the maximum norm defined by

$$\|(w_1, \ldots, w_n)\| = \max_{1\le i\le n} \|w_i\|.$$

Lemma 10. *For each fixed* $n \in \mathbb{N}$, V_n *is a uniformly continuous function on each bounded subset of* X^n.

Proof. The proof of the continuity of V_n is quite straightforward. However, since X is an infinite dimensional space and there is no local compactness, the uniform continuity on a bounded set does not follows immediately. To see this, it is enough to show that V_n is uniformly continuous on each ball $B(0,r)$ of X^n. Let $w=(w_1,\dots,w_n)\in B(0,r)$ and $\Delta w=(\Delta w_1,\dots,\Delta w_n)\in X^n$. We will show that for fixed $r>0$ and n and for each $\epsilon>0$ there exists $\delta>0$ depending only on ϵ such that if $\|\Delta w\|<\delta$, then

$$\|V_n(w+\Delta w)-V_n(w)\|\le\epsilon.$$

Let

$$C(w)=\min\left\{\overline{\mathrm{dist}}(w_i,\mathrm{span}\{w_j\}_{i<j\le n})\mid 1\le i\le n\right\}$$

and $R=2^{n+1}r^n/\epsilon$. We consider two cases.

Case 1. We assume $C(w)\ge\frac{r}{R}$. Let $\delta=\min\left\{r,\frac{r}{2nR(1+R)^{n-1}},\frac{r}{n^2R^2(1+R)^{n-1}}\right\}$. We first note that there exists $\{a_{ij}\}_{1\le i\le j\le n}\subset\mathbb{R}$ such that

$$a_{ii}=1\text{ and }\overline{\mathrm{dist}}(w_i,\mathrm{span}\{w_j\}_{i<j})=\|\sum_{j=i}^{n}a_{ij}w_j\|.$$

Since $\|\sum_{j=i+1}^{n}a_{ij}w_j\|\le\overline{\mathrm{dist}}(w_i,\mathrm{span}\{w_j\}_{i<j})+\|w_i\|\le 2r$, by using Lemma 9, we have that

$$|a_{ij}|\le\frac{2r}{\mathrm{dist}(w_j,\mathrm{span}\{w_k\}_{k\ne j})}\le\frac{2r}{\left(\frac{C(w)}{r+C(w)}\right)^{n-1}C(w)}\le 2R(1+R)^{n-1}.$$

For any Δw such that $\|\Delta w\| < \delta$, we have $w + \Delta w \in B(0, 2r)$ and for $1 \leq i \leq n$,

$$\begin{aligned}
&\overline{\text{dist}}(w_i + \Delta w_i, \text{span}\{w_j + \Delta w_j\}_{i<j\leq n}) - \overline{\text{dist}}(w_i, \text{span}\{w_j\}_{i<j\leq n}) \\
&= \overline{\text{dist}}(w_i + \Delta w_i, \text{span}\{w_j + \Delta w_j\}_{i<j\leq n}) - \|\sum_{j=i}^{n} a_{ij} w_j\| \\
&\leq \|\sum_{j=i}^{n} a_{ij}(w_j + \Delta w_j) - \sum_{j=i}^{n} a_{ij} w_j\| \leq \|\sum_{j=i}^{n} a_{ij} \Delta w_j\| \\
&< \sum_{j=i}^{n} \|a_{ij}\| \delta \leq 2(1+R)^{n-1} R n \delta.
\end{aligned} \tag{11}$$

Similarly, there exists $\{a'_{ij}\}_{1\leq i\leq j\leq n} \subset \mathbb{R}$ such that

$$a'_{ii} = 1 \text{ and } \overline{\text{dist}}(w_i + \Delta w_i, \text{span}\{w_j + \Delta w_j\}_{i<j\leq n}) = \|\sum_{j=i}^{n} a'_{ij}(w_j + \Delta w_j)\|.$$

Let

$$M'_i = \max\{|a'_{ij}|\}_{i\leq j\leq n} \text{ and } M' = \max\{|a'_{ij}|\}_{1\leq i,j\leq n}.$$

Then, by Lemma 9 and the choice of δ, we have

$$\begin{aligned}
2r \geq &\overline{\text{dist}}(w_i + \Delta w_i, \text{span}\{w_j + \Delta w_j\}_{i<j\leq n}) \\
\geq &\|\sum_{j=i}^{n} a'_{ij} w_j\| - \|\sum_{j=i}^{n} a'_{ij} \Delta w_j\| \\
\geq &M'_i \min\left\{\text{dist}(w_j, \text{span}\{w_k\}_{k\neq j})\right\}_{1\leq j\leq n} - nM'_i\delta \\
\geq &M'_i \left(\frac{C(w)}{r + C(w)}\right)^{n-1} C(w) - nM'_i\delta \\
\geq &\frac{M'_i r}{2R(1+R)^{n-1}},
\end{aligned}$$

which implies

$$M' = \max\{M'_i\}_{1\leq i\leq n} \leq 4R(1+R)^{n-1}.$$

Thus, for $1 \leq i \leq n$, we obtain

$$
\begin{aligned}
&\overline{\text{dist}}(w_i, \text{span}\{w_j\}_{i<j\leq n}) - \overline{\text{dist}}(w_i + \Delta w_i, \text{span}\{w_j + \Delta w_j\}_{i<j\leq n}) \\
=&\overline{\text{dist}}(w_i, \text{span}\{w_j\}_{i<j\leq n}) - \|\sum_{j=i}^{n} a'_{ij}(w_j + \Delta w_j)\| \\
\leq&\|\sum_{j=i}^{n} a'_{ij} w_j - \sum_{j=i}^{n} a'_{ij}(w_j + \Delta w_j)\| \\
=&\|\sum_{j=i}^{n} a'_{ij} \Delta w_j\| \\
<&\sum_{j=i}^{n} M'\delta \leq 4R(1+R)^{n-1} n\delta.
\end{aligned} \tag{12}
$$

Therefore, from (11) and (12), it follows that for any $1 \leq i \leq n$

$$
|\overline{\text{dist}}(w_i, \text{span}\{w_j\}_{i<j\leq n}) - \overline{\text{dist}}(w_i + \Delta w_i, \text{span}\{w_j + \Delta w_j\}_{i<j\leq n})| < \frac{\epsilon}{n 2^{n-1} r^{n-1}}.
$$

So, we have that

$$
\begin{aligned}
&\|V_n(w + \Delta w) - V_n(w)\| \\
=&\Big\|\prod_{i=1}^{n} \overline{\text{dist}}(w_i + \Delta w_i, \text{span}\{w_j + \Delta w_j\}_{i<j\leq n}) - V_n(w)\Big\| \\
\leq&\Big\|\prod_{i=1}^{n} [\overline{\text{dist}}(w_i, \text{span}\{w_j\}_{i<j\leq n}) + K] - V_n(w)\Big\| \\
\leq& n(2r)^{n-1} K < \epsilon,
\end{aligned}
$$

where $K = \max_{1\leq i\leq n} \|\overline{\text{dist}}(w_i, \text{span}\{w_j\}_{i<j\leq n}) - \overline{\text{dist}}(w_i + \Delta w_i, \text{span}\{w_j + \Delta w_j\}_{i<j\leq n})\|$.

Case 2. Assume that $C(w) < \frac{r}{R}$. Choose $\delta = \min\left\{r, \frac{r}{nR(1+R)^{n-1}}, \frac{r}{nR^2(1+R)^{n-1}}\right\}$. We claim that if $\|\Delta w\| < \delta$, then

$$
C(w + \Delta w) \leq \frac{\epsilon}{2^n r^{n-1}}.
$$

Using this claim, we obtain

$$\begin{aligned}
&\|V_n(w+\Delta w) - V_n(w)\| \leq \|V_n(w+\Delta w)\| + \|V_n(w)\| \\
&\leq C(w+\Delta w)(2r)^{n-1} + C(w)r^{n-1} < \epsilon.
\end{aligned}$$

We prove this claim by contradiction. Suppose that

$$C(w+\Delta w) > \frac{\epsilon}{2^n r^{n-1}} = \frac{2r}{R}.$$

As in Case 1, we choose $\{a_{ij}\}_{1\leq i\leq j\leq n} \subset \mathbb{R}$ such that

$$a_{ii} = 1 \text{ and } \overline{\text{dist}}(w_i, \text{span}\{w_j\}_{i<j\leq n}) = \|\sum_{j=i}^{n} a_{ij} w_j\|.$$

Let

$$M_i = \max\{|a_{ij}|\}_{i\leq j\leq n} \text{ and } M = \max\{|a_{ij}|\}_{1\leq i,j\leq n}.$$

Then, by Lemma 9, we have

$$\begin{aligned}
r \geq& \overline{\text{dist}}(w_i, \text{span}\{w_j\}_{i<j\leq n}) \\
\geq& \|\sum_{j=i}^{n} a_{ij}(w_j + \Delta w_j)\| - \|\sum_{j=i}^{n} a_{ij}\Delta w_j\| \\
\geq& M_i \min\left\{\text{dist}(w_j + \Delta w_j, \text{span}\{w_k + \Delta w_k\}_{k\neq j})\right\}_{1\leq j\leq n} - nM_i\delta \\
\geq& M_i \left(\frac{C(w+\Delta w)}{2r + C(w+\Delta w)}\right)^{n-1} C(w+\Delta w) - nM_i\delta \\
\geq& \frac{M_i r}{R(1+R)^{n-1}},
\end{aligned}$$

which implies

$$M = \max\{M_i\}_{1\leq i\leq n} \leq R(1+R)^{n-1}.$$

Thus for $1 \leq i \leq n$,

$$
\begin{aligned}
&\overline{\text{dist}}(w_i + \Delta w_i, \text{span}\{w_j + \Delta w_j\}_{i<j\leq n}) - \overline{\text{dist}}(w_i, \text{span}\{w_j\}_{i<j\leq n}) \\
&= \overline{\text{dist}}(w_i + \Delta w_i, \text{span}\{w_j + \Delta w_j\}_{i<j\leq n}) - \| \sum_{j=i}^{n} a_{ij} w_j \| \\
&\leq \| \sum_{j=i}^{n} a_{ij}(w_j + \Delta w_j) - \sum_{j=i}^{n} a_{ij} w_j \| \\
&= \| \sum_{j=i}^{n} a_{ij} \Delta w_j \| \leq \sum_{j=i}^{n} M\delta \leq R(1+R)^{n-1} n\delta \leq \frac{r}{R}.
\end{aligned}
$$

Therefore $C(w) \geq C(w+\Delta w) - \frac{r}{R} \geq \frac{r}{R}$, which contradicts to the assumption $C(w) < r/R$. This completes the proof of the theorem.

$\square$

From the proof of Lemma 10, we also obtain the following

Lemma 11. *Let $r > 0$, $n \in \mathbb{N}$, and $0 < C < r$ be fixed. For each $\{w_i\}_{1\leq i\leq n} \in B(0,r)$ satisfying*

$$\min \left\{ \overline{\text{dist}}\big(w_i, \text{span}\{w_j\}_{i<j\leq n}\big) \right\}_{1\leq i\leq n} > C$$

and each $\epsilon > 0$, there exists a $\delta > 0$ depending only on r, n, C, and ϵ such that if $\|\Delta w_i\| < \delta$, then

$$\left| \overline{\text{dist}}(w_i, \text{span}\{w_j\}_{i<j\leq n}) - \overline{\text{dist}}(w_i + \Delta w_i, \text{span}\{w_j + \Delta w_j\}_{i<j\leq n}) \right| < \epsilon, \ 1 \leq i \leq n.$$

4.2 Change of Volume Under Linear Transformations.

Let $T \in L(X, X)$, the space of all bounded linear operators from a Banach space X to itself. We define

$$V_n(T) = \sup_{\substack{\|w_i\|=1 \\ 1\le i\le n}} V_n(Tw_1, Tw_2, \ldots, Tw_n). \tag{13}$$

Lemma 12. *The following holds*

$$V_n(T) = \sup_{V_n(w_1,\ldots,w_n)=1} V_n(Tw_1, \ldots, Tw_n) = \sup_{V_n(w_1,\ldots,w_n)\neq 0} \frac{V_n(Tw_1, \ldots, Tw_n)}{V_n(w_1, \ldots, w_n)}. \tag{14}$$

Proof. First, from (8) we have

$$\sup_{V_n(w_1,\ldots,w_n)=1} V_n(Tw_1, \ldots, Tw_n) = \sup_{V_n(w_1,\ldots,w_n)\neq 0} \frac{V_n(Tw_1, \ldots, Tw_n)}{V_n(w_1, \ldots, w_n)}.$$

Let

$$V_n'(T) = \sup_{V_n(w_1,\ldots,w_n)=1} V_n(Tw_1, \ldots, Tw_n).$$

We want to show $V_n(T) = V_n'(T)$. By the definition of $V_n'(T)$, for any $\epsilon > 0$, there exists $\{w_i\}_{1\le i\le n}$ such that

$$V_n(w_1, \ldots, w_n) = 1$$

and

$$V_n(Tw_1, Tw_2, \ldots, Tw_n) > V_n'(T) - \epsilon.$$

Then for any $1 \le i \le n$, we have $w_i' \in \operatorname{span}\{w_j\}_{i\le j\le n}$ such that $\|w_i'\| = 1$ and

$$w_i - \overline{\operatorname{dist}}(w_i, \operatorname{span}\{w_j\}_{i<j\le n}) w_i' \in \operatorname{span}\{w_j\}_{i<j\le n}.$$

Therefore,

$$V_n(w_1',\ldots,w_n') = V_n(\frac{w_1}{\mathrm{dist}(w_1,\mathrm{span}\{w_j\}_{1<j\leq n})},\ldots,\frac{w_n}{\|w_n\|}) = \frac{V_n(w_1,\ldots,w_n)}{V_n(w_1,\ldots,w_n)} = 1$$

and

$$\begin{aligned}
&V_n(Tw_1',\ldots,Tw_n')\\
&= V_n(\frac{Tw_1}{\mathrm{dist}(w_1,\mathrm{span}\{w_j\}_{1<j\leq n})},\ldots,\frac{Tw_n}{\|w_n\|})\\
&= \frac{V_n(Tw_1,\ldots,Tw_n)}{V_n(w_1,\ldots,w_n)}\\
&= V_n(Tw_1,\ldots,Tw_n)\\
&> V_n'(T) - \epsilon.
\end{aligned}$$

Hence,

$$V_n(T) \geq V_n'(T).$$

On the other hand, there exist linearly independent unit vectors $\{v_i\}_{1\leq i\leq n}$ such that

$$V_n(Tv_1,\ldots,Tv_n) > V_n(T) - \epsilon.$$

Since $0 < V_n(v_1,\ldots,v_n) \leq 1$, we have

$$\frac{V_n(Tv_1,\ldots,Tv_n)}{V_n(v_1,\ldots,v_n)} > V_n(T) - \epsilon.$$

Thus,

$$V_n'(T) \geq V_n(T).$$

So $V_n(T) = V_n'(T)$. This completes the proof of the lemma. $\square$

The next lemma is on the monotonicity of $\frac{1}{n}\log V_n(T)$.

Lemma 13. *Let $T \in L(X,X)$. Then,*

$$\frac{1}{n+1}\log V_{n+1}(T) \leq \frac{1}{n}\log V_n(T).$$

Proof. From the definition of $V_n(T)$, for any $\epsilon > 0$ there are $w_1, w_2, \ldots, w_{n+1}$ such that

$$\|w_i\| = 1\ ,\ i = 1, \ldots, n+1$$

and

$$V_{n+1}(Tw_1, \ldots, Tw_{n+1}) \geq V_{n+1}(T) - \epsilon.$$

Let $k \in \{1, \ldots, n+1\}$ such that

$$\overline{\text{dist}}\big(Tw_k, \text{span}\{Tw_j\}_{k<j\leq n+1}\big) = \min_{i=1,\ldots,n+1}\big\{\overline{\text{dist}}(Tw_i, \text{span}\{Tw_j\}_{i<j\leq n+1})\big\},$$

which is denoted by a^{ϵ}_{n+1}. Thus,

$$\begin{aligned}
V_{n+1}(T) &\leq \epsilon + a^{\epsilon}_{n+1}\prod_{\substack{i=1\\ i\neq k}}^{n+1}\overline{\text{dist}}(Tw_i, \text{span}\{Tw_j\}_{i<j\leq n+1})\\
&= \epsilon + a^{\epsilon}_{n+1}\prod_{i=k+1}^{n+1}\overline{\text{dist}}\big(Tw_i, \text{span}\{Tw_j\}_{i<j\leq n+1}\big)\cdot\prod_{i=1}^{k-1}\overline{\text{dist}}(Tw_i, \text{span}\{Tw_j\}_{i<j\leq n+1}\big)\\
&\leq \epsilon + a^{\epsilon}_{n+1}\prod_{i=k+1}^{n+1}\overline{\text{dist}}\big(Tw_i, \text{span}\{Tw_j\}_{i<j\leq n+1}\big)\cdot\prod_{i=1}^{k-1}\overline{\text{dist}}\big(Tw_i, \text{span}\{Tw_j\}_{i<j\leq n+1,\, j\neq k}\big)\\
&= \epsilon + a^{\epsilon}_{n+1}V_n(Tw_1, Tw_3, \ldots, Tw_{k-1}, Tw_{k+1}, \ldots, Tw_{n+1}).
\end{aligned}$$

Since $a^{\epsilon}_{n+1} \leq (V_{n+1}(T))^{\frac{1}{n+1}}$, we have

$$V_{n+1}(T) \leq \epsilon + (V_{n+1}(T))^{\frac{1}{n+1}}V_n(T),$$

thus

$$V_{n+1}(T) \leq (V_{n+1}(T))^{\frac{1}{n+1}} V_n(T),$$

which yields that

$$\frac{1}{n+1} \log V_{n+1}(T) \leq \frac{1}{n} \log V_n(T).$$

The proof of the lemma is complete.

$\square$

Lemma 13 implies that $\frac{1}{n} \log V_n(T)$ has a limit. The following lemma gives a upper bound for this limit in terms of the measure of noncompactness of operator T.

Lemma 14.

$$\lim_{n \to +\infty} \frac{1}{n} \log V_n(T) \leq \log 2r (= -\infty \text{ when } r = 0)$$

where $r = \|T\|_\alpha$.

Proof. From Lemma 13, $\frac{1}{n} \log V_n(T)$ is a non-increasing sequence. Thus, $\lim_{n \to +\infty} \frac{1}{n} \log V_n(T)$ exists, which is denoted by A.

We prove this lemma by a contradiction. Suppose $A > \log 2r$. Then, for sufficiently small $\epsilon > 0$, $A - 3\epsilon > \log(2 + \epsilon)r$. Note that there exists $N > 0$ such that if $n > N$, then

$$A \leq \frac{1}{n} \log V_n(T) \leq A + \epsilon,$$

which means

$$e^{nA} \leq V_n(T) \leq e^{n(A+\epsilon)}.$$

For any $m > N$, there exist unit vectors $\{w_i\}_{i=1,\dots,2m}$ such that

$$\frac{1}{2m} \log V_{2m}(Tw_1, \dots, Tw_{2m}) \geq \frac{1}{2m} \log V_{2m}(T) - \epsilon \geq A - \epsilon.$$

Let σ be a permutation on $\{1,\dots,2m\}$ such that for any $1 \le i < j \le 2m$,

$$\overline{\mathrm{dist}}(Tw_{\sigma^{-1}(i)}, \mathrm{span}\{Tw_k\}_{\sigma^{-1}(i)<k\le 2m}) \ge \overline{\mathrm{dist}}(Tw_{\sigma^{-1}(j)}, \mathrm{span}\{Tw_l\}_{\sigma^{-1}(j)<l\le 2m}).$$

Then,

$$\begin{aligned}
A-\epsilon &\le \frac{1}{2m}\log V_{2m}(Tw_1,\dots,Tw_{2m}) \\
&= \frac{1}{2m}\Big[\sum_{i=1}^{2m}\log\overline{\mathrm{dist}}\big(Tw_{\sigma^{-1}(i)}, \mathrm{span}\{Tw_k\}_{k>\sigma^{-1}(i)}\big)\Big] \\
&\le \frac{1}{2}\Big[\frac{1}{m}\sum_{i=1}^{m}\log\overline{\mathrm{dist}}\big(Tw_{\sigma^{-1}(i)}, \mathrm{span}\{Tw_k\}_{k>\sigma^{-1}(i)}\big) \\
&\qquad + \log\overline{\mathrm{dist}}\big(Tw_{\sigma^{-1}(m+1)}, \mathrm{span}\{Tw_k\}_{k>\sigma^{-1}(m+1)}\big)\Big] \\
&\le \frac{1}{2}\Big[\frac{1}{m}\log\prod_{i=1}^{m}\overline{\mathrm{dist}}\big(Tw_{\sigma^{-1}(i)}, \mathrm{span}\{Tw_k\}_{\substack{k>\sigma^{-1}(i)\\ k\in\sigma^{-1}\{1,\dots,m\}}}\big) \\
&\qquad + \log\overline{\mathrm{dist}}\big(Tw_{\sigma^{-1}(m+1)}, \mathrm{span}\{Tw_k\}_{k>\sigma^{-1}(m+1)}\big)\Big] \\
&\le \frac{1}{2}\Big[\frac{1}{m}\log V_m(T) + \log\overline{\mathrm{dist}}\big(Tw_{\sigma^{-1}(m+1)}, \mathrm{span}\{Tw_k\}_{k>\sigma^{-1}(m+1)}\big)\Big] \\
&\le \frac{1}{2}[A+\epsilon+\log\overline{\mathrm{dist}}\big(Tw_{\sigma^{-1}(m+1)}, \mathrm{span}\{Tw_k\}_{k>\sigma^{-1}(m+1)}\big)],
\end{aligned}$$

which implies that

$$\overline{\mathrm{dist}}\big(Tw_{\sigma^{-1}(m+1)}, \mathrm{span}\{Tw_k\}_{k>\sigma^{-1}(m+1)}\big) \ge e^{A-3\epsilon}.$$

Thus, by the definition of σ we have

$$\overline{\mathrm{dist}}\big(Tw_{\sigma^{-1}(i)}, \mathrm{span}\{Tw_k\}_{k>\sigma^{-1}(i)}\big) \ge e^{A-3\epsilon}$$

for $1 \le i \le m+1$. Then, letting

$$S_m = \{Tw_{\sigma^{-1}(i)}\}_{i=1,\dots,m+1},$$

we have

$$S_m \in \overline{T[B(0,1)]}.$$

Let D be a subset of Banach space X, define

$$N(D, \epsilon) = \inf\{N |\, D \text{ can be covered by } N \text{ balls whose radius are less than } \epsilon\}.$$

Then

$$N(S_m, \frac{1}{2+\epsilon} e^{A-3\epsilon}) \geq m+1,$$

which goes to infinite as $m \to \infty$. On the other hand, since $\|T\|_\alpha = r$ and $\frac{1}{2+\epsilon} e^{A-3\epsilon} > r$, we have

$$N\left(\overline{T[B(0,1)]}, \frac{1}{2+\epsilon} e^{A-3\epsilon}\right) < +\infty,$$

which gives a contradiction. Hence

$$\lim_{n\to\infty} \frac{1}{n} \log V_n(T) \leq \log 2r.$$

This completes the proof of this lemma. □

Lemma 15. *For any $n \in \mathbb{N}$ and bounded operators T_1, T_2, we have*

$$V_n(T_2 T_1) \leq V_n(T_2) V_n(T_1).$$

Proof. This directly follows from (14). In fact, we have that

$$\begin{aligned} V_n(T_2T_1) &= \sup_{V_n(v_1,\ldots,v_n)\neq 0} \frac{V_n(T_2T_1v_1,\ldots,T_2T_1v_n)}{V_n(v_1,\ldots,v_n)} \\ &\leq \sup_{V_n(v_1,\ldots,v_n)\neq 0} \frac{V_n(T_1v_1,\ldots,T_1v_n)}{V_n(v_1,\ldots,v_n)} \sup_{V_n(T_1v_1,\ldots,T_1v_n)\neq 0} \frac{V_n(T_2T_1v_1,\ldots,T_2T_1v_n)}{V_n(T_1v_1,\ldots,T_1v_n)} \\ &\leq V_n(T_1)V_n(T_2). \end{aligned}$$

$\square$

From the definition of $V_n(T)$, we have

Lemma 16. $V_n(T) \leq \|T\|^n$.

5 Gap and Distance Between Closed Linear Subspaces

In this section, we introduce the gap and distance between two closed subspaces of a Banach space X and give their basic properties. We first give the concepts of gap and distance taken from Kato [12], pp 197.

Let M and N be the linear subspaces of the Banach space X. Let S_M denote the unit sphere of M. Set

$$\delta(M, N) = \sup_{u \in S_M} \operatorname{dist}(u, N), \tag{15}$$

$$\hat{\delta}(M, N) = \max[\delta(M, N), \delta(N, M)]. \tag{16}$$

Note that when $M = 0$, $\delta(M, N)$ is not defined by (15). In this case, we define $\delta(0, N) = 0$. Clearly, when $\dim(M) \geq 1$, $\delta(M, 0) = 1$. $\hat{\delta}$ is called the gap between M and N. Note that $\hat{\delta}$ is not a distance.

Define for linear subspaces $M \neq 0, N \neq 0$

$$d(M, N) = \sup_{u \in S_M} \operatorname{dist}(u, S_N), \tag{17}$$

$$\hat{d}(M, N) = \max[d(M, N), d(N, M)], \tag{18}$$

and set $d(0, N) = 0$ for any N and $d(M, 0) = 2$ for $M \neq 0$.

Then, the set of closed linear subspaces of X is a complete metric space if the distance between M, N is given by $\hat{d}(M, N)$, which is denoted by $\mathbf{K}$. The following estimates also hold

$$\delta(M,N) \leq d(M,N) \leq 2\delta(M,N), \hat{\delta}(M,N) \leq \hat{d}(M,N) \leq 2\hat{\delta}(M,N), \tag{19}$$

which follows from that for any unit vector $u \in X$,

$$\operatorname{dist}(u, S_M) \leq 2\operatorname{dist}(u, M). \tag{20}$$

For details, see [12], pp 198.

We denote by K_m the set of m-dimensional subspaces of X endowed with the metric $\hat{d}$. In addition to the above concepts from [12], we also define

$$\begin{aligned} d'(M,N) &= \inf_{u \in S_M} \operatorname{dist}(u, S_N), \quad \text{for } M, N \neq 0, \\ d'(0,N) &= d'(M,0) = 0. \end{aligned} \tag{21}$$

Lemma 17. *Let M, N, and Z be finite-dimensional linear subspaces. Then, we have*

$$d'(M,N) = d'(N,M), \tag{22}$$

$$d'(M,N) \geq d'(M,Z) - d(N,Z). \tag{23}$$

Proof. We first prove that (22) holds. Since M is finite-dimensional, for each $v \in S_N$, there exists a $u \in S_M$ such that $\operatorname{dist}(v, S_M) = \|v - u\|$. On the other hand, $\|v - u\| \geq \operatorname{dist}(u, S_N) \geq d'(M,N)$. Thus, $d'(N,M) \geq d'(M,N)$. Switching the positions of M and N gives (22).

Next, we show (23). Since M, N, and Z are finite-dimensional, for each $u \in S_M$, there exist $v \in S_N$ and $w \in S_Z$ such that $\operatorname{dist}(u, S_N) = \|u - v\|$ and $\operatorname{dist}(v, S_Z) = \|v - w\|$. So, $\operatorname{dist}(u, S_N) \geq \|u - w\| - \|v - w\|$. Using the definitions of d and d', we have $\|v - w\| \leq d(N,Z)$ and $\|u - w\| \geq \operatorname{dist}(u, S_Z) \geq d'(M,Z)$, which yield (23).

This completes the proof of this lemma. $\square$

Remark 2. Lemma 17 is also true for closed subspaces M, N, Z. The proof needs only a minor modification.

If M and N are finite-dimensional subspaces, then $d'(M, N) > 0$ if and only if $M \cap N = \{0\}$. For $m, n \in \mathbb{N}$, set

$$K_{(m,n)} = \{(M, N) |\ M \in K_m,\ N \in K_n\}$$

endowed with the metric

$$\hat{d}_{(m,n)}((M_1, N_1), (M_2, N_2)) = \hat{d}(M_1, M_2) + \hat{d}(N_1, N_2).$$

Lemma 18. *Let* $K^+_{(m,n)} = \left\{(M, N) \in K_{(m,n)} \mid d'(M, N) > 0\right\}$ *and define* $\Sigma : K^+_{(m,n)} \to K_{m+n}$ *as* $\Sigma(M, N) = M \oplus N$*, then* Σ *is continuous.*

Proof. Let $\epsilon > 0$ and $(M, N) \in K^+_{(m,n)}$. For a unit vector v in $M \oplus N$, we may write v as $v = v_1 + v_2$ where $v_1 \in M$ and $v_2 \in N$. Let $C = d'(M, N) > 0$. Using (20), we have

$$1 \geq \operatorname{dist}(v_1, N) \geq \frac{1}{2}\|v_1\| d'(M, N) \tag{24}$$

and

$$1 \geq \operatorname{dist}(v_2, M) \geq \frac{1}{2}\|v_2\| d'(N, M), \tag{25}$$

which give

$$\max\{\|v_1\|, \|v_2\|\} \leq \frac{2}{d'(N, M)} = \frac{2}{C}. \tag{26}$$

Take $\delta = \min\{\frac{C}{4}, \frac{C\epsilon}{16}\}$. Then for any $(E, F) \in K_{(m,n)}$, if $\hat{d}_{(m,n)}((M, N), (E, F)) < \delta$, by (17) we have

$$d'(F,M) = d'(M,F) \geq d'(M,N) - d(F,N) \geq C - \delta \geq \frac{3}{4}C,$$
$$d'(E,N) = d'(N,E) \geq d'(N,M) - d(E,M) \geq C - \delta \geq \frac{3}{4}C.$$

Thus

$$d'(E,F) \geq d'(E,N) - d(F,N) \geq \frac{3}{4}C - \delta \geq \frac{1}{2}C. \tag{27}$$

For v_1 and v_2, using the definition of d and (26), there are $v_1' \in E$ and $v_2' \in F$ such that

$$\|v_1 - v_1'\| \leq \|v_1\| d(M,E) \leq \frac{2\delta}{C} < \frac{\epsilon}{4}$$

and

$$\|v_2 - v_2'\| \leq \|v_2\| d(N,F) \leq \frac{2\delta}{C} < \frac{\epsilon}{4}.$$

Therefore,

$$\mathrm{dist}(v_1 + v_2, E \oplus F) < \frac{\epsilon}{2}.$$

Then, using (19), we obtain

$$d(M \oplus N, E \oplus F) \leq 2\delta(M \oplus N, E \oplus F) < \epsilon.$$

Similarly, for a unit vector $u \in E \oplus F$, there exist $u_1 \in E$ and $u_2 \in F$ such that $u = u_1 + u_2$. In the same fashion as for v_1 and v_2, using (27), we have

$$\max\{\|u_1\|, \|u_2\|\} \leq \frac{2}{d'(E,F)} \leq \frac{4}{C}.$$

Furthermore, there are $u_1' \in M$ and $u_2' \in N$ such that

$$\|u_1 - u_1'\| \leq \|u_1\| d(E,M) \leq \frac{4\delta}{C} \leq \frac{\epsilon}{4}$$

and

$$\|u_2 - u_2'\| \leq \|u_2\| d(F, N) \leq \frac{4\delta}{C} \leq \frac{\epsilon}{4}.$$

Thus,

$$d(E \oplus F, M \oplus N) \leq 2\delta(E \oplus F, M \oplus N) \leq \epsilon.$$

Hence,

$$\hat{d}(E \oplus F, M \oplus N) \leq \epsilon.$$

This completes the proof of the lemma. □

Remark 3. From the proof one can obtain that $K^+_{(m,n)}$ is an open set in $K_{(m,n)}$.

As a special case of Lemma 18, we have

Corollary 19. *Let $n \geq 1$ and*

$$K^+_{1\times n} = \left\{(E, F) \in K_1 \times K_n \;\middle|\; d(E, F) > 0\right\}.$$

Then, the map $\Psi :\; K^+_{1\times n} \to K_{1+n}$ defined by

$$\Psi(E, F) = E \oplus F$$

is a continuous map.

From the definition of $\hat{d}$ it follows that if $e(\omega) \neq 0$ is measurable from Ω to X, then span$\{e(\omega)\}$ is measurable from Ω to K_1. Thus, by using Corollary 19, we have

Lemma 20. *Let $e_1(\omega), \cdots, e_n(\omega)$ be linearly independent and measurable. Then*

$$span\big\{e_1(\omega), \cdots, e_n(\omega)\big\}$$

is measurable from Ω to K_n.

6 Lyapunov Exponents and Oseledets Spaces

In this section we establish the principal Lyapunov exponent and its associated Oseledets space. For the sake of simplicity, we assume that the metric dynamical system $(\Omega, \mathcal{F}, P, \{\theta^n\}_{n\in\mathbb{Z}})$ is ergodic. The standard example of an ergodic metric dynamical system is the one generated by a Wiener process. When Ω can be decomposed into ergodic components (see Cornfeld, Fomin, and Sinai [5]), we simply restrict our study to each component. For general non-ergodic metric dynamical systems, the results can be proved in the same way with some modifications which are given in Appendix B.

For the remainder of this thesis, we assume

$$\log^+ ||S(\cdot)|| \in L^1(\Omega, \mathcal{F}, P). \tag{28}$$

We will prove Theorem 4 by an induction procedure. We assume that there exists an invariant splitting of the phase space

$$X = E(\omega) \oplus G(\omega),$$

where $E(\omega)$ is a finitely dimensional linear subspace of dimension $d \geq 0$ and $G(\omega)$ is a linear subspace of finite codimension, such that

(B1) $S(\omega)(E(\omega)) = E(\theta\omega), \quad S(\omega)(G(\omega)) \subset G(\theta\omega)$;

(B2) Mappings $\omega \to E(\omega)$ is measurable and $\omega \to \pi(\omega)$ is strongly measurable, where $\pi(\omega)$ is the associated projection from X onto $G(\omega)$;

(B3) $\|\pi(\omega)\|$ is tempered.

We consider the restriction of $S(\omega)$ onto $G(\omega)$. We denote

$$T(\omega) = S(\omega)\pi(\omega).$$

Since $S(\omega)$ is injective, $T(\omega)|_{G(\omega)}$ is injective. For any $v \in S(\theta^{-1}\omega)G(\theta^{-1}\omega)$, we denote by $T^{-1}(\omega)v$ the vector $v' \in G(\theta^{-1}\omega)$ such that $S(\theta^{-1}\omega)v' = v$. Let

$$T^n(\omega) = T(\theta^{n-1}\omega)\cdots T(\omega).$$

Note that $T^n(\omega)u = \Phi(n,\omega)u$ for $u \in G(\omega)$. For any $w \in T^n(\theta^{-n}\omega)G(\theta^{-n}\omega)$ we also denote by $T^{-n}(\omega)w$ the vector $w' \in G(\theta^{-n}\omega)$ such that $T^n(\theta^{-n}\omega)w' = w$. We summarize the properties of $T(\omega)$ as follows

(T1) $T(\omega) \in L(X,X)$ and $T(\omega)G(\omega) \subset G(\theta\omega)$;

(T2) $T(\omega)$ is strongly measurable;

(T3) $T(\omega)x = 0, \quad x \in E(\omega)$;

(T4) $T(\omega)\big|_{G(\omega)}$ is injective;

(T5) $\log^+ \|T(\cdot)\big|_{G(\cdot)}\| \in L^1(\Omega, \mathcal{F}, P)$.

These properties are the conditions under which our results in this section and next section hold.

For $\lambda \in \mathbb{R}$, we let

$$E^\lambda(\omega) = \left\{ v \in G(\omega) |\ T^{-n}(\omega)v \text{ exists for all } n \geq 0 \text{ and } \lim_{n\to+\infty}\sup \frac{1}{n}\log\|T^{-n}(\omega)\cdot v\| \leq -\lambda \right\}. \tag{29}$$

Our main results of this section are the following.

Theorem 21. *If $\lambda > l_\alpha$, then*

$$\dim E^{\lambda}(\omega) < +\infty, \quad a.s..$$

Theorem 22. *If $\kappa(T) > l_\alpha$, then*

$$\dim E^{\kappa(T)}(\omega) = m \geq 1, \quad a.s.,$$

where m is a constant. Furthermore, $E^{\kappa(T)}(\cdot):\ \Omega \to K_m$ is measurable.

Here the largest Lyapunov exponent $\kappa(T)$ and the essential exponent l_α are given in Lemma 25.

Remark 4. If we replace $\pi(\cdot)$ and $G(\cdot)$ by identity operator I and X respectively, then $\kappa(T)$ becomes to be $\kappa(\Phi)$ and in lemma 25, for such T, we will obtain that

$$l_\alpha = \lim_{n\to\infty} \frac{1}{n} \log \|T^n(\omega)\|_\alpha \text{ a.e..}$$

Thus, the properties of $E^{\kappa(\Phi)}(\omega)$ follows from these propositions.

Theorem 23. *There exists a θ-invariant subset $\tilde{\Omega} \subset \Omega$ of full measure such that for each $\omega \in \tilde{\Omega}$*

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|T^n(\omega)v\| = \kappa(T)$$

for every $v(\neq 0) \in E^{\kappa(T)}(\omega)$. Furthermore,

$$\lim_{n\to\infty} \frac{1}{n} \log \|T^{-n}(\omega)\big|_{E^{\kappa(T)}(\omega)}\| = -\kappa(T)$$

The proofs of these theorems are based on a number of lemmas and propositions, which will be given in the following four subsections.

6.1 Exponential Growth Rates.

In this section, we study several exponential growth rates. The first lemma gives the measurability of $\|T(\cdot)\|_\alpha$.

Lemma 24. *If $T : \Omega \to L(X, X)$ is strongly measurable, where X is a separable Banach space, then $\|T(\cdot)\|_\alpha$ is measurable.*

Proof. For each $r > 0$, we denote

$$\Omega_r = \{\omega \mid \|T(\omega)\|_\alpha < r\}.$$

We want to show that Ω_r is measurable. Since X is separable, there exist countably many vectors $\{x_i\}_{i\geq 1}$ and $\{y_i\}_{i\geq 1}$ such that $\{x_i\}_{i\geq 1}$ is dense in X and $\{y_i\}_{i\geq 1}$ is dense in $B(0, 1)$. Let U be the set of all finite subsets of $\{x_i\}_{i\geq 1}$. It is easy to see that U contains countable many elements. So we can assume that $U = \{U_i\}_{i\geq 1}$.
Let

$$\Omega'_r = \bigcup_{n=2}^{+\infty}\bigcup_{i=1}^{+\infty}\bigcap_{j=1}^{+\infty}\left\{\omega \,\middle|\, T(\omega)y_j \in \bigcup_{x\in U_i} B(x, r - \frac{r}{n})\right\}.$$

Since $T(\cdot)$ is strongly measurable, Ω'_r is measurable. We will show that $\Omega'_r = \Omega_r$.

First, we show that $\Omega'_r \subset \Omega_r$. For any $\omega \in \Omega'_r$, there exist $n \geq 2$ and $i \geq 1$ such that

$$\{T(\omega)y_j\}_{j\geq 1} \subset \bigcup_{x\in U_i} B(x, r - \frac{r}{n}).$$

Since for any $y \in B(0, 1)$, there exists y_j such that $\|y_j - y\| < \frac{r}{2n\|T(\omega)\|}$. Then there exists $x \in U_i$ such that $T(\omega)y \in B(x, r - \frac{r}{2n})$. Thus

$$T(\omega)B(0,1) \subset \bigcup_{x \in U_i} B(x, r - \frac{r}{2n}).$$

So $\|T(\omega)\|_\alpha < r$, which implies $\omega \in \Omega_r$.

Next, we prove that $\Omega_r \subset \Omega'_r$. For any $\omega \in \Omega_r$, we have $\|T(\omega)\|_\alpha < r$. Note that $\|T(\omega)\|_\alpha = r'$. Then, there exists $n > 1$ such that $r - \frac{r}{n} > r'$. Thus, there exists a finite set $\{x'_i\}_{1 \le i \le N} \subset X$ such that

$$T(\omega)B(0,1) \in \bigcup_{i=1}^{N} B(x'_i, r - \frac{r}{n}).$$

Since $\{x_i\}_{i \ge 1}$ is dense in X, for each x'_i there exists $x_{i'}$ such that $\|x'_i - x_{i'}\| < \frac{r}{2n}$. So $B(x'_i, r - \frac{r}{n}) \subset B(x_{i'}, r - \frac{r}{2n})$, therefore

$$T(\omega)B(0,1) \subset \bigcup_{i=1}^{N} B(x_{i'}, r - \frac{r}{2n}).$$

Thus $\omega \in \Omega'_r$. The proof is complete. $\square$

The next lemma is about the existence of the growth rates associated with the norm of operator, the Kuratowski measure of operator, and the volume function.

Lemma 25. *There exists a θ-invariant subset $\tilde{\Omega} \subset \Omega$ of full measure such for each*

$\omega \in \Omega$ the following limits exist

$$\lim_{n \to +\infty} \frac{1}{n} \log V_k(T^n(\theta^{-n}(\omega))) = l_k(T), \tag{30}$$

$$\lim_{n \to +\infty} \frac{1}{n} \log \|T^n(\omega)\| = \kappa(T), \tag{31}$$

$$\lim_{n \to +\infty} \frac{1}{n} \log \|T^n(\omega)\|_\alpha = l_\alpha, \tag{32}$$

$$\lim_{n \to +\infty} \frac{1}{n} \log \|T^n(\theta^{-n}\omega)\|_\alpha = l'_\alpha, \tag{33}$$

$$\lim_{n \to +\infty} \frac{1}{n} \log \|T^n(\theta^{-n}\omega)\| = \kappa'(T), \tag{34}$$

where $l_k(T)$, $\kappa(T)$, l_α, l'_α, and $\kappa'(T)$ are either constants or $-\infty$.

Proof. We first prove (30). Since

$$V_k(T^n(\theta^{-n}(\omega))|_{G(\theta^{-n}\omega)}) \leq V_k(T^n(\theta^{-n}(\omega))) \leq V_k(T^n(\theta^{-n}(\omega))|_{G(\theta^{-n}\omega)})||\pi(\theta^{-n}\omega)||^k$$

and $||\pi(\cdot)||$ is tempered, it is sufficient to show that

$$\lim_{n \to +\infty} \frac{1}{n} \log V_k(T^n(\theta^{-n}(\omega))|_{G(\theta^{-n}\omega)}) = l_k(T) \quad a.s..$$

Using Lemma 15 and the fact that X is separable, we have $V_k(T^n(\theta^{-n}(\omega))|_{G(\theta^{-n}\omega)})$ is measurable in ω. In order to use Kingman's subadditive ergodic theorem, we let

$$F_n^k(\omega) = \log V_k(T^n(\theta^{-n}(\omega))|_{G(\theta^{-n}\omega)}).$$

From Lemma 16 it follows that

$$F_1^{k+}(\omega) \leq k \log^+ \|T(\theta^{-1}(\omega))\big|_{G(\theta^{-1}\omega)}\|.$$

Then, by property (T5) and 15 we have

$$F_1^{k+} \in L^1(\Omega, \mathcal{F}, P)$$

and

$$\begin{aligned}
F_{m+n}^k(\omega) &= \log V_k(T^{m+n}(\theta^{-m-n}(\omega))|_{G(\theta^{-m-n}(\omega))}) \\
&\leq \log \big(V_k(T^m(\theta^{-m-n}(\omega))|_{G(\theta^{-m-n}(\omega))}) \cdot V_k(T^n(\theta^{-n}(\omega))|_{G(\theta^{-n}(\omega))}) \big) \\
&= \log V_k(T^m(\theta^{-m-n}(\omega))|_{G(\theta^{-m-n}(\omega))}) + \log V_k(T^n(\theta^{-n}(\omega))|_{G(\theta^{-n}(\omega))}) \\
&= F_m^k\big(\theta^{-n}(\omega)\big) + F_n^k(\omega).
\end{aligned}$$

Then by Kingman's subadditive ergodic theorem and ergodicity of θ, we have

$$\lim_{n \to +\infty} \frac{1}{n} \log V_k(T^n(\theta^{-n}(\omega))|_{G(\theta^{-n}\omega)}) = l_k(T) \quad a.s.,$$

where $l_k(T)$ is either a constant or $-\infty$.

The proof of (31) is directly from Kingman's subadditive ergodic theorem. To show (32), we notice that

$$\begin{aligned}
&\log \|T(\omega)\|_\alpha = \log \|S(\omega)\|_\alpha \leq \log^+ \|S(\omega)\|, \\
&\log \|T^{n+m}(\omega)\|_\alpha \leq \log \|T^n(\omega)\|_\alpha + \log \|T^m(\theta^n \omega)\|_\alpha.
\end{aligned}$$

Thus, using Lemma 24 and Kingman's subadditive ergodic theorem we have

$$\lim_{n \to +\infty} \frac{1}{n} \log \|T^n(\omega)\|_\alpha = l_\alpha \quad a.s.,$$

where l_α is a constant or $-\infty$. Similarly, we can show (33) and (34). This completes the proof of this lemma. $\square$

The next lemma is borrowed from Mane [23].

Lemma 26. *Let* $(\Lambda, \mathcal{G}, \mu)$ *be a probability space and* $\vartheta : \Lambda \to \Lambda$ *be a* μ*-measure preserving transformation. Then for any measurable function* $f : \Lambda \to \mathbb{R}$ *we have*

$$\liminf_{n\to+\infty} \frac{1}{n} f(\vartheta^n(x)) \leq 0 \quad a.s..$$

Moreover, if there exists $F \in L^1(\Lambda, \mathcal{G}, \mu)$ *such that*

$$f(\vartheta(x)) - f(x) \leq F(x) \quad a.s.,$$

or there exists $G \in L^1(\Lambda, \mathcal{G}, \mu)$ *such that*

$$f(\vartheta(x)) - f(x) \geq G(x) \quad a.s.,$$

then

$$f \circ \vartheta - f \in L^1(\Lambda, \mathcal{G}, \mu)$$

and

$$\lim_{n\to+\infty} \frac{1}{n} f(\vartheta^n(x)) = 0 \quad a.s..$$

Moreover, if ϑ *is invertible, then*

$$\lim_{n\to-\infty} \frac{1}{n} f(\vartheta^n(x)) = 0 \quad a.s..$$

In the next lemma, we show that the limits given by (32) and (33) are the same.

Lemma 27.

$$l'_\alpha = l_\alpha.$$

Proof. We first prove that $l'_\alpha \leq l_\alpha$. We prove it by contradiction. Suppose it is not true. Then, for sufficiently small $\epsilon > 0$, $l'_\alpha - \epsilon > l_\alpha$. Let

$$C_\epsilon(\omega) = \sup\left\{\frac{e^{n(l'_\alpha-\epsilon)}}{\|T^n(\theta^{-n}\omega)\|_\alpha}\right\}_{n\geq 0}.$$

Then, $C_\epsilon(\omega)$ is measurable by using Lemma 24. Furthermore, we have

$$\begin{aligned}
C_\epsilon(\theta\omega) =& \sup\left\{\frac{e^{n(l'_\alpha-\epsilon)}}{\|T^n(\theta^{-n}\theta\omega)\|_\alpha}\right\}_{n\geq 0} \\
=& \max\left\{\sup\left\{\frac{e^{n(l'_\alpha-\epsilon)}}{\|T(\omega)T^{n-1}(\theta^{-(n-1)}\omega)\|_\alpha}\right\}_{n\geq 1}, 1\right\} \\
\geq& \sup\left\{\frac{e^{(n-1)(l'_\alpha-\epsilon)}}{\|T^{n-1}(\theta^{-(n-1)}\omega)\|_\alpha}\frac{e^{l'_\alpha-\epsilon}}{\|T(\omega)\|_\alpha}\right\}_{n\geq 1} \\
\geq& \frac{e^{l'_\alpha-\epsilon}}{\|T(\omega)\|_\alpha}C_\epsilon(\omega),
\end{aligned}$$

which implies

$$\begin{aligned}
\log C_\epsilon(\theta\omega) - \log C_\epsilon(\omega) &\geq l'_\alpha - \epsilon - \log\|T(\omega)\|_\alpha \\
&\geq l'_\alpha - \epsilon - \log^+\|T(\omega)\big|_{G(\omega)}\|.
\end{aligned}$$

Since $l'_\alpha - \epsilon - \log^+\|T(\omega)\big|_{G(\omega)}\|$ is a L^1 function, by Lemma 26, we have

$$\lim_{n\to+\infty}\frac{1}{n}\log C_\epsilon(\theta^n\omega) = 0.$$

Hence,

$$\begin{aligned}
l_\alpha &= \lim_{n\to+\infty}\frac{1}{n}\log\|T^n(\omega)\|_\alpha = \lim_{n\to+\infty}\frac{1}{n}\log\|T^n(\theta^{-n}\theta^n\omega)\|_\alpha \\
&\geq \lim_{n\to+\infty}\frac{1}{n}\log\frac{e^{n(l'_\alpha-\epsilon)}}{C_\epsilon(\theta^n\omega)} \geq l'_\alpha - \epsilon - \lim_{n\to+\infty}\frac{1}{n}\log C_\epsilon(\theta^n\omega) \\
&= l'_\alpha - \epsilon > l_\alpha,
\end{aligned}$$

which gives a contradiction. This also implies that if $l_\alpha = -\infty$, then $l'_\alpha = -\infty$. Next, we show that $l_\alpha \le l'_\alpha$. Suppose this is not true. Then, for sufficiently small $\epsilon > 0$, we have $l_\alpha - \epsilon > l'_\alpha$. Let

$$C'_\epsilon(\omega) = \sup \left\{ \frac{\|T^n(\theta^{-n}\omega)\|_\alpha}{e^{n(l_\alpha-\epsilon)}} \right\}_{n\ge 0}.$$

Then, using Lemma 24, we have $C'_\epsilon(\omega)$ is measurable and

$$\begin{aligned}
C'_\epsilon(\theta\omega) &= \sup \left\{ \frac{\|T^n(\theta^{-n}\theta\omega)\|_\alpha}{e^{n(l_\alpha-\epsilon)}} \right\}_{n\ge 0} \\
&= \max\left\{ \sup \left\{ \frac{\|T(\omega)T^{n-1}(\theta^{-(n-1)}\omega)\|_\alpha}{e^{n(l_\alpha-\epsilon)}} \right\}_{n\ge 1}, 1 \right\} \\
&\le \max\left\{ \sup \left\{ \frac{\|T^{n-1}(\theta^{-(n-1)}\omega)\|_\alpha}{e^{(n-1)(l_\alpha-\epsilon)}} \frac{\|T(\omega)\|_\alpha}{e^{l_\alpha-\epsilon}} \right\}_{n\ge 1}, 1 \right\} \\
&= \max\left\{ \frac{\|T(\omega)\|_\alpha}{e^{l_\alpha-\epsilon}} C'_\epsilon(\omega), 1 \right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\log C'_\epsilon(\theta\omega) - \log C'_\epsilon(\omega) &\le \max\left\{ \log \frac{\|T(\omega)\|_\alpha}{e^{l_\alpha-\epsilon}}, \log \frac{1}{C'_\epsilon(\omega)} \right\} \\
&\le \max\left\{ \log \|T(\omega)\|_\alpha - (l_\alpha - \epsilon), 0 \right\} \\
&\le \max\left\{ \log^+ \|T(\omega)\big|_{G(\omega)}\| - (l_\alpha - \epsilon), 0 \right\},
\end{aligned}$$

which is a L^1 function. Thus by Lemma 26, we have

$$\lim_{n\to+\infty} \frac{1}{n} \log C'_\epsilon(\theta^n\omega) = 0.$$

Hence,

$$
\begin{aligned}
l_\alpha &= \lim_{n\to+\infty} \frac{1}{n}\log\|T^n(\omega)\|_\alpha = \lim_{n\to+\infty}\frac{1}{n}\log\|T^n(\theta^{-n}\theta^n\omega)\|_\alpha \\
&\le \lim_{n\to+\infty}\frac{1}{n}\log(e^{n(l_\alpha-\epsilon)}C'_\epsilon(\theta^n\omega)) \le l_\alpha-\epsilon+\lim_{n\to+\infty}\frac{1}{n}\log C_\epsilon(\theta^n\omega) \\
&= l_\alpha-\epsilon<l_\alpha,
\end{aligned}
$$

which gives a contradiction. Therefore, $l'_\alpha = l_\alpha$. This completes the proof of this lemma. □

Lemma 28. $\kappa(T)=\kappa'(T)$.

Proof. For each $v\in E^{\kappa(T)}(\omega)$, by the definition of $E^{\kappa(T)}(\omega)$, we have

$$\limsup_{n\to\infty}\frac{1}{n}\log\|T^{-n}(\omega)v\|\le-\kappa(T).$$

Then, using the cocycle property, we obtain

$$\kappa'(T)=\lim_{n\to+\infty}\frac{1}{n}\log\|T^n(\theta^{-n}\omega)\|\ge\kappa(T).$$

Next, we show $\kappa'(T)\le\kappa(T)$. We prove this by contradiction. Suppose that this is not true. Then, we can choose $\epsilon>0$ such that

$$\kappa'(T)-2\epsilon>\kappa(T).$$

Let

$$C_\epsilon(\omega)=\sup\left\{\frac{e^{n(\kappa'(T)-\epsilon)}}{\|T^n(\theta^{-n}\omega)|_{G(\theta^{-n}\omega)}\|}\right\}_{n\ge0}.$$

Then $C_\epsilon(\omega)$ is measurable and by (34)

$$1 \le C_\epsilon(\omega) < +\infty \ a.s. \ .$$

We estimate

$$\begin{aligned}
C_\epsilon(\theta\omega) &= \sup\left\{\frac{e^{n(\kappa'(T)-\epsilon)}}{\|T^n(\theta^{-n+1}\omega)|_{G(\theta^{-n+1}\omega)}\|}\right\}_{n\ge 0} \\
&= \max\left\{\sup\left\{\frac{e^{n(\kappa'(T)-\epsilon)}}{\|T(\omega)T^{n-1}(\theta^{-(n-1)}\omega)|_{G(\theta^{-n+1}\omega)}\|}\right\}_{n\ge 1}, 1\right\} \\
&\ge \frac{e^{\kappa'(T)-\epsilon}}{\|T(\omega)\big|_{G(\omega)}\|}\sup\left\{\frac{e^{(n-1)(\kappa'(T)-\epsilon)}}{\|T^{n-1}(\theta^{-(n-1)}\omega)|_{G(\theta^{-n+1}\omega)}\|}\right\}_{n\ge 1} \\
&\ge \frac{e^{\kappa'(T)-\epsilon}}{\|T(\omega)\big|_{G(\omega)}\|}C_\epsilon(\omega).
\end{aligned}$$

Then,

$$\begin{aligned}
&\log C_\epsilon(\theta\omega) - \log C_\epsilon(\omega) \\
&\ge \kappa'(T) - \epsilon - \log \|T(\omega)\big|_{G(\omega)}\| \\
&\ge \kappa'(T) - \epsilon - \log^+ \|T(\omega)\big|_{G(\omega)}\|
\end{aligned}$$

Note that $\kappa'(T) - \epsilon - \log^+ \|T(\omega)\big|_{G(\omega)}\|$ is a L^1 function. Hence, by Lemma 26, we obtain

$$\lim_{n\to+\infty}\frac{1}{n}\log C_\epsilon(\theta^n\omega) = 0, \quad a.s..$$

We also notice that from the definition of $C_\epsilon(\omega)$, we have that for any $n \in \mathbb{N}$

$$\begin{aligned}
\|T^n(\omega)\| &\ge \|T^n(\theta^{-n}\theta^n\omega)|_{G(\omega)}\| \\
&\ge C_\epsilon^{-1}(\theta^n\omega)e^{n(\kappa'(T)-\epsilon)} \\
&\ge C_\epsilon^{-1}(\theta^n\omega)e^{n(\kappa(T)+\epsilon)}.
\end{aligned}$$

Hence,

$$\begin{aligned}\kappa(T) &= \lim_{n\to+\infty} \frac{1}{n}\log \|T^n(\omega)\| \\ &\geq \kappa(T) + \epsilon - \lim_{n\to+\infty} \frac{1}{n}\log C_\epsilon(\theta^n\omega) \\ &= \kappa(T) + \epsilon,\end{aligned}$$

which gives a contradiction. This completes the proof of the lemma. □

Next we will prove that

Lemma 29.

$$\lim_{k\to+\infty} \frac{1}{k} l_k(T) \leq l_\alpha,$$

where $l_k(T)$ and l_α are given by (30) and (32).

Proof. By Lemma 13, we have that $\frac{1}{k}l_k$ is a decreasing sequence. Thus, $\lim_{k\to+\infty}\frac{1}{k}l_k$ exists. By Lemma 14, we have that for any bounded operator T,

$$\lim_{n\to+\infty} \frac{1}{n}\log V_n(T) \leq \log 2 + \log \|T\|_\alpha.$$

For $n \geq 1,\ m \geq 1$, and a small $\epsilon > 0$, we set

$$A^{n,m} = \left\{\omega \in \Omega \middle| \ \frac{1}{n}\log V_n(T^m(\omega)|_{G(\omega)}) < \log 2 + m(l_\alpha + \epsilon)\right\}$$

and

$$A^m = \left\{\omega \in \Omega \middle| \ \frac{1}{m}\log \|T^m(\omega)|_{G(\omega)}\|_\alpha < l_\alpha + \epsilon\right\}.$$

We notice that $\|T^m(\omega)\|_\alpha = \|T^m(\omega)|_{G(\omega)}\|_\alpha$. From the above definitions, we have

$$\lim_{n\to+\infty} A^{n,m} \supset A^m$$

and

$$\begin{aligned}
\frac{1}{k}l_k &= \lim_{n\to+\infty}\frac{1}{k}\cdot\frac{1}{n}\log V_k(T^n(\theta^{-n}\omega)|_{G(\theta^{-n}\omega)}) \\
&= \lim_{n\to+\infty}\frac{1}{k}\cdot\frac{1}{nm}\log V_k(T^{nm}(\theta^{-nm}\omega)|_{G(\theta^{-nm}\omega)}) \\
&\le \lim_{n\to+\infty}\frac{1}{k}\cdot\frac{1}{nm}\log\prod_{i=1}^{n}V_k(T^m(\theta^{-im}\omega)|_{G(\theta^{-im}\omega)}) \\
&= \lim_{n\to+\infty}\frac{1}{km}\cdot\frac{1}{n}\sum_{i=1}^{n}\log V_k(T^m(\theta^{-im}\omega)|_{G(\theta^{-im}\omega)}).
\end{aligned}$$

By the Birkhoff ergodic theorem, we have

$$\int_\Omega \lim_{n\to+\infty}\frac{1}{km}\cdot\frac{1}{n}\sum_{i=1}^{n}\log V_k(T^m(\theta^{-im}\omega)|_{G(\theta^{-im}\omega)}) = \frac{1}{km}\int_\Omega \log V_k(T^m(\omega)|_{G(\omega)})dP.$$

Hence,

$$\frac{1}{k}l_k \le \frac{1}{m}\int_\Omega \frac{1}{k}\log V_k(T^m(\omega)|_{G(\omega)})dP.$$

Since

$$\lim_{n\to+\infty}\frac{1}{n}\log\|T^n(\omega)\|_\alpha = l_\alpha \quad a.s.,$$

for any fixed $\epsilon > 0$, $\lim_{m\to+\infty}P(A^m) = 1$. Thus, we can choose a sufficiently large m such that $\frac{1}{m}\log 2 < \epsilon$ and $P(A^m) > 1 - \delta/2$, where $\delta > 0$ is chosen such that for any measurable set F, if $P(F) < \delta$, then

$$\int_F \log^+\|T(\omega)\big|_{G(\omega)}\| < \epsilon.$$

Then for such m, we choose k large enough such that $P(A^{k,m}) > 1 - \delta$. Thus,

$$\begin{aligned}
&\frac{1}{mk}\int_{\Omega} \log V_k(T^m(\omega)\big|_{G(\omega)})dP \\
&= \int_{\Omega - A^{k,m}} \frac{1}{km} \log V_k(T^m(\omega)\big|_{G(\omega)})dP + \int_{A^{k,m}} \frac{1}{km} \log V_k(T^m(\omega)\big|_{G(\omega)})dP \\
&\leq \int_{\Omega - A^{k,m}} \frac{1}{mk} \log^+ \|T^m(\omega)\big|_{G(\omega)}\|^k dP + \frac{1}{m}\log 2 + l_\alpha + \epsilon \\
&\leq \int_{\Omega - A^{k,m}} \frac{1}{m} \sum_{i=0}^{m-1} \log^+ \|T(\theta^i\omega)\big|_{G(\theta^i\omega)}\| dP + \frac{1}{m}\log 2 + l_\alpha + \epsilon \\
&\leq \frac{1}{m} \sum_{i=0}^{m-1} \int_{\theta^i(\Omega - A^{k,m})} \log^+ \|T(\omega)\big|_{G(\omega)}\| dP + \frac{1}{m}\log 2 + l_\alpha + \epsilon \\
&\leq l_\alpha + 3\epsilon.
\end{aligned}$$

Since ϵ can be arbitrary small, we have

$$\lim_{k\to+\infty} \frac{1}{k} l_k \leq l_\alpha.$$

$\square$

6.2 Oseledets Spaces.

We are now ready to prove Theorem 21.

Proof of Theorem 21.

We prove this proposition by contradiction. Suppose that

$$\dim E^\lambda(\omega) = +\infty.$$

Then for any $d > 0$, there exists a set of unit vectors $\{v_i^0\}_{1\leq i\leq d} \subset E_\omega^\lambda$ such that

$$\mathrm{dist}(v_{i+1}^0, F_i^0) = 1$$

where $F_i^0 = \mathrm{span}(v_1^0, \ldots, v_i^0)$. For each $k \in \mathbb{N}$, let

$$F_i^k = \mathrm{span}(T^{-k}(\omega)v_1^0, \ldots, T^{-k}(\omega)v_i^0)$$

and choose unit vectors $v_i^k \in F_i^k,\ 1 \leq i \leq d$, such that

$$\mathrm{dist}(v_{i+1}^k, F_i^k) = 1 \qquad \text{for any } 1 \leq i \leq d-1.$$

We also define $F_0^k = \{0\}$. Then, there exist real numbers $\{\lambda_i^k\}_{k\in\mathbb{N},1\leq i\leq d}$, such that

$$T(\theta^{-k}\omega)v_i^k = \lambda_i^k v_i^{k-1} + w \text{ for some } w \in F_{i-1}^{k-1}.$$

Hence, for each $p \geq 1$

$$T^p(\theta^{-k}\omega)v_i^k = \lambda_i^k \cdots \lambda_i^{k-p+1} v_i^{k-p} + w \text{ for some } w \in F_{i-1}^{k-p},$$

which yields that

$$\mathrm{dist}(T^k(\theta^{-k}\omega)v_i^k, \mathrm{span}\{T^k(\theta^{-k}\omega)v_j\}_{j<i}) = \prod_{l=1}^{k} |\lambda_i^l|.$$

Therefore,

$$V_d(T^k(\theta^{-k}\omega)v_d^k, T^k(\theta^{-k}\omega)v_{d-1}^k, \ldots, T^k(\theta^{-k}\omega)v_1^k) = \prod_{l=1}^{k}\prod_{i=1}^{d} |\lambda_i^l|,$$

which implies that

$$\frac{1}{k}\sum_{l=1}^{k}\sum_{i=1}^{d}\log|\lambda_i^l| \le \frac{1}{k}\log V_d\big(T^k(\theta^{-k}\omega)\big|_{G(\theta^{-k}\omega)}\big).$$

So,

$$\sum_{i=1}^{d}\liminf_{k\to+\infty}\frac{1}{k}\sum_{l=1}^{k}\log|\lambda_i^l| \le \lim_{k\to+\infty}\frac{1}{k}\log V_d\big(T^k(\theta^{-k}\omega)\big|_{G(\theta^{-k}\omega)}\big) = l_d(T) \quad \text{a.s.}. \tag{35}$$

On the other hand, from the construction of $\{v_i^k\}_{1\le i\le d\ \ 1\le k}$, we have

$$T^{-n}(\omega)v_i^0 = (\lambda_i^1,\ldots,\lambda_i^n)^{-1}\cdot v_i^n + w, \quad \text{for some} \quad w\in F_{i-1}^n, 1\le i\le d.$$

Thus, by using the property $\mathrm{dist}(v_{i+1}^n, F_i^n) = 1$, we have

$$\|T^{-n}(\omega)v_i^0\| \ge |\lambda_i^1,\ldots,\lambda_i^n|^{-1}.$$

Using the fact $v_i^0\in E^\lambda(\omega)$, namely,

$$\limsup_{n\to+\infty}\frac{1}{n}\log\|T^{-n}(\omega)v_i^0\| \le -\lambda,$$

we obtain

$$\liminf_{n\to+\infty}\frac{1}{n}\sum_{l=1}^{n}\log|\lambda_i^l| \ge \lambda,$$

which together with (35) gives

$$\frac{1}{d}l_d(T) \ge \frac{1}{d}\sum_{i=1}^{d}\liminf_{k\to+\infty}\frac{1}{k}\sum_{l=1}^{k}\log|\lambda_i^l| \ge \frac{1}{d}\sum_{i=1}^{d}\lambda = \lambda. \tag{36}$$

Thus, by Lemma 29, we have

$$\lambda \leq \lim_{d \to +\infty} \frac{1}{d}\, l_d(T) \leq l_\alpha.$$

This contradicts to the assumption $\lambda > l_\alpha$. This completes the proof.

$\square$.

In the next proposition, we will see that almost surely $E^\lambda(\omega)$ is not empty for $l_\alpha < \lambda < \kappa(T)$. We first notice that from Lemma 25, $l_k(T)$ is a constant almost surely and that $\frac{1}{k} l_k(T)$ is a nondecreasing sequence from Lemma 13. Thus, there exists a positive integer m such that

$$\begin{aligned} \frac{1}{k} l_k(T) &= l_1(T), \qquad \text{for } 1 \leq k \leq m, \\ \frac{1}{k} l_k(T) &< l_1(T), \qquad \text{for } k > m. \end{aligned} \tag{37}$$

Then, from (36), we have

$$\dim E^{\kappa(T)}(\omega) \leq m, \ a.s..$$

The following proposition gives that the space $E^\lambda(\omega)$ has at least dimension m.

Proposition 30. For $l_\alpha < \lambda \leq \kappa(T)$, we have

$$\dim E^\lambda(\omega) \geq m, \ a.s..$$

Furthermore, $\dim E^{\kappa(T)}(\omega) = m$.

The proof of this proposition is based on the following lemmas. We first borrow a lemma from Pliss, see [22].

Lemma 31. *For given $H_0 < \lambda_0$ and $\epsilon > 0$, there exist $N_0 = N_0(H_0, \lambda_0, \epsilon)$ and*

$\delta = \delta(H_0, \lambda_0, \epsilon)$ such that if $a_i \in \mathbb{R},\ i = 0, \ldots, N$ satisfy

$$N \geq N_0$$

$$a_n \geq H_0$$

$$\sum_{n=0}^{N} a_n \leq (\lambda_0 - \epsilon)(N+1),$$

then there exist $0 \leq n_1 < n_2 < \cdots < n_{j_0} \leq N$ such that

$$j_0 \geq \delta N$$

and

$$\sum_{n=n_j+1}^{k} a_n \leq (k - n_j)\lambda_0$$

for all $n_j < k \leq N,\ 1 \leq j \leq j_0$.

Let λ be fixed such that $l_\alpha < \lambda < \kappa(T)$. For each positive integer n, we use A_n^λ to denote the subset of Ω such that for each $\omega \in A_n^\lambda$ there exist m vectors $\{v_i\}_{1 \leq i \leq m} \subset X$ such that

$\{\pi(\omega)v_i\}_{1 \leq i \leq m}$ are independent; (38)

$$T^{-k}(\omega)\pi(\omega)v_i \text{ exists for } 1 \leq i \leq m, 1 \leq k \leq n; \tag{39}$$

$$\frac{V_m(T^{-k}(\omega)\pi(\omega)v_1, \ldots, T^{-k}(\omega)\pi(\omega)v_m)}{V_m(\pi(\omega)v_1, \ldots, \pi(\omega)v_m)} \leq e^{-km\lambda},\ 1 \leq k \leq n. \tag{40}$$

Lemma 32. *There exists a $\delta > 0$ such that*

$$P_{in}(A_n^\lambda) > \delta, \quad \textit{for all} \ \ n,$$

where $P_{in}(A_n^\lambda)$ *is the inner measure of* A_n^λ *and there exists a countable subset* S_d *of* $(l_\alpha, \kappa(T))$ *such that for any* $\lambda \in (l_\alpha, \kappa(T)) - S_d$*,* A_n^λ *is measurable and*

$$P(A_n^\lambda) \geq \delta.$$

Proof. We first show that if A_n^λ is measurable, then $P(A_n^\lambda) > \delta$. We note that from $\log^+ \|S(\cdot)\| \in L^1(\Omega, \mathcal{F}, P)$ it follows that for each $\epsilon > 0$, there exists a $H(\epsilon) > 0$ such that

$$\int_{E\left(H(\epsilon)\right)} \log^+ \|S(\omega)\| dP < \epsilon,$$

where $E\big(H(\epsilon)\big) = \big\{\omega \mid \log \|S(\omega)\| > H(\epsilon)\big\}$.

In order to apply Lemma 31, we choose ϵ, λ_0, and H_0 such that

$$\begin{aligned}
&0 < 2\epsilon < \kappa(T) - \lambda, \\
&\lambda_0 = -\lambda, \\
&H_0 = -H(\frac{\epsilon}{2}) < \lambda_0.
\end{aligned}$$

By using the Birkhoff ergodic theorem and Lemma 16, there exists a θ-invariant subset set $\tilde{\Omega} \subset \Omega$ of full measure such that for each $\omega \in \tilde{\Omega}$

$$\lim_{n \to +\infty} \frac{1}{mn} \sum_{k=0}^{n-1} \chi_{E\left(H(\frac{\epsilon}{2})\right)}(\theta^k \omega) \log V_m(T(\theta^k \omega)|_{G(\theta^k \omega)}) \leq \int_{E\left(H(\frac{\epsilon}{2})\right)} \log^+ \|S(\omega)\| dP < \frac{\epsilon}{2}. \tag{41}$$

To save on notation, we use Ω to denote $\tilde{\Omega}$. For $\omega \in \Omega$, from the definition of m, we have

$$\lim_{n \to +\infty} \frac{1}{mn} \log V_m(T^n(\omega)) = \kappa(T) > \lambda + 2\epsilon.$$

Thus, there exists $N_1(\omega) > 0$ such that for any $n > N_1(\omega)$,

$$V_m(T^n(\omega)) > e^{nm(\lambda+2\epsilon)}. \tag{42}$$

Using (152), there exists $N_2(\omega) > 0$ such that for any $n > N_2(\omega)$,

$$\frac{1}{mn}\sum_{k=0}^{n-1} \chi_{E\left(H(\frac{\epsilon}{2})\right)}(\theta^k\omega) \log V_m(T(\theta^k\omega)|_{G(\theta^k\omega)}) < \epsilon. \tag{43}$$

Let $N_0(H_0, \lambda_0, \epsilon)$ be the number given in Lemma 31.

Thus, for each $N \geq \max\{N_1, N_2, N_0(H_0, \lambda_0, \epsilon)\} + 1$, by using (154), there exist vectors $\{v_i\}_{1\leq i\leq m} \subset X$ such that $\{\pi(\omega)v_i\}_{1\leq i\leq m}$ are linearly independent and

$$V_m(T^{N+1}(\omega)\pi(\omega)v_1, \ldots, T^{N+1}(\omega)\pi(\omega)v_m) > e^{(N+1)m(\lambda+2\epsilon)}V_m(\pi(\omega)v_1, \ldots, \pi(\omega)v_m). \tag{44}$$

Let

$$a'_k = \frac{1}{m}\log\frac{V_m(T^{N-k}(\omega)\pi(\omega)v_1, \ldots, T^{N-k}(\omega)\pi(\omega)v_m)}{V_m(T^{N-k+1}\pi(\omega)v_1, \ldots, T^{N-k+1}\pi(\omega)v_m)} \quad 0 \leq k \leq N.$$

Then,

$$\sum_{k=0}^{N} a'_k = \frac{1}{m}\log\frac{V_m(\pi(\omega)v_1, \ldots, \pi(\omega)v_m)}{V_m(T^{N+1}(\omega)\pi(\omega)v_1, \ldots, T^{N+1}(\omega)\pi(\omega)v_m)} < -(N+1)(\lambda+2\epsilon).$$

Set

$$a_k = \begin{cases} a'_k & \text{if } a'_k \geq H_0 \\ 0 & \text{if } a'_k < H_0 \end{cases}$$

Since (14)

$$a'_k \geq -\frac{1}{m}\log V_m(T(\theta^{N-k}\omega)|_{G(\theta^{N-k}\omega)}).$$

Hence, if $a'_k < H_0$, then $\theta^{N-k}\omega \in E\big(H(\frac{\epsilon}{2})\big)$. Thus, using (155), we have

$$\begin{aligned}
0 &\geq \frac{1}{N+1}\left(\sum_{k=0}^{N} a'_k - \sum_{k=0}^{N} a_k\right) \\
&\geq \frac{1}{m(N+1)}\sum_{k=0}^{N} -\chi_{E\left(H(\frac{\epsilon}{2})\right)}(\theta^k\omega)\log V_m(T(\theta^k\omega)|_{G(\theta^k\omega)}) \\
&\geq -\epsilon,
\end{aligned}$$

which implies that

$$\begin{aligned}
\sum_{k=0}^{N} a_k \leq &\sum_{k=0}^{N} a'_k + (N+1)\epsilon \\
\leq &-(N+1)(\lambda+\epsilon) \\
= &(N+1)(\lambda_0-\epsilon),
\end{aligned}$$

here $\lambda_0 = -\lambda$ is used. By Lemma 31, there exist integers $0 < n_1 < n_2 < \cdots < n_{j_0} \leq N$ such that $j_0 \geq \delta N$ and for all $n_j \leq k \leq N$,

$$\begin{aligned}
(k-n_j)\lambda_0 &\geq \sum_{n=n_j+1}^{k} a_n \geq \sum_{n=n_j+1}^{k} a'_n \\
&= \sum_{n=n_j+1}^{k} \frac{1}{m}\log\frac{V_m(T^{N-n}(\omega)\pi(\omega)v_1,\ldots,T^{N-n}(\omega)\pi(\omega)v_m)}{V_m(T^{N-n+1}(\omega)\pi(\omega)v_1,\ldots,T^{N-n+1}(\omega)\pi(\omega)v_m)} \\
&= \frac{1}{m}\log\frac{V_m(T^{N-k}(\omega)\pi(\omega)v_1,\ldots,T^{N-k}(\omega)\pi(\omega)v_m)}{V_m(T^{N-n_j}(\omega)\pi(\omega)v_1,\ldots,T^{N-n_j}(\omega)\pi(\omega)v_m)}.
\end{aligned}$$

We note that $N - n_j \geq n$ when $j_0 - j > n$. From the definition of A_n^λ it follows that for $n + j \leq j_0$

$$\theta^{N-n_j}\omega \in A_n^\lambda.$$

Since $j_0 - n \geq \delta N - n$, we have

$$\begin{aligned}
&\frac{1}{N}\#\{0 < i \leq N |\ \theta^i\omega \in A_n^\lambda\} \\
&\geq \frac{1}{N}\#\{0 < i \leq N |\ i = N - n_j, 1 \leq j \leq j_0 - n\} \\
&\geq \frac{1}{N}(j_0 - n) \geq \delta - \frac{n}{N}
\end{aligned}$$

in which the lower bound will go to δ as N goes to $+\infty$. Thus, if A_n^λ is measurable, then by the Birkhoff ergodic theorem we obtain

$$P(A_n^\lambda) = \lim_{N\to+\infty} \frac{1}{N}\#\{0 < i \leq N |\ \theta^i\omega \in A_n^\lambda\} \geq \delta. \tag{45}$$

Next, we show that A_n^λ is measurable except for countably many λ. For each $n \geq 1$ and $w_1, \cdots w_m \in X$, We use $S_{k,\lambda}(w_1, \ldots, w_m)$ to denote the set of $\omega \in \Omega$ such that the following conditions hold

$$V_m(T^n(\theta^{-n}\omega)w_1, \ldots, T^n(\theta^{-n}\omega)w_m) \neq 0; \tag{46}$$

$$\frac{V_m(T^{n-k}(\theta^{-n}\omega)w_1, \ldots, T^{n-k}(\theta^{-n}\omega)w_m)}{V_m(T^n(\theta^{-n}\omega)w_1, \ldots, T^n(\theta^{-n}\omega)w_m)} \leq e^{-mk\lambda}, \tag{47}$$

for $1 \leq k \leq n,\ l_\alpha < \lambda < \kappa(T)$.

Since $T(\omega)$ is strongly measurable and $V_m :\ X^m \to \mathbb{R}$ is continuous, $S_{k,\lambda}(w_1, \ldots, w_m)$ is measurable. Let

$$D_\lambda(w_1, \ldots, w_m) = \bigcap_{k=1}^{n} S_{k,\lambda}(w_1 \ldots, w_m).$$

Then $D_\lambda(w_1, \ldots, w_m)$ is also measurable.

Since X is a separable Banach space, we have a countable dense set $\{v_i(\neq 0)\}_{i\geq 1}$

of X. We set

$$K_\lambda = \bigcap_{j=1}^{\infty} \bigcup_{(n_1,\ldots,n_m)\in\mathbb{N}^m} D_{\lambda-\frac{1}{j}}(v_{n_1},\ldots,v_{n_m}).$$

Then K_λ is measurable. By the definition of $S_{n,\lambda}(w_1,\cdots,w_m)$, we have that for each small $\epsilon > 0$

$$S_{n,\lambda}(w_1,\cdots,w_m) \subset S_{n,\lambda-\epsilon}(w_1,\cdots,w_m),$$

which yields

$$K_\lambda \subset K_{\lambda-\epsilon}$$

and

$$K_\lambda = \lim_{\epsilon\to 0^+} K_{\lambda-\epsilon} \text{ decreasingly.}$$

Since $P(K_\lambda) \le 1$, we have

$$P(K_\lambda) = \lim_{\epsilon\to 0^+} P(K_{\lambda-\epsilon}).$$

Next, we show

Claim: The inner measure of A_n^λ, $P_{in}(A_n^\lambda)$, is equal to its outer measure $P_{out}(A_n^\lambda)$, for each $\lambda \in (l_\alpha, \kappa(T)) - S_d$, where S_d is a countable set.

We first prove that for each small $\epsilon > 0$,

$$K_\lambda \subset A_n^{\lambda-\epsilon},$$

in other words

$$K_{\lambda+\epsilon} \subset A_n^\lambda.$$

Let $\omega \in K_\lambda$. Then, there exists $(n_1,\ldots,n_m) \in \mathbb{N}^m$ such that $\omega \in D_{\lambda-\epsilon}(v_{n_1},\ldots,v_{n_m})$

which means that

$$V_m(T^n(\theta^{-n}\omega)v_{n_1}, \ldots, T^n(\theta^{-n}\omega)v_{n_m}) \neq 0$$

and

$$\frac{V_m(T^{n-k}(\theta^{-n}\omega)v_{n_1}, \ldots, T^{n-k}(\theta^{-n}\omega)v_{n_m})}{V_m(T^n(\theta^{-n}\omega)v_{n_1}, \ldots, T^n(\theta^{-n}\omega)v_{n_m})} \leq e^{-mk(\lambda-\epsilon)}, \quad 1 \leq k \leq n.$$

By letting

$$v_i' = T^n(\theta^{-n}\omega)v_{n_i},$$

we obtain that

$$\omega \in A_n^{\lambda-\epsilon}.$$

Next, we prove that

$$A_n^\lambda \subset K_\lambda.$$

Let $\omega \in A_n^\lambda$. Then, there exist vectors $\{w_i\}_{1\leq i\leq m} \subset X$ such that $T^{-k}(\omega)\pi(\omega)w_i$ exists for $1 \leq i \leq m$, $1 \leq k \leq n$,

$$V_m(\pi(\omega)w_1, \ldots, \pi(\omega)w_m) \neq 0$$

and

$$\frac{V_m(T^{-k}(\omega)\pi(\omega)w_1, \ldots, T^{-k}(\omega)\pi(\omega)w_m)}{V_m(\pi(\omega)w_1, \ldots, \pi(\omega)w_m)} \leq e^{-km\lambda}, \; 1 \leq k \leq n.$$

For small $\epsilon > 0$, since $\{v_i\}_{i\geq 1}$ is a countable dense subset of X and $V_m : \; X \to \mathbb{R}$ is continuous, we have that there exists $(n_1, \ldots, n_m) \in \mathbb{N}^m$ such that $(\pi(\theta^{-n}\omega)v_{n_1}, \ldots, \pi(\theta^{-n}\omega)v_{n_m})$ is close enough to $(T^{-n}(\omega)\pi(\omega)w_1, \ldots, T^{-n}(\omega)\pi(\omega)w_m)$ such that

$$\frac{V_m(T^{n-k}(\theta^{-n}\omega)v_{n_1}, \ldots, T^{n-k}(\theta^{-n}\omega)v_{n_m})}{V_m(T^n(\theta^{-n}\omega)v_{n_1}, \ldots, T^n(\theta^{-n}\omega)v_{n_m})} \leq e^{-km(\lambda-\epsilon)}, \; 1 \leq k \leq n.$$

So

$$\omega \in D_{\lambda-\epsilon}(v_{n_1}, \ldots, v_{n_m}).$$

Therefore,

$$\omega \in K_{\lambda-\epsilon}.$$

Since $\epsilon > 0$ can be arbitrary small, and by the definition of K_λ, we have

$$\omega \in K_\lambda,$$

hence

$$A_n^\lambda \subset K_\lambda.$$

Summarizing the above discussion, we have that for any $\epsilon > 0$,

$$K_{\lambda+\epsilon} \subset A_n^\lambda \subset K_\lambda.$$

Since $P(K_\lambda)$ is a monotone function with respect to λ, it has at most countable many discontinuous points. We use S_d to denote the set of these discontinuous points. Thus for any $\lambda \in (l_\alpha, \kappa(T)) - S_d$, we have

$$\lim_{\epsilon \to 0^+} P(K_{\lambda+\epsilon}) = P(K_\lambda),$$

which implies that

$$P_{in}(A_n^\lambda) = P_{out}(A_n^\lambda).$$

Therefore, A_n^λ is measurable. Then, by using (45), we have that for any $\lambda \notin S_d$ and $l_\alpha < \lambda < \kappa(T)$,

$$P(A_n^\lambda) \geq \delta,$$

which implies for each $l_\alpha < \lambda < \kappa(T)$,

$$P_{in}(A_n^\lambda) \geq \delta.$$

Then

$$P_{in}\left(\bigcap_{n=1}^{+\infty} A_n^{\lambda}\right) \geq \delta.$$

This completes the proof of the lemma.

$\square$

Lemma 33. *There exists a constant C depends only on m such that for any small number $\epsilon > 0$ satisfying $\kappa(T) - C\epsilon > l_\alpha$, if $\omega \in \bigcap_{n=1}^{+\infty} A_n^{\kappa(T)-\epsilon}$, then* $\dim E^{\kappa(T)-C\epsilon}(\omega) \geq$ *m.*

Proof. Let $\epsilon > 0$ such that $\kappa(T) - \epsilon > l_\alpha$ and $\omega \in \bigcap_{n=1}^{+\infty} A_n^{\kappa(T)-\epsilon}$. Then, $\omega \in A_j^{\kappa(T)-\epsilon}$ for all $j \geq 1$. Thus, there exists $\{w_{ij}\}_{1\leq i\leq m, 1\leq j} \in X$ such that for any $j \geq 1$, $\{\pi(\omega)w_{ij}\}_{1\leq i\leq m}$ are linear independent and

$$\frac{V_m(T^{-k}(\omega)\pi(\omega)w_{1j},\ldots,T^{-k}(\omega)\pi(\omega)w_{mj})}{V_m(\pi(\omega)w_{1j},\ldots,\pi(\omega)w_{mj})} \leq e^{-km(\kappa(T)-\epsilon)},\ 1 \leq k \leq j.$$

Let

$$C_\epsilon(\omega) = \sup\left\{\frac{\|T^n(\omega)|_{G(\omega)}\|}{e^{n(\kappa(T)+\epsilon)}}\right\}_{n\geq 0}.$$

By the definition of $\kappa(T)$, we have that $C_\epsilon(\omega)$ is measurable and is finite almost surely. We note that

$$\begin{aligned}
C_\epsilon(\omega) &= \sup\left\{\frac{\|T^n(\omega)|_{G(\omega)}\|}{e^{n(\kappa(T)+\epsilon)}}\right\}_{n\geq 0} \\
&= \max\left\{\sup\left\{\frac{\|(T^{n-1}(\theta\omega)|_{G(\theta\omega)})(T(\omega)|_{G(\omega)})\|}{e^{n(\kappa(T)+\epsilon)}}\right\}_{n\geq 1}, 1\right\} \\
&\leq \max\left\{\sup\left\{\frac{\|(T^{n-1}(\theta\omega)|_{G(\theta\omega)})\|\|T(\omega)|_{G(\omega)}\|}{e^{n(\kappa(T)+\epsilon)}}\right\}_{n\geq 1}, 1\right\} \\
&\leq \max\left\{1, \frac{\|T(\omega)|_{G(\omega)}\|}{e^{\kappa(T)+\epsilon}}C_\epsilon(\theta\omega)\right\} \\
&\leq C_\epsilon(\theta\omega)\max\left\{1, \frac{\|T(\omega)|_{G(\omega)}\|}{e^{\kappa(T)+\epsilon}}\right\}.
\end{aligned}$$

Then

$$\log C_\epsilon(\omega) - \log C_\epsilon(\theta\omega) \le \max\{0, \log^+ \|T(\omega)|_{G(\omega)}\| - \kappa(T) - \epsilon\},$$

which is a L^1 function. Thus, using Lemma 26, we obtain

$$\lim_{n\to\pm\infty} \frac{1}{n} \log C_\epsilon(\theta^n\omega) = 0.$$

Therefore, for a fixed ω, there exists a positive integer $N_1(\omega)$ such that if $n \ge N_1(\omega)$, then

$$C_\epsilon(\theta^{-n}\omega) \le e^{n\epsilon}.$$

Thus

$$\|T^n(\theta^{-n}\omega)|_{G(\omega)}\| \le e^{n(\kappa(T)+2\epsilon)},\ n \ge N_1(\omega). \tag{48}$$

For any $j \ge 1$, let

$$v_{mj} = \frac{\pi(\omega)w_{mj}}{\|\pi(\omega)w_{mj}\|}$$

and choose unit vectors v_{ij}, $1 \le i \le m-1$ such that

$$\pi(\omega)w_{ij} - \operatorname{dist}(\pi(\omega)w_{ij}, \operatorname{span}\{\pi(\omega)w_{kj}\}_{i<k\le m})v_{ij} \in \operatorname{span}\{\pi(\omega)w_{kj}\}_{i<k\le m}.$$

Thus,

$$v_{ij} \in G(\omega) \text{ and } \|v_{ij}\| = \overline{\operatorname{dist}}(v_{ij}, \operatorname{span}\{v_{kj}\}_{i<k\le m}) = 1,\ 1 \le i \le m. \tag{49}$$

Therefore,

$$V_m(v_{1j}, \dots, v_{mj}) = 1 \tag{50}$$

and for any $1 \le k \le j$,

$$
\begin{aligned}
&V_m(T^{-k}(\omega)v_{1j}, \ldots, T^{-k}(\omega)v_{mj}) \\
&= V_m\left(T^{-k}(\omega)\frac{\pi(\omega)w_{1j}}{\mathrm{dist}(\pi(\omega)w_{1j}, \mathrm{span}\{\pi(\omega)w_{lj}\}_{1<l\le m})}, \ldots, T^{-k}(\omega)\frac{\pi(\omega)w_{mj}}{\|\pi(\omega)w_{mj}\|}\right) \\
&= \frac{V_m(T^{-k}(\omega)\pi(\omega)w_{1j}, \ldots, T^{-k}(\omega)\pi(\omega)w_{mj})}{V_m(\pi(\omega)w_{1j}, \ldots, \pi(\omega)w_{mj})} \\
&\le e^{-mk(\kappa(T)-\epsilon)}.
\end{aligned}
$$

By modifying $\{T^{-k}(\omega)v_{ij}\}_{1\le i\le m}$ in the same way as above, we obtain unit vectors $\{v_{ij}^k\}_{1\le i\le m} \subset G(\theta^{-k}\omega)$ such that for $1 \le i \le m$,

$$
\|v_{ij}^k\| = \overline{\mathrm{dist}}(v_{ij}^k, \mathrm{span}\{v_{lj}^k\}_{i<l\le m}) = 1,
$$

$$
T^{-k}(\omega)v_{ij} - \overline{\mathrm{dist}}\big(T^{-k}(\omega)v_{ij}, \mathrm{span}\{T^{-k}(\omega)v_{lj}\}_{i<l\le m}\big)v_{ij}^k \in \mathrm{span}\{T^{-k}(\omega)v_{lj}\}_{i<l\le m}. \quad (51)
$$

Hence,

$$
\begin{aligned}
&V_m(T^k(\theta^{-k}\omega)v_{1j}^k, \ldots, T^k(\theta^{-k}\omega)v_{mj}^k) \\
&= V_m\left(T^k(\theta^{-k}\omega)\frac{T^{-k}(\omega)v_{1j}}{\mathrm{dist}(T^{-k}(\omega)v_{1j}, \mathrm{span}\{T^{-k}(\omega)v_{ij}\}_{1<i\le m})}, \ldots, T^k(\theta^{-k}\omega)\frac{T^{-k}(\omega)v_{mj}}{\|T^{-k}(\omega)v_{mj}\|}\right) \\
&= \frac{V_m(v_{1j}, \ldots, v_{mj})}{V_m(T^{-k}(\omega)v_{1j}, \ldots, T^{-k}(\omega)v_{mj})} \\
&\ge e^{mk(\kappa(T)-\epsilon)}.
\end{aligned}
$$

Using (48), we have that for $1 \le i \le m$, $N_1(\omega) < k \le j$,

$$
\begin{aligned}
&\overline{\mathrm{dist}}(T^k(\theta^{-k}\omega)v_{ij}^k, \mathrm{span}\{T^k(\theta^{-k}\omega)v_{lj}^k\}_{i<l\le m}) \\
&\ge \frac{V_m(T^k(\theta^{-k}\omega)v_{1j}^k, \ldots, T^k(\theta^{-k}\omega)v_{mj}^k)}{\prod_{1\le l\le m, l\ne i}\|T^k(\theta^{-k}\omega)v_{lj}^k\|} \\
&\ge \frac{e^{mk(\kappa(T)-\epsilon)}}{e^{k(m-1)(\kappa(T)+2\epsilon)}} = e^{k[\kappa(T)-(3m-2)\epsilon]}.
\end{aligned} \quad (52)
$$

For each $1 \leq i \leq m$, using (51), we write

$$T^k(\theta^{-k}\omega)v_{ij}^k = \lambda_i^k v_{ij} + \pi(\omega)w_{i,j}^k,$$

where $\pi(\omega)w_{i,j}^k \in \mathrm{span}\{T^k(\theta^{-k}\omega)v_{lj}^k\}_{i<l\leq m}$. Using (49) and (51), we have that

$$|\lambda_i^k| = \mathrm{dist}(T^k(\theta^{-k}\omega)v_{ij}^k, \mathrm{span}\{T^k(\theta^{-k}\omega)v_{lj}^k\}_{i<l\leq m}).$$

Thus, from (52) and (48) we obtain

$$\|\lambda_i^k\| \geq e^{k[\kappa(T)-(3m-2)\epsilon]}$$

and

$$\begin{aligned}
\|\pi(\omega)w_{i,j}^k\| &\leq \|T^k(\theta^{-k}\omega)v_{ij}^k\| + \|\lambda_i^k v_{ij}\| \\
&\leq e^{k(\kappa(T)+2\epsilon)} + |\lambda_i^k| \\
&\leq 2e^{k(\kappa(T)+2\epsilon)}.
\end{aligned}$$

Thus, for $\pi(\omega)w_{i,j}^k \neq 0$, we have

$$\begin{aligned}
\|T^{-k}(\omega)v_{ij}\| &= \left\| \frac{v_{ij}^k}{\lambda_i^k} - \frac{\|\pi(\omega)w_{i,j}^k\|}{\lambda_i^k} T^{-k}(\omega)\frac{\pi(\omega)w_{i,j}^k}{\|\pi(\omega)w_{i,j}^k\|} \right\| \\
&\leq \left\| \frac{v_{ij}^k}{\lambda_i^k} \right\| + \left\| \frac{\|\pi(\omega)w_{i,j}^k\|}{\lambda_i^k} T^{-k}(\omega)\frac{\pi(\omega)w_{i,j}^k}{\|\pi(\omega)w_{i,j}^k\|} \right\| \\
&\leq e^{-k[\kappa(T)-(3m-2)\epsilon]} + 2e^{3mk\epsilon} \left\| T^{-k}(\omega)\frac{\pi(\omega)w_{i,j}^k}{\|\pi(\omega)w_{i,j}^k\|} \right\| \\
&\leq e^{-k[\kappa(T)-(3m-2)\epsilon]} + 2^m e^{3mk\epsilon} \sum_{l=i+1}^{m} \|T^{-k}(\omega)v_{lj}\|,
\end{aligned}$$

here we used the fact that if $v = \sum_{i=1}^n a_i v_i$, $||v|| = ||v_i|| = 1$, and $V_n(v_1, \cdots, v_n) = 1$, then $|a_i| \leq 2^{n-1}$. For $\pi(\omega)w_{i,j}^k = 0$, we have

$$\|T^{-k}(\omega)v_{i,j}\| \leq e^{-k[\kappa(T)-(3m-2)\epsilon]}.$$

From (48) and (52), we also have that

$$\|T^{-k}(\omega)v_{mj}\| = \|\lambda_m^k\|^{-1} \leq e^{-k[\kappa(T)-(3m-2)\epsilon]}.$$

Then by induction, there are two positive constants C, C' which depend only on m such that for any $1 \leq i \leq m$ and $N_1(\omega) < k \leq j$

$$\|T^{-k}(\omega)v_{ij}\| \leq Ce^{-k(\kappa(T)-C'\epsilon)}. \tag{53}$$

For example, we can take $C' = 3m^2$ and $C = 2^{m^2}$.

Thus we can choose $\epsilon > 0$ small enough such that $\kappa(T) - C'\epsilon > l_\alpha$ and let $\lambda = \kappa(T) - C'\epsilon$.

Next, we claim that for each nonnegative integer n and for each sequence of unit vectors $\{v_j \in G(\omega)\}_{j>0}$ satisfying

$$\|T^{-k}(\omega)v_j\| \leq Ce^{-k\lambda}, \quad N_1(\omega) < k \leq j, \tag{54}$$

$\{T^{-n}(\omega)v_j\}_{j>n}$ has a convergence subsequence.

Before proving this claim, we show how to use it to prove

$$\dim\,(E^\lambda(\omega)) \geq m$$

by applying it to the unit vectors v_{ij}. We first consider the case $n = 0$. Using (53) and this claim, there exists a subsequence $\{j_l^0\}_{l\geq 1}$ in $\mathbb{N}$ such that for any $1 \leq i \leq m$,

$T^0(\omega)v_{ij_l^0}$ converges. We denote the limit by u_i^0. Since v_{ij} are unit vectors, $\|u_i^0\| = 1$. Furthermore, from (50), $V_m(u_1^0, \ldots, u_m^0) = 1$.

Next, we apply the claim to $n = 1$. Thus, by using (53), there exists a subsequence $\{j_l^1\}_{l\geq 1}$ of $\{j_l^0\}_{l\geq 1}$ such that for each $1 \leq i \leq m$, $T^{-1}(\omega)v_{ij_l^1}$ converges to u_i^1 as $l \to \infty$. By the continuity of $T(\theta^{-1}\omega)$, we have

$$T(\theta^{-1}\omega)u_i^1 = \lim_{l\to+\infty} T(\theta^{-1}\omega)T^{-1}(\omega)v_{ij_l^1} = \lim_{l\to+\infty} v_{ij_l^1} = \lim_{l\to+\infty} v_{ij_l^0} = u_i^0,$$

which gives

$$T^{-1}(\omega)u_i^0 = u_i^1.$$

By induction, for a positive integer n, there exists a subsequence $\{j_l^n\}_{l\geq 1}$ of $\{j_l^{n-1}\}_{l\geq 1}$ such that for any $1 \leq i \leq m$, $T^{-n}(\omega)v_{ij_l^n}$ converges to u_i^n. By the continuity of $T^n(\theta^{-n}\omega)$ we have

$$T^n(\theta^{-n}\omega)u_i^n = \lim_{l\to+\infty} T^n(\theta^{-n}\omega)T^{-n}(\omega)v_{ij_l^n} = \lim_{l\to+\infty} v_{ij_l^n} = \lim_{l\to+\infty} v_{ij_l^0} = u_i^0.$$

So

$$T^{-n}(\omega)u_i^0 = u_i^n.$$

Since for $N_1 < k \leq j_l^n$

$$\|T^{-k}(\omega)v_{ij_l^n}\| \leq Ce^{-k\lambda},$$

we have

$$\|T^{-k}(\omega)u_i^0\| = \|u_i^k\| \leq Ce^{-k\lambda}.$$

Hence, we have that $T^{-n}(\omega)u_i^0$ exists for all positive integer n and

$$\|T^{-n}(\omega)u_i^0\| \leq Ce^{-n\lambda} \text{ when } n > N_1,$$

which yields

$$u_i^0 \in E^\lambda(\omega),\ 1 \le i \le m.$$

Since

$$V_m(u_1^0, \dots, u_m^0) = 1,$$

we have

$$\dim E^\lambda(\omega) \ge m.$$

We now prove the claim. Let n be a nonnegative integer. Since $\lambda > l_\alpha = l'_\alpha$, we have that for any $\epsilon > 0$ such that $l'_\alpha + 2\epsilon < \lambda$ there exists a integer $N > N_1$ such that for $l > N$

$$\|T^l(\theta^{-(n+l)}\omega)\|_\alpha < e^{l(l'_\alpha+\epsilon)}. \tag{55}$$

Let

$$G_l = \{T^{-(l+n)}(\omega)v_j |\ j > l+n\}, \quad \text{for } l \ge 0.$$

Then,

$$T^l(\theta^{-(n+l)}\omega)G_l \subset G_0$$

and $G_0 - T^l(\theta^{-(n+l)}\omega)G_l$ is a finite set.

Recall that $\alpha(B)$ is the smallest nonnegative real number such that for any $r' > \alpha$, the set $B \subset X$ can be covered by a finite number of balls of radius r' (not necessarily centered on B). Let

$$\|B\| = \sup\{\|v\| |\ v \in B\}$$

.

Then for any $l > N$, using (54) and (55), we obtain

$$\begin{aligned}\alpha(G_0) =&\alpha\left(T^l(\theta^{-(n+l)}\omega)G_l \cup [G_0 - T^l(\theta^{-(n+l)}\omega)G_l]\right)\\ \leq& \max\left\{\alpha\left(T^l(\theta^{-(n+l)}\omega)G_l\right), \alpha\left(G_0 - T^l(\theta^{-(n+l)}\omega)G_l\right)\right\}\\ =&\alpha\left(T^l(\theta^{-(n+l)}\omega)G_l\right) \leq \|T^l(\theta^{-(n+l)}\omega)\|_\alpha \|G_l\|\\ \leq& e^{l(l'_\alpha+\epsilon)}Ce^{-(n+l)\lambda} = Ce^{-n\lambda}e^{l(l'_\alpha+\epsilon-\lambda)} \leq e^{-n\lambda}e^{-l\epsilon}.\end{aligned}$$

Since l can be arbitrarily large,

$$\alpha(G_0) = 0,$$

which implies G_0 is precompact, thus there exists a converging subsequence of $\{T^{-n}(\omega)v_j | \, n_j > n\}$. This completes the proof of the lemma.

□

Proof of Proposition 30. It is sufficient to show that the lemma holds for $l_\alpha < \lambda < \kappa(T)$. Since $E^\lambda(\omega)$ is a decreasing sequence of finite dimensional subspaces, we have

$$\dim E^{\kappa(T)}(\omega) = \dim \bigcap_{\lambda_\alpha<\lambda<\kappa(T)} E^\lambda(\omega). \tag{56}$$

Letting λ be fixed such that $l_\alpha < \lambda < \kappa(T)$, by Lemma 32, there exists a countable set S of $(l_\alpha, \kappa(T))$ such that for each $\lambda_0 \in (l_\alpha, \kappa(T)) - S$, $\bigcap_{n=1}^\infty A_n^{\lambda_0}$ is measurable and

$$P(\bigcap_{n=1}^\infty A_n^{\lambda_0}) \geq \delta.$$

Choose $\epsilon > 0$ such that $\lambda < \kappa(T) - C\epsilon$ and $\kappa(T) - \epsilon \in (l_\alpha, \kappa(T)) - S$. Then, $\bigcap_{n=1}^\infty A_n^{\kappa(T)-\epsilon}$ is measurable and

$$P(\bigcap_{n=1}^\infty A_n^{\kappa(T)-\epsilon}) \geq \delta.$$

By Lemma 33, we have that for each $\omega \in \bigcap_{n=1}^{\infty} A_n^{\kappa(T)-\epsilon}$,

$$\dim(E^{\kappa(T)-C\epsilon}(\omega)) \geq m.$$

Since $\lambda < \kappa(T) - C\epsilon$, $E^{\kappa(T)-C\epsilon}(\omega) \subset E^{\lambda}(\omega)$. Thus,

$$\dim(E^{\lambda}(\omega)) \geq m.$$

Since $E^{\kappa(T)-C\epsilon}(\omega)$ is invariant and $T(\omega)|_{G(\omega)}$ is injective, $\dim(E^{\kappa(T)-C\epsilon}(\theta^n\omega)) \geq m$ for all $n \in \mathbb{Z}$. Let

$$\mathcal{A}^{\kappa(T)-\epsilon} = \bigcup_{j\in\mathbb{Z}} \theta^j \left(\bigcap_{n=1}^{+\infty} A_n^{\kappa(T)-\epsilon} \right).$$

Then, $\mathcal{A}^{\kappa(T)-\epsilon}$ is a θ-invariant measurable set of positive measure and $\dim(E^{\kappa(T)-C\epsilon}(\omega)) \geq m$ for all $\omega \in \mathcal{A}^{\kappa(T)-\epsilon}$. By the ergodicity of θ, we obtain

$$P\big(\mathcal{A}^{\kappa(T)-\epsilon}\big) = 1.$$

This completes the proof of the proposition.

□

6.3 Measurability of Oseledets Spaces.

In this subsection, we prove the measurability of $E^{\kappa(T)}(\omega)$. We will use a modified version of the following theorem of measurable selection taken from [3]

Theorem 34. *Let Y be a complete separable metric space, $(\mathbf{T}, \mathcal{L})$ be a measurable space, and Γ be a multifunction from $\mathbf{T}$ to a closed non-empty subset of Y. If for any open set U in Y, $\Gamma^{-}(U)(= \{t|\ \Gamma(t) \cap U \neq \emptyset\})$ belongs to $\mathcal{L}$, then Γ admits a measurable selection.*

The following is a modification of the above theorem.

Corollary 35. *Let Y be a complete separable metric space, $(\mathbf{T}, \mathcal{L})$ be a measurable space, and Γ be a multifunction from $\mathbf{T}$ to a non-empty subset of Y. If for any open set U in Y, $\Gamma^-(U)(=\{t|\ \Gamma(t) \cap U \neq \emptyset\})$ belongs to $\mathcal{L}$, then $\overline{\Gamma}$ admits a measurable selection, where $\overline{\Gamma}$ is defined as $\overline{\Gamma}(t) =$ closure $\Gamma(t)$ for any $t \in \mathbf{T}$.*

Proof. Since $\overline{\Gamma}(t)$ is closed for every $t \in \mathbf{T}$ and $\Gamma(t) \subset \overline{\Gamma}(t)$, then for any open set $U \subset Y$,

$$\Gamma^-(U) \subset \overline{\Gamma}^-(U) = \{t|\ \overline{\Gamma}(t) \cap U \neq \emptyset\}.$$

For any $t \in \overline{\Gamma}^-(U)$, there exists a $x \in \overline{\Gamma}(t) \cap U$. If $x \in \Gamma(t)$, then $t \in \Gamma^-(U)$. Otherwise there exists a real number $r > 0$ and $x' \in \Gamma(t)$ such that the ball $B(x, r) \subset U$ and $x' \in B(x, r)$. Thus, $t \in \Gamma^-(U)$. Therefore,

$$\overline{\Gamma}^-(U) \subset \Gamma^-(U).$$

Hence

$$\overline{\Gamma}^-(U) = \Gamma^-(U),$$

which is measurable. By the theorem of measurable selection, we have that $\overline{\Gamma}$ admits a measurable selection. This completes the proof. $\square$

Now we are ready to prove that

Proposition 36. $E^{\kappa(T)}(\omega):\ \Omega \to K_m$ is a measurable function, where K_m is the metric space of all m-dimensional linear subspaces of X introduced in Section 5.

Proof. From (37), we have $\kappa(T) = l_1 > l_{m+1}/(m+1)$ and $\kappa(T) = l_m/m$. Let $\epsilon > 0$ satisfy $(m+1)\kappa(T) - (m^2 + 3m + 4)\epsilon > l_{m+1}(T)$. Using (30), we have

$$\lim_{n \to +\infty} \frac{V_m(T^n(\theta^{-n}\omega)|_{G(\theta^{-n}\omega)})}{e^{mn(\kappa(T)-\epsilon)}} = +\infty \quad a.s.. \tag{57}$$

We use Ω_0 denote the exceptional set of zero measure on which (57)does not hold. Let

$$Q_n = \left\{\omega \;\middle|\; V_m(T^n(\theta^{-n}\omega)|_{G(\theta^{-n}\omega)}) > e^{mn(\kappa(T)-\epsilon)}\right\}.$$

Since $V_m(T^n(\theta^{-n}\omega)|_{G(\theta^{-n}\omega)})$ is measurable as we have seen in the proof of Lemma 25, Q_n is measurable. Using (57) and Egoroff's theorem, we obtain

$$\lim_{n\to+\infty} P(Q_n) = 1.$$

Since X is separable, we let $D = \{(v_1^i, \ldots, v_m^i)\}_{i\geq 1}$ be a countable dense subset of the unit ball $B(0,1) \subset X^m$. In order to apply Corollary 35, we define $\Gamma_n(\omega):\ Q_n \to 2^{X^m}$ as

$$\Gamma_n(\omega) = \left\{u \;\middle|\; V_m(T^n(\theta^{-n}\omega)u) > e^{mn(\kappa(T)-\epsilon)}, u = (u_1, \ldots, u_m) \in D\right\},$$

where $T^n(\theta^{-n}\omega)u$ denotes $(T^n(\theta^{-n}\omega)u_1, \cdots, T^n(\theta^{-n}\omega)u_m)$. Let $(\mathbf{T}, \mathcal{L}) = (Q_n, \mathcal{F}|_{Q_n})$ and $Y = X^m$, then for any open set $U \subset X^m$ we have

$$\begin{aligned}
&\Gamma_n^-(U) = \left\{\omega|\ \Gamma_n(\omega) \cap U \neq \emptyset\right\} \\
=&\left\{\omega|\ \exists\, u \in D\bigcap U \text{ such that } V_m(T^n(\theta^{-n}\omega)u) > e^{mn(\kappa(T)-\epsilon)}\right\} \\
=&\bigcup_{v\in D\cap U} \left\{\omega \;\middle|\; V_m(T^n(\theta^{-n}\omega)v_1^i, \ldots, T^n(\theta^{-n}\omega)v_m^i) > e^{mn(\kappa(T)-\epsilon)}\right\}.
\end{aligned}$$

Thus using the facts that $T(\omega)$ is strongly measurable and V_m is continuous, we have that $\Gamma_n^-(U)$ is measurable. Hence, by Corollary 35, $\overline{\Gamma}_n$ admits a measurable selection $\tilde{\sigma}_n(\omega) = (\tilde{\sigma}_n^1(\omega), \cdots, \tilde{\sigma}_n^m(\omega))$ such that

$$T^n(\theta^{-n}\omega)\tilde{\sigma}_n^1(\omega), \cdots, \tilde{T}^n(\theta^{-n}\omega)\sigma_n^m(\omega)$$

are linearly independent and for $\omega \in Q_n$

$$V_m\left(T^n(\theta^{-n}\omega)\tilde{\sigma}_n^1(\omega), \cdots, T^n(\theta^{-n}\omega)\tilde{\sigma}_n^m(\omega)\right) \geq e^{mn(\kappa(T)-\epsilon)}. \tag{58}$$

Let

$$\sigma_n = \operatorname{span}\left\{T^n(\theta^{-n}\omega)\tilde{\sigma}_n^1(\cdot), \cdots, T^n(\theta^{-n}\omega)\tilde{\sigma}_n^m(\cdot)\right\}.$$

By Lemma 20, σ_n is measurable from Q_n to K_m. Note that $\sigma_n(\omega) \subset G(\omega)$ for $\omega \in Q_n$. Furthermore, since Q_n is measurable, we can extend σ_n to a measurable map from Ω to K_m with a constant extension on $\Omega - Q_n$.

Next we will prove that

$$\hat{d}(\sigma_n(w), E^{\kappa(T)}(\omega)) \to 0 \text{ as } n \to +\infty \ a.s.,$$

which yields that $E^{\kappa(T)}(\omega)$ is measurable.

For each fixed $\omega \in \Omega - \Omega_0$, from (57) there exists a $N = N(\omega)$ such that for $n \geq N(\omega)$, $\omega \in Q_n$. Let v be a unit vector in $T^{-n}(\sigma_n(\omega))$ and choose unit vectors $\{u_i\}_{1\leq i\leq m} \subset E^{\kappa(T)}(\omega)$ such that

$$\overline{\operatorname{dist}}(u_i, \operatorname{span}\{u_j\}_{i<j\leq m}) = 1, \quad 1 \leq i \leq m. \tag{59}$$

Since

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|T^{-n}(\omega)u_i\| \leq -\kappa(T), \ 1 \leq i \leq m,$$

there exists a $N_0(\omega) \geq N(\omega)$ such that if $n > N_0(\omega)$, then

$$\|T^{-n}(\omega)u_i\| < e^{n(-\kappa(T)+\frac{\epsilon}{m})}.$$

Hence,

$$
\begin{aligned}
&V_m\left(T^n(\theta^{-n}\omega)\Big(\frac{T^{-n}(\omega)u_1}{\|T^{-n}(\omega)u_1\|}\Big),\dots,T^n(\theta^{-n}\omega)\Big(\frac{T^{-n}(\omega)u_m}{\|T^{-n}(\omega)u_m\|}\Big)\right)\\
&=V_m(u_1,\dots,u_m)\left(\prod_{i=1}^{m}\|T^{-n}(\omega)u_i\|\right)^{-1}\\
&\geq e^{nm\kappa(T)-n\epsilon}.
\end{aligned}
\tag{60}
$$

Since

$$
\lim_{n\to+\infty}\frac{1}{n}\log V_{m+1}\left(T^n(\theta^{-n}\omega)\right)=l_{m+1},
$$

there exists a positive integer $N_1(\omega)\geq N_0(\omega)$ such that for $n>N_1(\omega)$

$$
\frac{1}{n}\log V_{m+1}\left(T^n(\theta^{-n}\omega)\right)<l_{m+1}+\epsilon.
$$

Thus, for $n>N_1(\omega)$ we obtain

$$
\begin{aligned}
&\frac{1}{n}\log V_{m+1}\left(T^n(\theta^{-n}\omega)v,T^n(\theta^{-n}\omega)\left(\frac{T^{-n}(\omega)u_1}{\|T^{-n}(\omega)u_1\|}\right),\cdots,T^n(\theta^{-n}\omega)\left(\frac{T^{-n}(\omega)u_m}{\|T^{-n}(\omega)u_m\|}\right)\right)\\
&=\frac{1}{n}\log\operatorname{dist}\left(T^n(\theta^{-n}\omega)v,E^{\kappa(T)}(\omega)\right)\\
&\quad+\frac{1}{n}\log V_m\left(T^n(\theta^{-n}\omega)\left(\frac{T^{-n}(\omega)u_i}{\|T^{-n}(\omega)u_1\|}\right),\cdots,T^n(\theta^{-n}\omega)\left(\frac{T^{-n}(\omega)u_i}{\|T^{-n}(\omega)u_m\|}\right)\right)\\
&\leq l_{m+1}+\epsilon.
\end{aligned}
\tag{61}
$$

By Lemma 28, we have

$$
\lim_{n\to+\infty}\frac{1}{n}\log\|T^n(\theta^{-n}\omega)\|=\kappa(T),
$$

which implies that there exists a $N_2(\omega)>N_1(\omega)$ such that for $n>N_2(\omega)$

$$
\frac{1}{n}\log\|T^n(\theta^{-n}\omega)\|<\kappa(T)+\frac{1}{m}\epsilon. \tag{62}
$$

By (58), there exist unit vectors $\{v_i^n\}_{1\le i\le m} \subset T^{-n}(\sigma_n(\omega))$ such that

$$V_m(T^n(\theta^{-n}\omega)v_1^n, \ldots, T^n(\theta^{-n}\omega)v_m^n) \geq e^{nm(\kappa(T)-\epsilon)}. \tag{63}$$

From (62), we have

$$\|T^n(\theta^{-n}\omega)v_i^n\| < e^{n(\kappa(T)+\frac{1}{m}\epsilon)}, \quad 1 \leq i \leq m. \tag{64}$$

Therefore, for any $1 \leq i \leq m$,

$$\begin{aligned}
&\overline{\text{dist}}\left(T^n(\theta^{-n}\omega)v_i^n, \text{span}\{T^n(\theta^{-n}\omega)v_j^n\}_{i<j\le m}\right) \\
&\geq \frac{V_m(T^n(\theta^{-n}\omega)v_1^n, \ldots, T^n(\theta^{-n}\omega)v_m^n)}{\prod_{1\le j\le m, j\ne i}\|T^n(\theta^{-n}\omega)v_i^n\|} \\
&\geq \frac{e^{nm(\kappa(T)-\epsilon)}}{e^{n(m-1)(\kappa(T)+\frac{1}{m}\epsilon)}} > e^{n(\kappa(T)-(m+1)\epsilon)},
\end{aligned}$$

which together with (64) gives

$$e^{n(\kappa(T)-(m+1)\epsilon)} < \|T^n(\theta^{-n}\omega)v_i^n\| < e^{n(\kappa(T)+\frac{1}{m}\epsilon)}. \tag{65}$$

Let

$$w_i^n = \frac{T^n(\theta^{-n}\omega)v_i^n}{\|T^n(\theta^{-n}\omega)v_i^n\|}, \ 1 \leq i \leq m.$$

Then

$$\begin{aligned}
&\overline{\text{dist}}(w_i^n, \text{span}\{w_j^n\}_{i<j\le m}) \\
&= \overline{\text{dist}}\left(\frac{T^n(\theta^{-n}\omega)v_i^n}{\|T^n(\theta^{-n}\omega)v_i^n\|}, \text{span}\{T^n(\theta^{-n}\omega)v_j^n\}_{i<j\le m}\right) \\
&> \frac{e^{n(\kappa(T)-(m+1)\epsilon)}}{e^{n(\kappa(T)+\frac{1}{m}\epsilon)}} > e^{-n(m+2)\epsilon}.
\end{aligned} \tag{66}$$

Then for any $(a_1, \ldots, a_m) \in \mathbb{R}^m$ satisfying $\| \sum_{i=1}^m a_i w_i^n \| = 1$, using (66) we have

$$|a_1| \leq e^{n(m+2)\epsilon},$$
$$|a_i| \leq e^{n(m+2)\epsilon}\big(1 + \sum_{j=1}^{i-1} |a_j|\big), \ 1 < i \leq m,$$

which implies that for $1 \leq k \leq m$

$$|a_k| \leq k! e^{kn(m+2)\epsilon}.$$

For any $1 \leq i \leq m$, using (60), (61), and (65), we have

$$\begin{aligned} &\frac{1}{n} \log \operatorname{dist}(w_i^n, E^{\kappa(T)}(\omega)) = \frac{1}{n} \log \operatorname{dist}\left(\frac{T^n(\theta^{-n}\omega)v_i^n}{\|T^n(\theta^{-n}\omega)v_i^n\|}, E^{\kappa(T)}(\omega)\right) \\ &\leq l_{m+1} + \epsilon - (l_m - \epsilon) - \big(\kappa(T) - (m+1)\epsilon\big) \\ &= l_{m+1} - (m+1)\kappa(T) + (m+3)\epsilon. \end{aligned}$$

Thus, for any unit vector $\sum_{i=1}^m a_i w_i^n$, we have

$$\begin{aligned} &\operatorname{dist}(\sum_{i=1}^m a_i w_i^n, E^{\kappa(T)}(\omega)) \leq \sum_{i=1}^m |a_i| \operatorname{dist}(w_i^n, E^{\kappa(T)}(\omega)) \\ &\leq e^{n(l_{m+1} - (m+1)\kappa(T) + (m+3)\epsilon)} \sum_{i=1}^m |a_i| \\ &< e^{n(l_{m+1} - (m+1)\kappa(T) + (m+3)\epsilon)} m! \, m e^{mn(m+2)\epsilon} \\ &\leq m! \, m e^{n\left(l_{m+1} - (m+1)\kappa(T) + (m^2+3m+3)\epsilon\right)} \\ &\leq m! \, m e^{-n\epsilon}, \end{aligned}$$

here we used the fact $(m+1)\kappa(T) - (m^2+3m+4)\epsilon > l_{m+1}(T)$. Thus, by using (20), we obtain

$$d(\sigma_n(\omega), E^{\kappa(T)}(\omega)) \leq 2m! \, m e^{-n\epsilon}, \ n > N_2(\omega). \tag{67}$$

Similarly, for any $1 \le i \le m$ and $n > N_2(\omega)$, we have

$$\frac{1}{n}\log V_{m+1}\left(T^n(\theta^{-n}\omega)\left(\frac{T^{-n}(\omega)u_i}{\|T^{-n}(\omega)u_i\|}\right), T^n(\theta^{-n}\omega)v_1^n, \cdots, T^n(\theta^{-n}\omega)v_m^n\right) \le l_{m+1} + \epsilon.$$

Recall that

$$V_m(T^n(\theta^{-n}\omega)v_1^n, \ldots, T^n(\theta^{-n}\omega)v_m^n) \ge e^{nm(\kappa(T)-\epsilon)}$$

and

$$\|T^{-n}(\omega)u_i\| < e^{n(-\kappa(T)+\frac{\epsilon}{m})}.$$

Therefore,

$$\begin{aligned}\mathrm{dist}(u_i, \sigma_n(\omega)) &\le e^{n(l_{m+1}+\epsilon-m(\kappa(T)-\epsilon)+(-\kappa(T)+\frac{\epsilon}{m}))}\\ &\le e^{n(l_{m+1}-(m+1)\kappa(T)+(m+2)\epsilon)}.\end{aligned}$$

Let $(b_1, \ldots, b_m) \in \mathbb{R}^m$ such that $\|\sum_{i=1}^m b_i u_i\| = 1$. Using (59) then

$$\begin{aligned}&|b_1| \le 1,\\ &|b_i| \le 1 + \sum_{j=1}^{i-1}|b_j|,\ 1 \le i \le m,\end{aligned}$$

which implies that for $1 \le k \le m$

$$|b_k| \le 2^{k-1}.$$

Hence, for any unit vector $\sum_{i=1}^m b_i u_i$, we have

$$
\begin{aligned}
&\mathrm{dist}(\sum_{i=1}^m b_i u_i, \sigma_n(\omega)) \le \sum_{i=1}^m |b_i| \mathrm{dist}(u_i, \sigma_n(\omega)) \\
&\le 2^m e^{n(l_{m+1}-(m+1)\kappa(T)+(m+2)\epsilon)} \\
&\le 2^m e^{-n\epsilon}.
\end{aligned}
$$

So,

$$d(E^{\kappa(T)}(\omega), \sigma_n(\omega)) \le 2^{m+1} e^{-n\epsilon}, \ n > N_2(\omega).$$

Thus

$$\hat{d}(\sigma_n(\omega), E^{\kappa(T)}(\omega)) \le 2m!\, m e^{-n\epsilon}, \ n > N_2(\omega),$$

which implies that

$$\hat{d}(\sigma_n(\omega), E^{\kappa(T)}(\omega)) \to 0, \text{ as } n \to +\infty.$$

This completes the proof of the proposition $\square$

Remark 5. In fact we can choose ϵ so small that the rating in which $\sigma_n(\omega)$ converging to $E^{\kappa(T)}(\omega)$ as closed to $e^{n(l_{m+1}-(m+1)\kappa(T))}$ as we need.

6.4 Principal Lyapunov Exponents.

In this subsection, we establish the principal Lyapunov exponent and prove Theorem 23.

Proof of Theorem 23.

We first prove that there exists a θ-invariant subset $\tilde{\Omega} \subset \Omega$ of full measure such that for each $\omega \in \tilde{\Omega}$

$$\liminf_{n\to+\infty} \frac{1}{n} \log \|T^n(\omega)v\| \ge \kappa(T) \tag{68}$$

for every $v(\neq 0) \in E^{\kappa(T)}(\omega)$.

Let $\epsilon > 0$ and define

$$C_\epsilon(\omega) = \sup \left\{ \frac{\|T^{-n}(\omega)|_{E^{\kappa(T)}(\omega)}\|}{e^{n(-\kappa(T)+\epsilon)}} \right\}_{n \geq 0}.$$

Since $E^{\kappa(T)}(\omega)$ is a m-dimensional space, we can choose m unit vectors $\{u_i\}_{1 \leq i \leq m}$ of $E^{\kappa(T)}(\omega)$ such that

$$\overline{\text{dist}}(u_i, \text{span}\{u_j\}_{i<j\leq m}) = 1, \; 1 \leq i \leq m. \tag{69}$$

Let $\tilde{\Omega}$ be the θ-invariant set of full measure such that $\dim\,(E^{\kappa(T)}(\omega)) = m$ for $\omega \in \tilde{\Omega}$. For $u_i \in E^{\kappa(T)}(\omega)$,

$$\limsup_{n \to +\infty} \frac{1}{n} \log \|T^{-n}(\omega) u_i\| \leq -\kappa(T).$$

Then,

$$\sup \left\{ \frac{\|T^{-n}(\omega) u_i\|}{e^{n(-\kappa(T)+\epsilon)}} \right\}_{n \geq 0} < +\infty.$$

For any $(a_1, \ldots, a_m) \in \mathbb{R}^m$ such that $\|\sum_{i=1}^m a_i u_i\| = 1$, since u_i are unit vectors such that $\overline{\text{dist}}(u_i, \text{span}\{u_j\}_{i<j\leq m}) = 1, \; 1 \leq i \leq m$, we have

$$\|a_i\| < 2^m, 1 \leq i \leq m.$$

Thus

$$
\begin{aligned}
C_\epsilon(\omega) =& \sup\left\{\frac{\|T^{-n}(\omega)|_{E^{\kappa(T)}(\omega)}\|}{e^{n(-\kappa(T)+\epsilon)}}\right\}_{n\geq 0} \\
=& \sup_{n\geq 0} \sup_{\|\sum_{i=1}^m a_iu_i\|=1} \left\{\frac{\|T^{-n}(\omega)\sum_{i=1}^m a_iu_i\|}{e^{n(-\kappa(T)+\epsilon)}}\right\} \\
\leq& \sup_{n\geq 0} \sup_{\|\sum_{i=1}^m a_iu_i\|=1} \left\{\frac{\sum_{i=1}^m |a_i|\|T^{-n}(\omega)u_i\|}{e^{n(-\kappa(T)+\epsilon)}}\right\} \\
\leq& m2^m \sup_{n\geq 0} \max_{1\leq i\leq m} \left\{\frac{\|T^{-n}(\omega)u_i\|}{e^{n(-\kappa(T)+\epsilon)}}\right\} \\
<& +\infty.
\end{aligned}
$$

Since $E^{\kappa(T)}(\omega)$ is measurable and finite dimensional, by using the theorem of measurable selections, $E^{\kappa(T)}(\omega)$ has a measurable basis $u_i(\omega), 1 \leq i \leq m$, details will be given in section 7. Thus, by using that $T(\omega)$ is strongly measurable, we have that

$$
\inf_{(a_1,\cdots,a_m)\in\mathbb{Q}^m} \frac{\|T^n(\theta^{-n}\omega)(a_1u_1 + \cdots + a_mu_m)\|}{\|(a_1u_1 + \cdots + a_mu_m)\|} \tag{70}
$$

is measurable. This implies that $\|T^{-n}(\omega)\big|_{E^k(\omega)}\|$ is measurable. Thus, $C_\epsilon(\omega)$ is a measurable. Then, for each $\omega \in \tilde{\Omega}$ and any $v \in E^{\kappa(T)}(\omega)$,

$$
\|T^n(\omega)v\| \geq \|T^{-n}(\theta^n\omega)|_{E^{\kappa(T)}(\theta^n\omega)}\|^{-1}\|v\| \geq C_\epsilon^{-1}(\theta^n\omega)e^{n(\kappa(T)-\epsilon)}\|v\|.
$$

Hence

$$
\liminf_{n\to+\infty} \frac{1}{n}\log\|T^n(\omega)v\| \geq \kappa(T) - \epsilon - \limsup_{n\to+\infty}\frac{1}{n}\log C_\epsilon(\theta^n\omega).
$$

Since ϵ can be arbitrarily small, it is enough to show that for any $\epsilon > 0$,

$$
\lim_{n\to+\infty}\frac{1}{n}\log C_\epsilon(\theta^n\omega) = 0 \quad a.s..
$$

We estimate

$$
\begin{aligned}
C_\epsilon(\omega) &= \sup\left\{\frac{\|T^{-n}(\omega)|_{E^{\kappa(T)}(\omega)}\|}{e^{n(-\kappa(T)+\epsilon)}}\right\}_{n\geq 0} \\
&= \sup\left\{\frac{\|T^{-(n+1)}(\theta\omega)T(\omega)|_{E^{\kappa(T)}(\omega)}\|}{e^{n(-\kappa(T)+\epsilon)}}\right\}_{n\geq 0} \\
&\leq \sup\left\{\frac{\|T^{-(n+1)}(\theta\omega)|_{E^{\kappa(T)}(\theta\omega)}\|}{e^{(n+1)(-\kappa(T)+\epsilon)}}\frac{\|T(\omega)\big|_{G(\omega)}\|}{e^{\kappa(T)-\epsilon}}\right\}_{n\geq 0} \\
&\leq \frac{\|T(\omega)\big|_{G(\omega)}\|}{e^{\kappa(T)-\epsilon}}C_\epsilon(\theta\omega).
\end{aligned}
\tag{71}
$$

Then,

$$
\log C_\epsilon(\omega) - \log C_\epsilon(\theta\omega) \leq \log^+\|T(\omega)\big|_{G(\omega)}\| - \kappa(T) + \epsilon
$$

which is an L^1 function. Thus by lemma 26, we have

$$
\lim_{n\to\pm\infty}\frac{1}{n}\log C_\epsilon(\theta^n\omega) = 0 \quad a.s.,
$$

which together with (31) gives Theorem 23 for $n \to +\infty$.

Next, we show that Theorem 23 holds for $n \to -\infty$. By the definition of $E^{\kappa(T)}(\omega)$, it is sufficient to prove that for almost every $\omega \in \Omega$ and every $v(\neq 0) \in E^{\kappa(T)}(\omega)$,

$$
\liminf_{n\to+\infty}\frac{1}{n}\log\|T^{-n}(\omega)v\| \geq -\kappa(T).
$$

Given $\epsilon > 0$, define

$$
C'_\epsilon(\omega) = \sup\left\{\frac{\|T^{n}(\omega)|_{E^{\kappa(T)}(\omega)}\|}{e^{n(\kappa(T)+\epsilon)}}\right\}_{n\geq 0}.
$$

By the definition of $\kappa(T)$, $C'_\epsilon(\omega)$ is a bounded measurable function since $E^{\kappa(T)}(\omega)$ is measurable. Then, for $v \in E^{\kappa(T)}(\omega)$ we obtain

$$
\|v\| = \|T^n(\theta^{-n}\omega)T^{-n}(\omega)v\| \leq C'_\epsilon(\theta^{-n}\omega)e^{n(\kappa(T)+\epsilon)}\|T^{-n}(\omega)v\|,
$$

which gives

$$\liminf_{n\to+\infty} \frac{1}{n} \log \|T^{-n}(\omega)v\| \geq -\kappa(T) - \epsilon - \limsup_{n\to+\infty} \frac{1}{n} \log C'_\epsilon(\theta^{-n}\omega).$$

Therefore, the proof is completed if we can show for all $\epsilon > 0$.

$$\lim_{n\to+\infty} \frac{1}{n} \log C'_\epsilon(\theta^{-n}\omega) = 0 \quad a.s..$$

For $n \geq 1$, we have that

$$\begin{aligned}
&\|T^n(\theta^{-1}\omega)|_{E^{\kappa(T)}(\theta^{-1}\omega)}\| \\
&\leq \|T(\theta^{-1}\omega)|_{E^{\kappa(T)}(\theta^{-1}\omega)}\| \cdot \|T^{n-1}(\omega)|_{E^{k(T)}(\omega)}\| \\
&\leq \|T(\theta^{-1}\omega)|_{E^{\kappa(T)}(\theta^{-1}\omega)}\| C'_\epsilon(\omega) e^{(n-1)(\kappa(T)+\epsilon)} \\
&= \left[\|T(\theta^{-1}\omega)|_{E^{\kappa(T)}(\theta^{-1}\omega)}\| e^{-\kappa(T)-\epsilon}\right] \left[C'_\epsilon(\omega) e^{n(\kappa(T)+\epsilon)}\right].
\end{aligned}$$

Therefore

$$\begin{aligned}
C'(\theta^{-1}\omega) \leq& \max\left\{1, \sup\left\{\frac{\|T^n(\theta^{-1}\omega)|_{E^{\kappa(T)}(\theta^{-1}\omega)}\|}{e^{n(\kappa(T)+\epsilon)}}\right\}_{n\geq 1}\right\} \\
\leq& \max\left\{1, \|T(\theta^{-1}\omega)|_{E^{\kappa(T)}(\theta^{-1}\omega)}\| e^{-\kappa(T)-\epsilon} C'_\epsilon(\omega)\right\} \\
&\text{(as } C'_\epsilon(\omega) \geq 1 \text{ for all } \omega \in \Omega) \\
\leq& C'_\epsilon(\omega) \sup\left\{1, \|T(\theta^{-1}\omega)|_{E^{\kappa(T)}(\theta^{-1}\omega)}\| e^{-\kappa(T)-\epsilon}\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\log C'(\theta^{-1}\omega) - \log C'_\epsilon(\omega) \\
&\leq \max\left\{0, \log \|T(\theta^{-1}\omega)|_{E^{\kappa(T)}(\theta^{-1}\omega)}\| - \kappa(T) - \epsilon\right\} \\
&\leq \max\{0, \log^+ \|T(\theta^{-1}\omega)\big|_{G(\theta^{-1}\omega)}\| - \kappa(T) - \epsilon\}
\end{aligned}$$

which is an L^1 function. By Lemma 26, we have

$$\lim_{n\to+\infty} \frac{1}{n} \log C'_\epsilon(\theta^{-n}\omega) = 0 \quad a.s..$$

Therefore,

$$\lim_{n\to-\infty} \frac{1}{n} \log \|T^n(\omega)v\| = \kappa(T), \tag{72}$$

which together with (68) implies that

$$\liminf_{n\to\infty} \frac{1}{n} \log \|T^{-n}(\omega)\big|_{E^{\kappa(T)}(\omega)}\| \geq -\kappa(T).$$

For each unit vector v, we write $v = a_1u_1 + \cdots + a_mu_m$, where $u_i \in E^{\kappa(T)(\omega)}$ are unit basis satisfying (69). Thus, we have

$$\begin{aligned} &\|T^{-n}(\omega)\big|_{E^{\kappa(T)}(\omega)}\| = \sup_{\|a_1u_1+\cdots+a_mu_m\|=1} \|T^{-n}(\omega)(a_1u_1 + \cdots + a_mu_m)\| \\ &\leq m2^m \max_{1\leq i\leq m} \|T^{-n}(\omega)(u_i)\|. \end{aligned}$$

Then, by using (72) and (68), we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \log \|T^{-n}(\omega)\big|_{E^{\kappa(T)}(\omega)}\| \leq -\kappa(T).$$

This completes the proof of this Theorem.

□

7 Measurable Random Invariant Complementary Subspaces

In this section, we establish the existence of measurable invariant subspaces. Our main result is following.

Theorem 37. *If $\kappa(T) > l_\alpha$, then there exists a θ-invariant full measure subset $\tilde{\Omega}$ in Ω such that for any $\omega \in \tilde{\Omega}$ there exists a subspace $F(\omega)$ with codimension $m + \dim E(\omega)$ satisfying following properties:*

(i) $T(\omega)(F(\omega)) \subset F(\theta\omega)$;

(ii) $E^{\kappa(T)}(\omega) \oplus F(\omega) = G(\omega)$ *and the associated projection operator* $\pi_1(\omega) : X \to F(\omega)$ *is strongly measurable and* $\|\pi_1(\omega)\|$ *is tempered;*

(iii) $\kappa(T|F) := \limsup_{n\to+\infty} \frac{1}{n} \log \|T^n(\omega)\big|_{F(\omega)}\| < \kappa(T)$;

(iv) *For every* $v(\neq 0) \in F(\omega)$ *such that* $T^{-n}v$ *exists for all* $n \geq 0$, *we have*

$$\liminf_{n\to+\infty} \frac{1}{n} \log \|T^{-n}v\| > -\kappa(T);$$

(v) *For any* $\epsilon > 0$, *letting*

$$K(\omega) = \sup\left\{\frac{\|T^n(\omega)\pi_1(\omega)\|}{e^{n(\kappa(T|F)+\epsilon)}}\right\}_{n\geq 0},$$

then $K(\cdot) : \Omega \to [1, +\infty)$ *is a tempered function.*

Before proving this theorem, we need the following lemmas and propositions.

Let $E: \Omega \to K_m$ be a measurable map and ϵ be fixed such that $\frac{1}{2} > \epsilon > 0$. Define a multifunction $\Gamma: \Omega \to K_1 \times K_{m-1}$ as

$$\Gamma(\omega) = \{(E_1, E_{m-1}) \in K_1 \times K_{m-1} |\ E_1 \subset E(\omega),\ E_{m-1} \subset E(\omega) \text{ and } \delta(E_1, E_{m-1}) > 1-\epsilon\}.$$

Lemma 38. $\overline{\Gamma}(\omega)$ *admits a measurable selection.*

Proof. First we notice that $\Gamma(\omega)$ is not empty since there are m unit vectors $u_i \in E(\omega)$ such that

$$\mathrm{dist}\big(u_1, \mathrm{span}\{u_i\}_{1<i\leq m}\big) = 1$$

In order to apply Corollary 35, we need to show that for any open set $U \subset K_1 \times K_{m-1}$, $\Gamma^-(U) = \{\omega |\ \Gamma(\omega) \cap U \neq \emptyset\}$ is a measurable set. For $\omega \in \Gamma^-(U)$, take $(E_1(\omega), E_{m-1}(\omega)) \in \Gamma(\omega) \cap U$. Since U is open, there exists $0 < \epsilon'(\omega) < \frac{\epsilon}{2}$ such that

$$B(E_1(\omega), \frac{1}{2}\epsilon'(\omega)) \times B(E_{m-1}(\omega), \frac{1}{2}\epsilon'(\omega)) \subset U.$$

We choose unit vectors $v_1 \in E_1(\omega)$ and $\{v_i\}_{2\leq i\leq m} \subset E_{m-1}(\omega)$ such that

$$\overline{\mathrm{dist}}(v_i, \mathrm{span}\{v_j\}_{i<j\leq m}) = 1,\ 2 \leq i \leq m.$$

By Lemma 11, there exists a $\delta'(\omega) > 0$ such that for any set of vectors $\{w_i\}_{1\leq i\leq m}$ satisfying

$$\|w_i - v_i\| < \delta'(\omega),\ 1 \leq i \leq m,$$

we have

$$\overline{\mathrm{dist}}(w_i, \mathrm{span}\{w_j\}_{i<j\leq m}) > 1 - \epsilon,\ 1 \leq i \leq m. \tag{73}$$

Set

$$\delta^*(\omega) = \min\left\{\frac{\delta'(\omega)}{2}, \frac{\epsilon'(\omega)}{2m3^m}\right\}$$

and

$$G(U) = \bigcup_{\omega \in \Gamma^-(U)} B\big(E(\omega), \delta^*(\omega)\big).$$

Since $G(U)$ is open, $E^{-1}(G(U))$ is measurable. In order to prove that $\Gamma^-(U)$ is measurable, it is sufficient to show $E^{-1}(G(U)) = \Gamma^-(U)$.

We first notice that $\Gamma^-(U) \subset E^{-1}(G(U))$. Hence, we need only to show $E^{-1}(G(U)) \subset \Gamma^-(U)$.

For any $\omega' \in E^{-1}(G(U))$, there exists $\omega \in \Gamma^-(U)$ such that $E(\omega') \in B(E(\omega), \delta^*(\omega))$. Associated with $\omega \in \Gamma^-(U)$, let $(E_1(\omega), E_{m-1}(\omega)) \in \Gamma(\omega) \cap U$, $0 < \epsilon'(\omega) < \frac{\epsilon}{2}$, $\{v_i\}_{1\le i\le m}$ be given as the above. Thus, there exist m unit vectors $\{v_i'\}_{1\le i\le m} \subset E(\omega')$ such that

$$\|v_i - v_i'\| \le \delta^*(\omega), \ 1 \le i \le m.$$

Then, by (73), we have

$$\overline{\mathrm{dist}}(v_i', \mathrm{span}\{v_j'\}_{i<j\le m}) > 1 - \epsilon, \quad 1 \le i \le m. \tag{74}$$

Let $E_1(\omega') = \mathrm{span}\{v_1'\}$ and $E_{m-1}(\omega') = \mathrm{span}\{v_i'\}_{2\le i\le m}$. For any unit vector $v \in E_{m-1}(\omega')$, there exists $\{a_i\}_{2\le i\le m} \subset \mathbb{R}$ such that

$$v = \sum_{i=2}^{m} a_i v_i'.$$

Since (74), by Lemma 9, we have

$$|a_i| \le \frac{(2-\epsilon)^{m-2}}{(1-\epsilon)^{m-1}} < 3^{m-1}.$$

Therefore

$$\operatorname{dist}(v, E_{m-1}(\omega)) \leq \| \sum_{i=2}^{m} a_i(v_i' - v_i)\| \leq m3^{m-1}\delta^*(\omega) < \frac{\epsilon'}{2}.$$

Thus, by using (20), we have

$$d(E_{m-1}(\omega'), E_{m-1}(\omega)) \leq 2\delta(E_{m-1}(\omega'), E_{m-1}(\omega)) < \epsilon'(\omega).$$

Similarly, $d(E_{m-1}(\omega), E_{m-1}(\omega')) < \epsilon'(\omega)$. So,

$$(E_1(\omega'), E_{m-1}(\omega')) \in B(E_1(\omega), \epsilon'(\omega)) \times B(E_{m-1}(\omega), \epsilon'(\omega)) \subset U.$$

Thus $\omega' \in \Gamma^-(U)$. This completes the proof of this lemma. □

Corollary 39. *Let $E(\omega)$ be a measurable m-dimensional subspace of X. Then, $E(\omega)$ has a measurable unit basis $\{e_1(\omega), \cdots, e_m(\omega)\}$ which satisfies*

$$\overline{\operatorname{dist}}(e_i(\omega), \operatorname{span}\{e_j(\omega)\}_{i<j\leq m}) \geq 1 - \epsilon, \ 1 \leq i \leq m \tag{75}$$

where $0 < \epsilon < 1/2$.

This corollary together with Lemma 20 gives that a m-dimensional space $E(\omega)$ is measurable if and only if there exists a measurable basis.

Proof. We first show that for a measurable one-dimensional space $\hat{E}_1(\omega)$ there is a measurable function $e_1(\cdot) : \Omega \to X$ such that $\hat{E}_1(\omega) = \operatorname{span}\{e_1(\omega)\}$. Define a multifunction $\Gamma : \ \Omega \to S_X$ as

$$\Gamma(\omega) = \{u \in S_X \mid u \in E_1(\omega)\}.$$

Since $\hat{E}_1(\omega)$, $\Gamma(\omega)$ is not empty and closed. Let U be open set of S_X, we want to show $\Gamma^-(U)\big(= \{\omega | \ \Gamma(\omega) \cap U \neq \emptyset\}\big)$ is a measurable set. For $\omega \in \Gamma^-(U)$, take $u_0(\omega) \in \Gamma(\omega) \cap U$. Since U is open, there exists $0 < r(\omega) < 1/2$ such that

$$B\big(u_0(\omega), r(\omega)\big) \subset U.$$

Set

$$G(U) = \bigcup_{\omega \in \Gamma^-(U)} B\big(u_0(\omega), r(\omega)\big).$$

For each $u \in G(U)$, let $E_{1,u}$ denote the space spanned by u. Set

$$\tilde{G}(U) = \big\{E_{1,u} \ \big| \ u \in G(U)\big\}$$

Using the definition of the metric d, we have

$$\tilde{G}(U) = \bigcup_{\omega \in \Gamma^-(U)} B\big(E_{1,u_0}, r(\omega)\big)$$

Since $\hat{E}_1(\omega)$ is measurable and $\tilde{G}(U)$ is open, $\hat{E}_1^{-1}\big(\tilde{G}(U)\big)$ measurable. Furthermore,

$$\hat{E}_1^{-1}\big(\tilde{G}(U)\big) = \Gamma^-(U).$$

Thus, by Corollary 35, Γ admits a measurable selection $e_1(\omega)$. Thus, using Lemma 38, we have a measurable basis for $E(\omega)$. This completes the proof of the Lemma.

□

The following result is an extension of the Hahn-Banach theorem to measurable functionals.

Proposition 40. Let X be a separable Banach space and $E : \ \Omega \to K_n$ be a n-dimensional measurable space with measurable basis $\{e_1(\omega), \cdots, e_n(\omega)\}$. Let $f(\omega)$ be a bounded linear functional on $E(\omega)$ such that functions $f(\cdot)e_i(\cdot) : \Omega \to \mathbb{R},\ 1 \le i \le n$, and $\|f(\cdot)\|$ are measurable. Then $f(\omega)$ can be extend to a bounded linear functional $\phi(\omega)$ on X such that $\|\phi(\omega)\| = \|f(\omega)\|$ and the map $\phi(\omega)$ is strongly measurable.

Proof. Let $\{x_i\}_{i\ge 1}$ be a dense set in X. We first construct the extension $\psi(\omega)$ of $f(\omega)$ satisfying

(i) For any $i \in \mathbb{N},\ \psi(\omega)(x_i) :\ \Omega \to \mathbb{R}$ is a measurable function;

(ii) for any $l \in \mathbb{N},\ \psi(\omega) = f_l(\omega)$, where $f_l(\omega)$ is a bounded linear functional on $\operatorname{span}\{x_1, \ldots, x_l, E(\omega)\}$, which satisfies $\|f_l(\omega)\| = \|f(\omega)\|$ and $f_l(\omega)$ is an extension of $f_{l-1}(\omega)$.

We will construct $\psi(\omega)$ by induction. For any integer $l \ge 0$, we define a $l+1$-tuple s_l as following

(1) For $l = 0$, $s_0 = (\emptyset)$;

(2) For $l \ge 1$, $s_l = (\emptyset, y_1, \ldots, y_l)$ where $y_j = \{x_j\}$ or $\emptyset$ for $1 \le j \le l$.

We say $s_l < s_k$ if and only if $l < k$ and $s_k = (s_l, y_{l+1}, \ldots, y_k)$. We use $\check{s}_l$ to denote $\cup_{i=1}^{l} y_i$ and denote the component y_i by s_l^i, where $s_l = (\emptyset, y_1, \ldots, y_l)$. Set

$$\tilde{s}_l^i = \begin{cases} 0, & \text{when } s_l^i = \emptyset, \\ x_i, & \text{when } s_l^i = \{x_i\}. \end{cases}$$

We note that the set of all such $l+1$-tuples, which we denote by S_l, contains 2^l elements only. We define a function $\Omega(\cdot)$ on these tuples by induction.

(a) For 1-tuple, $\Omega((\emptyset)) = \Omega$;

(b) For 2-tuples:

$$\Omega((\emptyset, \emptyset)) = \{\omega \mid \operatorname{dist}(x_1, E(\omega)) = 0\},$$
$$\Omega((\emptyset, \{x_1\})) = \{\omega \mid \operatorname{dist}(x_1, E(\omega)) > 0\}.$$

(c) Assume that $\Omega(s_l)$ is defined for all $l+1$ tuples. For s_{l+1}, we write $s_{l+1} = (s_l, s_{l+1}^{l+1})$.

$$\Omega(s_{l+1}) = \{\omega \in \Omega_l(s_l) \mid x_{l+1} \in \operatorname{span}\{\breve{s}_l, E(\omega)\}\}, \quad \text{for } s_{l+1}^{l+1} = \emptyset,$$
$$\Omega(s_{l+1}) = \{\omega \in \Omega_l(s_l) \mid \operatorname{dist}(x_{l+1}, \operatorname{span}\{\breve{s}_l, E(\omega)\}) > 0\}, \quad \text{for } s_{l+1}^{l+1} = \{x_{l+1}\}.$$

From this definition it follows that for any integers $0 \le i < j$, $s_i < s_j$ if and only if $\Omega(s_i) \supset \Omega(s_j)$.

To show (i) and (ii), we first prove the following claim by induction and denote $f(\omega)$ by $f_0(\omega)$.

Claim. *For each integer $l \ge 1$, we have*

(C1) $\Omega(s_l)$ is a measurable set for any $s_l \in S_l$;

(C2) $\operatorname{span}\{\breve{s}_l, E(\omega)\}$ is measurable on $\Omega(s_l)$;

(C3) $\breve{s}_l \cup \{e_i(\omega)\}_{1 \le i \le n}$ is a measurable basis on $\Omega(s_l)$;

(C4) There exists an extension $f_l(\omega)$ of $f(\omega)$ to $\operatorname{span}\{x_1, \cdots, x_l, E(\omega)\}$ such that $\|f_l(\omega)\| = \|f_{l_1}(\omega)\|$ and $f_l(\cdot)(x_i)$ is measurable for $1 \le i \le l$.

We first consider the case $l = 1$. Since E is measurable and $\operatorname{dist}(x_1, E(\omega))$ depends on $E(\omega)$ continuously for a fixed $x_1 \in X$, $\operatorname{dist}(x_1, E(\omega))$ is a measurable function. Thus, both $\Omega((\emptyset, \emptyset))$ and $\Omega((\emptyset, \{x_1\}))$ are measurable sets. For each $\omega \in \Omega((\emptyset, \{x_1\}))$,

$\dim\big(\operatorname{span}\{x_1, E(\omega)\}\big) = n+1$. Hence, (C1)-(C3) hold. Next, we extend $f(\omega)$ to $f_1(\omega)$ on $\operatorname{span}\{x_1, E(\omega)\}$ by

$$f_1(\omega)(tx_1 + y) = t\beta_1(\omega) + f(\omega)(y),\ t \in \mathbb{R},\ y \in E(\omega),$$

where $\beta_1(\omega) = \sup\{f(\omega)(y) - \|f(\omega)\|\| - x_1 + y\| \mid y \in E(\omega)\}$.

Since $\{\sum_{i=1}^n a_i e_i(\omega)|\ (a_1, \ldots, a_n) \in \mathbb{Q}^n\}$ is a countable dense set of $E(\omega)$, we have

$$\begin{aligned}
&\beta_1(\omega)\\
&= \sup\Big\{f(\omega)(y) - \|f(\omega)\|\| - x_1 + y\| \ \Big|\ y \in \Big\{\sum_{i=1}^n a_i e_i(\omega)|\ (a_1, \ldots, a_n) \in \mathbb{Q}^n\Big\}\Big\}\\
&= \sup\Big\{\sum_{i=1}^n a_i f(\omega)(e_i(\omega)) - \|f(\omega)\|\| - x_1 + \sum_{i=1}^n a_i e_i(\omega)\| \ \Big|\ \{(a_1, \ldots, a_n) \in \mathbb{Q}^n\}\Big\},
\end{aligned}$$

which is measurable.

For $\omega \in \Omega((\emptyset, \emptyset))$, we have $x_1 \in E(\omega)$. Then,

$$\begin{aligned}
&f(\omega)(x_1)\\
&= f(\omega)(y) - f(\omega)(y - x_1)\\
&= \sup\{f(\omega)(y) - \|f(\omega)\|\| - x_1 + y\| \mid y \in E(\omega)\}\\
&= \sup\left\{f(\omega)(y) - \|f(\omega)\|\| - x_1 + y\| \ \middle|\ y \in \left\{\sum_{i=1}^n a_i e_i(\omega)|\ (a_1, \ldots, a_n) \in \mathbb{Q}^n\right\}\right\}\\
&= \sup\left\{\sum_{i=1}^n a_i f(\omega)(e_i(\omega)) - \|f(\omega)\|\| - x_1 + \sum_{i=1}^n a_i e_i(\omega)\| \ \middle|\ \{(a_1, \ldots, a_n) \in \mathbb{Q}^n\}\right\},
\end{aligned}$$

which is $\beta_1(\omega)$ and is measurable. We define $f_1(\omega) = f(\omega)$ on $\Omega((\emptyset, \emptyset))$. Thus, we have that $f_1(\omega)x_1$ is measurable, $f_1(\omega)$ is a bounded linear functional on $\operatorname{span}\big\{x_1, E(\omega)\big\}$. The definition of $\beta(\omega)$ implies that $\|f_1(\omega)\| = \|f(\omega)\|$. By Corollary 19, we have that $\operatorname{span}\{x_1, E(\omega)\}$ is measurable on $\Omega((\emptyset, \emptyset))$ and $\Omega((\emptyset, \{x_1\}))$ respectively. And we also note that $\{x_1\} \cup \{e_i(\omega)\}_{1 \le i \le n}$ and $\emptyset \cup \{e_i(\omega)\}_{1 \le i \le n}$ are measurable basis of $\Omega(\emptyset, \{x_1\}))$

and $\Omega((\emptyset, \emptyset))$, respectively.

Next, we assume that (C1)-(C4) hold for $l = k-1$. We want to show that (C1)-(C4) are true for $l = k$. It is easy to see that (C1)-(C3) are all true, which come from Corollary 19 and induction hypothesis.

We consider s_k and take $s_{k-1} < s_k$. If $s_k^k = \emptyset$, then $x_k \in \mathrm{span}\{\breve{s}_{k-1}, E(\omega)\}$ for $\omega \in \Omega(s_k)$. Define

$$f_k(\omega) = f_{k-1}(\omega), \text{ for } \omega \in \Omega(s_k).$$

Thus $\|f_k(\omega)\| = \|f(\omega)\|$, for $\omega \in \Omega(s_k)$. Let

$$D(\omega) = \Big\{ \sum_{i=1}^{n} a_i e_i(\omega) + \sum_{j=1}^{k-1} a_{n+j} \tilde{s}_{k-1}^j | \ (a_1, \ldots, a_{n+k-1}) \in \mathbb{Q}^{n+k-1} \Big\}$$

be a countable dense set of $\mathrm{span}\{\breve{s}_{k-1}, E(\omega)\}$. We notice that

$$\begin{aligned}
& f_{k-1}(\omega)(x_k) = f_{k-1}(\omega)(y) - f_{k-1}(\omega)(y - x_k) \\
& = \sup \Big\{ f_{k-1}(\omega)(y) - \|f_{k-1}(\omega)\| \| - x_k + y\| \mid y \in \mathrm{span}\{\breve{s}_{k-1}, E(\omega)\} \Big\} \\
& = \sup \Big\{ f_{k-1}(\omega)(y) - \|f_{k-1}(\omega)\| \| - x_k + y\| \ \Big| \ y \in D(\omega) \Big\} \\
& = \sup \Big\{ \sum_{i=1}^{n} a_i f_{k-1}(\omega)(e_i(\omega)) + \sum_{j=1}^{k-1} a_{n+j} f_{k-1}(\omega)(\tilde{s}_{k-1}^j) \\
& - \|f(\omega)\| \Big\| \sum_{i=1}^{n} a_i e_i(\omega) + \sum_{j=1}^{k-1} a_{n+j} \tilde{s}_{k-1}^j - x_k \Big\| \ \Big| \ \{(a_1, \cdots, a_{n+k-1}) \in \mathbb{Q}^{n+k-1}\} \Big\}.
\end{aligned} \tag{76}$$

The last term is measurable on $\Omega(s_{k-1})$ from the induction hypotheses, which implies that $f_k(\omega)x_k$ is measurable on $\Omega(s_k)$.

If $s_k^k = \{x_k\}$, then $x_k \notin \mathrm{span}\{\breve{s}_{k-1}, E(\omega)\}$. We extend $f_{k-1}(\omega)$ to $f_k(\omega)$ on $\Omega(s_k)$ from $\mathrm{span}\{\breve{s}_{k-1}, E(\omega)\}$ to $\mathrm{span}\{\breve{s}_k, E(\omega)\}$ by defining

$$f_k(\omega)(tx_k + y) = t\beta_k(\omega) + f_{k-1}(\omega)(y), \ t \in \mathbb{R}, \ y \in \mathrm{span}\{\breve{s}_{k-1}, E(\omega)\},$$

where $\beta_k(\omega) = \sup\{f_{k-1}(\omega)(y) - \|f_{k-1}(\omega)\|\|-x_k + y\| \mid y \in \operatorname{span}\{\breve{s}_{k-1}, E(\omega)\}\}$. From (76), we have

$$\begin{aligned}
&\beta_k(\omega) \\
&= \sup\Big\{\sum_{i=1}^{n} a_i f_{k-1}(\omega)(e_i(\omega)) + \sum_{j=1}^{k-1} a_{n+j} f_{k-1}(\omega)(\tilde{s}^j_{k-1}) \\
&- \|f(\omega)\|\Big\|\sum_{i=1}^{n} a_i e_i(\omega) + \sum_{j=1}^{k-1} a_{n+j}\tilde{s}^j_{k-1} - x_k\Big\| \;\Big|\; \{(a_1, \ldots, a_{n+k-1}) \in \mathbb{Q}^{n+k-1}\}\Big\},
\end{aligned}$$

which is measurable on $\Omega(s_{k-1})$. Thus $f_k(\omega)(x_l)$ is measurable on $\Omega(s_k)$. The choice of $\beta(\omega)$ gives $\|f_k(\omega)\| = \|f_{k-1}(\omega)\| = \|f(\omega)\|$, for $\omega \in \Omega(s_k)$. Since Ω is the union of disjoint sets $\Omega(s_k), s_k \in S_k$, we have that $f_k(\omega)$ is a bounded linear functional on $\operatorname{span}\big\{x_1, \cdots, x_k, E(\omega)\big\}$, $\|f_k(\omega)\| = \|f(\omega)\|$, and $f_k(\omega)x_k$ is measurable. This completes the proof of the claim.

Using this claim, we obtain a functional $\psi(\omega)$ defined on a dense set and $\psi(\omega) = f_l(\omega)$ on $\operatorname{span}\big\{x_1, \cdots, x_l, E(\omega)\big\}$ for each $l \geq 0$, which satisfies (i) and (ii) given at the beginning of the proof. At the remaining points of X, the functional is defined by continuity. For each $x \in X$, there exists a subsequence of $\{x_i\}_{i\geq 1}$, denoted by $\{x_{n_i}\}_{i\geq 1}$, such that

$$\lim_{i\to+\infty} x_{n_i} = x.$$

Since for $i < j$ we have

$$\|\psi(\omega)(x_{n_i}) - \psi(\omega)(x_{n_j})\| = \|f_{n_j}(\omega)(x_{n_i}) - f_{n_j}(\omega)(x_{n_j})\| \leq \|f(\omega)\|\|x_{n_i} - x_{n_j}\|,$$

$\psi(\omega)(x_{n_i})$ is a Cauchy sequence. Thus, $\lim_{i\to+\infty}\psi(\omega)(x_{n_i})$ exists. We define

$$\phi(\omega)(x) = \lim_{i\to+\infty}\psi(\omega)(x_{n_i}).$$

It is easy to see that $\phi(\omega)(x)$ is well defined.

Next, we show that $\phi(\omega)$ is a bounded linear functional with $\|\phi(\omega)\| = \|\psi(\omega)\|$. For any $x_1, x_2 \in X$, there exist subsequences $\{x_{n_i}\}_{i\geq 1}$ and $\{x_{m_i}\}_{i\geq 1}$ such that converge to x_1, x_2, respectively. Thus, for $a, b \in \mathbb{R}$

$$\begin{aligned}\phi(\omega)(ax_1 + bx_2) = &\lim_{i\to+\infty} \psi(\omega)(ax_{n_i} + bx_{m_i})\\ =&a \lim_{i\to+\infty} \psi(\omega)(x_{n_i}) + b \lim_{i\to+\infty} \psi(\omega)(x_{m_i})\\ =&a\phi(x_1) + b\phi(x_2),\end{aligned}$$

which means that $\phi(\omega)$ is linear. Furthermore,

$$\begin{aligned}\|\phi(\omega)\| = \sup_{\|x\|=1} \{\|\phi(\omega)(x)\|\} &= \sup\{\|\phi(\omega)(\frac{x_i}{\|x_i\|})\|\}_{i\geq 1}\\ = \sup\{\|\psi(\omega)(\frac{x_i}{\|x_i\|})\|\}_{i\geq 1} &= \|f(\omega)\|.\end{aligned}$$

Thus $\phi(\omega)$ is a bounded linear functional on X with norm $\|f(\omega)\|$. Since $\phi(\omega)x_i = f_i(\omega)x_i$ is measurable and $\{x_i\}_{i\geq 1}$ is dense in X, $\phi(\omega)x$ is measurable for all $x \in X$. This completes the proof of the proposition □

By using Corollary 39, $E^{\kappa(T)}(\omega)$ has a measurable unit basis $\{e_i(\omega)\}_{1\leq i\leq m}$ satisfying

$$\overline{\mathrm{dist}}(e_i(\omega), \mathrm{span}\{e_j(\omega)\}_{i<j\leq m}) \geq 1-\epsilon,\ 1 \leq i \leq m, \tag{77}$$

for $0 < \epsilon < 1/2$. By lemma 9 we have that

$$\mathrm{dist}(e_i(\omega), \mathrm{span}\{e_j(\omega)\}_{1\leq j\leq m, j\neq i}) \geq \frac{1-\epsilon}{(1+\frac{1}{1-\epsilon})^{m-1}},\ 1 \leq i \leq m. \tag{78}$$

For each $x \in E^{\kappa(T)}(\omega)$, we write

$$x = \sum_{i=1}^{m} x_i e_i(\omega),$$

where $x_i \in \mathbb{R}, 1 \le i \le m$.

Lemma 41. *For each integer $1 \le i \le m$, there exists a bounded functional $\phi_i(\omega)$ such that*

(i) $\phi_i : \Omega \to X^*$ is strongly measurable;

(ii) $\phi_i(\omega)x = x_i$ for all $x \in E^{\kappa(T)}(\omega)$;

(iii) The norm of $\phi_i(\omega)$ is given by

$$\|\phi_i(\omega)\| = \frac{1}{\operatorname{dist}(e_i(\omega), \operatorname{span}\{e_j(\omega)\}_{1 \le j \le m, j \ne i})}.$$

Proof. For $x \in E^{\kappa(T)}(\omega)$ and $1 \le i \le m$, we define a functional $f_i(\omega)$ on $E^{\kappa(T)}(\omega)$ by $f_i(\omega)(x) = x_i$, where x_i is given by $x = \sum_{i=1}^{m} x_i e_i(\omega)$. Clearly, $f_i(\omega)$ is a linear functional. Next, we show that $f_i(\omega)$ is bounded. For each unit vector x, using (78), we have

$$\|x_i\| \le \frac{1}{\operatorname{dist}(e_i(\omega), \operatorname{span}\{e_j(\omega)\}_{1 \le j \le m, j \ne i})} \le \frac{(1 + \frac{1}{1-\epsilon})^{m-1}}{1 - \epsilon},$$

which implies that

$$\|f_i(\omega)\| \le \frac{1}{\operatorname{dist}(e_i(\omega), \operatorname{span}\{e_j(\omega)\}_{1 \le j \le m, j \ne i})} \le \frac{(1 + \frac{1}{1-\epsilon})^{m-1}}{1 - \epsilon}, \ 1 \le i \le m. \tag{79}$$

Since $E^{\kappa(T)}(\omega)$ is finitely dimensional, there exists a unit vector $e_i'(\omega) \in E^{\kappa(T)}(\omega)$ such that

$$e_i(\omega) - \operatorname{dist}(e_i(\omega), \operatorname{span}\{e_j(\omega)\}_{1 \le j \le m, j \ne i}) e_i'(\omega) \in \operatorname{span}\{e_j(\omega)\}_{1 \le j \le m, j \ne i}, \tag{80}$$

which yields that

$$|f_i(\omega)e_i'(\omega)| = \frac{1}{\mathrm{dist}(e_i(\omega), \mathrm{span}\{e_j(\omega)\}_{1\le j\le m, j\ne i})}.$$

This together with (79) gives

$$\|f_i(\omega)\| = \frac{1}{\mathrm{dist}(e_i(\omega), \mathrm{span}\{e_j(\omega)\}_{1\le j\le m, j\ne i})}.$$

Thus, $\|f_i(\omega)\|$ is measurable. Since $f_i(\omega)e_j(\omega) = \delta_{ij},\ 1 \le i,j \le m,\ f_i(\cdot)e_j(\cdot):\ \Omega \to \mathbb{R}$ are measurable. By using Proposition 40, we can extend $f_i(\omega)$ to $\phi_i(\omega) \in X^*$ which satisfies the conditions of Lemma 41. This completes the proof of the lemma.

□

To prove Theorem 37, we need following lemma.

Lemma 42. *Let $f : \Omega \to (0, +\infty)$ be tempered and $\gamma : \Omega \to (0, +\infty)$ be a θ-invariant random variable. Then, there is a tempered random variable $R(\omega)$ such that*

i $\frac{1}{R(\omega)} \le f(\omega) \le R(\omega)$

ii $e^{-\gamma(\omega)|n|}R(\omega) \le R(\theta^n\omega) \le e^{\gamma(\omega)|n|}R(\omega).$

This lemma is a consequence of Proposition 4.3.3([1],page 187).

Proof of Theorem 37 We prove this theorem in four steps.

Step 1. We construct complementary spaces.

We first define $\Pi''(\omega) :\ X \to E^{\kappa(T)}(\omega)$ as

$$\Pi''(\omega)(x) = \sum_{i=1}^{m} \phi_i(\omega)(x)e_i(\omega),$$

for $x \in X$, where ϕ_i is given by Lemma 41. For each $x \in E^{\kappa(T)}(\omega)$, there exists $(a_1, \ldots, a_m) \in \mathbb{R}^m$ such that $x = \sum_{i=1}^m a_i e_i(\omega)$. Then

$$\begin{aligned}\Pi''(\omega)(x) =& \Pi''(\omega)(\sum_{i=1}^m a_i e_i(\omega)) = \sum_{j=1}^m \phi_j(\omega)(\sum_{i=1}^m a_i e_i(\omega))e_j(\omega)\\ =& \sum_{i=1}^m(\sum_{i=1}^m a_i\delta_{ij})e_j(\omega) = \sum_{i=1}^m a_i e_i(\omega) = x.\end{aligned}$$

This implies that $\Pi''(\omega)$ is a projection on $E^{\kappa(T)}(\omega)$. Furthermore, using (78), we obtain

$$\|\Pi''(\omega)\| \leq \sum_{i=1}^m \|\phi_i(\omega)\| \leq \frac{(1+\frac{1}{1-\epsilon})^{m-1}m}{1-\epsilon}.$$

Since X is a separable Banach space, $\Pi''(\omega)$ is strongly measurable.

Let $\Pi'(\omega) = \Pi''(\omega)\pi(\omega)$ and $\Pi(\omega) = \pi(\omega) - \Pi'(\omega)$. Then $\Pi(\omega)$ and $\Pi'(\omega)$ are projection operators satisfying

$$\begin{aligned}\|\Pi'(\omega)\| &\leq \|\pi(\omega)\|\frac{(1+\frac{1}{1-\epsilon})^{m-1}m}{1-\epsilon},\\ \|\Pi(\omega)\| &\leq \|\pi(\omega)\|\big(\frac{(1+\frac{1}{1-\epsilon})^{m-1}m}{1-\epsilon}+1\big).\end{aligned}$$

Since $\pi(\omega)$ is tempered, $\|\Pi'(\omega)\|$ and $\|\Pi(\omega)\|$ are tempered. We also note that $\|\Pi'(\omega)|_{G(\omega)}\| \leq \|\Pi''\|$ and $\|\Pi(\omega)|_{G(\omega)}\| \leq 1 + \|\Pi''\|$, which are uniformly bounded on Ω. Let $G_1(\omega) = \Pi(\omega)(X)$. Then $G(\omega) = E^{\kappa(T)} \oplus G_1(\omega)$. Hence, G_1 is a complementary space of $E(\omega) \oplus E^{\kappa(T)}(\omega)$. From the measurability of $\pi(\cdot)$ and $\Pi''(\cdot)$ it follows that both $\Pi'(\cdot)$ and $\Pi(\cdot)$ are strongly measurable.

Step 2. We study the properties of T on $G_1(\omega)$.

Define $\tilde{T}(\omega) = \Pi(\theta\omega)T(\omega)$. Then $\tilde{T}(\cdot) : \Omega \to L(X,X)$ is strongly measurable. It is easy to see that $\tilde{T}(\omega)$ is injective on $G_1(\omega)$ because $T(\omega)$ is injective on $G(\omega)$.

From the definition of $\Pi(\omega)$, we have that

$$\tilde{T}(\omega)G_1(\omega) \subset G_1(\theta\omega) \quad \text{and} \quad \tilde{T}(\omega)x = 0 \;\text{ for }\; x \in E(\omega) \oplus E^{\kappa(T)}(\omega).$$

Since $T(\omega)G(\omega) \subset G(\theta\omega)$ and $\|\Pi(\omega)\big|_{G(\omega)}\| \leq 1 + \|\Pi''\|$ which is uniformly bounded,

$$\|\tilde{T}(\omega)\big|_{G_1(\omega)}\| \leq \|\Pi(\theta\omega)\big|_{G(\theta\omega)}\| \|T(\omega)\big|_{G(\omega)}\|,$$

which implies that $\log^+ \|\tilde{T}(\cdot)\big|_{G_1(\cdot)}\| \in L^1(\Omega, \mathcal{F}, P)$. Thus, the results obtained in Section 6 can be applied to $\tilde{T}$.

By using Lemma 21,22 and 23, we have $\kappa(\tilde{T})$, a measurable subspace $E^{\kappa(\tilde{T})}(\omega)$, and a positive integer m' such that

$$\dim E^{\kappa(\tilde{T})}(\omega) = m'; \tag{81}$$

$$E^{\kappa(\tilde{T})}(\omega) \subset G_1(\omega); \tag{82}$$

$$\tilde{T}(E^{\kappa(\tilde{T})}(\omega)) = E^{\kappa(\tilde{T})}(\theta\omega); \tag{83}$$

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|\tilde{T}^n(\omega)|_{E^{\kappa(\tilde{T})}(\omega)}\| = \kappa(\tilde{T}). \tag{84}$$

We claim that $\kappa(\tilde{T}) < \kappa(T)$.

We prove it by a contradiction. Suppose that $\kappa(\tilde{T}) \geq \kappa(T)$. Let $E'(\omega) = E^{\kappa(T)}(\omega) \oplus E^{\kappa(\tilde{T})}(\omega)$. Then, using (83), we have that $T(\omega)E'(\omega) = E'(\theta\omega)$. We will show that for any $v(\neq 0) \in E^{\kappa(\tilde{T})}(\omega)$

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|T^{-n}(\omega)v\| \leq -\kappa(T), \tag{85}$$

which implies that $v \in E^{\kappa(T)}(\omega)$, thus $v = 0$, a contradiction.

For the sake of simplicity, for a bounded linear operator L from a Banach space

Y to a Banach space Z, we denote

$$\|L\|^- = \inf_{\|v\|=1}\{\|Lv\|\}.$$

It is easy to see that

$$\|L\|^- \leq \|L\|$$

and if L^{-1} exists, then $\|L\|^- = \|L^{-1}\|^{-1}$.

First we have that

$$\lim_{n\to+\infty}\frac{1}{n}\log\|\tilde{T}^n(\theta^{-n})|_{E^{\kappa(\tilde{T})}(\theta^{-n}\omega)}\|^- = \kappa(\tilde{T}), \tag{86}$$

which follows

$$\lim_{n\to+\infty}\frac{1}{n}\log\|\tilde{T}^n(\theta^{-n})\omega)|_{E^{\kappa(\tilde{T})}(\theta^{-n}\omega)}\|^- = \lim_{n\to+\infty}\frac{1}{n}\log\frac{1}{\|\tilde{T}^{-n}(\omega)|_{E^{\kappa(\tilde{T})}(\omega)}\|} = \kappa(\tilde{T}).$$

For any $v(\neq 0) \in E^{\kappa(\tilde{T})}(\omega)$, we have

$$\begin{aligned}
v =& T^n(\theta^{-n}\omega)T^{-n}(\omega)v \\
=& T^n(\theta^{-n}\omega)\Pi'(\theta^{-n}\omega)T^{-n}(\omega)v \\
&+ T^n(\theta^{-n}\omega)\Pi(\theta^{-n}\omega)T^{-n}(\omega)v \\
=& T^n(\theta^{-n}\omega)\Pi'(\theta^{-n}\omega)T^{-n}(\omega)v \quad (1) \\
&+ \Pi'(\omega)T^n(\theta^{-n}\omega)\Pi(\theta^{-n}\omega)T^{-n}(\omega)v \quad (2) \\
&+ \Pi(\omega)T^n(\theta^{-n}\omega)\Pi(\theta^{-n}\omega)T^{-n}(\omega)v \quad (3).
\end{aligned}$$

Note that (1) + (2) = 0, because $v \in E^{\kappa(\tilde{T})}(\omega)$. So

$$\|v\| = \|\Pi(\omega)T^n(\theta^{-n}\omega)\Pi(\theta^{-n}\omega)T^{-n}(\omega)v\|,$$

which implies that

$$\|v\| \geq \|\tilde{T}^n(\theta^{-n})|_{E^{\kappa(\tilde{T})}(\theta^{-n}\omega)}\|^- \|\Pi(\theta^{-n}\omega)T^{-n}(\omega)v\|.$$

Thus

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|\Pi(\theta^{-n}\omega)T^{-n}(\omega)v\| \leq -\kappa(\tilde{T}). \tag{87}$$

Since $\|\Pi'(\omega)|_{G(\omega)}\|$ is bounded and

$$\lim_{n\to+\infty} \frac{1}{n} \log \|T^n(\theta^{-n}\omega)\| = \kappa(T) \leq \kappa(\tilde{T}),$$

then

$$\begin{aligned}
&\limsup_{n\to+\infty} \frac{1}{n} \log \|\Pi'(\omega)T^n(\theta^{-n}\omega)\Pi(\theta^{-n}\omega)T^{-n}(\omega)v\| \\
\leq &\limsup_{n\to+\infty} \frac{1}{n} \log \left(\|\Pi'(\omega)T^n(\theta^{-n}\omega)\|\|\Pi(\theta^{-n}\omega)T^{-n}(\omega)v\|\right) \\
\leq &\limsup_{n\to+\infty} \frac{1}{n} \log \|\Pi'(\omega)T^n(\theta^{-n}\omega)\| + \limsup_{n\to+\infty} \frac{1}{n} \log \|\Pi(\theta^{-n}\omega)T^{-n}(\omega)v\| \\
\leq &\kappa(T) - \kappa(\tilde{T}) \leq 0.
\end{aligned}$$

Therefore,

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|\Pi'(\omega)T^n(\theta^{-n}\omega)\Pi'(\theta^{-n}\omega)T^{-n}(\omega)v\| \leq 0.$$

Note that

$$\lim_{n\to+\infty} \frac{1}{n} \log \|T^n(\theta^{-n}\omega)|_{E^{\kappa(T)}(\theta^{-n}\omega)}\|^- = \lim_{n\to+\infty} \frac{1}{n} \log \frac{1}{\|T^{-n}(\omega)|_{E^{k(T)}(\omega)}\|} = \kappa(T). \tag{88}$$

Since

$$\|T^n(\theta^{-n}\omega)\Pi'(\theta^{-n}\omega)T^{-n}(\omega)v\| \geq \|T^n(\theta^{-n}\omega)|_{E^{\kappa(T)}(\theta^{-n}\omega)}\|^- \|\Pi'(\theta^{-n}\omega)T^{-n}(\omega)v\|,$$

we have that

$$\begin{aligned}
&\limsup_{n\to+\infty} \frac{1}{n} \log \|\Pi'(\theta^{-n}\omega)T^{-n}(\omega)v\| \\
\leq &\limsup_{n\to+\infty} \frac{1}{n} \log \|\Pi'(\omega)T^n(\theta^{-n}\omega)\Pi'(\theta^{-n}\omega)T^{-n}(\omega)v\| \\
&- \lim_{n\to+\infty} \frac{1}{n} \log \|T^n(\theta^{-n}\omega)|_{E^{\kappa(T)}(\theta^{-n}\omega)}\|^- \\
\leq &0 - \kappa(T) = -\kappa(T).
\end{aligned}$$

Combining (87), we can obtain that

$$\begin{aligned}
&\limsup_{n\to+\infty} \frac{1}{n} \log \|T^{-n}(\omega)v\| \\
= &\limsup_{n\to+\infty} \frac{1}{n} \log \|\Pi'(\theta^{-n}\omega)T^{-n}(\omega)v + \Pi(\theta^{-n}\omega)T^{-n}(\omega)v\| \\
\leq &\limsup_{n\to+\infty} \frac{1}{n} \log \left(\|\Pi'(\theta^{-n}\omega)T^{-n}(\omega)v\| + \|\Pi(\theta^{-n}\omega)T^{-n}(\omega)v\|\right) \\
\leq &-\kappa(T),
\end{aligned}$$

which gives (85).

Step 3. We construct an invariant complementary space.

We now construct an invariant complementary space $F(\omega)$ of $E(\omega) \oplus E^{\kappa(T)}(\omega)$, which is given by the graph of a strongly measurable map $\Psi(\cdot): \ \Omega \to L(X, X)$ such that

$$\Psi(\omega) = \Psi(\omega)\Pi(\omega), \tag{89}$$

$$\Psi(\omega)G_1(\omega) \subset E^{\kappa(T)}(\omega), \tag{90}$$

$$T(\omega)\mathcal{G}(\Psi(\omega)) = \mathcal{G}(\Psi(\theta\omega)), \tag{91}$$

where $\mathcal{G}(\Psi(\omega))\big(= \big\{v + \Psi(\omega)(v) \bigm| v \in G_1(\omega)\big\}\big)$ is the graph of $\Psi(\omega)$ over $G_1(\omega)$.

We first observe that the following statements are equivalent

$$T(\omega)\big(v + \Psi(\omega)v\big) = v' + \Psi(\theta\omega)v', \quad \text{where } v \in G_1(\omega),\ v' \in G_1(\theta\omega), \tag{92}$$

$$\Pi'(\theta\omega)T(\omega)v + T(\omega)\Psi(\omega)v = \Psi(\theta\omega)v', \qquad \tilde{T}(\omega)v = v', \tag{93}$$

$$\Pi'(\theta\omega)T(\omega)v + T(\omega)\Psi(\omega)v = \Psi(\theta\omega)\tilde{T}(\omega)v, \tag{94}$$

$$\begin{aligned} &\big(T(\theta\omega)|_{E^{\kappa(T)}(\omega)}\big)^{-1}\Psi(\theta\omega)\left(\tilde{T}(\omega)|_{G_1(\omega)}\right) - \Psi(\omega) \\ &= \big(T(\theta\omega)|_{E^{\kappa(T)}(\omega)}\big)^{-1}\Pi'(\theta\omega)(T(\omega)|_{G_1(\omega)}). \end{aligned} \tag{95}$$

Thus, we define

$$\Psi(\omega) = -\sum_{n=0}^{+\infty} T^{-(n+1)}(\theta^{(n+1)}\omega)\Pi'(\theta^{n+1}\omega)T(\theta^n\omega)\Pi(\theta^n\omega)T^n(\omega)\Pi(\omega). \tag{96}$$

We will show that the above infinite series converges absolutely and is strongly measurable.

Since $\kappa(T) > \kappa(\tilde{T})$, we let ϵ satisfy $0 < 3\epsilon < \kappa(T) - \kappa(\tilde{T})$. Set

$$C(\omega) = \sup\left\{\frac{\|T^{-n}(\omega)|_{E^{\kappa(T)}(\omega)}\|}{e^{n(-\kappa(T)+\epsilon)}}\right\}_{n\geq 0},$$

and

$$C_1(\omega) = \sup\left\{\frac{\|\tilde{T}^n(\omega)|_{G(\omega)}\|}{e^{n(\kappa(\tilde{T})+\epsilon)}}\right\}_{n\geq 0}.$$

By the definitions of $\kappa(T)$ and $\kappa(\tilde{T})$, we have that $1 \leq C(\omega) < +\infty$ and $1 \leq C_1(\omega) < +\infty$. Using the same argument as before, we have that $C(\omega)$ and $C_1(\omega)$ are tempered random variables. Then, for each $x \in X$, we have

$$\begin{aligned}
&\|\Psi(\omega)x\| \\
\leq &\sum_{n=0}^{+\infty} \|T^{-(n+1)}(\theta^{n+1}\omega)|_{E^{\kappa(T)}(\theta^{n+1}\omega)}\| \, \|\Pi'(\theta^{n+1}\omega)\| \, \|T(\theta^n\omega)\big|_{G(\theta^n\omega)}\| \, \|\tilde{T}^n(\omega)\| \, \|\Pi(\omega)x\| \\
\leq &\sum_{n=0}^{+\infty} C(\theta^{n+1}\omega)e^{(n+1)(-\kappa(T)+\epsilon)}C_1(\omega)e^{n(\kappa(\tilde{T})+\epsilon)}\|\Pi'(\theta^{n+1}\omega)\| \, \|T(\theta^n\omega)\big|_{G(\theta^n\omega)}\| \, \|\Pi(\omega)x\| \\
\leq &e^{-\kappa(T)+\epsilon}C_1(\omega)\sum_{n=0}^{+\infty} C(\theta^{n+1}\omega)e^{n(-\kappa(T)+\epsilon+\kappa(\tilde{T})+\epsilon)}\|\Pi'(\theta^{n+1}\omega)\| \, \|T(\theta^n\omega)\big|_{G(\theta^n\omega)}\| \, \|\Pi(\omega)x\| \\
\leq &e^{-\kappa(T)+\epsilon}C_1(\omega)\sum_{n=0}^{+\infty} C(\theta^{n+1})e^{-n\epsilon}\|\Pi'(\theta^{n+1}\omega)\| \, \|T(\theta^n\omega)\big|_{G(\theta^n\omega)}\| \, \|\Pi(\omega)x\|.
\end{aligned}$$

Since $C(\omega)$, $\|\Pi'(\omega)\|$, and $\|T(\omega)\big|_{G(\omega)}\|$ are tempered, by Lemma 42, there exists a tempered random variable $R(\omega)$ such that

$$\frac{1}{R(\omega)} \leq \max\{\|\Pi'(\omega)\|, C(\omega), \|T(\omega)\big|_{G(\omega)}\|\} \leq R(\omega)$$

and

$$e^{-|n|\frac{\epsilon}{4}}R(\omega) \leq R(\theta^n\omega) \leq e^{|n|\frac{\epsilon}{4}}R(\omega).$$

Hence,

$$\begin{aligned}
&\|\Psi(\omega)x\| \\
\leq&\|\Pi(\omega)\| \, \|x\|e^{-\kappa(T)+\epsilon}C_1(\omega)\sum_{n=0}^{+\infty} C(\theta^{n+1})e^{-n\epsilon}\|\Pi'(\theta^{n+1}\omega)\|\|T(\theta^n\omega)\big|_{G(\theta^n\omega)}\| \\
\leq&\|\Pi(\omega)\| \, \|x\|C_1(\omega)e^{-\kappa(T)+\frac{3}{2}\epsilon}\sum_{n=0}^{+\infty} R^3(\omega)e^{-\frac{n}{4}\epsilon} \\
=&\|\Pi(\omega)\| \, \|x\|C_1(\omega)R^3(\omega)\frac{e^{-\kappa(T)+\frac{3}{2}\epsilon}}{1-e^{-\frac{\epsilon}{4}}},
\end{aligned} \tag{97}$$

which yields that (96) is absolutely convergence. To show that $\Psi(\omega)$ is strongly measurable, it is enough to prove that $T^{-1}(\theta\omega)\Pi'(\theta\omega)$ is strongly measurable. Let $e_1(\omega), \cdots e_m(\omega)$ be the measurable basis of $E^{\kappa(T)(\omega)}$. Then $T(\omega)e_1(\omega), \cdots T(\omega)e_m(\omega)$ is also a basis of $E^{\kappa(T)}(\theta\omega)$. For each $x \in X$, since $\Pi'(\omega)x \in E^{\kappa(T)}(\omega)$, there are $a_1(\omega), \cdots, a_m(\omega) \in \mathbb{R}$ such that

$$\Pi'(\theta\omega)x = a_1(\omega)T(\omega)e_1(\omega) + \cdots + a_m(\omega)T(\omega)e_m(\omega).$$

Since $\Pi'(\omega)x$ and $T(\omega)e_i(\omega)$ are measurable, by using Proposition 40, we have that $a_i(\omega), 1 \leq i \leq m$, are measurable. Thus, $T^{-1}(\theta\omega)\Pi'(\theta\omega)x = a_1(\omega)e_1(\omega) + \cdots + a_m(\omega)e_m(\omega)$ is measurable.

Let

$$\pi_1(\omega) = \Pi(\omega) + \Psi(\omega), \quad \pi_1'(\omega) = I - \pi_1(\omega), \tag{98}$$

and

$$F(\omega) = \mathcal{G}(\Psi(\omega)) \tag{99}$$

Then, we have $F(\omega) = \pi_1(\omega)X$ and

$$T(\omega)F(\omega) \subset F(\theta\omega)$$

which gives (i) in Theorem 37. We also have that π_1 and π_1' are strongly measurable.

For any $u \in F(\omega)$, by the definition, there exists $u' \in G_1(\omega)$ such that

$$u = u' + \Psi(\omega)u'.$$

Then

$$\begin{aligned}\pi_1(\omega)u =& \left(\Pi(\omega) + \Psi(\omega)\right)\left(u' + \Psi(\omega)u'\right)\\ =& \left(\Pi(\omega) + \Psi(\omega)\right)u' + \left(\Pi(\omega) + \Psi(\omega)\right)\Psi(\omega)u'\\ =& u' + \Psi(\omega)u' + 0 = u.\end{aligned}$$

Hence, $\pi_1(\omega)$ is a projection on $F(\omega)$. And for any $v \in E(\omega) \oplus E^{\kappa(T)}(\omega)$, we have

$$\pi_1'(\omega)v = (I - \Pi(\omega))v - \Psi(\omega)v = (I - \Pi(\omega))v = v.$$

Therefore, we have an invariant splitting of X

$$X = \left(E(\omega) \oplus E^{\kappa(T)}(\omega)\right) \oplus F(\omega).$$

Since $C_1(\omega)$ and $R(\omega)$ are tempered, using the estimate (97), we have that $\Psi(\omega)$ is tempered from above. Since

$$1 \le \pi_1(\omega) \le \|\Pi(\omega)\| + \|\Psi(\omega)\|,$$

$\|\pi_1(\omega)\|$ is tempered. Hence, the property (ii) in Theorem 37 holds.

Step 4. We establish the exponential rate of $T(\omega)$ in $F(\omega)$.

We first show that $\pi_1(\omega)|_{G_1(\omega)}$ is an isomorphism from $G_1(\omega)$ to $F(\omega)$. From the definition of $F(\omega)$ we have $\pi_1(\omega)G_1(\omega) = F(\omega)$, which means that $\pi_1(\omega)|_{G_1(\omega)}$ is surjective. For any $v_1, v_2 \in G_1(\omega)$, if $\pi_1(\omega)v_1 = \pi_1(\omega)v_2$, then $v_1 - v_2 = \Psi(\omega)v_2 - \Psi(\omega)v_1$. Since $v_1 - v_2 \in G_1(\omega), \Psi(\omega)v_2 - \Psi(\omega)v_1 \in E^{\kappa(T)}(\omega)$ and $G_1(\omega) \cap E^{\kappa(T)}(\omega) = \{0\}$. So $v_1 = v_2$, which implies that $\pi_1(\omega)|_{G_1(\omega)}$ is injective. Thus $\pi_1(\omega)|_{G_1(\omega)}$ is a one-to-one map between $F(\omega)$ and $G_1(\omega)$. We also note that $\pi_1(\omega)$ is a bounded linear operator. Furthermore, $\| \left(\pi_1(\omega)|_{G_1(\omega)}\right)^{-1} \| = \|\Pi(\omega)|_{F(\omega)}\|$. Hence, $\pi_1(\omega)|_{G_1(\omega)}$

is an isomorphism. We notice that

$$T^n(\omega)|_{F(\omega)} = \pi_1(\theta^{n+1}\omega)\tilde{T}^n(\omega)\Pi(\omega)|_{F(\omega)}.$$

Since both $\|\pi_1(\theta^{n+1}\omega)\|$ and $\|\Pi(\omega)\|$ are tempered, we obtain

$$\limsup_{n\to+\infty} \frac{1}{n}\log \|T^n(\omega)|_{F(\omega)}\| \le \lim_{n\to+\infty} \frac{1}{n}\log \|\tilde{T}^n(\omega)\| < \kappa(T),$$

which gives the property (iii) in Theorem 37 .

For any $v(\neq 0) \in F(\omega)$ such that $T^{-n}(\omega)v$ exists for all $n \ge 0$, we have

$$\begin{aligned}\|v\| &= \|T^n(\theta^{-n}\omega)T^{-n}(\omega)v\| = \|\pi_1(\omega)\tilde{T}^n(\theta^{-n}\omega)\Pi(\theta^{-n}\omega)T^{-n}(\omega)v\| \\ &\le \|\Pi(\theta^{-n}\omega)\|(1+\|\Psi(\omega)\|)\|\tilde{T}^n(\theta^{-n}\omega)\big|_{G(\theta^{-n}\omega)}\| \; \|T^{-n}(\omega)v\|.\end{aligned}$$

Then, as long as $\|\Pi(\cdot)\|$ and $\|\Psi(\cdot)\|$ are tempered, we have that

$$\begin{aligned}&\liminf_{n\to+\infty} \frac{1}{n}\log \|T^{-n}(\omega)v\| \\ \ge &\liminf_{n\to+\infty} \frac{1}{n}\log \frac{\|v\|}{\|\Pi(\theta^{-n}\omega)\|(1+\|\Psi(\omega)\|)\|\tilde{T}^n(\theta^{-n}\omega)\big|_{G(\theta^{-n}\omega)}\|} \\ = &-\limsup_{n\to+\infty} \frac{1}{n}\log \|\tilde{T}^n(\theta^{-n}\omega)\big|_{G(\theta^{-n}\omega)}\| \\ = &-\kappa(\tilde{T}) > -\kappa(T),\end{aligned}$$

which gives the property (iv).

For any $\epsilon > 0$, let

$$K'(\omega) = \sup\left\{\frac{\|T^n(\omega)|_{F(\omega)}\|}{e^{n(\kappa(T|F)+\epsilon)}}\right\}_{n\ge 0}.$$

Then, we have

$$1 \leq K(\omega) \leq K'(\omega)\|\pi_1(\omega)\|. \tag{100}$$

Since for $n \geq 1$,

$$\|T^n(\omega)|_{F(\omega)}\| = \left\|\left(T^{n-1}(\theta\omega)|_{F(\theta\omega)}\right)\left(T(\omega)|_{F(\omega)}\right)\right\| \leq \|T^{n-1}(\theta\omega)|_{F(\theta\omega)}\|\|S(\omega)\|,$$

we have

$$\log K'(\omega) - \log K'(\theta\omega) \leq \max\{\log^+ \|S(\omega)\| - \kappa(T|F) - \epsilon, 0\}.$$

By Lemma 26, we have

$$\lim_{n\to\pm\infty} \frac{1}{n} \log K'(\theta^n\omega) = 0.$$

Therefore, combining with (100), we have

$$\limsup_{n\to\pm\infty} \frac{1}{n} \log K(\theta^n\omega) \leq \limsup_{n\to\pm\infty} \frac{1}{n} \log K'(\theta^n\omega) + \limsup_{n\to\pm\infty} \frac{1}{n} \log \|\pi_1(\theta^n\omega)\| = 0.$$

Hence

$$\lim_{n\to\pm\infty} \frac{1}{n} \log K(\theta^n\omega) = 0.$$

The proof of Theorem 37 is complete. □

8 Proof of Multiplicative Ergodic Theorem

We prove Theorem 4 by using Theorems 21, 22, 23, 37 and prove it by induction. First, we let $E(\omega) = 0$. We assume that $\kappa(\Phi) > l(\Phi)$. It is clear that the conditions (B1)-(B3) in Section 6 are satisfied and $T = \Phi$. By Theorems 21, 22, 23, and 37, we obtain $\lambda_1 = \kappa(T)$, $E_1(\omega) = E^{\kappa(T)}(\omega)$, $F_1(\omega) = F(\omega)$ and $\Pi_1(\omega) = \Pi(\omega)$ which satisfy all the conditions of Theorem 4.

Next, we introduce the induction assumption: There exist $k(\omega)$ numbers $\lambda_1(\omega) > \ldots > \lambda_{k(\omega)}(\omega) > l_\alpha(\omega)$ and a splitting

$$X = E_1(\omega) \oplus \cdots \oplus E_{k(\omega)}(\omega) \oplus F_k(\omega)$$

of finite dimensional linear subspaces $E_j(\omega)$ and finite codimension linear subspace $F_k(\omega)$ such that

1) Invariance: $k(\theta\omega) = k(\omega)$, $\lambda_i(\theta\omega) = \lambda_i(\omega)$, $S(\omega)E_j(\omega) = E_j(\theta\omega)$ and $S(\omega)F_k(\omega) \subset F_k(\theta\omega)$;

2) Lyapunov Exponents:

$$\lim_{n\to+\infty} \frac{1}{n} \log \|\Phi(n,\omega)v\| = \lambda_j(\omega) \text{ for all } v(\neq 0) \in E_j(\omega), 1 \le j \le k;$$

3) Exponential Decay Rate on $F(\omega)$:

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|\Phi(n,\omega)|_{F_k(\omega)}\| < \lambda_{k(\omega)}(\omega)$$

and if $v(\neq 0) \in F_k(\omega)$ and $(\Phi(n,\theta^{-n}\omega))^{-1}v$ exists for all $n \ge 0$, which is denoted by $\Phi(-n,\omega)v$, then

$$\liminf_{n\to+\infty}\frac{1}{n}\log\|\Phi(-n,\omega)v\| > -\lambda_{k(\omega)}(\omega);$$

4) Tempered Projections: The projection operators associated with the decompositions

$$X=\Big(\bigoplus_{i=1}^{j}E_i(\omega)\Big)\oplus\Big(\big(\bigoplus_{i=j+1}^{k(\omega)}E_i(\omega)\big)\oplus F_k(\omega)\Big)=\Big(\bigoplus_{i=1}^{k(\omega)}E_i(\omega)\Big)\oplus F_k(\omega)$$

are tempered;

5) Measurability: $k(\omega),\lambda_i(\omega)$, and $E_j(\omega)$ are measurable and the projection operators are strongly measurable.

We have two cases. If $\kappa\big(\Phi\big|_{F_k(\omega)}\big)=l_\alpha$, then we have (II) in Theorem 4. Note that $\kappa\big(\Phi\big|_{F_k(\omega)}\big)\geq l_\alpha$. If $\kappa\big(\Phi\big|_{F_k(\omega)}\big)>l_\alpha$, we let

$$E(\omega)=E_1(\omega)\oplus\cdots\oplus E_{k(\omega)}(\omega),$$
$$G(\omega)=F_k(\omega),\quad \text{and } \pi(\omega)=\Pi_{k(\omega)}(\omega).$$

Then, by applying Theorems 21, 22, 23, and 37 to $T(\omega)=S(\omega)\pi(\omega)$, we can obtain that $\lambda_{k+1}=\kappa(T)$, $E_{k+1}(\omega)=E^{\kappa(T)}(\omega)$, $F_{k(\omega)+1}(\omega)=F(\omega)$ and $\Pi_{k+1}(\omega)=\pi_1(\omega)$ which satisfy all the above conditions by replacing k by $k+1$. If $\kappa\big(\Phi\big|_{F_k(\omega)}\big)>l_\alpha$ for all positive integer k, then (III) holds. Otherwise, (II) holds. $\lim_{k\to\infty}\lambda_k=l_\alpha$ follows from Theorem 21. The proof is complete.

$\square$

9 Stable and Unstable Manifolds

In this section, we establish random stable and unstable manifolds of a nonuniformly hyperbolic random invariant set $\mathcal{A}(\omega)$ for infinite dimensional random dynamical systems in a Banach space X.

9.1 Nonuniformly Hyperbolic Linear RDS.

We consider a linear random dynamical system $\Phi(n, \omega)$ in a Banach space X.

Definition 4. $\Phi(n, \omega)$ is said to be nonuniformly hyperbolic if for almost every $\omega \in \Omega$, there exists a splitting

$$X = E^u(\omega) \oplus E^s(\omega)$$

of closed subspaces with associated projections $\Pi_u(\omega)$ and $\Pi_s(\omega)$ such that

(i) The splitting is invariant: $\Phi(n, \omega)E^u(\omega) = E^u(\theta^n\omega)$ and $\Phi(n, \omega)E^s(\omega) \subset E^s(\theta^n\omega)$.

(ii) $\Phi(1, \omega)\big|_{E^u(\omega)} : E^u(\omega) \to E^u(\theta\omega)$ is an isomorphism.

(iii) There is a θ-invariant random variable $\beta : \Omega \to (0, \infty)$, a tempered random variable $K(\omega) : \Omega \to [1, \infty)$ such that

$$||\Phi(n, \omega)\Pi_s(\omega)|| \le K(\omega)e^{-\beta(\omega)n} \quad \text{for } n \ge 0 \tag{101}$$

$$||\Phi(n, \omega)\Pi_u(\omega)|| \le K(\omega)e^{\beta(\omega)n} \quad \text{for } n \le 0 \tag{102}$$

Remark 6. If $K(\omega)$ is uniformly bounded, then we call this dynamical system uniformly hyperbolic.

Remark 7. When $\Phi(n, \omega)$ satisfies the conditions of Theorem A and has no zero Lyapunov exponents in which $l_\alpha(\omega) < 0$, we may divide all the Lyapunov exponents into two group based on their signs. Let

$$\sigma^u(\omega) := \{\lambda_i(\omega) > 0\} \quad \text{and} \quad \sigma^s(\omega) := \{\lambda_i(\omega) < 0\}$$

and denote

$$E^u(\omega) := \bigoplus_{\lambda_i(\omega)\in\sigma_u(\omega)} E_i(\omega) \quad \text{and} \quad E^s(\omega) := \bigoplus_{\lambda_i(\omega)\in\sigma_s(\omega)} E_i(\omega).$$

Then

$$X = E^u(\omega) \oplus E^s(\omega).$$

We call $E^u(\omega)$ the unstable Oseledets subspace and $E^s(\omega)$ the stable Oseledets subspace. $\beta(\omega)$ may be chosen as

$$\beta(\omega) = \frac{1}{2}\min\{|\lambda_i(\omega)|\}.$$

As ω varies, $\beta(\omega)$ may be arbitrarily small and $K(\omega)$ may be arbitrarily large. However, along each orbit $\theta^n\omega$, $\beta(\omega)$ is a constant and $K(\omega)$ can increase only at a subexponential rate, which together with conditions (101) and (102) imply that the linear system $\Phi(n,\omega)$ is nonuniformly hyperbolic in the sense of Pesin.

9.2 Nonuniformly Hyperbolic Random Sets.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, X be a separable Banach space, $(\theta^n)_{n\in\mathbb{Z}}$ be a metric dynamical system, and ϕ be a random dynamical system (or a cocycle) on X over θ^n. For $\mathcal{A} \in \mathcal{F}\otimes\mathcal{B}(X)$, we call $\mathcal{A}(\omega) := \{x \in X \mid (\omega, x) \in \mathcal{A}\} \in \mathcal{B}(X)$ is the ω-section of $\mathcal{A}$.

Definition 5. Let $\mathcal{A} \subset \Omega \times X$ be a measurable set.

(a) $\mathcal{A}$ is called forward invariant if for any $n > 0$

$$\phi(n, \omega)\mathcal{A}(\omega) \subset (\mathcal{A}(\theta^n \omega)) \quad P - a.s.,$$

(b) $\mathcal{A}$ is called invariant if for each $x_0 \in \mathcal{A}(\omega)$ and each $n > 0$ there exists a unique $x \in \mathcal{A}(\omega^{-n})$ such that $\phi(n, \theta^{-n}\omega)(x) = x_0$ and for all $n \in \mathbb{Z}$

$$\phi(n, \omega)\mathcal{A}(\omega) = (\mathcal{A}(\theta^n \omega)) \quad P - a.s.,$$

where $\phi(-n, \omega)(x)$ denotes $\phi^{-1}(n, \theta^{-n}\omega)(x)$ for $n > 0$.

We now define the nonuniform hyperbolicity of a nonlinear dynamical system ϕ on an invariant set $\mathcal{A}$.

Definition 6. ϕ is said to be nonuniformly hyperbolic on $\mathcal{A}$ if for almost every $\omega \in \Omega$ and $x \in \mathcal{A}(\omega)$, there exists a splitting

$$X = E^u(\omega, x) \oplus E^s(\omega, x)$$

of closed subspaces with associated projections $\Pi^u(\omega, x)$ and $\Pi^s(\omega, x)$ such that

(i) The splitting is invariant:

$$D_x\phi(n, \omega)(x)E^u(\omega, x) = E^u(\theta^n\omega, \phi(n, \omega)(x))$$

and

$$D_x\phi(n, \omega)(x)E^s(\omega, x) \subset E^s(\theta^n\omega, \phi(n, \omega)(x)).$$

(ii) $D_x\phi(1, \omega)(x)\big|_{E^u(\omega,x)} : E^u(\omega, x) \to E^u(\theta\omega, \phi(1, \omega)(x))$ is an isomorphism.

(iii) There is a (θ, ϕ)-invariant random variable $\beta : \mathcal{A} \to (0, \infty)$, a tempered random variable $K(\omega, x) : \mathcal{A} \to [1, \infty)$ such that

$$||D_x\phi(n,\omega)(x)\Pi_s(\omega,x)|| \leq K(\omega,x)e^{-\beta(\omega,x)n} \quad \text{for } n \geq 0 \tag{103}$$

$$||D_x\phi(n,\omega)(x)\Pi_u(\omega,x)|| \leq K(\omega,x)e^{\beta(\omega,x)n} \quad \text{for } n \leq 0 \tag{104}$$

Another definition is about pseudo-hyperbolicity,

Definition 7. ϕ is said to be nonuniformly pseudo-hyperbolic on $\mathcal{A}$ if for almost every $\omega \in \Omega$ and $x \in \mathcal{A}(\omega)$, there exists a splitting

$$X = E^u(\omega, x) \oplus E^s(\omega, x)$$

of closed subspaces with associated projections $\Pi^u(\omega, x)$ and $\Pi^s(\omega, x)$ such that

(i) The splitting is invariant:

$$D_x\phi(n,\omega)(x)E^u(\omega,x) = E^u(\theta^n\omega, \phi(n,\omega)(x))$$

and

$$D_x\phi(n,\omega)(x)E^s(\omega,x) \subset E^s(\theta^n\omega, \phi(n,\omega)(x)).$$

(ii) $D_x\phi(1,\omega)(x)\big|_{E^u(\omega,x)} : E^u(\omega,x) \to E^u(\theta\omega, \phi(1,\omega)(x))$ is an isomorphism.

(iii) There are (θ, ϕ)-invariant random variables $\alpha < \beta$, a tempered random variable $K(\omega, x) : \mathcal{A} \to [1, \infty)$ such that

$$||D_x\phi(n,\omega)(x)\Pi_s(\omega,x)|| \leq K(\omega,x)e^{-\beta(\omega,x)n} \quad \text{for } n \geq 0 \tag{105}$$

$$||D_x\phi(n,\omega)(x)\Pi_u(\omega,x)|| \leq K(\omega,x)e^{-\alpha(\omega,x)n} \quad \text{for } n \leq 0 \tag{106}$$

Remark 8. Assume that there is an invariant set $\mathcal{A}$ of ϕ. Define map $\Theta^t : \mathcal{A} \to \mathcal{A}$ by

$$\Theta^t(\omega, x) = (\theta^t\omega, \phi(t, \omega)(x)) \quad t \in \mathbb{T}.$$

Also assume that there is a Θ^t-invariant probability measure μ on $(\mathcal{A}, \mathcal{F} \otimes \mathcal{B}(X)|_{\mathcal{A}})$ such that $\mu(\cup_{\omega \in F}\{\omega \times \mathcal{A}(\omega)\}) = P(F)$ for any $F \in \mathcal{F}$. Denote the metric dynamical system $(\mathcal{A}, \mathcal{F} \otimes \mathcal{B}(X)|_{\mathcal{A}}, \mu, \Theta)$ by $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \Theta)$ and let

$$S(\tilde{\omega}) = D_x\phi(1, \omega)(x)$$

where $\tilde{\omega} = (\omega, x) \in \tilde{\Omega}$. Then $S(\cdot)$ generates a linear random dynamical system Φ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. If we assume that $S(\cdot)$ satisfies the conditions of Theorem 4, then we have the nonuniform pseudo-hyperbolicity of the invariant set $\mathcal{A}$.

Remark 9. If $\beta(\omega) > 0$, then $E^s(\omega)$ is a stable random invariant subspace of $D_x\phi$. If $\alpha(\omega) < 0$, then $E^u(\omega)$ is an unstable random invariant subspace of $D_x\phi$.

For the remainder of this thesis, we will assume that $\mathcal{A}$ is invariant and ϕ is nonuniformly pseudo-hyperbolic on $\mathcal{A}$, We will consider two cases: $\alpha(\omega) < \beta(\omega), \beta(\omega) > 0$ and $\beta(\omega) > \alpha(\omega), \alpha(\omega) < 0$.

9.3 Stable and Unstable Manifolds.

We first write ϕ as

$$\phi(1, \omega, x) = \phi(1, \omega)(x) = D_x\phi(1, \omega)(x) + f(1, \omega, x).$$

We assume that the nonlinear term f satisfies

Hypothesis H1: *ϕ is a $C^N, N > 1$, random dynamical system and there are tempered functions $\rho_0, B_k : \mathcal{A} \to (0, +\infty),\ 0 \le k \le N$ such that for almost every $(\omega, x) \in \mathcal{A}$,*

$$\sup_{y \in U(\omega,x)} ||D^k f(1, \omega, y)|| \le B_k(\omega, x), \quad \textit{for all } 0 \le k \le N,$$

where $U(\omega, x) = B(x, \rho_0(\omega, x)) = \{y \in X |\ \|y - x\| < \rho_0(\omega, x)\}$, which is called a tempered ball.

Our main result is following:

Theorem 43. *(Stable and Unstable Manifolds) Assume that ϕ is nonuniformly pseudo-hyperbolic on $\mathcal{A}$ and Hypothesis H1 holds.*

(i) If $\alpha(\omega) < \beta(\omega), \beta(\omega) > 0$, then for almost every point $(\omega, x) \in \mathcal{A}$, there exists a C^N local stable manifold for ϕ given by

$$W^s_{loc}(\omega, x) = \{y \in B(x, \rho(\omega, x)) | ||\phi(n, \omega, y) - \phi(n, \omega, x)\| e^{\gamma(\omega)n} \to 0,\ \textit{as } n \to +\infty\},$$

where $\gamma(\omega) > 0, \alpha(\omega) < \gamma(\omega) < \beta(\omega)$, $B(x, \rho(\omega, x))$ is a tempered ball.

(ii) If $\beta(\omega) > \alpha(\omega), \alpha(\omega) < 0$, then for almost every point $(\omega, x) \in \mathcal{A}$, there exists a C^N local unstable manifold for ϕ given by

$$W^u_{loc}(\omega, x) = \{y \in B(x, \rho(\omega, x)) | ||\phi(n, \omega, y) - \phi(n, \omega, x)\| e^{-\gamma(\omega)n} \to 0,\ \textit{as } n \to -\infty\},$$

where $\gamma(\omega) > 0, \alpha(\omega) < -\gamma(\omega) < \beta(\omega)$, $B(x, \rho(\omega, x))$ is a tempered ball.

Remark 10. We note that this theorem holds for a $C^{k,\alpha}$ random dynamical system for $k \ge 1$ and $0 < \alpha \le 1$.

9.4 Proof of Theorem 43.

We will prove this theorem by using the standard Lyapunov and Perron approach. It is sufficient to show the existence of the stable or unstable manifold along a fixed orbit of ϕ in $\mathcal{A}$. Let

$$\Theta^n(\tilde{\omega}) = (\theta^n\omega, \phi(n, \omega, x)), \text{ where } \tilde{\omega} = (\omega, x) \in \mathcal{A}.$$

For $(\omega, y_0) \in \mathcal{A}$, we also write $\Theta^i(\omega, y_0) = (\theta^i\omega, y_i)$, where $\{y_i\}_{i\in\mathbb{Z}} \subset X$. By the invariance of $\mathcal{A}$, $(\theta^i\omega, y_i) \in \mathcal{A}$, for all $i \in \mathbb{Z}$. Note that $\{y_n\}_{n\in\mathbb{Z}}$ is an orbit given by $\phi(n, \omega, y_0)$. Let $\{z_n\}_{n\in\mathbb{Z}}$ be another orbit ϕ and $x_n = z_n - y_n$. Then for any $n \in \mathbb{Z}$

$$z_{n+1} = \phi(1, \theta^n\omega)(z_n) = D\phi(1, \theta^n\omega, y_n)x_n + \phi(1, \theta^n\omega, y_n) + f(1, \theta^n\omega, y_n, x_n)$$

where $f(1, \theta^n\omega, y_n, x_n) = \phi(1, \theta^n\omega, z_n) - \phi(1, \theta^n\omega, y_n) - D\phi(1, \theta^n\omega, y_n)x_n$ satisfying

$$f(1, \theta^n\omega, y_n, 0) = 0 \text{ and } Df(1, \theta^n\omega, y_n, 0) = 0.$$

Thus, by using the fact $y_{n+1} = \phi(n, \omega, y_n)$, we have that x_n satisfy that for all $n \in \mathbb{Z}$

$$x_{n+1} = D\phi(1, \Theta^n\tilde{\omega})x_n + F(\Theta^n\tilde{\omega}, x_n), \tag{107}$$

where $F(\Theta^n\tilde{\omega}, x_n) = f(1, \Theta^n\tilde{\omega}, x_n)$ and $\tilde{\omega} = (\omega, y_0)$. We denote the linear random dynamical system generated by $D\phi(1, \tilde{\omega})$ by

$$\Phi(i, \tilde{\omega}) = \begin{cases} D\phi(1, \Theta^{i-1}\tilde{\omega}) \circ \cdots \circ D\phi(1, \tilde{\omega}) & \text{when } i > 0, \\ I & \text{when } i = 0. \end{cases}$$

By the assumptions of Theorem 43, $\Phi(i, \tilde{\omega})$ is nonuniformly pseudo-hyperbolic.

For any number $\gamma \geq 0$, define the following Banach space

$$C_\gamma^- = \{\mathbf{x} = \{x_n\}_{n \leq 0} | x_n \in X,\ \sup_{n \leq 0} \|x_n\| e^{-n\gamma} < +\infty\}$$

with the norm

$$\|\mathbf{x}\|_{C_\gamma^-} = \sup_{n \leq 0} \|x_n\| e^{-n\gamma}$$

and

$$C_\gamma^+ = \{\mathbf{x} = \{x_n\}_{n \geq 0} | x_n \in X,\ \sup_{n \geq 0} \|x_n\| e^{n\gamma} < +\infty\}$$

with the norm

$$\|\mathbf{x}\|_{C_\gamma^+} = \sup_{n \geq 0} \|x_n\| e^{n\gamma}.$$

Similarly, for any integer $j > 0$, we also define Banach space

$$C_\gamma^{j-} = \{\mathbf{x} = \{x_n\}_{n \leq 0} | x_n \in L^j(X, X),\ \sup_{n \leq 0} \|x_n\| e^{-n\gamma} < +\infty\}$$

with the norm

$$\|\mathbf{x}\|_{C_\gamma^{j-}} = \sup_{n \leq 0} \|x_n\| e^{-n\gamma}$$

and

$$C_\gamma^{j+} = \{\mathbf{x} = \{x_n\}_{n \geq 0} | x_n \in L^j(X, X),\ \sup_{n \geq 0} \|x_n\| e^{n\gamma} < +\infty\}$$

with the norm

$$\|\mathbf{x}\|_{C_\gamma^{j+}} = \sup_{n \geq 0} \|x_n\| e^{n\gamma},$$

where $L^j(X, X)$ is the regular j-form.

Let $\{x_n\}_{n\geq 0}$ satisfies equation (107), then we have that

$$x_n = \Phi(n-k, \Theta^k\tilde{\omega})x_k + \sum_{i=k}^{n-1} \Phi(n-1-i, \Theta^{i+1}\tilde{\omega})F(\Theta^i\tilde{\omega}, x_i). \tag{108}$$

We first consider the stable manifold. Here we assume that $\alpha(\omega) < \beta(\omega), \beta(\omega) > 0$.

Lemma 44. *Let $\gamma(\tilde{\omega}) \in (\alpha(\tilde{\omega}), \beta(\tilde{\omega}))$ be positive and Θ-invariant. Then $\mathbf{x} = \{x_n\}_{n\geq 0} \in C_\gamma^+$ satisfies equation (108) if and only if $\{x_n\}_{n\geq 0} \in C_\gamma^+$ satisfies the following equations*

$$\begin{aligned} x_n = {} & \Phi_s(n, \tilde{\omega})x_0^s + \sum_{i=0}^{n-1} \Phi_s(n-1-i, \Theta^{i+1}\tilde{\omega})F_s(\Theta^i\tilde{\omega}, x_i) \\ & - \sum_{i=n}^{+\infty} \Phi_u(n-1-i, \Theta^{i+1}\tilde{\omega})F_u(\Theta^i\tilde{\omega}, x_i), \quad \textit{for} \;\; n \geq 1; \\ x_0 = {} & x_0^s - \sum_{i=0}^{+\infty} \Phi_u(-1-i, \Theta^{i+1}\tilde{\omega})F_u(\Theta^i\tilde{\omega}, x_i), \end{aligned} \tag{109}$$

where $\Phi_u = \Phi|_{E^u}$, $\Phi_s = \Phi|_{E^s}$, $F_u = \Pi_u F$, and $F_s = \Pi_s F$.

Proof. Consider $\mathbf{x} \in C_\gamma^+$. Suppose that $\{x_n\}_{n\geq 0}$ satisfies equation 108, then we have

$$x_n = \Phi(n-k, \Theta^k\tilde{\omega})x_k + \sum_{i=k}^{n-1} \Phi(n-1-i, \Theta^{i+1}\tilde{\omega})F(\Theta^i\tilde{\omega}, x_i).$$

By setting k=0, we have that its stable part satisfies

$$x_n^s = \Phi_s(n, \tilde{\omega})x_0^s + \sum_{i=0}^{n-1} \Phi_s(n-1-i, \Theta^{i+1}\tilde{\omega})F_s(\Theta^i\tilde{\omega}, x_i). \tag{110}$$

Applying the unstable projection Π_u on equation (108) and switching k and n,we have

$$x_n^u = \Phi_u^{-1}(k-n, \Theta^n\tilde{\omega})x_k^u - \sum_{i=n}^{k-1} \Phi_u(n-1-i, \Theta^{i+1}\tilde{\omega})F_u(\Theta^i\tilde{\omega}, x_i). \tag{111}$$

Since $\{x_n\}_{n\geq 0} \in C_\gamma^+$, then

$$\|\Phi_u^{-1}(k-n, \Theta^n\tilde{\omega})x_k^u\| \leq K(\Theta^k\tilde{\omega})e^{(k-n)\alpha(\tilde{\omega})-k\gamma}\|\mathbf{x}\|_{C_\gamma^+} \to 0, \text{ as } k \to +\infty,$$

where $\lim_{k\to+\infty} \frac{1}{k}\log K(\Theta^k\tilde{\omega}) = 0$ was used. Let $k \to +\infty$ in equation (111). Then we can obtain that

$$x_n^u = \sum_{i=n}^{+\infty} \Phi_u(n-1-i, \Theta^{i+1}\tilde{\omega})F_u(\Theta^i\tilde{\omega}, x_i). \tag{112}$$

Adding equation (110) and (112) together, we get equation (109). The converse follows from a straightforward computation. This complete the proof.

□

Proposition 45. Assume that ϕ is nonuniformly pseudo-hyperbolic on $\mathcal{A}$ and Hypothesis H1 holds, and $\alpha(\omega) < \beta(\omega), \beta(\omega) > 0$. Then for each positive invariant random variable $\gamma(\cdot) \in (\alpha(\cdot), \beta(\cdot))$, equation (109) has a unique solution $\mathbf{x}(\eta, \tilde{\omega}) = \{x_n(\eta, \tilde{\omega})\} \in C_{\gamma(\tilde{\omega})}^+$ with $x_0^s = \eta$ for each $\eta \in E^s(\tilde{\omega}) \cap B(0, \rho(\tilde{\omega}))$, where $\rho(\tilde{\omega})$ is a tempered function. Furthermore,

(i) For each $\tilde{\omega} \in \Omega$, $\mathbf{x}(\cdot, \tilde{\omega})$ is C^N from $E^s(\tilde{\omega}) \cap B(0, \rho(\tilde{\omega}))$ to $C_{\gamma(\tilde{\omega})}^+$, $\mathbf{x}(0, \tilde{\omega}) = 0$, and $D\mathbf{x}(0, \tilde{\omega}) = 0$;

(ii) There are random variables tempered from above, $K_i(\tilde{\omega})$, $1 \leq i \leq N$ when $N < \infty$ and $i \geq 1$ when $N = \infty$, such that

$$||D^i\mathbf{x}(\eta, \tilde{\omega})|| \leq K_i(\tilde{\omega}).$$

(iii) There are random variables tempered from above, $K_i(\tilde{\omega})$, $1 \leq i < N$ when

$N < \infty$ and $i \geq 1$ when $N = \infty$, such that for any $\eta, \eta_0 \in E^s(\tilde{\omega}) \cap B(0, \frac{1}{2}\rho(\tilde{\omega}))$

$$||D^i \mathbf{x}(\eta, \tilde{\omega}) - D^i \mathbf{x}(\eta_0, \tilde{\omega})|| \leq K_i(\tilde{\omega})|\eta - \eta_0|.$$

Proof. **Step 1**. *The Existence of a Lipschitz Continuous Solution.*

Fix positive Θ-invariant random variables $\{\gamma_i(\cdot)\}_{1 \leq i \leq N}$ such that $\alpha(\tilde{\omega}) < \gamma_N(\tilde{\omega}) < \ldots < \gamma_1(\tilde{\omega}) < \beta(\tilde{\omega})$. Let $\eta \in E^s(\tilde{\omega})$. For each $\mathbf{x} = \{x_n\}_{n \leq 0} \in C^+_{\gamma_1(\tilde{\omega})}$ with $x_0^s = \eta$, we define a map $\mathbf{y} = \{y_n\}_{n \leq 0} = \mathcal{J}^s(\mathbf{x}, \eta, \tilde{\omega})$ by

$$\begin{aligned} y_n &= \Phi_s(n, \tilde{\omega})\eta + \sum_{i=0}^{n-1} \Phi_s(n-1-i, \Theta^{i+1}\tilde{\omega}) F_s(\Theta^i \tilde{\omega}, x_i) \\ &\quad - \sum_{i=n}^{+\infty} \Phi_u(n-1-i, \Theta^{i+1}\tilde{\omega}) F_u(\Theta^i \tilde{\omega}, x_i), \quad \text{for } \ n \geq 1; \\ y_0 &= \eta - \sum_{i=0}^{+\infty} \Phi_u(-1-i, \Theta^{i+1}\tilde{\omega}) F_u(\Theta^i \tilde{\omega}, x_i). \end{aligned}$$

Let $\rho(\tilde{\omega}) = \min\left\{\rho_0(\tilde{\omega}), \frac{L(\tilde{\omega})}{B_2(\tilde{\omega})K(\Theta\tilde{\omega})}\right\}, \ i \geq 0$, where

$$L(\tilde{\omega}) = \min\left\{\frac{1}{3}\left(\frac{e^{\gamma_i(\tilde{\omega})}}{1 - e^{\gamma_i(\tilde{\omega}) - \beta(\tilde{\omega})}} + \frac{e^{\alpha(\tilde{\omega})}}{1 - e^{-(\gamma_i(\tilde{\omega}) - \alpha(\tilde{\omega}))}}\right)^{-1}\right\}_{1 \leq i \leq N}$$

is a Θ-invariant function. Thus $\rho(\tilde{\omega})$ is tempered. By Lemma 42, there exists a positive tempered function $R(\tilde{\omega})$ such that $\rho(\Theta^n\tilde{\omega}) \geq 2R(\tilde{\omega})e^{-n\gamma_N(\tilde{\omega})}, \ n \geq 0$. Let

$$A(\tilde{\omega}) = \left\{\mathbf{x} \in C^+_{\gamma_1(\tilde{\omega})} \mid \|x_0^s\| \leq \min\left\{\frac{R(\tilde{\omega})}{3K(\tilde{\omega})}, \frac{R(\tilde{\omega})}{3}\right\}, \ \|\mathbf{x}\|_{C^+_{\gamma_1(\tilde{\omega})}} \leq R(\tilde{\omega})\right\}.$$

Thus $\mathbf{x} \in A(\tilde{\omega})$ implies that for any $n \geq 0$, $x_n \in B(0, \frac{1}{2}\rho(\Theta^n\tilde{\omega}))$. We first show that

$\mathcal{J}^s(\cdot,\eta,\tilde{\omega})$ maps $A(\tilde{\omega})$ into itself as long as $\eta < \frac{R(\tilde{\omega})}{3K(\tilde{\omega})}$. When $n > 1$, we have that

$$\begin{aligned}
&\|\mathcal{J}^s(\mathbf{x},\eta,\tilde{\omega})_n e^{n\gamma_1(\tilde{\omega})}\| \\
\leq &\|\Phi_s(n,\tilde{\omega})\eta e^{n\gamma_1(\tilde{\omega})}\| + \sum_{i=0}^{n-1} \|\Phi_s(n-1-i,\Theta^{i+1}\tilde{\omega})F_s(\Theta^i\tilde{\omega},x_i)\| e^{n\gamma_1(\tilde{\omega})} \\
&\quad + \sum_{i=n}^{+\infty} \|\Phi_u(n-1-i,\Theta^{i+1}\tilde{\omega})F_u(\Theta^i\tilde{\omega},x_i)\| e^{n\gamma_1(\tilde{\omega})}
\end{aligned}$$

(by pseudo-hyperbolicity and H1, we have)

$$\begin{aligned}
&\leq K(\tilde{\omega})\|\eta\| e^{n(\gamma_1(\tilde{\omega})-\beta(\tilde{\omega}))} + \sum_{i=0}^{n-1} B_2(\Theta^i\tilde{\omega}) e^{n\gamma_1(\tilde{\omega})}\|x_i\| K(\Theta^{i+1}) e^{-(n-1-i)\beta(\tilde{\omega})}\rho(\Theta^i\tilde{\omega}) \\
&\quad + \sum_{i=n}^{+\infty} B_2(\Theta^i\tilde{\omega}) e^{n\gamma_1(\tilde{\omega})}\|x_i\| e^{-(n-1-i)\alpha(\tilde{\omega})} K(\Theta^{i+1}\tilde{\omega})\rho(\Theta^i\tilde{\omega})
\end{aligned}$$

(by the definition of L, we have)

$$\begin{aligned}
&\leq K(\tilde{\omega})\|\eta\| e^{n(\gamma_1(\tilde{\omega})-\beta(\tilde{\omega}))} + \sum_{i=0}^{n-1} L(\tilde{\omega})\|\mathbf{x}\|_{C^+_{\gamma_1(\tilde{\omega})}} e^{(n-i)\gamma_1(\tilde{\omega})} e^{-(n-1-i)\beta(\tilde{\omega})} \\
&\quad + \sum_{i=n}^{+\infty} L(\tilde{\omega})\|\mathbf{x}\|_{C^+_{\gamma_1(\tilde{\omega})}} e^{(n-i)\gamma_1(\tilde{\omega})} e^{-(n-1-i)\alpha(\tilde{\omega})} \\
&= K(\tilde{\omega})\|\eta\| e^{n(\gamma_1(\tilde{\omega})-\beta(\tilde{\omega}))} + L(\tilde{\omega})\|\mathbf{x}\|_{C^+_{\gamma_1(\omega)}} e^{\gamma_1(\tilde{\omega})} \frac{1-e^{n(\gamma_1(\tilde{\omega})-\beta(\tilde{\omega}))}}{1-e^{(\gamma_1(\tilde{\omega})-\beta(\omega))}} \\
&\quad + L(\tilde{\omega})\|\mathbf{x}\|_{C^+_{\gamma_1(\tilde{\omega})}} \frac{e^{\alpha(\tilde{\omega})}}{1-e^{-(\gamma_1(\omega)-\alpha(\tilde{\omega}))}} \\
&\leq K(\tilde{\omega})\|\eta\| + L(\tilde{\omega})\left(\frac{e^{\gamma_1(\tilde{\omega})}}{1-e^{(\gamma_1(\tilde{\omega})-\beta(\tilde{\omega}))}} + \frac{e^{\alpha(\tilde{\omega})}}{1-e^{-(\gamma_1(\tilde{\omega})+\lambda_k(\tilde{\omega}))}}\right)\|\mathbf{x}\|_{C^+_{\gamma_1(\tilde{\omega})}} \\
&\leq \frac{2}{3}R(\tilde{\omega}).
\end{aligned}$$

Similarly, when $n = 0$ we have

$$\|\mathcal{J}^s(\mathbf{x}, \eta, \tilde{\omega})_0\| \le \|\eta\| + \frac{L(\tilde{\omega})e^{\alpha(\tilde{\omega})}}{1 - e^{-(\gamma_1(\tilde{\omega}) - \alpha(\tilde{\omega}))}} \|\mathbf{x}\|_{C^+_{\gamma_1(\tilde{\omega})}} \le \frac{2}{3} R(\tilde{\omega}),$$

and $\Pi_s(\tilde{\omega})\mathcal{J}^s(\mathbf{x}, \eta, \tilde{\omega})_0 = \eta$. Hence, $\mathcal{J}^s(\cdot, \eta, \tilde{\omega})$ maps $A(\tilde{\omega})$ into itself when $\eta < \frac{R(\tilde{\omega})}{3K(\tilde{\omega})}$.

Next, we show that $\mathcal{J}^s(\mathbf{x}, \eta, \tilde{\omega})$ is a uniform contraction in $\mathbf{x}$ respect to η on $A(\tilde{\omega})$. For each $\mathbf{x},\ \mathbf{x}' \in A(\tilde{\omega})$, when $n > 1$, by straightforward computations, we have that

$$\begin{aligned}
&\|\mathcal{J}^s(\mathbf{x}, \eta, \tilde{\omega})_n - \mathcal{J}^s(\mathbf{x}', \eta, \tilde{\omega})_n\| e^{n\gamma_1(\tilde{\omega})} \\
&\le \sum_{i=0}^{n-1} L(\tilde{\omega}) \|\mathbf{x} - \mathbf{x}'\|_{C^+_{\gamma_1(\tilde{\omega})}} e^{(n-i)\gamma_1(\tilde{\omega})} e^{-(n-1-i)\beta(\tilde{\omega})} \\
&\quad + \sum_{i=n}^{+\infty} L(\tilde{\omega}) \|\mathbf{x} - \mathbf{x}'\|_{C^+_{\gamma_1(\tilde{\omega})}} e^{(n-i)\gamma_1(\tilde{\omega})} e^{-(n-1-i)\alpha(\tilde{\omega})} \\
&= L(\tilde{\omega}) \|\mathbf{x} - \mathbf{x}'\|_{C^+_{\gamma_1(\tilde{\omega})}} e_1^{\gamma}(\tilde{\omega}) \frac{1 - e^{n(\gamma_1(\tilde{\omega}) - \beta(\tilde{\omega}))}}{1 - e^{\gamma_1(\tilde{\omega}) - \beta(\tilde{\omega})}} \\
&\quad + L(\tilde{\omega}) \|\mathbf{x} - \mathbf{x}'\|_{C^+_{\gamma_1(\tilde{\omega})}} \frac{e^{\alpha(\tilde{\omega})}}{1 - e^{-(\gamma_1(\tilde{\omega}) - \alpha(\tilde{\omega}))}} \\
&\le L(\tilde{\omega}) \left(\frac{e^{\gamma_1(\tilde{\omega})}}{1 - e^{\gamma_1(\tilde{\omega}) - \beta(\tilde{\omega})}} + \frac{e^{\alpha(\tilde{\omega})}}{1 - e^{-(\gamma_1(\tilde{\omega}) - \alpha(\tilde{\omega}))}} \right) \|\mathbf{x} - \mathbf{x}'\|_{C^+_{\gamma_1(\tilde{\omega})}} \\
&= \frac{1}{3} \|\mathbf{x} - \mathbf{x}'\|_{C^+_{\gamma_1(\tilde{\omega})}}.
\end{aligned}$$

Similarly, when $n = 0$ we have

$$\begin{aligned}
&\|\mathcal{J}^s(\mathbf{x}, \eta, \tilde{\omega})_0 - \mathcal{J}^s(\mathbf{x}', \eta, \tilde{\omega})_0\| \\
&\le L(\tilde{\omega}) \left(\frac{e^{\gamma_1(\tilde{\omega})}}{1 - e^{\gamma_1(\tilde{\omega}) - \beta(\tilde{\omega})}} + \frac{e^{\alpha(\tilde{\omega})}}{1 - e^{-(\gamma_1(\tilde{\omega}) - \alpha(\tilde{\omega}))}} \right) \|\mathbf{x} - \mathbf{x}'\|_{C^+_{\gamma_1(\tilde{\omega})}} \\
&\le \frac{1}{3} \|\mathbf{x} - \mathbf{x}'\|_{C^+_{\gamma_1(\tilde{\omega})}}.
\end{aligned}$$

Hence, we obtain that

$$\|\mathcal{J}^s(\mathbf{x},\eta,\tilde{\omega}) - \mathcal{J}^s(\mathbf{x}',\eta,\tilde{\omega})\|_{C^+_{\gamma_1(\tilde{\omega})}} \le \frac{1}{3}\|\mathbf{x}-\mathbf{x}'\|_{C^+_{\gamma_1(\tilde{\omega})}}.$$

Thus $\mathcal{J}^s(\mathbf{x},\eta,\tilde{\omega})$ is a uniform contraction in $\mathbf{x}$ respect to η on $A(\tilde{\omega})$. So there exists a unique solution of equation (109) for fixed $x_0^s = \eta$. Let $\mathbf{x}(\eta_1,\tilde{\omega})$ and $\mathbf{x}(\eta_2,\tilde{\omega})$ be two solutions in $A(\tilde{\omega})$. By the definition of $A(\tilde{\omega})$, we have that for $n \ge 1$

$$\|\mathbf{x}(\eta_1,\tilde{\omega})_n - \mathbf{x}(\eta_2,\tilde{\omega})_n\| \le \rho(\Theta^n\tilde{\omega}).$$

For sake of convenience, we use x_i and x_i' to represent $\mathbf{x}(\eta_1,\tilde{\omega})_i$ and $\mathbf{x}(\eta_2,\tilde{\omega})_i$ respectively. Then, for $n \ge 1$, by simple computations, we have

$$\begin{aligned}
&\|\mathbf{x}(\eta_1,\tilde{\omega})_n - \mathbf{x}(\eta_2,\tilde{\omega})_n\| e^{n\gamma_1(\tilde{\omega})} \\
&\le K(\tilde{\omega})\|\eta_1-\eta_2\| \\
&\quad + L(\tilde{\omega})\left(\frac{e^{\gamma_1(\tilde{\omega})}}{1-e^{\gamma_1(\tilde{\omega})-\beta(\tilde{\omega})}} + \frac{e^{\alpha(\tilde{\omega})}}{1-e^{-(\gamma_1(\tilde{\omega})-\alpha(\tilde{\omega}))}}\right)\|\mathbf{x}(\eta_1,\tilde{\omega}) - \mathbf{x}(\eta_2,\tilde{\omega})\|_{C^+_{\gamma_1(\tilde{\omega})}} \\
&= K(\tilde{\omega})\|\eta_1-\eta_2\| + \frac{1}{3}\|\mathbf{x}(\eta_1,\tilde{\omega}) - \mathbf{x}(\eta_2,\tilde{\omega})\|_{C^+_{\gamma_1(\tilde{\omega})}}.
\end{aligned}$$

Similar computation gives that

$$\|\mathbf{x}(\eta_1,\tilde{\omega})_0 - \mathbf{x}(\eta_2,\tilde{\omega})_0\| \le K(\tilde{\omega})\|\eta_1-\eta_2\| + \frac{1}{3}\|\mathbf{x}(\eta_1,\tilde{\omega}) - \mathbf{x}(\eta_2,\tilde{\omega})\|_{C^+_{\gamma_1(\tilde{\omega})}}.$$

Thus, we have that

$$\|\mathbf{x}(\eta_1,\tilde{\omega}) - \mathbf{x}(\eta_2,\tilde{\omega})\|_{C^+_{\gamma_1(\tilde{\omega})}} \le K(\tilde{\omega})\|\eta_1-\eta_2\| + \frac{1}{3}\|\mathbf{x}(\eta_1,\tilde{\omega}) - \mathbf{x}(\eta_2,\tilde{\omega})\|_{C^+_{\gamma_1(\tilde{\omega})}}.$$

So

$$\|\mathbf{x}(\eta_1,\tilde{\omega}) - \mathbf{x}(\eta_2,\tilde{\omega})\|_{C^+_{\gamma_1(\tilde{\omega})}} \le \frac{3}{2}K(\tilde{\omega})\|\eta_1-\eta_2\|, \tag{113}$$

which implies that the solutions are Lipschitz continuous on η.

Step 2. $\mathbf{x}(\cdot,\tilde{\omega})$ is C^1.

In order to show that $\mathbf{x}(\eta,\tilde{\omega})$ is C^N, we show that $\mathbf{x}(\cdot,\tilde{\omega})$ is C^i from $E^s(\tilde{\omega})\cap B(0,\rho(\tilde{\omega}))$ to $C^+_{\gamma_i(\tilde{\omega})}$ for any $1\le i\le N$ by induction.

For each $\eta\in E^s(\tilde{\omega})\cap B(0,\frac{R(\tilde{\omega})}{3K(\tilde{\omega})})$, define a linear operator

$$\mathcal{G}_\eta:\ C^+_{\gamma_1(\tilde{\omega})}\to C^+_{\gamma_1(\tilde{\omega})}$$

by

$$\begin{aligned}(\mathcal{G}_\eta\mathbf{v})_n &= \sum_{i=0}^{n-1}\Phi_s(n-1-i,\Theta^{i+1}\tilde{\omega})DF_s(\Theta^i\tilde{\omega},x_i(\eta,\tilde{\omega}))v_i\\ &\quad-\sum_{i=n}^{+\infty}\Phi_u(n-1-i,\Theta^{i+1}\tilde{\omega})DF_u(\Theta^i\tilde{\omega},x_i(\eta,\tilde{\omega}))v_i,\ \text{for } n\ge 1; \qquad (114)\\ (\mathcal{G}_\eta\mathbf{v})_0 &= -\sum_{i=0}^{+\infty}\Phi_u(-1-i,\Theta^{i+1}\tilde{\omega})DF_u(\Theta^i\tilde{\omega},x_i(\eta,\tilde{\omega}))v_i,\end{aligned}$$

where $\mathbf{v}=\{v_n\}_{n\ge0}\in C^+_{\gamma_1(\tilde{\omega})}$. We also define a bounded linear operator $\mathcal{S}$ from $E^s(\tilde{\omega})$ to $C^+_{\gamma_1(\tilde{\omega})}$ by

$$\mathcal{S}(\eta)=\{\Phi(n,\tilde{\omega})\}_{n\ge0}.$$

Note that for $n > 0$

$$
\begin{aligned}
&\|(\mathcal{G}_\eta \mathbf{v})_n\| e^{n\gamma_1(\tilde{\omega})} \\
&= \sum_{i=0}^{n-1} \|\Phi_s(n-1-i, \Theta^{i+1}\tilde{\omega})\| \|DF_s(\Theta^i\tilde{\omega}, x_i(\eta, \tilde{\omega}))\| \|v_i\| e^{n\gamma_1(\tilde{\omega})} \\
&\quad + \sum_{i=n}^{+\infty} \|\Phi_u(n-1-i, \Theta^{i+1}\tilde{\omega})\| \|DF_u(\Theta^i\tilde{\omega}, x_i(\eta, \tilde{\omega}))\| \|v_i\| e^{n\gamma_1(\tilde{\omega})} \\
&\leq \sum_{i=0}^{n-1} K(\Theta^{i+1}\tilde{\omega}) e^{-(n-1-i)\beta(\tilde{\omega})} B_2(\Theta^i\tilde{\omega}) \rho(\Theta^i\tilde{\omega}) \|\mathbf{v}\|_{C^+_{\gamma(\tilde{\omega})}} e^{(n-i)\gamma_1(\tilde{\omega})} \\
&\quad + \sum_{i=n}^{+\infty} K(\Theta^{i+1}\tilde{\omega}) e^{-(n-1-i)\alpha(\tilde{\omega})} B_2(\Theta^i\tilde{\omega}) \rho(\Theta^i\tilde{\omega}) \|\mathbf{v}\|_{C^+_{\gamma_1(\tilde{\omega})}} e^{(n-i)\gamma_1(\tilde{\omega})} \\
&\leq L(\tilde{\omega}) \left(\frac{e^{\gamma_1(\tilde{\omega})}}{1 - e^{\gamma_1(\tilde{\omega}) - \beta(\tilde{\omega})}} + \frac{e^{\alpha(\tilde{\omega})}}{1 - e^{-(\gamma_1(\tilde{\omega}) - \alpha(\tilde{\omega}))}} \right) \|\mathbf{v}\|_{C^+_{\gamma_1(\tilde{\omega})}} \\
&\leq \frac{1}{3} \|\mathbf{v}\|_{C^+_{\gamma_1(\tilde{\omega})}}.
\end{aligned}
$$

Similarly, we have that

$$
\|(\mathcal{G}_\eta \mathbf{v})_0\| \leq \frac{1}{3} \|\mathbf{v}\|_{C^+_{\gamma_1(\tilde{\omega})}}.
$$

Thus, $\mathcal{G}_\eta$ is a bounded linear operator from $C^+_{\gamma_1(\tilde{\omega})}$ into itself and its norm is bounded by $\frac{1}{3}$, which implies that $(Id - \mathcal{G}_\eta)$ has a bounded inverse in $L(C^+_{\gamma_1(\tilde{\omega})}, C^+_{\gamma_1(\tilde{\omega})})$.
For $\eta, \eta_0 \in E^s(\tilde{\omega}) \cap B\left(0, \frac{R(\tilde{\omega})}{6K(\tilde{\omega})}\right)$, we set

$$
\begin{aligned}
I_n = &\sum_{i=0}^{n-1-i} \Phi_s(n-1, \Theta^{i+1}\tilde{\omega}) \\
&\Big(F_s(\Theta^i\tilde{\omega}, x_i(\eta, \tilde{\omega})) - F_s(\Theta^i\tilde{\omega}, x_i(\eta_0, \tilde{\omega})) - DF_s(\Theta^i\tilde{\omega}, x_i(\eta_0, \tilde{\omega}))(x_i(\eta, \tilde{\omega}) - x_i(\eta_0, \tilde{\omega})) \Big) \\
&- \sum_{i=n}^{+\infty} \Phi_u(n-1-i, \Theta^{i+1}\tilde{\omega}) \Big(F_u(\Theta^i\tilde{\omega}, x_i(\eta, \tilde{\omega})) \\
&\quad - F_u(\Theta^i\tilde{\omega}, x_i(\eta_0, \tilde{\omega})) - DF_u(\Theta^i\tilde{\omega}, x_i(\eta_0, \tilde{\omega}))(x_i(\eta, \tilde{\omega}) - x_i(\eta_0, \tilde{\omega})) \Big), \text{ for } n \geq 1; \\
I_0 = &- \sum_{i=0}^{+\infty} \Phi_u(-1-i, \Theta^{i+1}\tilde{\omega}) \Big(F_u(\Theta^i\tilde{\omega}, x_i(\eta, \tilde{\omega})) - F_u(\Theta^i\tilde{\omega}, x_i(\eta_0, \tilde{\omega})) \\
&- DF_u(\Theta^i\tilde{\omega}, x_i(\eta_0, \tilde{\omega}))(x_i(\eta, \tilde{\omega}) - x_i(\eta_0, \tilde{\omega})) \Big).
\end{aligned}
$$

By using (113), we have

$$
\begin{aligned}
&\|F_u(\Theta^i\tilde{\omega}, x_i(\eta,\tilde{\omega})) - F_u(\Theta^i\tilde{\omega}, x_i(\eta_0,\tilde{\omega})) - DF_u(\Theta^i\tilde{\omega}, x_i(\eta_0,\tilde{\omega}))(x_i(\eta,\tilde{\omega}) - x_i(\eta_0,\tilde{\omega}))\| \\
&\le \frac{1}{2}B_2(\Theta^i\tilde{\omega})\|x_i(\eta,\tilde{\omega}) - x_i(\eta_0,\tilde{\omega})\|^2 \\
&\le \frac{9}{8}K^2(\tilde{\omega})e^{-2i\gamma_1(\tilde{\omega})}\|\eta-\eta_0\|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\|F_s(\Theta^i\tilde{\omega}, x_i(\eta,\tilde{\omega})) - F_s(\Theta^i\tilde{\omega}, x_i(\eta_0,\tilde{\omega})) - DF_s(\Theta^i\tilde{\omega}, x_i(\eta_0,\tilde{\omega}))(x_i(\eta,\tilde{\omega}) - x_i(\eta_0,\tilde{\omega}))\| \\
&\le \frac{1}{2}B_2(\Theta^i\tilde{\omega})\|x_i(\eta,\tilde{\omega}) - x_i(\eta_0,\tilde{\omega})\|^2 \\
&\le \frac{9}{8}K^2(\tilde{\omega})e^{-2i\gamma_1(\tilde{\omega})}\|\eta-\eta_0\|^2.
\end{aligned}
$$

Then $\|I\|_{C^+_{\gamma_1(\tilde{\omega})}} = o(\|\eta-\eta_0\|)$ as $\eta \to \eta_0$. Thus,

$$
\begin{aligned}
&\mathbf{x}(\eta,\tilde{\omega}) - \mathbf{x}(\eta_0,\tilde{\omega}) - \mathcal{G}_{\eta_0}(\mathbf{x}(\eta,\tilde{\omega}) - \mathbf{x}(\eta_0,\tilde{\omega})) \\
&= \mathcal{S}(\eta-\eta_0) + I \\
&= \mathcal{S}(\eta-\eta_0) + o(\|\eta-\eta_0\|), \text{ as } \eta \to \eta_0,
\end{aligned}
\tag{115}
$$

which yields

$$
\mathbf{x}(\eta,\tilde{\omega}) - \mathbf{x}(\eta_0,\tilde{\omega}) = (Id - \mathcal{G}_{\eta_0})^{-1}\mathcal{S}(\eta-\eta_0) + o(\|\eta-\eta_0\|).
$$

Hence, $\mathbf{x}(\eta,\tilde{\omega})$ is differentiable in η and its derivative satisfies

$$
D\mathbf{x}(\eta,\tilde{\omega}) \in L(E^s(\tilde{\omega}), C^+_{\gamma_1(\tilde{\omega})})
$$

and

$$\|D\mathbf{x}(\eta,\tilde{\omega})\|_{L(E^s(\tilde{\omega}),C^+_{\gamma_1(\tilde{\omega})})} \leq \|(Id-\mathcal{G}_{\eta_0})^{-1}\|_{L(C^+_{\gamma(\tilde{\omega})},C^+_{\gamma(\tilde{\omega})})}\|\mathcal{S}\|_{L(E^s(\tilde{\omega}),C^+_{\gamma(\tilde{\omega})})} \leq \frac{3}{2}K(\tilde{\omega}). \tag{116}$$

Furthermore, we can obtain

$$\begin{aligned} Dx_n(\eta,\tilde{\omega}) &= \sum_{i=0}^{n-1}\Phi_s(n-1-i,\Theta^{i+1}\tilde{\omega})DF_s(\Theta^i\tilde{\omega},x_i(\eta,\tilde{\omega}))Dx_i(\eta,\tilde{\omega}) \\ &\qquad -\sum_{i=n}^{+\infty}\Phi_u(n-1-i,\Theta^{i+1}\tilde{\omega})DF_u(\Theta^i\tilde{\omega},x_i(\eta,\tilde{\omega}))Dx_i(\eta,\tilde{\omega}), \text{ for } n\geq 1; \\ Dx_0(\eta,\tilde{\omega}) &= -\sum_{i=0}^{+\infty}\Phi_u(-1-i,\Theta^{i+1}\tilde{\omega})DF_u(\Theta^i\tilde{\omega},x_i(\eta,\tilde{\omega}))Dx_i(\eta,\tilde{\omega}). \end{aligned} \tag{117}$$

By using (116), we have that for any $v\in E^s(\tilde{\omega})$ and $n>0$

$$\begin{aligned} &\|Dx_n(\eta,\tilde{\omega})v\|e^{n\gamma_1(\tilde{\omega})} \\ &\leq \sum_{i=0}^{n-1}\|\Phi_s(n-1-i,\Theta^{i+1}\tilde{\omega})\|\|DF_s(\Theta^i\tilde{\omega},x_i(\eta,\tilde{\omega}))\|\|Dx_i(\eta,\tilde{\omega})v\|e^{n\gamma_1(\tilde{\omega})} \\ &\quad +\sum_{i=n}^{+\infty}\|\Phi_u(n-1-i,\Theta^{i+1}\tilde{\omega})\|\|DF_u(\Theta^i\tilde{\omega},x_i(\eta,\tilde{\omega}))\|\|Dx_i(\eta,\tilde{\omega})v\|e^{n\gamma_1(\tilde{\omega})} \\ &\leq \sum_{i=0}^{n-1}K(\Theta^{i+1}\tilde{\omega})e^{-(n-1-i)\beta(\tilde{\omega})}B_2(\Theta^i\tilde{\omega})\rho(\Theta^i\tilde{\omega})\|D\mathbf{x}(\eta,\tilde{\omega})\|_{L(E^s(\tilde{\omega}),C^+_{\gamma_1(\tilde{\omega})})}\|v\|e^{(n-i)\gamma_1(\tilde{\omega})} \\ &\quad +\sum_{i=n}^{+\infty}K(\Theta^{i+1}\tilde{\omega})e^{-(n-1-i)\alpha(\tilde{\omega})}B_2(\Theta^i\tilde{\omega})\rho(\Theta^i\tilde{\omega})\|D\mathbf{x}(\eta,\tilde{\omega})\|_{L(E^s(\tilde{\omega}),C^+_{\gamma_1(\tilde{\omega})})}\|v\|e^{(n-i)\gamma_1(\tilde{\omega})} \\ &\leq L(\tilde{\omega})\left(\frac{e^{\gamma_1(\tilde{\omega})}}{1-e^{\gamma_1(\tilde{\omega})-\beta(\tilde{\omega})}}+\frac{e^{\alpha(\tilde{\omega})}}{1-e^{-(\gamma_1(\tilde{\omega})-\alpha(\tilde{\omega}))}}\right)\|D\mathbf{x}(\eta,\tilde{\omega})\|_{L(E^s(\tilde{\omega}),C^+_{\gamma_1(\tilde{\omega})})}\|v\| \\ &\leq \frac{1}{2}K(\tilde{\omega})\|v\|. \end{aligned}$$

Similarly,

$$\|Dx_0(\eta,\tilde{\omega})v\| \leq \frac{1}{2}K(\tilde{\omega})\|v\|.$$

Thus for $\eta \in E^s(\tilde{\omega}) \cap B(0, \frac{R(\tilde{\omega})}{3K(\tilde{\omega})})$

$$\|D\mathbf{x}(\eta, \tilde{\omega})\|_{C^+_{\gamma_1(\tilde{\omega})}} \leq \frac{1}{2} K(\tilde{\omega}). \tag{118}$$

Using (113), we obtain that for $\eta_1, \eta_2 \in E^s(\tilde{\omega}) \cap B(0, \frac{R(\tilde{\omega})}{6K(\tilde{\omega})})$ and $i \geq 0$

$$\begin{aligned}
&\|DF_s(\Theta^i\tilde{\omega}, x_i(\eta_1, \tilde{\omega}))Dx_i(\eta_1, \tilde{\omega}) - DF_s(\Theta^i\tilde{\omega}, x_i(\eta_2, \tilde{\omega}))Dx_i(\eta_2, \tilde{\omega})\| \\
&\leq \|(DF_s(\Theta^i\tilde{\omega}, x_i(\eta_1, \tilde{\omega})) - DF_s(\Theta^i\tilde{\omega}, x_i(\eta_2, \tilde{\omega})))Dx_i(\eta_1, \tilde{\omega})\| \\
&\quad + \|DF_s(\Theta^i\tilde{\omega}, x_i(\eta_2, \tilde{\omega})))(Dx_i(\eta_1, \tilde{\omega}) - Dx_i(\eta_2, \tilde{\omega})\| \\
&\leq B_2(\Theta^i\tilde{\omega})\|x_i(\eta_1, \tilde{\omega}) - x_i(\eta_2, \tilde{\omega})\|\|Dx_i(\eta_1, \tilde{\omega})\| \\
&\quad + B_2(\Theta^i\tilde{\omega})\rho(\Theta^i\tilde{\omega})\|(Dx_i(\eta_1, \tilde{\omega}) - Dx_i(\eta_2, \tilde{\omega})\| \\
&\leq \frac{3}{4}K^2(\tilde{\omega})B_2(\Theta^i\tilde{\omega})e^{-2i\gamma_1(\tilde{\omega})}\|\eta_1 - \eta_2\| \\
&\quad + B_2(\Theta^i\tilde{\omega})\rho(\Theta^i\tilde{\omega})e^{-i\gamma(\tilde{\omega})}\|D\mathbf{x}(\eta_1, \tilde{\omega}) - D\mathbf{x}(\eta_2, \tilde{\omega})\|_{C^+_{\gamma_1(\tilde{\omega})}}.
\end{aligned}$$

Thus, for any $n \geq 0$

$$
\begin{aligned}
&\|Dx_n(\eta_1,\tilde{\omega}) - Dx_n(\eta_2,\tilde{\omega})\|e^{n\gamma_1(\tilde{\omega})} \\
&\leq \sum_{n=0}^{n-1} \|\Phi_s(n-1-i,\Theta^{i+1}\tilde{\omega})\| \cdot \\
&\quad \|DF_s(\Theta^i\tilde{\omega}, x_i(\eta_1,\tilde{\omega}))Dx_i(\eta_1,\tilde{\omega}) - DF_s(\Theta^i\tilde{\omega}, x_i(\eta_2,\tilde{\omega}))Dx_i(\eta_2,\tilde{\omega})\|e^{n\gamma_1(\tilde{\omega})} \\
&\quad + \sum_{i=n}^{+\infty} \|\Phi_u(n-1-i,\Theta^{i+1}\tilde{\omega})\| \cdot \\
&\quad \|DF_s(\Theta^i\tilde{\omega}, x_i(\eta_1,\tilde{\omega}))Dx_i(\eta_1,\tilde{\omega}) - DF_s(\Theta^i\tilde{\omega}, x_i(\eta_2,\tilde{\omega}))Dx_i(\eta_2,\tilde{\omega})\|e^{n\gamma_1(\tilde{\omega})} \\
&\leq \sum_{n=0}^{n-1} K(\Theta^{i+1})e^{-(n-i-1)\beta(\tilde{\omega})}\frac{3}{4}K^2(\tilde{\omega})B_2(\Theta^i\tilde{\omega})e^{-2i\gamma_1(\tilde{\omega})}\|\eta_1-\eta_2\|e^{n\gamma_1(\tilde{\omega})} \\
&\quad + \sum_{n=0}^{n-1} K(\Theta^{i+1})e^{-(n-i-1)\beta(\tilde{\omega})} \cdot \\
&\quad B_2(\Theta^i\tilde{\omega})\rho(\Theta^i\tilde{\omega})e^{-i\gamma_1(\tilde{\omega})}\|D\mathbf{x}(\eta_1,\tilde{\omega}) - D\mathbf{x}(\eta_2,\tilde{\omega})\|_{C^+_{\gamma_1(\tilde{\omega})}}e^{n\gamma_1(\tilde{\omega})} \\
&\quad + \sum_{i=n}^{+\infty} K(\Theta^{i+1})e^{-(n-i-1)\alpha(\tilde{\omega})}\frac{3}{4}K^2(\tilde{\omega})B_2(\Theta^i\tilde{\omega})e^{-2i\gamma_1(\tilde{\omega})}\|\eta_1-\eta_2\|e^{n\gamma_1(\tilde{\omega})} \\
&\quad + \sum_{i=n}^{+\infty} K(\Theta^{i+1})e^{-(n-i-1)\alpha(\tilde{\omega})} \cdot \\
&\quad B_2(\Theta^i\tilde{\omega})\rho(\Theta^i\tilde{\omega})e^{-i\gamma_1(\tilde{\omega})}\|D\mathbf{x}(\eta_1,\tilde{\omega}) - D\mathbf{x}(\eta_2,\tilde{\omega})\|_{C^+_{\gamma_1(\tilde{\omega})}}e^{n\gamma_1(\tilde{\omega})} \\
&\leq \frac{1}{3}\|D\mathbf{x}(\eta_1,\tilde{\omega}) - D\mathbf{x}(\eta_2,\tilde{\omega})\|_{C^+_{\gamma_1(\tilde{\omega})}} + K_1'(\tilde{\omega})\|\eta_1-\eta_2\|,
\end{aligned}
$$

where

$$
\begin{aligned}
K_1'(\tilde{\omega}) = &\sum_{i=0}^{n-1} \frac{3}{4}K^2(\tilde{\omega})K(\Theta^{i+1})B_2(\Theta^i\tilde{\omega})e^{-(n-i-1)\beta(\tilde{\omega})}e^{(n-2i)\gamma_1(\tilde{\omega})} \\
&+ \sum_{i=n}^{+\infty} \frac{3}{4}K^2(\tilde{\omega})K(\Theta^{i+1})B_2(\Theta^i\tilde{\omega})e^{-(n-i-1)\alpha(\tilde{\omega})}e^{(n-2i)\gamma_1(\tilde{\omega})}.
\end{aligned}
$$

Then

$$
\|D\mathbf{x}(\eta_1,\tilde{\omega}) - D\mathbf{x}(\eta_2,\tilde{\omega})\|_{C^+_{\gamma_1(\tilde{\omega})}} \leq \frac{3}{2}K_1'(\tilde{\omega})\|\eta_1-\eta_2\|, \tag{119}
$$

where $\eta, \eta_0 \in E^s(\tilde{\omega}) \cap B(0, \frac{R(\tilde{\omega})}{6K(\tilde{\omega})})$. By the definition of $\rho(\tilde{\omega})$, $L(\tilde{\omega})$ and $R(\tilde{\omega})$, we have that

$$B_2(\Theta^i\tilde{\omega})K(\Theta^{i+1}\tilde{\omega}) \leq \frac{L(\tilde{\omega})}{\rho(\Theta^i\tilde{\omega})} \leq \frac{L(\tilde{\omega})e^{i\gamma_1(\tilde{\omega})}}{2R(\tilde{\omega})}.$$

Then

$$1 \leq K'(\tilde{\omega}) \leq \frac{K^2(\tilde{\omega})}{8R(\tilde{\omega})}.$$

Thus $K'(\tilde{\omega})$ is tempered.

Step 3. $\mathbf{x}(\eta, \tilde{\omega})$ is C^N.

Let $2 \leq m \leq N$. By the induction hypothesis, we have that $\mathbf{x}(\cdot, \tilde{\omega})$ is C^j from $E^s(\tilde{\omega}) \cap B(0, \frac{R(\tilde{\omega}))}{3K(\tilde{\omega})})$ to $C^+_{\gamma_j(\tilde{\omega})}$ for all $1 \leq j \leq m-1$ and there exists tempered functions $K_j(\tilde{\omega})$ such that

$$\|D^j\mathbf{x}(\eta, \tilde{\omega})\|_{C^{j+}_{\gamma_j(\tilde{\omega})}} \leq K_j(\tilde{\omega}), \tag{120}$$

$$\|D^j\mathbf{x}(\eta, \tilde{\omega})\|_{L(E^s(\tilde{\omega}), C^{j-1+}_{\gamma_j(\tilde{\omega})})} \leq K_j(\tilde{\omega}), \tag{121}$$

$$\|D^j\mathbf{x}(\eta, \tilde{\omega}) - D^j\mathbf{x}(\eta_0, \tilde{\omega})\|_{C^{j+}_{\gamma_j(\tilde{\omega})}} \leq K_j(\tilde{\omega})\|\eta - \eta_0\|, \tag{122}$$

where $\eta, \eta_0 \in E^s(\tilde{\omega}) \cap B(0, \frac{R(\tilde{\omega})}{6K(\tilde{\omega})})$. Here $L(E^s(\tilde{\omega}), C^{j-1+}_{\gamma_j(\tilde{\omega})})$ is Banach space of bounded linear operators from $E^s(\tilde{\omega})$ to $C^{j-1+}_{\gamma_j(\tilde{\omega})}$. We want to show that $\mathbf{x}(\cdot, \tilde{\omega})$ is C^m and $D^m\mathbf{x}(\cdot, \tilde{\omega})$ is Lipschitz continuous from $E^s(\tilde{\omega}) \cap B(0, \frac{R(\tilde{\omega})}{6K(\tilde{\omega})})$ to $C^{m+}_{\gamma_m(\tilde{\omega})}$ when $m < N$.

Note that $D^{m-1}\mathbf{x}(\cdot,\tilde{\omega})$ satisfies the following equations

$$
\begin{aligned}
&D^{m-1}x_n(\eta,\tilde{\omega}) \\
&= \sum_{i=0}^{n-1}\Phi_s(n-1-i,\Theta^{i+1}\tilde{\omega})DF_s(\Theta^i\tilde{\omega},x_i(\eta,\tilde{\omega}))D^{m-1}x_i(\eta,\tilde{\omega}) \\
&\quad -\sum_{i=n}^{+\infty}\Phi_u(n-1-i,\Theta^{i+1}\tilde{\omega})DF_u(\Theta^i\tilde{\omega},x_i(\eta,\tilde{\omega}))D^{m-1}x_i(\eta,\tilde{\omega}) \\
&+\sum_{i=0}^{n-1}\Phi_s(n-1-i,\Theta^{i+1}\tilde{\omega})R^s_{m-1,i}(\eta,\tilde{\omega}) \\
&\quad -\sum_{i=n}^{+\infty}\Phi_u(n-1-i,\Theta^{i+1}\tilde{\omega})R^u_{m-1,i}(\eta,\tilde{\omega}), \quad \text{for } \ n\geq 1;
\end{aligned}
\tag{123}
$$

$$
\begin{aligned}
&D^{m-1}x_0(\eta,\tilde{\omega}) \\
&= -\sum_{i=n}^{+\infty}\Phi_u(-1-i,\Theta^{i+1}\tilde{\omega})DF_u(\Theta^i\tilde{\omega},x_i(\eta,\tilde{\omega}))D^{m-1}x_i(\eta,\tilde{\omega}) \\
&\quad -\sum_{i=n}^{+\infty}\Phi_u(-1-i,\Theta^{i+1}\tilde{\omega})R^u_{m-1,i}(\eta,\tilde{\omega})
\end{aligned}
\tag{124}
$$

where

$$
R^\tau_{m-1,i}(\eta,\tilde{\omega}) = \sum_{l=0}^{m-3}\begin{pmatrix} m-2 \\ l \end{pmatrix} D_\eta^{m-2-l}\big(D_xF_\tau(\Theta^i\tilde{\omega},x_i(\eta,\tilde{\omega}))\big)D^{l+1}x_i(\eta,\tilde{\omega}),
$$

for $\tau=u,s$. Applying the chain rule to

$$
D_\eta^{m-2-l}\big(D_xF_\tau(\Theta^i\tilde{\omega},x_i(\eta,\tilde{\omega}))\big).
$$

Then each term in $R^\tau_{m-1,i}(\eta,\tilde{\omega})$ contains factors: $D_x^{l_1}F_\tau(\Theta^i\tilde{\omega},x_i(\eta,\tilde{\omega}))$ for some $2\leq l_1\leq m-2$ and at least two derivatives $D_\eta^{l_2}x_i(\eta,\tilde{\omega})$ and $D_\eta^{l_3}x_i(\eta,\tilde{\omega})$ for some $l_2,l_3\in\{1,\cdots,m-2\}$. Since $D_\eta^l\mathbf{x}(\eta,\tilde{\omega})\in C^{l+}_{\gamma_l(\tilde{\omega})}$ for $l=1,\cdots,m-1$ and F is C^N,

$R^s_{m-1,i}(\cdot,\tilde{\omega}) : E^s(\tilde{\omega})\cap B(0,\frac{R(\tilde{\omega})}{3K(\tilde{\omega})}) \to L^{m-1}\big(E^s(\tilde{\omega}), E^s(\Theta^i\tilde{\omega})\big)$ and $R^u_{m-1,i}(\cdot,\tilde{\omega}) : E^s(\tilde{\omega})\cap B(0,\frac{R(\tilde{\omega})}{3K(\tilde{\omega})}) \to L^{m-1}\big(E^s(\tilde{\omega}), E^u(\Theta^i\tilde{\omega})\big)$ are C^1. Furthermore, by using (120) and (122), we have that for $\tau = u, s$

$$||R^\tau_{m-1,i}(\eta,\tilde{\omega})|| \leq \tilde{B}_{m-1}(\Theta^i\tilde{\omega})\tilde{K}_{m-2}(\tilde{\omega})e^{-2i\gamma_{m-2}(\tilde{\omega})} \tag{125}$$

$$||R^\tau_{m-1,i}(\eta,\tilde{\omega}) - R^\tau_{m-1,i}(\eta_0,\tilde{\omega})|| \leq \tilde{B}_m(\Theta^i\tilde{\omega})\tilde{K}_{m-2}(\tilde{\omega})e^{-2i\gamma_{m-2}(\tilde{\omega})}\|\eta-\eta_0\| \tag{126}$$

$$||D_\eta R^\tau_{m-1,i}(\eta,\tilde{\omega})|| \leq \tilde{B}_m(\Theta^i\tilde{\omega})\tilde{K}_{m-1}(\tilde{\omega})e^{-2i\gamma_{m-1}(\tilde{\omega})} \tag{127}$$

where $\tilde{B}_m(\tilde{\omega}) = \max_{1\leq i\leq m} B_i(\tilde{\omega})$ and $\tilde{K}_m(\tilde{\omega})$ is m-th order polynomial of $K_1(\tilde{\omega}),\cdots,$ $K_m(\tilde{\omega})$ with positive integer coefficients. Thus, $\tilde{B}_m(\Theta^i\tilde{\omega})\tilde{K}_{m-1}(\tilde{\omega})$ is tempered. If $m < N$, we have that for $\tau = u, s$

$$||D_\eta R^\tau_{m-1,i}(\eta,\tilde{\omega}) - D_\eta R^\tau_{m-1,i}(\eta_0,\tilde{\omega})|| \leq \tilde{B}_{m+1}(\Theta^i\tilde{\omega})\tilde{K}_m(\tilde{\omega})e^{-2i\gamma_{m-1}(\tilde{\omega})}\|\eta-\eta_0\|. \tag{128}$$

Set $J(\eta,\tilde{\omega}) = \{J_n(\eta,\tilde{\omega})\}_{n\geq 0}$ with

$$\begin{aligned}
J_n(\eta,\tilde{\omega}) &= \sum_{i=0}^{n-1}\Phi_s(n-1-i,\Theta^{i+1}\tilde{\omega})R^s_{m-1,i}(\eta,\tilde{\omega}) \\
&\qquad - \sum_{i=n}^{+\infty}\Phi_u(n-1-i,\Theta^{i+1}\tilde{\omega})R^u_{m-1,i}(\eta,\tilde{\omega}), \quad \text{for } \; n\geq 1, \\
J_0(\eta,\tilde{\omega}) &= -\sum_{i=n}^{+\infty}\Phi_u(-1-i,\Theta^{i+1}\tilde{\omega})R^u_{m-1,i}(\eta,\tilde{\omega}).
\end{aligned}$$

Note that for $0 \le i \le n-1$

$$
\begin{aligned}
&||\Phi_s(n-1-i, \Theta^{i+1}\tilde{\omega}) D_\eta R^s_{m-1,i}(\eta, \tilde{\omega})|| \\
&\le K(\Theta^{i+1}\tilde{\omega}) \tilde{B}_{m+1}(\Theta^i\tilde{\omega}) \tilde{K}_{m+1}(\tilde{\omega}) e^{-\beta(\tilde{\omega})(n-1-i)} e^{-2i\gamma_{m-1}(\tilde{\omega})} \qquad (129) \\
&= \Big(K(\Theta^{i+1}\tilde{\omega}) \tilde{B}_{m+1}(\Theta^i\tilde{\omega}) e^{-i\gamma_{m-1}(\tilde{\omega})} \Big) \tilde{K}_{m+1}(\tilde{\omega}) e^{\beta(\tilde{\omega})(n-1-i)} e^{-i\gamma_{m-1}(\tilde{\omega})},
\end{aligned}
$$

where $K(\Theta^{i+1}\tilde{\omega}) \tilde{B}_{m+1}(\Theta^i\tilde{\omega}) \} e^{-i\gamma_{m-1}(\tilde{\omega})}$ is bounded by a tempered function $K^*(\tilde{\omega})$.

Similarly, for $i \ge n$

$$
||\Phi_u(n-1-i, \Theta^{i+1}\tilde{\omega}) D_\eta R^u_{m-1,i}(\eta, \tilde{\omega})|| \le K^*(\tilde{\omega}) \tilde{K}_{m+1}(\tilde{\omega}) e^{-\alpha(\tilde{\omega})(n-1-i)} e^{-i\gamma_{m-1}(\tilde{\omega})}. \qquad (130)
$$

Estimates (125),(126),(127), (130), (129) yield that for all $n \ge 0$ $J_n(\eta, \tilde{\omega})$ in η is C^1 and

$$
\|D_\eta J_n(\eta, \tilde{\omega})\| e^{n\gamma_{m-1}(\tilde{\omega})} \le \hat{K}(\tilde{\omega}), \qquad (131)
$$

where $\hat{K}(\tilde{\omega})$ is tempered.

We consider a linear operator $\mathcal{G}_\eta$ from $C^{m-1+}_{\gamma_{m-1}(\tilde{\omega})}$ into itself that is given by (114) with $\mathbf{v} \in L^{m-1}(E^s(\tilde{\omega}), C^+_{\gamma_{m-1}(\tilde{\omega})})$ and set $I^m = \{I^m_n\}_{n \ge 0}$ with

$$
\begin{aligned}
&I^m_n \\
&= \sum_{i=0}^{n-1} \Phi_s(n-1-i, \Theta^{i+1}\tilde{\omega}) \Big(DF_s(\Theta^i\tilde{\omega}, x_i(\eta, \tilde{\omega})) - DF_s(\Theta^i\tilde{\omega}, x_i(\eta_0, \tilde{\omega})) \Big) D^{m-1} x_i(\eta, \tilde{\omega}) \\
&- \sum_{i=n}^{+\infty} \Phi_u(n-1-i, \Theta^{i+1}\tilde{\omega}) \Big(DF_u(\Theta^i\tilde{\omega}, x_i(\eta, \tilde{\omega})) - DF_u(\Theta^i\tilde{\omega}, x_i(\eta_0, \tilde{\omega})) \Big) D^{m-1} x_i(\eta, \tilde{\omega}) \\
&I^m_0 \\
&= -\sum_{i=n}^{+\infty} \Phi_u(-1-i, \Theta^{i+1}\tilde{\omega}) \Big(DF_u(\Theta^i\tilde{\omega}, x_i(\eta, \tilde{\omega})) - DF_u(\Theta^i\tilde{\omega}, x_i(\eta_0, \tilde{\omega})) \Big) D^{m-1} x_i(\eta, \tilde{\omega}).
\end{aligned}
$$

We also set $H^m(\eta-\eta_0) = \{H^m_n(\eta-\eta_0)\}_{n\geq 0}$ with

$$
\begin{aligned}
&H^m_n(\eta-\eta_0)\\
&=\sum_{i=0}^{n-1}\Phi_s(n-1-i,\Theta^{i+1}\tilde{\omega})D^2F_s(\Theta^i\tilde{\omega},x_i(\eta_0,\tilde{\omega}))Dx_i(\eta_0,\tilde{\omega})(\eta-\eta_0)D^{m-1}x_i(\eta_0,\tilde{\omega})\\
&-\sum_{i=n}^{+\infty}\Phi_u(n-1-i,\Theta^{i+1}\tilde{\omega})D^2F_u(\Theta^i\tilde{\omega},x_i(\eta_0,\tilde{\omega}))Dx_i(\eta_0,\tilde{\omega})(\eta-\eta_0)D^{m-1}x_i(\eta_0,\tilde{\omega})\\
&H^m_0(\eta-\eta_0)\\
&=-\sum_{i=n}^{+\infty}\Phi_u(-1-i,\Theta^{i+1}\tilde{\omega})D^2F_u(\Theta^i\tilde{\omega},x_i(\eta_0,\tilde{\omega}))Dx_i(\eta_0,\tilde{\omega})(\eta-\eta_0)D^{m-1}x_i(\eta_0,\tilde{\omega}).
\end{aligned}
$$

Obviously, H^m depends on $\tilde{\omega}$ and is a bounded linear operator from $E^s(\tilde{\omega})$ to $L^{m-1}(E^s(\tilde{\omega}), C^+_{\gamma_{m-1}(\tilde{\omega})})$ with the norm bounded by a tempered function, denoted by $\hat{K}'(\tilde{\omega})$. Then, equation (143) can be written as

$$
\begin{aligned}
&\big(D^{m-1}\mathbf{x}(\eta,\tilde{\omega})-D^{m-1}\mathbf{x}(\eta_0,\tilde{\omega})\big)-\mathcal{G}_{\eta_0}\big(D^{m-1}\mathbf{x}(\eta,\tilde{\omega})-D^{m-1}\mathbf{x}(\eta_0,\tilde{\omega})\big)\\
&=J(\eta,\tilde{\omega})-J(\eta_0,\tilde{\omega})+I^m.
\end{aligned}
\tag{132}
$$

We note that for $n>0$

$$
\begin{aligned}
&\|(\mathcal{G}_\eta\mathbf{v})_n\|e^{n\gamma_{m-1}(\tilde{\omega})}\\
&\leq\sum_{i=0}^{n-1}K(\Theta^{i+1}\tilde{\omega})e^{-(n-1-i)\beta(\tilde{\omega})}B_2(\Theta^i\tilde{\omega})\rho(\Theta^i\tilde{\omega})\|\mathbf{v}\|_{C^{m-1+}_{\gamma_{m-1}(\tilde{\omega})}}e^{(n-i)\gamma_{m-1}(\tilde{\omega})}\\
&\quad+\sum_{i=n}^{+\infty}K(\Theta^{i+1}\tilde{\omega})e^{-(n-1-i)\alpha(\tilde{\omega})}B_2(\Theta^i\tilde{\omega})\rho(\Theta^i\tilde{\omega})\|\mathbf{v}\|_{C^{m-1+}_{\gamma_{m-1}(\tilde{\omega})}}e^{(n-i)\gamma_{m-1}(\tilde{\omega})}\\
&\leq L(\tilde{\omega})\left(\frac{e^{\gamma_{m-1}(\tilde{\omega})}}{1-e^{\gamma_{m-1}(\tilde{\omega})-\beta(\tilde{\omega})}}+\frac{e^{\alpha(\tilde{\omega})}}{1-e^{-(\gamma_{m-1}(\tilde{\omega})-\alpha(\tilde{\omega}))}}\right)\|\mathbf{v}\|_{C^{m-1+}_{\gamma_{m-1}(\tilde{\omega})}}\\
&\leq\frac{1}{3}\|\mathbf{v}\|_{C^{m-1+}_{\gamma_{m-1}(\tilde{\omega})}}.
\end{aligned}
$$

Similarly, we have that

$$\|(\mathcal{G}_\eta \mathbf{v})_0\| \leq \frac{1}{3}\|\mathbf{v}\|_{C^{m-1+}_{\gamma_{m-1}(\tilde{\omega})}}.$$

Thus, $\mathcal{G}_\eta$ is a bounded linear operator from $C^{m-1+}_{\gamma_{m-1}(\tilde{\omega})}$ into itself and its norm is bounded by $\frac{1}{3}$, which implies that $(Id-\mathcal{G}_\eta)$ has a bounded inverse in $L(C^{m-1+}_{\gamma_{m-1}(\tilde{\omega})}, C^{m-1+}_{\gamma_{m-1}(\tilde{\omega})})$. Thus,

$$\begin{aligned}
&D^{m-1}\mathbf{x}(\eta,\tilde{\omega}) - D^{m-1}\mathbf{x}(\eta_0,\tilde{\omega}) \\
&= \big(Id-\mathcal{G}\big)^{-1}(DJ(\eta_0,\tilde{\omega}) + H^m)(\eta-\eta_0) \\
&+ \big(Id-\mathcal{G}\big)^{-1}\big(J(\eta,\tilde{\omega}) - J(\eta_0,\tilde{\omega}) - DJ(\eta_0,\tilde{\omega})(\eta-\eta_0)\big) \\
&+ \big(Id-\mathcal{G}\big)^{-1}\big(I^m - H^m(\eta-\eta_0)\big).
\end{aligned} \tag{133}$$

First, we will show that

$$\|I^m - H^m(\eta-\eta_0)\|_{C^{m+}_{\gamma_m(\tilde{\omega})}} = o(\|\eta-\eta_0\|), \text{ as } \eta \to \eta_0. \tag{134}$$

For $n \geq 1$, by using (122), (120) and straight forward computations, we have that

$$\begin{aligned}
&\|I^m_n - H^m_n(\eta-\eta_0)\|e^{n\gamma_m(\tilde{\omega})} \\
&\leq \sum_{i=0}^{n-1} K(\Theta^{i+1}\tilde{\omega})e^{-(n-1-i)\beta(\tilde{\omega})}K_{m-1}(\tilde{\omega})e^{-i\gamma_{m-1}(\tilde{\omega})}e^{n\gamma_m(\tilde{\omega})}\|\Delta^2 DF_s(\eta_0,\eta,\Theta^i\tilde{\omega})\| \\
&+ \sum_{i=n}^{+\infty} K(\Theta^{i+1}\tilde{\omega})e^{-(n-1-i)\alpha(\tilde{\omega})}K_{m-1}(\tilde{\omega})e^{-i\gamma_{m-1}(\tilde{\omega})}e^{n\gamma_m(\tilde{\omega})}\|\Delta^2 DF_u(\eta_0,\eta,\Theta^i\tilde{\omega})\| \\
&+ D(\tilde{\omega})\|\eta-\eta_0\|^2,
\end{aligned} \tag{135}$$

where

$$\begin{aligned}
&\Delta^2 DF_s(\eta_0,\eta,\Theta^i\tilde{\omega}) \\
&= DF_s(\Theta^i\tilde{\omega}, x_i(\eta,\tilde{\omega})) - DF_s(\Theta^i\tilde{\omega}, x_i(\eta_0,\tilde{\omega})) - D^2F_s(\Theta^i\tilde{\omega}, x_i(\eta_0,\tilde{\omega}))Dx_i(\eta_0,\tilde{\omega})(\eta-\eta_0),
\end{aligned}$$

$$\Delta^2 DF_u(\eta_0, \eta, \Theta^i\tilde{\omega})$$
$$= DF_u(\Theta^i\tilde{\omega}, x_i(\eta, \tilde{\omega})) - DF_u(\Theta^i\tilde{\omega}, x_i(\eta_0, \tilde{\omega})) - D^2F_u(\Theta^i\tilde{\omega}, x_i(\eta_0, \tilde{\omega}))Dx_i(\eta_0, \tilde{\omega})(\eta - \eta_0),$$

and

$$\begin{aligned}
&D(\tilde{\omega}) \\
&= \sup\Bigg\{\sum_{i=0}^{n-1} K(\Theta^{i+1}\tilde{\omega})e^{-(n-1-i)\beta(\tilde{\omega})}K_{m-1}(\tilde{\omega})e^{-i\gamma_{m-1}(\tilde{\omega})}e^{n\gamma_m(\tilde{\omega})}B_2(\Theta^i\tilde{\omega})K_1(\tilde{\omega})e^{-i\gamma_1(\tilde{\omega})} \\
&\quad + \sum_{i=n}^{+\infty} K(\Theta^{i+1}\tilde{\omega})e^{-(n-1-i)\alpha(\tilde{\omega})}K_{m-1}(\tilde{\omega})e^{-i\gamma_{m-1}(\tilde{\omega})}e^{n\gamma_m(\tilde{\omega})}B_2(\Theta^i\tilde{\omega})K_1(\tilde{\omega})e^{-i\gamma_1(\tilde{\omega})}\Bigg\}_{n\geq 0} \\
&\leq \sup\left\{\frac{K_{m-1}(\tilde{\omega})K_1(\tilde{\omega})e^{-i\gamma_i(\tilde{\omega})}}{3\rho(\Theta^i\tilde{\omega})}\right\}_{i\geq 0}.
\end{aligned}$$

Furthermore, $D(\tilde{\omega})$ is tempered and does not depend on n.

Also we have that for $i \geq 0$

$$\begin{aligned}
&\|\Delta^2 DF_s(\eta_0, \eta, \Theta^i\tilde{\omega})\| \\
&\leq \max_{0\leq t\leq 1}\|D^2F_s(\Theta^i\tilde{\omega}, tx_i(\eta, \tilde{\omega}) + (1-t)x_i(\eta_0, \tilde{\omega})) - D^2F_s(\Theta^i\tilde{\omega}, x_i(\eta_0, \tilde{\omega}))\| \\
&\quad \|x_i(\eta, \tilde{\omega}) - x_i(\eta_0, \tilde{\omega})\| + \|D^2F_s(\Theta^i\tilde{\omega}, x_i(\eta_0, \tilde{\omega}))\|K_1(\tilde{\omega})e^{-i\gamma_1(\tilde{\omega})}\|\eta - \eta_0\|^2 \qquad (136) \\
&\leq \max_{0\leq t\leq 1}\|D^2F_s(\Theta^i\tilde{\omega}, tx_i(\eta, \tilde{\omega}) + (1-t)x_i(\eta_0, \tilde{\omega})) - D^2F_s(\Theta^i\tilde{\omega}, x_i(\eta_0, \tilde{\omega}))\| \\
&\quad K_1(\tilde{\omega})e^{-i\gamma_1(\tilde{\omega})}\|\eta - \eta_0\| + \|D^2F_s(\Theta^i\tilde{\omega}, x_i(\eta_0, \tilde{\omega}))\|K_1(\tilde{\omega})e^{-i\gamma_1(\tilde{\omega})}\|\eta - \eta_0\|^2,
\end{aligned}$$

and

$$
\begin{aligned}
&\|\Delta^2 DF_u(\eta_0,\eta,\Theta^i\tilde{\omega})\| \\
&\le \max_{0\le t\le 1}\|D^2F_u(\Theta^i\tilde{\omega},tx_i(\eta,\tilde{\omega})+(1-t)x_i(\eta_0,\tilde{\omega}))-D^2F_u(\Theta^i\tilde{\omega},x_i(\eta_0,\tilde{\omega}))\| \\
&\quad \|x_i(\eta,\tilde{\omega})-x_i(\eta_0,\tilde{\omega})\|+\|D^2F_u(\Theta^i\tilde{\omega},x_i(\eta_0,\tilde{\omega}))\|K_1(\tilde{\omega})e^{-i\gamma_1(\tilde{\omega})}\|\eta-\eta_0\|^2 \qquad (137)\\
&\le \max_{0\le t\le 1}\|D^2F_u(\Theta^i\tilde{\omega},tx_i(\eta,\tilde{\omega})+(1-t)x_i(\eta_0,\tilde{\omega}))-D^2F_u(\Theta^i\tilde{\omega},x_i(\eta_0,\tilde{\omega})) \\
&\quad \|K_1(\tilde{\omega})e^{-i\gamma_1(\tilde{\omega})}\|\eta-\eta_0\|+\|D^2F_u(\Theta^i\tilde{\omega},x_i(\eta_0,\tilde{\omega}))\|K_1(\tilde{\omega})e^{-i\gamma_1(\tilde{\omega})}\|\eta-\eta_0\|^2.
\end{aligned}
$$

Note that for any $n-1\ge N$

$$
\begin{aligned}
&\sum_{i=N}^{n-1} 2B_2(\Theta^i\tilde{\omega})K(\Theta^{i+1}\tilde{\omega})e^{-(n-1-i)\beta(\tilde{\omega})}K_{m-1}(\tilde{\omega})e^{-i\gamma_{m-1}(\tilde{\omega})}e^{n\gamma_m(\tilde{\omega})}K_1(\tilde{\omega})e^{-i\gamma_1(\tilde{\omega})} \\
&\le \sup\left\{\frac{2K_{m-1}(\tilde{\omega})K_1(\tilde{\omega})e^{-i\gamma_i(\tilde{\omega})}}{3\rho(\Theta^i\tilde{\omega})}\right\}_{i\ge N},
\end{aligned}
$$

and for any $n\le N$

$$
\begin{aligned}
&\sum_{i=N}^{+\infty} 2B_2(\Theta^i\tilde{\omega})K(\Theta^{i+1}\tilde{\omega})e^{-(n-1-i)\alpha(\tilde{\omega})}K_{m-1}(\tilde{\omega})e^{-i\gamma_{m-1}(\tilde{\omega})}e^{n\gamma_m(\tilde{\omega})}K_1(\tilde{\omega})e^{-i\gamma_1(\tilde{\omega})} \\
&\le \sup\left\{\frac{2K_{m-1}(\tilde{\omega})K_1(\tilde{\omega})e^{-i\gamma_i(\tilde{\omega})}}{3\rho(\Theta^i\tilde{\omega})}\right\}_{i\ge N}.
\end{aligned}
$$

As long as $\rho(\tilde{\omega})$ is tempered, we have that for any $\epsilon>0$, there exists a $N(\epsilon,\tilde{\omega})>0$ such that for any $n-1\ge N(\epsilon,\tilde{\omega})$

$$
\sum_{i=N(\epsilon,\tilde{\omega})}^{n-1} 2B_2(\Theta^i\tilde{\omega})K(\Theta^{i+1}\tilde{\omega})e^{-(n-1-i)\beta(\tilde{\omega})}K_{m-1}(\tilde{\omega})e^{-i\gamma_{m-1}(\tilde{\omega})}e^{n\gamma_m(\tilde{\omega})}K_1(\tilde{\omega})e^{-i\gamma_1(\tilde{\omega})}\le\frac{1}{2}\epsilon \qquad (138)
$$

and for any $n\le N(\epsilon,\tilde{\omega})$

$$
\sum_{i=N(\epsilon,\tilde{\omega})}^{+\infty} 2B_2(\Theta^i\tilde{\omega})K(\Theta^{i+1}\tilde{\omega})e^{-(n-1-i)\alpha(\tilde{\omega})}K_{m-1}(\tilde{\omega})e^{-i\gamma_{m-1}(\tilde{\omega})}e^{n\gamma_m(\tilde{\omega})}K_1(\tilde{\omega})e^{-i\gamma_1(\tilde{\omega})}\le\frac{1}{2}\epsilon. \qquad (139)
$$

So for a fixed N and $\tau = s, u$

$$\lim_{\eta \to \eta_0} \max_{0 \le i \le N} \max_{0 \le t \le 1} \|D^2 F_\tau(\Theta^i \tilde{\omega}, t x_i(\eta, \tilde{\omega}) + (1-t) x_i(\eta_0, \tilde{\omega})) - D^2 F_\tau(\Theta^i \tilde{\omega}, x_i(\eta_0, \tilde{\omega}))\| = 0.$$

Combining inequalities (135), (136) and (137), we have that

$$\begin{aligned}
&\|I_n^m - H_n^m(\eta - \eta_0)\| e^{n\gamma_m(\tilde{\omega})} \\
&\le \sum_{i=0}^{n-1} K(\Theta^{i+1}\tilde{\omega}) e^{-(n-1-i)\beta(\tilde{\omega})} K_{m-1}(\tilde{\omega}) e^{-i\gamma_{m-1}(\tilde{\omega})} e^{n\gamma_m(\tilde{\omega})} K_1(\tilde{\omega}) e^{-i\gamma_1(\tilde{\omega})} \|\eta - \eta_0\| \\
&\quad \max_{0 \le t \le 1} \|D^2 F_s(\Theta^i \tilde{\omega}, t x_i(\eta, \tilde{\omega}) + (1-t) x_i(\eta_0, \tilde{\omega})) - D^2 F_s(\Theta^i \tilde{\omega}, x_i(\eta_0, \tilde{\omega}))\| \\
&+ \sum_{i=n}^{+\infty} K(\Theta^{i+1}\tilde{\omega}) e^{-(n-1-i)\alpha(\tilde{\omega})} K_{m-1}(\tilde{\omega}) e^{-i\gamma_{m-1}(\tilde{\omega})} e^{n\gamma_m(\tilde{\omega})} K_1(\tilde{\omega}) e^{-i\gamma_1(\tilde{\omega})} \|\eta - \eta_0\| \\
&\quad \max_{0 \le t \le 1} \|D^2 F_u(\Theta^i \tilde{\omega}, t x_i(\eta, \tilde{\omega}) + (1-t) x_i(\eta_0, \tilde{\omega})) - D^2 F_u(\Theta^i \tilde{\omega}, x_i(\eta_0, \tilde{\omega}))\| \\
&+ 2D(\tilde{\omega}) \|\eta - \eta_0\|^2
\end{aligned} \tag{140}$$

Then, by using inequalities (138), (139) and (140), we can obtain that for any $n \ge 1$

$$\lim_{\eta \to \eta_0} \max_{n \ge 1} \|I_n^m - H_n^m(\eta - \eta_0)\| e^{n\gamma_m(\tilde{\omega})} \le \epsilon.$$

Since ϵ can be arbitrarily small, we have

$$\lim_{\eta \to \eta_0} \max_{n \ge 1} \|I_n^m - H_n^m(\eta - \eta_0)\| e^{n\gamma_m(\tilde{\omega})} = 0.$$

Similarly we have that

$$\lim_{\eta \to \eta_0} \|I_0^m - H_0^m(\eta - \eta_0)\| = 0.$$

Therefore, (134) holds.

On the other hand, since $J(\eta, \tilde{\omega})$ is C^1 and (131) holds, by using the similar argument as in proving (134), we have that

$$\|J(\eta,\tilde{\omega})-J(\eta_0,\tilde{\omega})-DJ(\eta_0,\tilde{\omega})(\eta-\eta_0)\|_{C^{m-1+}_{\gamma_{m-1}(\tilde{\omega})}}=o(\|\eta-\eta_0\|),\ \text{as } \eta\to\eta_0. \quad (141)$$

With (134) and (141), one can obtain that $D^m\mathbf{x}(\eta,\tilde{\omega})$ exists and

$$\|D^m\mathbf{x}(\eta,\tilde{\omega})\|_{L(E^s,C^{m+}_{\gamma_m(\tilde{\omega})})}\le\frac{3}{2}(\hat{K}(\tilde{\omega})+\hat{K}'(\tilde{\omega})). \quad (142)$$

Furthermore, we also have that $D^m\mathbf{x}(\cdot,\tilde{\omega})$ satisfies the following equation

$$\begin{aligned}
D^m x_n(\eta,\tilde{\omega}) &= \sum_{i=0}^{n-1}\Phi_s(n-1-i,\Theta^{i+1}\tilde{\omega})DF_s(\Theta^i\tilde{\omega},x_i(\eta,\tilde{\omega}))D^m x_i(\eta,\tilde{\omega})\\
&\quad-\sum_{i=n}^{+\infty}\Phi_u(n-1-i,\Theta^{i+1}\tilde{\omega})DF_u(\Theta^i\tilde{\omega},x_i(\eta,\tilde{\omega}))D^m x_i(\eta,\tilde{\omega})\\
&\quad+\sum_{i=0}^{n-1}\Phi_s(n-1-i,\Theta^{i+1}\tilde{\omega})R^s_{m,i}(\eta,\tilde{\omega})\\
&\quad-\sum_{i=n}^{+\infty}\Phi_u(n-1-i,\Theta^{i+1}\tilde{\omega})R^u_{m,i}(\eta,\tilde{\omega}),\quad \text{for}\ \ n\ge 1;\\
D^m x_0(\eta,\tilde{\omega}) &= -\sum_{i=n}^{+\infty}\Phi_u(-1-i,\Theta^{i+1}\tilde{\omega})DF_u(\Theta^i\tilde{\omega},x_i(\eta,\tilde{\omega}))D^m x_i(\eta,\tilde{\omega})\\
&\quad-\sum_{i=n}^{+\infty}\Phi_u(-1-i,\Theta^{i+1}\tilde{\omega})R^u_{m,i}(\eta,\tilde{\omega})
\end{aligned} \quad (143)$$

where

$$R^\tau_{m,i}(\eta,\tilde{\omega})=\sum_{l=0}^{m-2}\begin{pmatrix} m-1\\ l\end{pmatrix}D_\eta^{m-1-l}\big(D_xF_\tau(\Theta^i\tilde{\omega},x_i(\eta,\tilde{\omega}))\big)D^{l+1}x_i(\eta,\tilde{\omega}),$$

for $\tau=u,s$. Applying the chain rule to

$$D_\eta^{m-1-l}\big(D_xF_\tau(\Theta^i\tilde{\omega},x_i(\eta,\tilde{\omega}))\big),$$

we have that each term in $R^{\tau}_{m-1,i}(\eta,\tilde{\omega})$ contains factors: $D^{l_1}_x F_{\tau}(\Theta^i\tilde{\omega}, x_i(\eta,\tilde{\omega}))$ for some $2 \le l_1 \le m-1$ and at least two derivatives $D^{l_2}_{\eta} x_i(\eta,\tilde{\omega})$ and $D^{l_3}_{\eta} x_i(\eta,\tilde{\omega})$ for some $l_2, l_3 \in \{1,\cdots,m-1\}$. Since $D^l_{\eta}\mathbf{x}(\eta,\tilde{\omega}) \in C^{l+}_{\gamma_l(\tilde{\omega})}$ for $l = 1,\cdots,m$ and F is C^N, $R^s_{m,i}(\cdot,\tilde{\omega}) : E^s(\tilde{\omega}) \cap B(0, \frac{R(\tilde{\omega})}{3K(\tilde{\omega})}) \to L^m\big(E^s(\tilde{\omega}), E^s(\Theta^i\tilde{\omega})\big)$ and $R^u_{m,i}(\cdot,\tilde{\omega}) : E^s(\tilde{\omega}) \cap B(0, \frac{R(\tilde{\omega})}{3K(\tilde{\omega})}) \to L^m\big(E^s(\tilde{\omega}), E^u(\Theta^i\tilde{\omega})\big)$ are C^1.

Furthermore, by using (120) and (122), we have that for $\tau = u, s$

$$||R^{\tau}_{m,i}(\eta,\tilde{\omega})|| \le \tilde{B}_m(\Theta^i\tilde{\omega})\tilde{K}_{m-1}(\tilde{\omega})e^{-2i\gamma_{m-1}(\tilde{\omega})} \tag{144}$$

and when $m < N$

$$||R^{\tau}_{m,i}(\eta,\tilde{\omega}) - R^{\tau}_{m,i}(\eta_0,\tilde{\omega})|| \le \tilde{B}_{m+1}(\Theta^i\tilde{\omega})\tilde{K}_{m-1}(\tilde{\omega})e^{-2i\gamma_{m-1}(\tilde{\omega})}\|\eta - \eta_0\|. \tag{145}$$

By using (144), (145) and the similar proof of (118), (119), one can obtain that when $m < N$ (120),(121),(122) holds for $j = m$. When $m = N < +\infty$, (122) may be not true. But since $R^s_{m,i}(\cdot,\tilde{\omega}) : E^s(\tilde{\omega}) \cap B(0, \frac{R(\tilde{\omega})}{3K(\tilde{\omega})}) \to L^m\big(E^s(\tilde{\omega}), E^s(\Theta^i\tilde{\omega})\big)$ and $R^u_{m,i}(\cdot,\tilde{\omega}) : E^s(\tilde{\omega}) \cap B(0, \frac{R(\tilde{\omega})}{3K(\tilde{\omega})}) \to L^m\big(E^s(\tilde{\omega}), E^u(\Theta^i\tilde{\omega})\big)$ are C^1, by using (144) and the similar argument as in the proof of (134), one can obtain that $D^m\mathbf{x}(\eta,\tilde{\omega})$ is continuous in η. The proof is done. $\square$

Now we are ready to prove the existence of the local stable manifold.

Theorem 46. *(***Local Stable Manifold Theorem***) Assume that ϕ is nonuniformly pseudo-hyperbolic on $\mathcal{A}$ and Hypothesis H1 holds, and $\alpha(\omega) < \beta(\omega), \beta(\omega) > 0$. Then, the local stable set $W^s_{loc}(\tilde{\omega})$ is a C^N manifold given by*

$$W^s_{loc}(\tilde{\omega}) = \{y_0 + \eta + h^s(\eta,\tilde{\omega}) | \eta \in B(0, \tilde{\rho}(\tilde{\omega})) \cap E^s(\tilde{\omega})\}$$

where $\tilde{\rho}(\cdot)$ is tempered and

$$h^s(\cdot,\tilde{\omega}) : B(0,\tilde{\rho}(\tilde{\omega})) \cap E^s(\tilde{\omega}) \to E^u(\tilde{\omega})$$

satisfies the following:

(i) $h^s(\eta,\tilde{\omega})$ is C^N in η with

$$Liph^s(\cdot,\tilde{\omega}) < 1,\ \ h^s(0,\tilde{\omega}) = 0,\ \ Dh^s(0,\tilde{\omega}) = 0.$$

(ii) $\|D^i h^s(\eta,\tilde{\omega})\| \le K_i(\tilde{\omega})$ for each $0 \le i \le N$ when $N < \infty$ and for all $i \ge 0$ when $N = \infty$, where $K_i(\tilde{\omega})$ are random variables tempered from above.

Proof. We define $h^s(\cdot,\tilde{\omega}) : B(0,\tilde{\rho}(\tilde{\omega})) \cap E^s(\tilde{\omega}) \to E^u(\tilde{\omega})$ by

$$h^s(\eta,\tilde{\omega}) = x_0^u(\eta,\tilde{\omega}) = -\sum_{i=0}^{+\infty} \Phi_u(-1-i,\Theta^{i+1}\tilde{\omega})F_u(\Theta^i\tilde{\omega},x_i).$$

Set

$$\tilde{\rho}(\tilde{\omega}) = \min\left\{\frac{1}{2}\rho(\tilde{\omega}), \frac{\rho(\tilde{\omega})}{3K(\tilde{\omega})}\right\}$$

where $\rho(\tilde{\omega})$ is the one in the proof of Proposition 45. So $\tilde{\rho}(\cdot)$ is tempered. Part (i) and (ii) in Proposition 45 imply that part (i) and (ii) in Theorem 46 except the estimate of the Liptchitz constant of h^s.

For each $\eta, \eta_0 \in B(0, \tilde{\rho}(\tilde{\omega})) \cap E^s(\tilde{\omega})\}$, we have

$$
\begin{aligned}
&\|h^s(\eta,\tilde{\omega}) - h^s(\eta_0,\tilde{\omega})\| \\
&\le \sum_{i=0}^{+\infty} K(\Theta^{i+1})e^{(1+i)\alpha(\tilde{\omega})}B_2(\Theta^i\tilde{\omega})\tilde{\rho}(\Theta^i\tilde{\omega})\frac{3}{2}K(\tilde{\omega})e^{-i\gamma_1(\tilde{\omega})}\|\eta-\eta_0\| \\
&\le \sum_{i=0}^{+\infty} K(\Theta^{i+1})e^{(1+i)\alpha(\tilde{\omega})}B_2(\Theta^i\tilde{\omega})\rho(\Theta^i\tilde{\omega})e^{-i\gamma_1(\tilde{\omega})}\|\eta-\eta_0\| \\
&\le \frac{1}{3}\|\eta-\eta_0\|.
\end{aligned}
$$

In above estimate we used the definition of $\rho(\tilde{\omega})$ and inequality (113).

Finally, for each $x \in W^s_{loc}$, since there exists $\eta \in E^s(\tilde{\omega})$ such that $x_0 = \eta + h^s(\eta,\tilde{\omega})$. By Proposition 45 we have that the orbit $\mathbf{x}$ satisfies

$$\mathbf{x}(\eta,\tilde{\omega}) = \{x_n\}_{n\ge 0} \in C^+_{\gamma_1(\tilde{\omega})},$$

which means that $x_n \to 0$ exponentially as $n \to +\infty$. This completes the proof. □

Theorem 47. *(***Local unstable Manifold Theorem***) Assume that ϕ is nonuniformly pseudo-hyperbolic on $\mathcal{A}$ and Hypothesis H1 holds, and $\alpha(\omega) < \beta(\omega), \alpha(\omega) < 0$. Then, the local unstable set $W^u_{loc}(\tilde{\omega})$ is a C^N manifold given by*

$$W^u_{loc}(\tilde{\omega}) = \{y_0 + \eta + h^u(\eta,\tilde{\omega}) | \eta \in B(0,\tilde{\rho}(\tilde{\omega})) \cap E^u(\tilde{\omega})\}$$

where $\tilde{\rho}(\cdot)$ is tempered and

$$h^u(\cdot,\tilde{\omega}) : B(0,\tilde{\rho}(\tilde{\omega})) \cap E^u(\tilde{\omega}) \to E^s(\tilde{\omega})$$

satisfies the following:

(i) $h^u(\eta, \tilde{\omega})$ *is* C^N *in* η *with*

$$Liph^u(\cdot, \tilde{\omega}) < 1, \ h^u(0, \tilde{\omega}) = 0, \ Dh^u(0, \tilde{\omega}) = 0.$$

(ii) $\|D^i h^u(\eta, \tilde{\omega})\| \leq K_i(\tilde{\omega})$ *for each* $0 \leq i \leq N$ *when* $N < \infty$ *and for all* $i \geq 0$ *when* $N = \infty$*, where* $K_i(\tilde{\omega})$ *are random variables tempered from above.*

This theorem can be proved in the same fashion as the stable manifold theorem with some modifications. Corresponding to $C^+_{\gamma_i(\tilde{\omega})}$, $1 \leq i \leq N$, we consider space $C^-_{\gamma_i(\tilde{\omega})}$ and the unstable set

$$W^u_{loc}(\tilde{\omega}) = \{x_0 \in B(0, \tilde{\rho}(\tilde{\omega})) \mid |x_n| e^{-\gamma n} \to 0, \text{ as } n \to -\infty\}$$

where $\tilde{\rho}(\tilde{\omega})$ is a tempered function and $\gamma > 0$, $-\beta < \gamma < -\alpha$. In order to show that $W^u_{loc}(\tilde{\omega})$ is given by the graph of C^N function, we first show that $\mathbf{x} = \{x_n\}_{n \leq 0} \in C^-_{\gamma(\tilde{\omega})}$ satisfies equation (107) if and only if $\{x_n\}_{n \leq 0} \in C^-_{\gamma(\tilde{\omega})}$ and satisfies the following equations:

$$\begin{aligned} x_n &= \Phi_u(n, \tilde{\omega}) x_0^u - \sum_{i=n}^{-1} \Phi_u(n-1-i, \Theta^{i+1}\tilde{\omega}) F_u(\Theta^i \tilde{\omega}, x_i) \\ &\qquad + \sum_{i=-\infty}^{n-1} \Phi_s(n-1-i, \Theta^{i+1}\tilde{\omega}) F_s(\Theta^i \tilde{\omega}, x_i), \quad \text{for } \ n \leq -1; \\ x_0 &= x_0^u + \sum_{i=-\infty}^{-1} \Phi_s(-1-i, \Theta^{i+1}\tilde{\omega}) F_s(\Theta^i \tilde{\omega}, x_i), \end{aligned} \tag{146}$$

where $\Phi_u = \Phi|_{E^u}$, $\Phi_s = \Phi|_{E^s}$, $F_u = \Pi_u F$, and $F_s = \Pi_s F$. Then by the same arguments as in the proofs of Theorem 45 and 46, we obtain this theorem.

Combining Theorem 46 and Theorem 47 gives Theorem 43.

Remark 11. Since $E^u(\tilde{\omega})$ are finite dimensional and measurable, by corollary 39, we can find a measurable basis $\{e_1(\tilde{\omega}), e_2(\tilde{\omega}), \ldots, e_{m_u(\tilde{\omega})}(\tilde{\omega})\}$ spanning $E^u(\tilde{\omega})$. For any $\eta \in B(0, \tilde{\rho}(\tilde{\omega})) \cap E^u(\tilde{\omega})$, we can write

$$\eta = \sum_{i=1}^{m_u(\tilde{\omega})} \eta_i e_i(\tilde{\omega})$$

where $(\eta_1, \ldots, \eta_{m_u(\tilde{\omega})}) \in \mathbb{R}^{m_u(\tilde{\omega})}$. The usual norm of $\mathbb{R}^{m_u(\tilde{\omega})}$ induces a norm on $E^u(\tilde{\omega})$. By using (75), we have that the induced norm is equivalent to the norm of X restricted on $E^u(\tilde{\omega})$ with a uniform controlling constant. By restricting our argument on a proper subset, $m_u(\tilde{\omega})$ becomes a constant. By using (146) and the same arguments as in the proofs of Theorem 45 and 46, one can verify that the assumptions of lemma 7.3.4 of [1] are satisfied. Thus we obtain that $h^u(\sum_{i=1}^{m_u(\tilde{\omega})} \eta_i e_i(\tilde{\omega}), \tilde{\omega})$ is measurable for fixed $(\eta_1, \ldots, \eta_{m_u(\tilde{\omega})}) \in \mathbb{R}^{m_u(\tilde{\omega})}$ on the set $\|\eta\| < \rho(\tilde{\omega})$.

A Non-ergodic Case

The multiplicative ergodic theorem in non-ergodic case can be proved in the same way with the following modifications. In this case, $l_k(T)$, $\kappa(T)$, l_α, l'_α, $\kappa'(T)$ are all measurable functions.

We first notice that $\frac{1}{k}l_k(T)(\omega)$ is a nondecreasing sequence from Lemma 13. Thus, there exists a positive integer function $m(\omega)$ such that

$$\begin{aligned} &\frac{1}{k}l_k(T)(\omega) = l_1(T)(\omega), \quad \text{for } 1 \leq k \leq m(\omega), \\ &\frac{1}{k}l_k(T)(\omega) < l_1(T)(\omega), \quad \text{for } k > m(\omega). \end{aligned} \tag{147}$$

Then, from (36), we have

$$\dim E^{\kappa(T)(\omega)}(\omega) \leq m(\omega),\ a.s..$$

We modify Proposition 30 as

Proposition 48. For $l_\alpha(\omega) < \lambda(\omega) \leq \kappa(T)(\omega)$, we have

$$\dim E^{\lambda(\omega)}(\omega) \geq m(\omega),\ a.s..$$

Furthermore, $\dim E^{\kappa(T)(\omega)}(\omega) = m(\omega)$.

Let

$$\Omega_m = \{\omega \mid m(\omega) = m\}.$$

Note that Ω_m is a θ-invariant measurable set on which $m(\omega)$ is a constant m and $l_a(\omega) \neq \kappa(T)(\omega)$. In the following, we will restrict our discussion on Ω_m and assume that $P(\Omega_m) = 1$.

Let

$$\lambda^t(\omega) = (1-t)\max\{l_\alpha(\omega),\ \kappa(T)(\omega)-1\} + t\kappa(T)(\omega),\ t\in(0,1),\ \omega\in\Omega_m,$$

which is measurable for fixed t. Note that on Ω_m, $\lambda^t < \kappa(T)$. Let

$$U_n = \{\omega\in\Omega_m | \lambda^t(\omega) < n\}, n\geq 1.$$

Then

$$\lim_{n\to\infty} P(\Omega_m - U_n) = 0.$$

We also note that for fixed t and small $\epsilon_0 > 0$, there exist $\lambda_0 > -\infty$, $\epsilon > 0$ and a subset $\Omega' \in \Omega_m$ such that $P(\Omega') > 1-\epsilon_0$ and for any $\omega\in\Omega'$,

$$\begin{aligned} &0 < 2\epsilon < \kappa(T)(\omega) - \lambda^t(\omega), \\ &\lambda^t(\omega) < -\lambda_0. \end{aligned} \tag{148}$$

For each positive integer n, we use $A_n^{\lambda^t}$ to denote the subset of Ω_m such that for each $\omega\in A_n^{\lambda^t}$ there exist m vectors $\{v_i\}_{1\leq i\leq m}\subset X$ such that

$\{\pi(\omega)v_i\}_{1\leq i\leq m}$ are independent;

$$\tag{149}$$

$$T^{-k}(\omega)\pi(\omega)v_i \text{ exists for } 1\leq i\leq m, 1\leq k\leq n; \tag{150}$$

$$\frac{V_m(T^{-k}(\omega)\pi(\omega)v_1,\ldots,T^{-k}(\omega)\pi(\omega)v_m)}{V_m(\pi(\omega)v_1,\ldots,\pi(\omega)v_m)} \leq e^{-km\lambda^t(\omega)},\ 1\leq k\leq n. \tag{151}$$

Lemma 49. *There exists a $\delta^t > 0$ such that*

$$P_{in}(A_n^{\lambda^t}) > \delta^t, \quad \textit{for all } n \textit{ and fixed } t,$$

where $P_{in}(A_n^{\lambda^t})$ *is the inner measure of* $A_n^{\lambda^t}$ *and there exists a countable subset* S_d *of* $(0,1)$ *such that for any* $t \in (0,1) - S_d$*,* $A_n^{\lambda^t}$ *is measurable and*

$$P(A_n^{\lambda^t}) \geq \delta^t.$$

Proof. We first show that if $A_n^{\lambda^t}$ is measurable, then $P(A_n^{\lambda^t}) > \delta^t$. We note that from $\log^+ \|S(\cdot)\| \in L^1(\Omega_m, \mathcal{F}, P)$ it follows that for each $\epsilon > 0$, there exists a $H(\epsilon) > 0$ such that

$$\int_{E\big(H(\epsilon)\big)} \log^+ \|S(\omega)\| dP < \epsilon,$$

where $E\big(H(\epsilon)\big) = \big\{\omega \mid \log \|S(\omega)\| > H(\epsilon)\big\}$.

Thus,by using (148), we have ϵ_0, $H(\frac{\epsilon\epsilon_0}{4})$, H_0, λ_0, ϵ, and Ω' such that $P(\Omega') > 1-\frac{1}{2}\epsilon_0$ and for any $\omega \in \Omega'$,

$$\begin{aligned}
&0 < 2\epsilon < \kappa(T)(\omega) - \lambda^t(\omega),\\
&\lambda^t(\omega) < -\lambda_0,\\
&H_0 = -H(\frac{\epsilon\epsilon_0}{4}) < \lambda_0 < -\lambda^t(\omega).
\end{aligned}$$

By using the Birkhoff ergodic theorem and Lemma 16, we have that

$$\lim_{n\to+\infty} \frac{1}{mn} \sum_{k=0}^{n-1} \chi_{E\big(H(\frac{\epsilon\epsilon_0}{4})\big)}(\theta^k\omega) \log V_m(T(\theta^k\omega)|_{G(\theta^k\omega)}) \text{ exists } a.s.,$$

and

$$\begin{aligned}
&\int \lim_{n\to+\infty} \frac{1}{mn} \sum_{k=0}^{n-1} \chi_{E\big(H(\frac{\epsilon\epsilon_0}{4})\big)}(\theta^k\omega) \log V_m(T(\theta^k\omega)|_{G(\theta^k\omega)}) dP\\
&\leq \int_{E\big(H(\frac{\epsilon\epsilon_0}{4})\big)} \log^+ \|S(\omega)\| dP < \frac{\epsilon\epsilon_0}{4}.
\end{aligned} \tag{152}$$

Let

$$\Omega'' = \left\{ \omega \in \Omega' \Big| \lim_{n\to+\infty} \frac{1}{mn} \sum_{k=0}^{n-1} \chi_{E\left(H(\frac{\epsilon\epsilon_0}{4})\right)}(\theta^k\omega) \log V_m(T(\theta^k\omega)|_{G(\theta^k\omega)}) < \frac{\epsilon}{2} \right\}.$$

By (152) we have that

$$P(\Omega'') > 1 - \epsilon_0. \tag{153}$$

For $\omega \in \Omega''$, from the definition of m, we have

$$\lim_{n\to+\infty} \frac{1}{mn} \log V_m(T^n(\omega)) = \kappa(T)(\omega) > \lambda^t(\omega) + 2\epsilon.$$

Thus, there exists $N_1(\omega) > 0$ such that for any $n > N_1(\omega)$,

$$V_m(T^n(\omega)) > e^{nm(\lambda^t(\omega)+2\epsilon)}. \tag{154}$$

Using (152), there exists $N_2(\omega) > 0$ such that for any $n > N_2(\omega)$,

$$\frac{1}{mn} \sum_{k=0}^{n-1} \chi_{E\left(H(\frac{\epsilon\epsilon_0}{4})\right)}(\theta^k\omega) \log V_m(T(\theta^k\omega)|_{G(\theta^k\omega)}) < \epsilon. \tag{155}$$

Let $N_0(H_0, \lambda^t(\omega), \epsilon)$ be the number given in Lemma 31.Thus, for each

$$N \geq \max\{N_1, N_2, N_0(H_0, \lambda^t(\omega), \epsilon)\} + 1,$$

by using (154), there exist vectors $\{v_i\}_{1\leq i\leq m} \subset X$ such that $\{\pi(\omega)v_i\}_{1\leq i\leq m}$ are linearly independent and

$$V_m(T^{N+1}(\omega)\pi(\omega)v_1, \ldots, T^{N+1}(\omega)\pi(\omega)v_m) > e^{(N+1)m(\lambda^t(\omega)+2\epsilon)} V_m(\pi(\omega)v_1, \ldots, \pi(\omega)v_m). \tag{156}$$

Let

$$a'_k = \frac{1}{m}\log\frac{V_m(T^{N-k}(\omega)\pi(\omega)v_1,\ldots,T^{N-k}(\omega)\pi(\omega)v_m)}{V_m(T^{N-k+1}\pi(\omega)v_1,\ldots,T^{N-k+1}\pi(\omega)v_m)} \quad 0\le k\le N.$$

Then,

$$\sum_{k=0}^{N} a'_k = \frac{1}{m}\log\frac{V_m(\pi(\omega)v_1,\ldots,\pi(\omega)v_m)}{V_m(T^{N+1}(\omega)\pi(\omega)v_1,\ldots,T^{N+1}(\omega)\pi(\omega)v_m)} < -(N+1)(\lambda^t(\omega)+2\epsilon).$$

Set

$$a_k = \begin{cases} a'_k & \text{if } a'_k \ge H_0,\\ 0 & \text{if } a'_k < H_0.\end{cases}$$

By(14), we have

$$a'_k \ge -\frac{1}{m}\log V_m(T(\theta^{N-k}\omega)|_{G(\theta^{N-k}\omega)})).$$

Hence, if $a'_k < H_0$, then $\theta^{N-k}\omega \in E\big(H(\frac{\epsilon\epsilon_0}{4})\big)$. Thus, using (154), we have

$$\begin{aligned} 0 &\ge \frac{1}{N+1}\left(\sum_{k=0}^{N} a'_k - \sum_{k=0}^{N} a_k\right)\\ &\ge \frac{1}{m(N+1)}\sum_{k=0}^{N} -\chi_{E\left(H(\frac{\epsilon\epsilon_0}{4})\right)}(\theta^k\omega)\log V_m(T(\theta^k\omega)|_{G(\theta^k\omega)})\\ &\ge -\epsilon, \end{aligned}$$

which implies that

$$\sum_{k=0}^{N} a_k \le \sum_{k=0}^{N} a'_k + (N+1)\epsilon \le -(N+1)(\lambda^t(\omega)+\epsilon). \tag{157}$$

By Lemma 31, there exist integers $0 < n_1 < n_2 < \cdots < n_{j_0} \le N$ such that $j_0 \ge$

$\delta'^t(\omega)N$ and for all $n_j \le k \le N$,

$$\begin{aligned}
(k-n_j)\lambda^t(\omega) &\ge \sum_{n=n_j+1}^{k} a_n \ge \sum_{n=n_j+1}^{k} a'_n \\
&= \sum_{n=n_j+1}^{k} \frac{1}{m}\log\frac{V_m(T^{N-n}(\omega)\pi(\omega)v_1,\dots,T^{N-n}(\omega)\pi(\omega)v_m)}{V_m(T^{N-n+1}(\omega)\pi(\omega)v_1,\dots,T^{N-n+1}(\omega)\pi(\omega)v_m)} \\
&= \frac{1}{m}\log\frac{V_m(T^{N-k}(\omega)\pi(\omega)v_1,\dots,T^{N-k}(\omega)\pi(\omega)v_m)}{V_m(T^{N-n_j}(\omega)\pi(\omega)v_1,\dots,T^{N-n_j}(\omega)\pi(\omega)v_m)}.
\end{aligned}$$

We note that $N-n_j \ge n$ when $j_0 - j > n$. From the definition of $A_n^{\lambda^t}$ it follows that for $n+j \le j_0$

$$\theta^{N-n_j}\omega \in A_n^{\lambda^t}.$$

Since $l-n \ge \delta'^t(\omega)N - n$, we have

$$\begin{aligned}
&\frac{1}{N}\#\{0<i\le N|\ \theta^i\omega \in A_n^{\lambda^t}\} \\
&\ge \frac{1}{N}\#\{0<i\le N|\ i = N-n_j, 1\le j\le j_0-n\} \\
&\ge \frac{1}{N}(j_0-n) \ge \delta'^t(\omega) - \frac{n}{N}
\end{aligned}$$

in which the lower bound will go to $\delta'^t(\omega)$ as N goes to $+\infty$. Note that $\delta'^t(\cdot)$ does not depends on n. Thus, if $A_n^{\lambda^t}$ is measurable, then by the Birkhoff ergodic theorem we obtain

$$P(A_n^{\lambda^t}) \ge \int_{\Omega''} \lim_{N\to+\infty}\frac{1}{N}\#\{0<i\le N|\ \theta^i\omega\in A_n^{\lambda^t}\} \ge \int_{\Omega''}\delta'^t dP(:=\delta^t) > 0. \qquad (158)$$

Next, we show that $A_n^{\lambda^t}$ is measurable except for countably many t. For each $n \ge 1$ and $w_1,\cdots w_m \in X$, we use $S_{n,t}(w_1,\dots,w_m)$ to denote the set of $\omega \in \Omega_m$ such that the following conditions hold

$$V_m(T^n(\theta^{-n}\omega)w_1,\ldots,T^n(\theta^{-n}\omega)w_m)\neq 0; \tag{159}$$

$$\frac{V_m(T^{n-k}(\theta^{-n}\omega)w_1,\ldots,T^{n-k}(\theta^{-n}\omega)w_m)}{V_m(T^n(\theta^{-n}\omega)w_1,\ldots,T^n(\theta^{-n}\omega)w_m)}\leq e^{-mk\lambda^t}, \tag{160}$$

for $1\leq k\leq n,\ 0<t<1$.

Since $T(\omega)$ is strongly measurable and $V_m:\ X^m\to\mathbb{R}$ is continuous, $S_{k,t}(w_1,\ldots,w_m)$ is measurable. Let

$$D_t(w_1,\ldots,w_m)=\bigcap_{k=1}^{n}S_{k,t}(w_1\ldots,w_m).$$

Then $D_t(w_1,\ldots,w_m)$ is also measurable.

Since X is a separable Banach space, we have a countable dense set $\{v_i(\neq 0)\}_{i\geq 1}$ of X. Set

$$K_t=\bigcap_{j=[1/t]+1}^{\infty}\ \bigcup_{(n_1,\ldots,n_m)\in\mathbb{N}^m}D_{t-\frac{1}{j}}(v_{n_1},\ldots,v_{n_m}).$$

Then K_t is measurable. By the definition of $S_{n,t}(w_1,\cdots,w_m)$, we have that for each small $\epsilon>0$

$$S_{n,t}(w_1,\cdots,w_m)\subset S_{n,t-\epsilon}(w_1,\cdots,w_m),$$

which yields

$$K_t\subset K_{t-\epsilon}$$

and

$$K_t=\lim_{\epsilon\to 0^+}K_{t-\epsilon}\text{ decreasingly.}$$

Since $P(K_t)\leq 1$, we have

$$P(K_t)=\lim_{\epsilon\to 0^+}P(K_{t-\epsilon}).$$

Next, we show

Claim: The inner measure of $A_n^{\lambda^t}$, $P_{in}(A_n^{\lambda^t})$, is equal to its outer measure $P_{out}(A_n^{\lambda^t})$ for each $t \in (0,1) - S_d$, where S_d is a countable set.

We first prove that for each small $\epsilon > 0$,

$$K_t \subset A_n^{\lambda^{t-\epsilon}},$$

in other words

$$K_{t+\epsilon} \subset A_n^{\lambda^t}.$$

Let $\omega \in K_t$. Then, there exists $(n_1, \ldots, n_m) \in \mathbb{N}^m$ such that $\omega \in D_{t-\epsilon}(v_{n_1}, \ldots, v_{n_m})$, which means that

$$V_m(T^n(\theta^{-n}\omega)v_{n_1}, \ldots, T^n(\theta^{-n}\omega)v_{n_m}) \neq 0$$

and

$$\frac{V_m(T^{n-k}(\theta^{-n}\omega)v_{n_1}, \ldots, T^{n-k}(\theta^{-n}\omega)v_{n_m})}{V_m(T^n(\theta^{-n}\omega)v_{n_1}, \ldots, T^n(\theta^{-n}\omega)v_{n_m})} \leq e^{-mk\lambda^{t-\epsilon}}, \quad 1 \leq k \leq n.$$

Set

$$v_i' = T^n(\theta^{-n}\omega)v_{n_i}.$$

Then

$$\omega \in A_n^{\lambda^{t-\epsilon}}.$$

Now, we prove that

$$A_n^{\lambda^t} \subset K_t.$$

Let $\omega \in A_n^{\lambda^t}$. Then, there exist vectors $\{w_i\}_{1 \leq i \leq m} \subset X$ such that $T^{-k}(\omega)\pi(\omega)w_i$ exists for $1 \leq i \leq m$, $1 \leq k \leq n$,

$$V_m(\pi(\omega)w_1, \ldots, \pi(\omega)w_m) \neq 0$$

and

$$\frac{V_m(T^{-k}(\omega)\pi(\omega)w_1, \ldots, T^{-k}(\omega)\pi(\omega)w_m)}{V_m(\pi(\omega)w_1, \ldots, \pi(\omega)w_m)} \leq e^{-km\lambda^t}, \ 1 \leq k \leq n.$$

For small $\epsilon > 0$, since $\{v_i\}_{i\geq 1}$ is a countable dense subset of X and $V_m : X \to \mathbb{R}$ is continuous, we have that there exists $(n_1, \ldots, n_m) \in \mathbb{N}$ such that $(\pi(\theta^{-n}\omega)v_{n_1}, \ldots, \pi(\theta^{-n}\omega)v_{n_m})$ is close enough to $(T^{-n}(\omega)\pi(\omega)w_1, \ldots, T^{-n}(\omega)\pi(\omega)w_m)$ and

$$\frac{V_m(T^{n-k}(\theta^{-n}\omega)v_{n_1}, \ldots, T^{n-k}(\theta^{-n}\omega)v_{n_m})}{V_m(T^{n}(\theta^{-n}\omega)v_{n_1}, \ldots, T^{n}(\theta^{-n}\omega)v_{n_m})} \leq e^{-km\lambda^{t-\epsilon}}, \ 1 \leq k \leq n.$$

So

$$\omega \in D_{t-\epsilon}(v_{n_1}, \ldots, v_{n_m}).$$

Therefore,

$$\omega \in K_{t-\epsilon}.$$

Since $\epsilon > 0$ can be arbitrary small, and by the definition of K_t, we have

$$\omega \in K_t,$$

hence

$$A_n^{\lambda^t} \subset K_t.$$

Summarizing the above discussion, we have that for any $\epsilon > 0$,

$$K_{t+\epsilon} \subset A_n^{\lambda^t} \subset K_t.$$

Since $P(K_t)$ is a monotone function with respect to t, it has at most countable many discontinuous points. We use S_d to denote the set of these discontinuous points. Thus for any $t \in (0,1) - S_d$, we have

$$\lim_{\epsilon \to 0^+} P(K_{t+\epsilon}) = P(K_t),$$

which implies that

$$P_{in}(A_n^{\lambda^t}) = P_{out}(A_n^{\lambda^t}).$$

Therefore, $A_n^{\lambda^t}$ is measurable. Then, by using (158), we have that for any $t \in (0,1) - S_d$,

$$P(A_n^{\lambda^t}) \geq \delta^t.$$

Then

$$P_{in}\left(\bigcap_{n=1}^{+\infty} A_n^{\lambda^t}\right) \geq \delta^t > 0.$$

This completes the proof of the lemma.

□

Remark 12. In the proof we have that for any $\omega \in \Omega''$, $\delta'^t(\omega)(>0)$ does not depends on n, so the frequency of which $\{\theta^n\omega\}_{n\geq 1}$ enters $\cap_{n=1}^{+\infty} A_n^{\lambda^t}$ is positive, which implies that

$$\Omega'' \subset \bigcup_{n\in\mathbb{Z}} \theta^n \left(\cap_{n=1}^{+\infty} A_n^{\lambda^t}\right).$$

Note that ϵ_0 can be arbitrary small and is independent of $P(A_n^{\lambda^t})$, although δ^t depends on ϵ_0. So we can obtain that

$$P\left(\bigcup_{n\in\mathbb{Z}} \theta^n \left(\cap_{n=1}^{+\infty} A_n^{\lambda^t}\right)\right) = 1.$$

Lemma 50. *There exists a constant C depends only on m such that for any small number $\epsilon > 0$ satisfying $C\epsilon < 1$, if $\omega \in \bigcap_{n=1}^{+\infty} A_n^{\lambda^{1-\epsilon}}$, then $\dim E^{\lambda^{1-C\epsilon}}(\omega) \geq m$.*

The proof will be exactly same as Lemma 33 if we restrict our discussion on invariant subsets $\Omega_m \cap \{\omega | \kappa(T)(\omega) > -n\}$, $n \geq 1$. Note that

$$\lim_{n \to +\infty} P(\Omega_m \cap \{\omega | \kappa(T)(\omega) > -n\}) = 1.$$

Proof of Proposition 48. It is sufficient to show the proposition holds for $t \in (0,1)$. Since $E^{\lambda^t}(\omega)$ is a decreasing sequence of finite dimensional subspaces, we have

$$\dim E^{\kappa(T)}(\omega) = \dim \bigcap_{0<t<1} E^{\lambda^t}(\omega). \tag{161}$$

Let t_0 be a fixed number in $(0,1)$. By Lemma 49, there exists a countable set S of $(0,1)$ such that for each $t \in (0,1) - S$, $\bigcap_{n=1}^{\infty} A_n^{\lambda^t}$ is measurable and

$$P(\bigcap_{n=1}^{\infty} A_n^{\lambda^t}) \geq \delta^t.$$

Choose $1 > \epsilon > 0$ such that $t_0 < 1 - C\epsilon$. Then, $\bigcap_{n=1}^{\infty} A_n^{\lambda^{1-\epsilon}}$ is measurable and

$$P(\bigcap_{n=1}^{\infty} A_n^{\lambda^{1-\epsilon}}) \geq \delta^{1-\epsilon}.$$

By Lemma 50, we have that for each $\omega \in \bigcap_{n=1}^{\infty} A_n^{\lambda^{1-\epsilon}}$,

$$\dim(E^{\lambda^{1-C\epsilon}}(\omega)) \geq m.$$

Since $t_0 < 1 - C\epsilon$, $E^{\lambda^{1-C\epsilon}}(\omega) \subset E^{\lambda^{t_0}}(\omega)$. Thus,

$$\dim(E^{\lambda^{t_0}}(\omega)) \geq m.$$

Since $E^{\lambda^{1-C\epsilon}}(\omega)$ is invariant and $T(\omega)|_{G(\omega)}$ is injective, $\dim(E^{\lambda^{1-C\epsilon}}(\theta^n\omega)) \geq m$ for all

$n \in \mathbb{Z}$. Let

$$\mathcal{A}^{\lambda^{1-\epsilon}} = \bigcup_{j \in \mathbb{Z}} \theta^j \left(\bigcap_{n=1}^{+\infty} A_n^{\lambda^{1-\epsilon}} \right).$$

Then, by Remark 12, we have that $\mathcal{A}^{\lambda^{1-\epsilon}}$ is a θ-invariant measurable set with full measure. This completes the proof of the proposition.

$\square$

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Department of Mathematics

Brigham Young University

Provo, UT 84602

$E-mail$: zenglian@math.byu.edu