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Student Teacher Knowledge and Its Impact on Task Design

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STUDENT TEACHER KNOWLEDGE AND ITS IMPACT ON TASK DESIGN

by

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ABSTRACT

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This study investigated how student teachers used their mathematical knowledge for teaching and pedagogical knowledge to design and modify mathematical tasks. It also examined the relationship between teacher knowledge and the cognitive demands of a task. The study relied heavily on the framework in Hill, Ball, and Shilling (2008), which describes the different domains of knowledge in mathematical knowledge for teaching, and the framework on the cognitive demands of mathematical tasks in Stein, Smith, Henningsen, and Silver (2000).

Results of the study indicated that the student teachers used their common content knowledge when they lacked sufficient knowledge in other domains, especially specialized content knowledge, to perform a particular job of teaching. There was often a decrease in the cognitive demands of a task when it was modified by the student teachers. These drops were often associated with a lack of specialized content knowledge.

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Introduction

Teachers' mathematical content knowledge impacts their classroom behavior which, in turn, indirectly affects student achievement (Fennema & Franke, 1992; Stein, Remillard, & Smith, 2007). However, establishing a direct link between teachers' knowledge and student achievement has proven elusive (Ball & Bass, 2000; Carpenter, Fennema, Peterson, & Carey, 1988; Fennema & Franke, 1992). Although a link has not yet been clearly established, it is likely that one exists and researchers continually seek to establish a connection.

The research that has attempted to ascertain a direct correlation between student achievement and teacher knowledge has proved unfruitful, at least in part, because researchers failed to account teacher classroom practices. Researchers often used global measures to assess knowledge, which removed knowledge from the practice of teaching (Ball, 1990b; Ma, 1999). In their review of relevant literature, Fennama and Franke (1992) cited several examples of how measuring mathematical knowledge according to the number of completed mathematics courses was invalid (see also National Research Council, 2001). This methodology implicitly assumed that mathematics coursework provided prospective teachers with all the necessary knowledge for teaching mathematics (Ball, 1990b). Other studies have defined the needed mathematical knowledge for teaching according to the overall content of the curriculum; similarly, this approach tacitly assumes that the only knowledge needed for teaching is that included in the curriculum, oversimplifying the situation (Ball & Bass, 2000). These methodologies are further flawed by their conception of knowledge. The notion that knowledge is an external object that can be acquired through transfer is inherent in the generation of lists of content to be known by preservice teachers. This conception of knowledge and the accompanying methodologies, do not explain the use and creation of knowledge in practice. If we are to gain a better understanding of how teachers use their knowledge in the act of teaching, a different epistemological theory and methodology should be considered.

Hill, Schilling, and Ball (2004) claimed that researchers should focus on how teachers use their knowledge rather than on quantities of possessed knowledge. By focusing on how teachers apply knowledge in different situations, researchers may learn how knowledge impacts teachers' behaviors. However, many of the studies focusing on the use of teacher knowledge in practice attempt to examine several aspects of teacher practice simultaneously: preparation, instruction, assessment and reflection (e.g., Kahan, Cooper, & Bethea, 2003; Tirosh, Even, & Robinson, 1998). Such an approach to studying teacher knowledge may be beneficial; however, there need to be more studies that examine only one aspect of teaching. Furthermore, by investigating only one area of teacher practice, researchers can look at teacher knowledge in greater depth. Thus, this study will focus on the aspect of preparation.

The National Council of Teachers of Mathematics (1991; 2007) suggested the importance of posing worthwhile mathematical tasks in the classroom. It has been suggested that the knowledge teachers use to implement mathematical tasks influences the cognitive demands of the tasks (Stein, Smith, Henningsen, & Silver, 2000). Furthermore, the cognitive demand of a task influences student learning (Stein et al., 2000). Examining how teacher knowledge impacts the cognitive demands of a task may provide understanding about how teacher knowledge impacts student learning. However, few studies have investigated how teachers use their knowledge to design and modify mathematical tasks. This is one area of teacher practice that needs to be researched in conjunction with teacher knowledge. Thus, this study will investigate the relationship between teacher knowledge and the design and modification of mathematical tasks.

Stein et al. (2000) claimed that student learning was greatest in high-level tasks. In order to increase student learning, teachers should strive to implement high-level tasks. By looking at the connection between teacher knowledge and the cognitive demands of a task, researchers can better describe the impact teacher knowledge has on classroom practice.

In studying teacher knowledge, it is valuable to study not only how it is used, but how it might develop. As such, it is important to establish a baseline of teacher knowledge by studying student teachers. Student teachers may not have the same knowledge base as more experienced teachers. The inexperience of novice teachers often means that they lack knowledge that experienced teachers might possess. Consequently student teachers will likely encounter more instances where they do not have the necessary knowledge. By researching student teachers instead of practicing teachers, these instances will be more frequent and allow the researcher to get a better understanding of how knowledge is used in designing and modifying tasks as well as the impact knowledge has on the cognitive demands of the tasks.

An additional problem prevalent in the research is the abundance of research on elementary teachers rather than secondary mathematics teachers. Even in studies examining both groups (e.g, Ball, 1990b), it is difficult to differentiate the conclusions drawn for the two different groups. The transfer of conclusions regarding knowledge and practice from elementary to secondary teachers is problematic because the groups likely have different knowledge bases due to differences in preparation, training, and practice.

Rationale

Given the difficulty of linking teacher knowledge to student achievement this research project was intended to contribute to the formation of a connection between the two variables by looking at how teachers used their knowledge in the classroom, specifically how student teachers used their knowledge to plan and modify worthwhile mathematical tasks. The research project chose to investigate the cognitive demands of tasks because of the importance of high levels of cognitive demand in student learning. Additionally, this research project intended to add to the sparse field of research on secondary mathematics preservice teachers' knowledge as well as on the impact such knowledge has on the cognitive demands of a task.

Research Questions

This study focused on how student teachers used their knowledge in the practice of designing and modifying mathematical tasks. The first question addressed how their knowledge was used; the second question addressed the impact their knowledge had on the cognitive demands of the task.

The specific research questions were:

- How does a student teacher use their mathematical knowledge for teaching and pedagogical knowledge to design or modify a mathematical task?
- How does the student teachers' use of their mathematical knowledge for teaching and pedagogical knowledge affect the cognitive demand of the task?

Literature Review and Theoretical Framework

The research area of teacher knowledge is discussed first in this chapter, looking at both what the research has found about mathematics teachers' knowledge as well as proposed theoretical frameworks for researching teacher knowledge. Next, the chapter addresses findings of and frameworks for researching mathematical tasks. The section integrating the two areas of research discusses the few studies investigating teacher knowledge and mathematical tasks as well as the rich research potential of integrating the two areas.

Teacher Knowledge

Teacher knowledge has long been considered critical to effective teaching. In *Education and Experience*, Dewey (1938) frequently discussed the role of the teacher as the more mature and knowledgeable individual in the classroom. In his discussion on creating a community of learners, Dewey (1938) stated,

> It requires thought and planning ahead. The educator is responsible for a knowledge of individuals and for a knowledge of subject-matter that will enable activities to be selected which lend themselves to social organization, an organization in which all individuals have an opportunity to contribute something, and in which the activities in which all participate are the chief carrier of control. (p. 56)

This quote specifically addresses how to establish a classroom environment where positive and productive discourse can occur, and the role teacher knowledge plays in such an endeavor. The knowledge Dewey mentioned can be generalized to all aspects of teaching; the teacher must have knowledge of students, in general and as individuals, as well as a rich knowledge of the content.

Teacher knowledge continues to receive attention as policy makers and researchers attempt to improve student achievement and learning. "Knowledge of teaching, of mathematics, and of students is an essential aspect of what a teacher needs to know to be successful" (NCTM, 2007, p. 16). In order to teach successfully, teachers need to have a firm knowledge base; however, research has indicated that many teachers lack the knowledge needed for teaching mathematics (Mewborn, 2003). In a review of literature, Mewborn (2003) found overwhelming evidence in the literature that teachers of mathematics had strong procedural knowledge, but possessed little or no conceptual knowledge of mathematics.

Many studies have pointed to a lack of teacher knowledge in preservice and practicing teachers. In a study on preservice elementary and secondary teachers, Ball (1990b) found that both sets of preservice teachers were unable to create a story representing fractional division. The secondary teachers were also unable to explain why division by zero is undefined. She found that the preservice teachers' content knowledge was compartmentalized and procedural. Even (1993) found similar results with respect to preservice secondary mathematics teachers and the concept of function. She found that some of the preservice teachers did not have an understanding of the modern definition of a function and many of the preservice teachers could not explain why univalence was an important aspect of the definition. Additionally, Tirosh et al. (1998) found that some practicing middle school teachers were unaware of common mistakes students make when solving equations, even after years of experience.

In order to gain a better understanding of the knowledge needed to teach mathematics, some researchers have compared the knowledge of Chinese teachers with U.S. teachers. Ma (1999) found that U.S. elementary teachers knowledge was compartmentalized and procedural compared to their Chinese counterparts whose knowledge was interconnected and rooted in the underlying mathematics. An, Kulm, and Zhonghe (2004) found similar differences in the knowledge of the two groups, and also found differences in the teaching practices of the two groups. They found that the Chinese teachers focused on developing both conceptual and procedural knowledge through the use of more traditional practices whereas the U.S. teachers advocated creativity and inquiry to help students develop mathematical understanding.

Although much of the research has pointed to a lack of teacher knowledge, one study indicated that there may be a correlation between teacher knowledge and practice. In a study comparing secondary preservice teachers' mathematical content knowledge to their lesson plans and transcripts of their lessons, Kahan, Cooper, and Bethea (2003) found that preservice teachers with strong mathematical content knowledge produced strong lessons. Similarly, strong mathematical content knowledge was also correlated with strong lesson transcripts.

Mathematical content knowledge is necessary and important for effective teaching; however, one study showed that teachers need more than just a strong knowledge of the content in order to teach mathematics. Thompson and Thompson (1996; 1994) found that although the teacher possessed a strong understanding of rate, he was unable to communicate that understanding in a way that would help a student develop a conceptual understanding of rate. The researchers found that the teacher often projected his understanding of rate into what the student's explanations.

Both Kahan et al. (2003) and Thompson et al. (1996; 1994) relied on the perspective that knowledge is situated in practice and can be inferred from the actions of teachers. In this situated cognition view, "knowing is viewed as the practices of a community and the abilities of individuals to participate in those practices; learning is the strengthening of those practices and participatory abilities" (Even & Tirosh, 2002, p. 232). Just as it is impossible to learn mathematics without doing what mathematicians do (Lave, 1997), it is impossible to learn to teach mathematics without doing what mathematics teachers do. Similarly, it is impossible to study how mathematics teachers use their knowledge without studying their use of that knowledge in the practice of teaching. While studies using methods segregating knowledge from the act of teaching to measure and understand a teacher's mathematical knowledge for teaching may provide some insights, studying a teacher's mathematical knowledge for teaching in situ will likely provide greater insights about their knowledge.

Several frameworks on teacher knowledge have been proposed. In addition to reviewing what the literature has said about the quantity and quality of teacher knowledge, this chapter discusses what the literature has said about how to study teacher knowledge. The next sections describe some of the more prominent frameworks on teacher knowledge.

Pedagogical Content Knowledge

In early research, pedagogical knowledge and content knowledge were considered separate entities. In simple terms, content knowledge included knowledge obtained in content-specific courses, while general pedagogical knowledge consisted of the subject matter taught in education classes (Ball & Bass, 2000). However, Shulman (1986) proposed that the two knowledge domains interacted to create another aspect of knowledge, termed *pedagogical content knowledge*.

In his framework of teacher knowledge, Shulman (1986) identified three categories: content knowledge, pedagogical content knowledge, and curricular knowledge. Content knowledge referred to the knowledge of the subject-matter the teacher was to teach. It extended beyond facts and principles of the subject and into the underlying structures of the subject, i.e. the organization of the subject and the establishment of truth in the subject.

Shulman (1986) described pedagogical content knowledge as subject matter knowledge that is pertinent to the act of teaching. "A second kind of knowledge is pedagogical knowledge, which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge *for teaching*" (Shulman, 1986, p. 9). This definition was refined and expanded through examples. According to Shulman, pedagogical content knowledge included knowledge of multiple representations in connection with the optimal representation needed to teach a specific topic, and knowledge of difficult and easy concepts for students as well as common misconceptions.

The final category of teacher knowledge was curricular knowledge. According to Shulman (1986), the curriculum included all the materials and resources available for the teaching of a subject, including full programs and supplementary materials. Additionally, the curriculum included the guides and suggestions of what should and should not be included in the instruction of a particular grade level. For example, the standards put forth by a state as well as resources from NCTM would be included as part of the curriculum. Shulman hypothesized that curricular knowledge needed for teaching extended beyond knowledge of the curriculum taught by the teacher to include knowledge of the other curricula their students were studying and the relationships among the different curricula. Similarly, teachers needed to have knowledge of the curriculum of previous and subsequent years. Curricular knowledge would assist teachers as they ordered their curriculum for the school year.

Shulman's (1986) conception of teacher knowledge has been adopted and refined by researchers in mathematics education. However, the detail and definitions used to study teacher knowledge have varied widely. Some mathematics education studies provided no explicit definition of teacher knowledge (e.g., Barnett, 1991; Steele, 2005). Others gave broad definitions of teacher knowledge along with a few supporting examples (e.g., Chinnappan & Lawson, 2005; Even, 1993; Llinares, 2000; A. G. Thompson & Thompson, 1996; Tirosh et al., 1998; Van der Valk & Broekman, 1999). For example, Chinnappan et al. (2005) provided the following definitions of teacher knowledge.

> *Mathematical content knowledge* includes information such as mathematical concepts, rules, and associated procedures for problem solving. *Pedagogical knowledge* refers to teachers' understanding of their students, and the processes involved in teaching. The *blend of content and pedagogical knowledge* includes understandings about why some children experience difficulties when learning a particular concept while others find it easy to assimilate, knowledge about useful ways to conceptualise and represent the chosen concept, the quality of explanations that teachers generate prior to and during instruction, and perceptions about the nature of mathematics. (p. 198)

These definitions exemplify the descriptions of teacher knowledge found in many other studies. They describe mathematical content knowledge as the content the teachers teach. General pedagogical knowledge refers to how students learn and strategies for teaching. Pedagogical content knowledge refers to the intersection of content and pedagogical knowledge. Such definitions are broad and lack detail.

Few researchers have elaborated more on Shulman's (1986) definition of pedagogical content knowledge and attempted to create a model or framework for examining teacher knowledge (e.g, An et al., 2004; Carpenter et al., 1988; Kahan et al., 2003). These models were often specific to the area of teacher practice the researchers were studying. Articles summarizing research on teacher knowledge also proposed models and frameworks for examining teacher knowledge (e.g., Even & Tirosh, 2002; Fennema & Franke, 1992; Graeber, 1999; NRC, 2001). These models often suggested studying subcategories of teacher knowledge as well as the interaction of the different categories. Commonly identified knowledge types were mathematical knowledge, knowledge of students, and knowledge of instructional practice (e.g., NRC, 2001).

Profound Understanding of Fundamental Mathematics

Ma (1999) provided another theory of mathematical teacher knowledge which has influenced the frameworks of other researchers in mathematics education. She described teachers' knowledge as consisting of *knowledge packages* (p. 113). Only when the teachers' knowledge was interconnected and rooted in the structure of mathematics were the knowledge packages adequate for teaching. She termed the mathematical knowledge needed for teaching as *profound understanding of fundamental mathematics*: "By profound understanding I mean an understanding of the terrain of fundamental mathematics that is deep, broad, and thorough" (Ma, 1999, p. 120). By depth, Ma was referring to conceptually connecting the mathematical topic to the structure of mathematics. On the other hand, the breadth of understanding referred to connecting the topic to similar ideas within the structure of mathematics. In terms of Shulman's (1986) framework, profound understanding of fundamental mathematics would best align with his category of content knowledge because profound understanding of fundamental mathematics referred exclusively to mathematical knowledge and did not include pedagogical knowledge.

Mathematical Knowledge for Teaching

Akin to profound understanding of fundamental mathematics, Ball (1990) identified three aspects of mathematics content knowledge. First, it included a correct knowledge of both procedures and concepts in mathematics. Second, mathematical content knowledge encompassed an understanding of the underlying principles related to the procedures and concepts. Finally, mathematical content knowledge included a network of connections relating different concepts and how each concept contributed to the whole of mathematics.

Ball, Lubienski, and Mewborn (2001) have elaborated on what Shulman (1986), Ma (1999), and Ball (1990) have discussed about teacher knowledge. M*athematical knowledge for teaching* refers to

> "Such knowledge is not something a mathematician would have by virtue of having studied advanced mathematics. Neither would it be part of a high school social studies teacher's knowledge by virtue of having teaching experience. Rather, it is knowledge special to the teaching of mathematics." (Ball et al., 2001, p. 448)

This definition was supported by many examples, both general and specific, of the knowledge teachers need to teach mathematics. These examples included "using curriculum materials judiciously, choosing and using representations and tools, skillfully interpreting and responding to their students' work, and designing useful homework assignments" (p. 433). Additional examples included responding to student questions and confusions and building on student thinking. According to Ball et al., mathematical knowledge for teaching extended beyond knowledge of mathematics and how to teach mathematics to include knowledge of the structure of mathematics. Additional examples of mathematical knowledge for teaching can be found in the literature (see Ball & Bass, 2000; Ball & Bass, 2002; Ball, Hill, & Bass, 2005; Hill, Rowan, & Ball, 2005; Hill, Sleep, Lewis, & Ball, 2007).

In their chapter on assessing teacher's mathematical knowledge, Hill et al. (2007) called for a better theoretical framework for studying and measuring mathematical knowledge for teaching. Hill, Ball, and Schilling (2008) suggested such a framework. They defined mathematical knowledge for teaching as "the mathematical knowledge that teachers use in classrooms to produce instruction and student growth" (p. 374). The framework divided teacher knowledge into six categories: *knowledge at the mathematical horizon*, *common content knowledge* (CCK), *specialized content knowledge* (SCK), *knowledge of content and students* (KCS), *knowledge of content and teaching* (KCT), and *curricular knowledge* (see Figure 1). The first three categories are knowledge of mathematics content; the last three categories are knowledge of content and pedagogy. Although this framework was developed to assist in writing assessment items, the authors suggested that the framework would apply to other investigations into teacher knowledge, including research on teachers' abilities to design tasks.

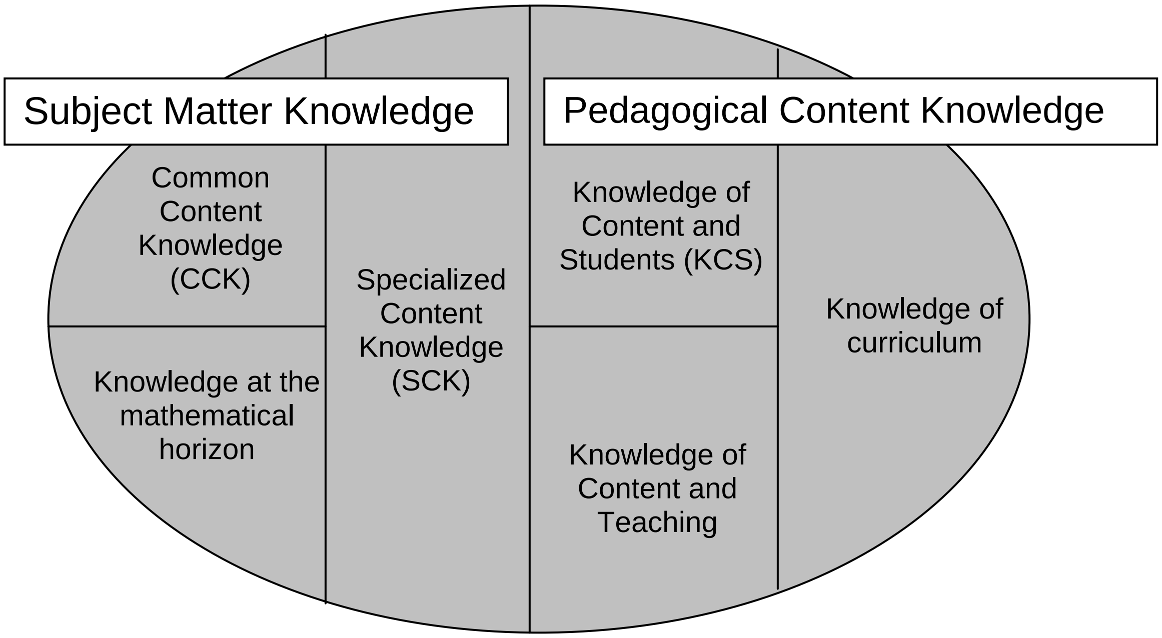

Figure 1. Mathematical knowledge for teaching. From "Unpacking Pedagogical Content Knowledge: Conceptualizing and Measuring Teachers' Topic-Specific Knowledge of Students," by H. C. Hill, D. L. Ball, and S. G. Schilling, 2008, Journal for Research in Mathematics Education, 39, p. 377. Copyright 2008 by NCTM. Adapted with permission.

Neither curricular knowledge nor knowledge at the mathematical horizon were defined in the text of the article (Hill et al., 2008). Given the strong influence of Shulman's (1986) theory on their work, it can be assumed that the curricular knowledge suggested in their framework is similar to what Shulman described in his.

In an article titled "With an Eye on the Mathematical Horizon: Dilemmas of Teaching Elementary School Mathematics", Ball (1993) discussed issues she encountered while trying to create mathematical experiences that built on her students' prior experiences, with an eye on the direction she wanted to take the class. In order to create meaningful experiences for the students, the teacher "must understand the specific mathematical content and its uses, bases, and history, as well as be actively ready to learn more about it through the eyes and experiences of [their] students" (p. 394). This article may provide insight into what Hill et al. (2008) meant by knowledge at the mathematical horizon.

Common content knowledge (CCK) and specialized content knowledge (SCK). In an attempt to distinguish between mathematical content knowledge and pedagogical content knowledge, Hill, Schilling, and Ball (2004) developed measures for determining a teacher's mathematical knowledge for teaching. They worked to develop test items dealing specifically with mathematical content knowledge that could be categorized as either CCK or SCK. CCK was determined by the mathematical knowledge an average adult should possess, beginning with elementary arithmetic through algebra and geometry concepts, as well as the mathematical knowledge of mathematicians and other professionals. NCTM's (2000) process standards, or things mathematicians do, would also be considered part of CCK. CCK also includes how an individual would solve a problem; for example, a teacher solving a problem for his/her own benefit would be CCK. In contrast, SCK is considered unique to a mathematics teacher. For example, a mathematics teacher should be able to create an ordering decimals problem that would address the mathematics of ordering decimals whereas the average adult or even mathematician would be unable to create such a task. SCK is deep in nature and includes performing error analysis, creating a mathematical explanation, and using representations (Ball et al., 2005). Furthermore, teachers not only need to be able to perform the procedures accurately, they also need to have a principled understanding of the mathematics (NRC, 2001). Hill et al. (2004) determined the nature of CCK and SCK to be related, but not equivalent, and concluded that SCK is a domain of knowledge related to, yet separate from, mathematics content knowledge and pedagogical content knowledge.

According to Hill et al. (2008), both CCK and SCK are mathematics content knowledge and do not involve any knowledge of students or teaching. In comparison to other research on teacher knowledge, CCK would align with Shulman's (1986) conception of content knowledge. SCK is likely a new construct not included in Shulman's theory (Hill et al., 2008). Ma's (1999) theory of knowledge packages would correlate well with the nature of the knowledge included in the SCK domain.

Knowledge of content and students (KCS). Hill et al. (2008) defined KCS as "content knowledge intertwined with knowledge of how students think about, know, or learn this particular content"(p. 6). It involves knowledge of common misconceptions and strategies, being able to assess student understanding, and knowing how students evolve in their mathematical thinking. KCS is distinct from CCK or SCK in that it requires knowledge of students. In relation to Shulman's (1986) framework, this knowledge domain would be a subcategory of pedagogical content knowledge.

Knowledge of content and teaching (KCT). KCT combines knowledge of content with knowledge of teaching. It involves knowing how to build on student thinking and strategies for addressing and correcting student misconceptions (Hill et al., 2008). In relation to Shulman (1986), KCT is another subset of pedagogical content knowledge, distinct from KCS in that it involves knowledge of teaching rather than knowledge of students. It is delineated from CCK and SCK in that it requires more than just knowledge of content.

Theoretical Framework of Teacher Knowledge

This study adapted the theoretical framework presented by Hill et al. (2008). This framework was selected because it was more developed and specific when compared to the other models considered. Additionally, it was the only model of teacher knowledge that separated mathematical content knowledge into two domains: CCK and SCK. The framework used in this study contained CCK, SCK, KCS, KCT, and curricular knowledge as defined above (Ball et al., 2005; Hill et al., 2008; Shulman, 1986). The framework omitted knowledge at the mathematical horizon because it was not defined well. Furthermore, Ball (1993) indicated that this type of knowledge was personal mathematical knowledge that could be gleaned from experiences with students whereas this study focused on the student teachers' knowledge in the planning phase, prior to their experiences with students.

As student teachers may use knowledge not related to mathematics in order to design or modify a task, the framework added an additional domain of pedagogical knowledge. *Pedagogical knowledge* refers to knowledge of students and teaching that is not directly related to a specific mathematical concept. This includes knowledge of how children learn in general (learning theories), child development, classroom management techniques such as establishing classroom norms, and other teaching practices not related to mathematics. Many of the frameworks reviewed in the literature did not consider the use of pedagogical knowledge when studying teacher knowledge. However, Webb (2006) found that preservice teachers spent a large amount of time anticipating nonmathematical responses during their lesson planning, which indicates that preservice teachers may use pedagogical rather than mathematical knowledge for teaching to make teaching decisions.

Figure 2 illustrates the author's conception of the different domains of teacher knowledge and is modified slightly from Hill et al. (2008). It shows the delineations among the different domains as well as how the framework relates to other research on teacher knowledge: the alignment of Shulman's (1986) content knowledge with CCK, Ma's (1999) profound understanding of fundamental mathematics with both CCK and SCK, and Shulman's pedagogical content knowledge with KCS and KCT. The alignment of the different theories is not perfect. For example, Shulman (1986) included knowledge of representations as part of pedagogical content knowledge; however Hill et al. (2008) placed knowledge of mathematical representations in SCK. Pedagogical knowledge is not shown in the figure because it is not considered part of mathematical knowledge for teaching.

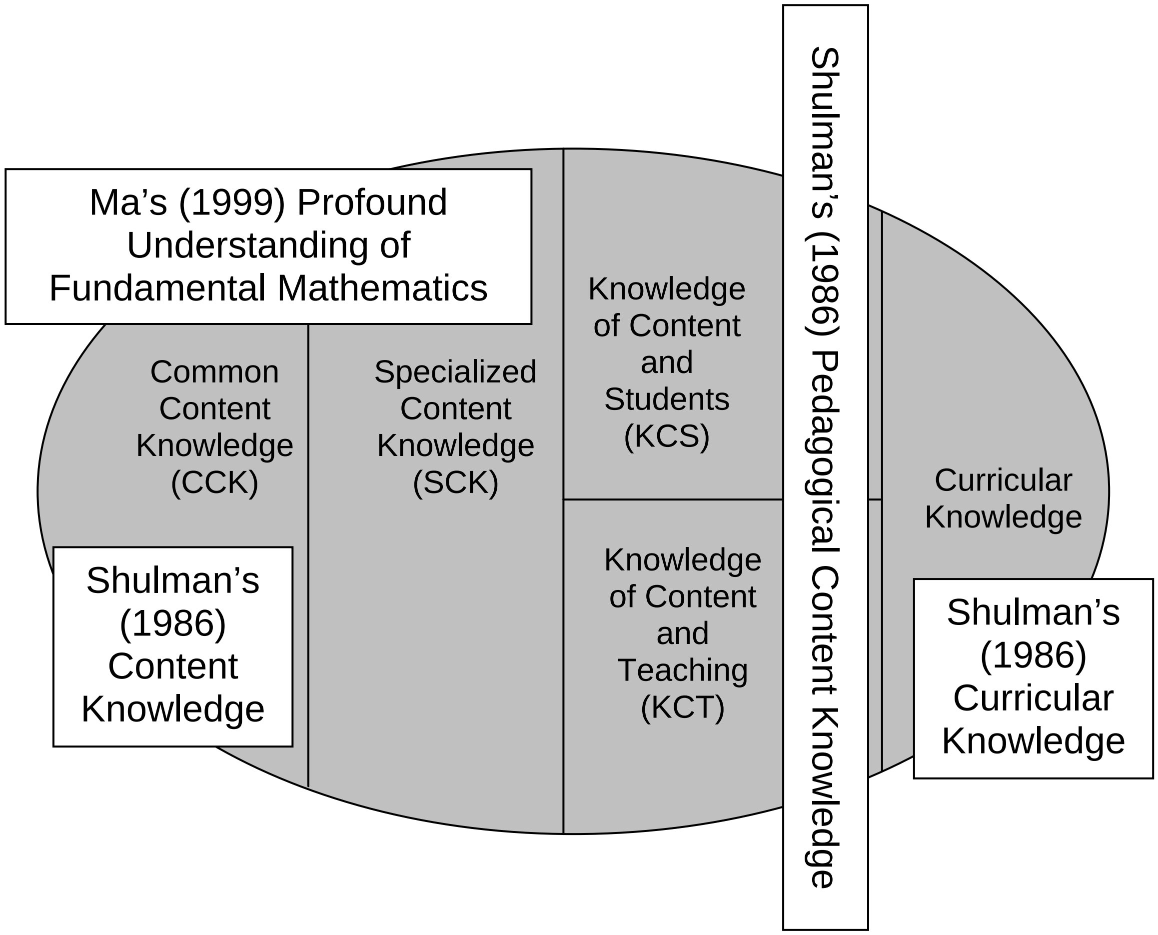

Figure 2. Integration of different theories of teacher knowledge. From "Unpacking Pedagogical Content Knowledge: Conceptualizing and Measuring Teachers' Topic-Specific Knowledge of Students," by H. C. Hill, D. L. Ball, and S. G. Schilling, 2008, Journal for Research in Mathematics Education, 39, p. 377. Copyright 2008 by NCTM. Adapted with permission.

Mathematical Tasks

Definition and Characteristics

Much of the research on academic tasks comes from Doyle and his colleagues (Doyle, 1988; Doyle & Carter, 1984). A task refers to the academic work students do in the classroom and consists of four components:

> (a) a goal state or end product to be achieved; (b) a problem space or set of conditions and resources available to accomplish the task, (c) the operations involved in assembling and using resources to reach the goal state or generate the product, and (d) the importance of the task in the overall work system of the class. (Doyle, 1988, p. 169)

Doyle et al. (1988) also described two different categories of tasks: novel and familiar. The categorization of a task as novel or familiar was dependent on the students' previous experiences. A novel task could become familiar if the teacher made the task routine in some way.

In order to improve the teaching and learning of mathematics, NCTM (1991; 2007) encouraged the use of *worthwhile mathematical tasks* (NCTM, 2007, p. 32). Not all tasks are created equal; some tasks engage students in the mathematics more than others. In order to facilitate student learning, a teacher must implement worthwhile mathematical tasks. NCTM (2007) provided a list of characteristics of such tasks:

> The teacher of mathematics should design learning experiences and pose tasks based on sound and significant mathematics and that---
>
> - engage students' intellect;
> - develop mathematical understandings and skills;
> - stimulate students to make connections and develop a coherent framework for mathematical ideas;
> - call for problem formulation, problem solving, and mathematical reasoning;
> - promote communication about mathematics;
> - represent mathematics as an ongoing human activity; and display sensitivity to, and draw on, students' diverse background experiences and dispositions. (pp. 32-33)

Tasks of such caliber appropriately challenge students to develop important mathematical concepts through communication and problem solving. Tasks that fail to meet the above description of a worthwhile mathematical task may not promote the desired mathematical learning.

The most research on mathematical tasks has been performed by Stein and her colleagues. Stein et al. (2000) defined a task as "a segment of classroom activity devoted to the development of a mathematical idea" (p. 7). "Tasks include expectations regarding what students are expected to produce, how they are expected to produce it, and the resources available for so doing" (Stein et al., 2007, p. 346).

Phases of a Task

Doyle et al. (1988) claimed that there a many different levels of a task. There was the task as it was launched by the teacher, the task as it was interpreted by the students, and the task as it appeared in the finished products of student thinking. They noted that a task may exhibit different features at different times.

Elaborating on Doyle et al.'s (1988) theory that a task changes at different levels of implementation, Stein et al. (2000) developed The Mathematical Tasks Framework which describes the phases of a task and it's impact on student learning (see also Stein et al., 2007; Stein & Smith, 1998). Stein et al. (2000) agreed with Doyle et al.'s (1988) theory that a task changes at it evolves through different levels of implementation; however, Stein et al. (2000) broadened the levels to include how the task appeared in the curriculum as well as how the teacher intended to use the task. The framework was based on research suggesting a significant difference between the task as it appears in the curricular resources and as it is implemented in the classroom (Stein et al., 2007). The first phase of a task is how it appears in the curriculum or other instructional materials or as created by the teacher. This was referred to as the *written* phase of a task. The next phase of a task is the *intended* phase, or how teachers plan to use the task in instruction. The final phase, the *enacted* phase, is how the task is actually implemented in the classroom by the students. Researchers need to investigate possible reasons for the changes in the task from one phase to another.

Stein et al. (2007) offered possible explanations for how and why tasks are transformed during instruction. The tasks may change between phases due to a teacher's knowledge, beliefs, or orientation to the curriculum. Structures and norms within the classroom, school, and community also hold explanatory power for the transformation of tasks.

Cognitive Demands of a Task

In order to better describe the differences between two tasks as well as the differences between two phases of one task, Doyle et al. (1988) identified two cognitive levels of academic tasks. The cognitive level of a task was defined as "the cognitive processes students are required to use in accomplishing [the task]" (Doyle, 1988, p. 170). Low-level tasks involved memorization, the use of formulas, and the use of search-and-match strategies. In contrast, high-level tasks involved decision making and interpretation. "The focus for tasks involving higher cognitive processes, then, is on comprehension, interpretation, flexible application or knowledge and skills, and assembly of information from several different sources to accomplish work" (Doyle, 1988, pp. 170-171).

Tasks requiring a high level of cognitive processes have features that distinguish them from low-level tasks. In a non-mathematical study of academic tasks, Doyle (1984) found key differences in the features of major (high-level) and minor (low-level) writing tasks. Major tasks were lengthy and ambiguous in nature and required more time for completion. Minor tasks were clearly defined and there was often an algorithm available for accomplishing the task.

Tasks require different kinds of thinking in order for students to successfully complete; the kinds of thinking are referred to as the *cognitive demands* of a task (Stein et al., 2000). Additionally, the cognitive demands of a task may change as the task passes through various phases. A task may require high levels of cognitive demand in the written phase and deteriorate to requiring low levels of cognitive demand during the enacted phase.

Stein et al. (2000) refined Doyle et al.'s (1988) theory on the cognitive levels of a task and developed four levels of cognitive demands for mathematical tasks. On the lower end of the spectrum were *memorization* and *procedures without connections*. Tasks requiring higher levels of demands were categorized as *procedures with connections* and *doing mathematics.* The descriptions of the different cognitive demands of tasks that follow come from The Task Analysis Guide (Stein et al., 2000, p. 16).

Memorization tasks. Memorization tasks require students to access previously learned facts, rules, definitions, formulas, etc. or commit them to memory. The use of a procedure is not practical either because a procedure does not exist or time limits prevent the use of a procedure. The expectations of the task are so clearly articulated that there is no ambiguity as to what the student is expected to do or reproduce. There is no connection to the underlying mathematics of the facts, rules, definitions, or formulas (Stein et al., 2000).

Procedures without connections tasks. The tasks in this lower-level category allow for the reproduction of a procedure. There is little ambiguity in what the students are expected to do: perform a series of steps to obtain the correct answer. If an explanation is required, the expected explanation focuses on the procedure used rather than connections to the underlying mathematics.

Procedures with connections tasks. The tasks in this category require the students to connect deeper mathematics to the procedure involved in the task. While a procedure may be implied in the instructions of the task, the procedure is closely tied to meaning and cannot be blindly applied. There are often multiple representations or pathways students may follow to arrive at a solution.

Doing mathematics tasks. Tasks in this high-level category require complex thinking accompanied by considerable effort and anxiety often due to the ambiguous nature of the task as well as the need to access relevant, time-removed knowledge. The students are required to explore mathematical relationships, monitoring their processes and solutions, considering constraints in the task that may limit possible solutions. Students are expected to justify their process and solution in terms of correct mathematical principles rather than procedures. Procedures with connections and doing mathematics tasks would likely be comparable to what NCTM (2007) described as a worthwhile mathematical task.

Superficial features of a task should not influence the analysis of the cognitive demands of a task. For example, many lower-level tasks could be considered higher-level tasks because they resemble reform-oriented tasks by the use of manipulatives, presence of multiple questions, or set in “real-world” context. Furthermore, the reverse is also true; high-level tasks can appear to be low-level if they resemble a traditional textbook problem.

Theoretical Framework of Tasks

The framework guiding this study was based on the notion that tasks often transform during different phases of instruction and that the written, intended, and enacted phases of a task will sometimes vary in subtle yet significant ways. This study adopted the framework and definitions of Stein et al. (2000) because of its specificity and detail, and focused primarily on the first two phases of a task: the written and intended phases. The task as it was given to and modified by the student teachers or developed by the student teachers was considered the written phase of the task. The intended phase of the task was how it appeared in the context of the lesson plan, including anticipated student thinking and acceptable student responses. The framework also relied heavily on the Task Analysis Guide (Stein et al., 2000).

Teacher Knowledge and Cognitive Demands of Tasks

This study sought to find a connection between teacher knowledge and the cognitive demand of a task. Although several researchers have surmised a connection between the two, the review of literature located only two studies investigating such a connection. Although Crespo (2003) did not study teacher knowledge explicitly, she found that how teachers launched tasks in the classroom changed as the teachers gained more teaching experience. In the beginning of the study, the teachers posed trivial, computational tasks and decreased the ambiguity of the task as they helped students. However, as the teachers gained more teaching experience the tasks posed to the students increased in complexity, being more open-ended and exploratory. Crespo hypothesized that teacher knowledge may hold explanatory power for the apparent change how teachers posed mathematical problems.

Another study explicitly examined the role of teacher knowledge in how teachers assessed the cognitive demands of mathematical tasks (Osana, Lacroix, Tucker, & Desrosiers, 2006). Osana et al. used Stein et al.'s (2000) definitions of cognitive demands of tasks to study whether teacher knowledge impacted preservice secondary mathematics teachers' abilities to sort the tasks correctly. They found that preservice teachers with stronger content knowledge, as measured by a standardized test were able to sort the tasks better than preservice teachers with weak content knowledge. Although this study investigated the connection between teacher knowledge and mathematical tasks, it differed from the present study in several ways. First, it measured content knowledge through the use of a standardized test rather than investigating the use of teacher knowledge in practice. Second, it only considered content knowledge and not the other domains of knowledge conceptualized by Hill et al. (2008). Additionally, it studied the impact of knowledge of preservice teachers' abilities to sort the tasks rather than plan the tasks.

Even though few studies investigating the impact of knowledge were found, many researchers have implied that such an investigation would prove fruitful. Dewey (1938) discussed the difficulty teachers face when attempting to plan educative experiences for their students.

> There is incumbent upon the educator the duty of instituting a much more intelligent, and consequently more difficult, kind of planning. He must survey the capacities and needs of the particular set of individuals with whom he is dealing and must at the same time arrange the conditions which provide the subject-matter or content for experiences that satisfy these needs and develop these capacities. The planning must be flexible enough to permit free play for individuality of experience and yet firm enough to give direction towards continuous development of power. (Dewey, 1938, p. 58)

Dewey implied the difficulty of creating educative experiences required a great amount of knowledge concerning the students who would engage in the experience and the subject-matter pertinent to the experience. The knowledge needed to create educative experiences could be comparable to the knowledge needed to create worthwhile mathematical tasks.

Stein et al. (2007) hypothesized that teacher knowledge, among other factors, contributed to the transformation of tasks between phases. They indicated that teacher knowledge likely impacts how tasks are used in instruction, implying that the impact of teacher knowledge on cognitive demands of tasks would be a fruitful area of study. Additionally, Hill et al. (2005) suggested investigating how mathematics teachers use their knowledge during planning. Combining the two suggestions, this study investigated how student teachers used their knowledge in the planning of tasks and the impact teacher knowledge had on the cognitive demands of the task. This study hoped that by studying student teachers there would be more opportunities to investigate the knowledge teachers rely on when they lack the most valuable knowledge for accomplishing a teaching job because student teachers may lack knowledge that experienced teachers possess.

Methodology

Structure of the Student Teaching Program

This study is part of a larger study involving secondary mathematics student teachers from a large university (see Galindo, Leatham, Peterson, & Wilson, 2008). As opposed to the traditional apprenticeship model of student teaching where the student teacher is assigned to a cooperating teacher and expected to learn how to teach by mimicking the actions of the teacher, the student teaching program for this study incorporated aspects of the Japanese model of student teaching which in turn uses aspects of Japanese lesson study (see Lewis, 2002).

The student teachers were assigned to clusters of four. Each cluster was divided into two pairs of student teachers. The pairs were assigned to different cooperating teachers. During weeks 3 through 5 and 14 of the 15-week student teaching experience, each pair of student teachers planned and taught one lesson. Each pair planned the lesson together and then each taught the lesson separately to different classes. The cooperating teacher, the other members of their cluster, and the university supervisor observed the two lessons taught by the pair of student teachers. Following the lessons, the observers and the student teachers discussed the teaching experiences in a reflection meeting, with a protocol encouraging the student teacher to take the primary role in the discussion. This sequence of events will be referred to as teach/observe/reflect sessions. Both the lesson and the reflection meeting were video recorded for later analysis.

Participants and Sampling

This study chose to investigate the knowledge of student teachers rather than the knowledge of practicing teachers. In Hill et al. (2008), the authors suggested that much of the KCS used by teachers was experiential rather than grounded in research. As student teachers do not have much experience with students and teaching mathematics, they may have to rely on different knowledge domains (e.g. CCK, SCK, KCS, KCT, etc.) than experienced teachers in order to design and modify tasks.

Additionally, student teachers may have more gaps in their mathematical knowledge for teaching than do experienced teachers. Instances where the student teachers lacked knowledge will be more frequent than they would be for experienced teachers. By studying student teachers, the researcher hoped to gain a better understanding of how student teachers use their knowledge in situations where they may lack the particular knowledge needed to teach the mathematics.

There were eight student teachers during the semester the data were collected. From the eight student teachers, one cluster of four student teachers was initially studied. Two of the student teachers were assigned to a cooperating teacher at a high school and taught pre-calculus; the other two student teachers were assigned to a cooperating teaching at a junior high and taught pre-algebra. A purposeful sample (Maxwell, 2005) was used to select the student teachers. This particular cluster was selected because of the traditional curricula encouraged by the school district as well as the mathematical teaching training of the cooperating teachers. The cooperating teacher at the high school had earned a master's degree in mathematics education; in contrast, the cooperating teacher at the junior high had only earned a minor in mathematics education. Although the knowledge of the cooperating teacher was not an integral part of this study, the cluster was initially selected to serve as a way to compare the pairs of student teachers in case the cooperating teacher's knowledge became an issue in the study.

The district-assigned curricula for the classes came from traditional textbooks. The high school teacher carefully followed the textbook and state core. The teacher at the junior high did not use the textbook adopted by the district, but used a collection of activities that he had gathered and felt aligned with the topics in the state core. The junior high classroom structure consisted of a 5-minute warm-up quiz, a problem-of-the-day presentation, and a task. The class period at the junior high was 45 minutes in length. A lesson at the high school lasted approximately 90 minutes and resembled more of a traditional classroom with the student teachers telling the students in a lecture how to do the mathematics rather than having the students explore the mathematics through a task.

Purposeful sampling was again used when deciding to omit the pair of high school student teachers. The structure of the high school classroom did not provide the necessary data for investigating teacher knowledge, especially that of how teachers use their knowledge to develop and modify tasks. Although the high school student teachers had the students work through examples, they did not have the students exploring mathematics through a task; the students worked on exercises instead of tasks. Consequently, the pair of student teachers assigned at the high school was not considered in this study.

The remaining pair of student teachers consisted of two females, Kristen and Abby (pseudonyms). Kristen and Abby are traditional university students in their early twenties. Kristen was married and expecting a baby soon after her student teaching experience. Abby took an 18-month sabbatical from school between her course work and her student teaching to serve a mission for her church. Both student teachers had taken university classes consisting of mathematics, education, and mathematics education courses. One objective of the university program, as well as the larger study, was to help preservice teachers learn to teach mathematics for understanding. The learning outcomes for the program included objectives in which preservice teachers engaged in mathematical inquiry and approached mathematics as a problem-solving activity, understood how students learn mathematics with understanding, and planned student-centered instruction that engaged students in mathematical inquiry.

As an ethnographic study, intended to describe and analyze the knowledge student teachers use to design and modify tasks, the researcher's primary objective in sampling was to sample a representative group (Mertens, 2005). The student teachers were representative of other mathematics student teachers. First, the student teachers' experiences as students were traditional in nature and likely resembled the past experiences of other student teachers (Ball, 1990a). Second, the student teachers were expected to teach mathematics differently from how they were taught and had received some training in the mathematics and pedagogy involved in teaching mathematics from a reform perspective.

This sample of two student teachers is also representative of the situation in which many student and practicing teachers find themselves. The student teachers did not have a reform textbook to serve as a reference of mathematical tasks and attempted to teach mathematics for understanding by replacing the traditional curriculum with tasks that they either created or modified. As not all teachers have access to task-based curricula, the situation of the student teachers is similar to the situations of other teachers.

This study initially planned to look at the enacted as well as written and intended phases of the tasks. However, the length of the 45-minute junior high classes did not allow the students to complete the task during the allotted time. The student teachers at the junior high would launch a task and give the students time to explore the task during the first 45-minute period. The classroom discussion of the task usually began at the end of the first 45-minute period and extended into the next day’s 45-minute period. The researcher assumed that the lessons would only last for one period and only planned to video record the first day’s instruction. Consequently, the entirety of the enacted phase of the task was not recorded and thus not analyzed in this study.

Unit of Analysis

This study focused on the knowledge teachers used in the design and modification of tasks. The units of analysis studied were the tasks in the four lesson plans created by the pair of student teachers. By examining the tasks, the researcher could analyze the cognitive demands of the tasks as well as infer, through the use of other data sources, the knowledge the student teachers used to design and modify the tasks. In the case of one task, the student teachers did not have time to give the entire task to the students; only the portion of the task included in the lesson itself was used in this study.

Data Collection

The data collected for use in this study came from multiple sources: lesson plans, lessons, reflection meetings, reflection papers, and interviews. The units of analysis were taken from the student teachers’ lesson plans; the other data sources were used to determine the knowledge associated with each task. The use of multiple data sources, in the form of two participants and multiple methods, provided for the triangulation of the data and increased the credibility of the study (Maxwell, 2005; Mertens, 2005).

Lesson plans. Van der Valk and Broekman (1999) showed that examining preservice teachers' lesson plans was an effective methodology for determining teachers' pedagogical content knowledge. Extrapolating these results to the assessment of mathematical knowledge for teaching, the use of lesson plans is one methodology for identifying mathematical knowledge for teaching. The student teachers were required to create a lesson plan as a pair for the four lessons they taught for the teach/observe/reflect sessions. A copy of each lesson plan was given to the lesson observers just prior to the lesson. Within the lesson plan, the student teachers described their teacher moves in the lesson, the anticipated student thinking, and formative assessment based on the anticipated thinking. The lesson plans contained the written and intended phases of the task. The lesson plan template the student teachers were expected to use can be found in Appendix A.

Lessons. Each lesson taught by the student teachers during weeks 3 through 5 and 14 were video recorded. Table 1 shows the dates for each recorded lesson.

Table 1

Dates of Data Collection for the Tasks

Data Source	Task 1	Task 2	Task 3	Task 4
Lesson Date	9/20/2007	9/25/2007	10/3/2007	12/6/2007
Reflection Meeting	9/20/2007	9/25/2007	10/3/2007	12/6/2007
Interview Date	9/24/2007	10/2/2007	10/10/2007	12/13/2007

The lessons were reviewed by the researcher and served as inspiration for the interview protocols. For example, instances in the lessons that may be related to a lack of knowledge were identified by the researcher as something to discuss during the interviews with the student teachers.

Reflection meetings. After both student teachers taught their lesson, the student teachers and observers met together in a reflection meeting to discuss the learning objectives and outcomes of the lessons. The reflection meetings always took place on the same day as the lessons. The reflection meetings were video recorded and transcribed. The student teachers discussed the tasks, how they planned to use the task in the classroom, and what they expected students to do with the task. Often, unexpected student thinking was discussed by the student teachers, providing insight into knowledge the student teachers did or did not use when designing or modifying the task. Table 1 shows the dates for each reflection meeting.

Reflection papers. Each week, the student teachers were required to write a reflection paper about their experiences. The student teachers were assigned a different focus for each reflection paper, sometimes the focus was on the lesson they had taught and other times it was on a lesson they observed. These papers occasionally added insight into the knowledge the student teachers used or did not use when planning or modifying the tasks. The reflection papers were not always written immediately following the lesson and reflection meeting, but were usually written a few days later.

Interviews. Interviews were used to increase the credibility of the study (Mertens, 2005), and to allow the researcher to better infer the knowledge the student teachers used to plan and modify the tasks. The lesson plans, lessons, and reflection meeting were reviewed for a few days following collection. During this time, literature regarding the topic of the task was consulted. Interview protocols were then created to further probe the knowledge of the student teachers. The interview protocols were reviewed by the researchers involved in the larger study, one of whom was the university supervisor, and modified accordingly. The structure of the interviews was flexible, closely following the interview protocol and asking additional questions to clarify and probe. In order to gain a broad understanding of the knowledge the student teachers used while teaching, questions were asked about the written, intended, and enacted phases of the task. During the interviews, the student teachers were shown clips of their lesson and were asked to interpret or evaluate student thinking. The interview protocols are included in Appendixes B through H.

The interviews were approximately 45 to 60 minutes in length. There was one interview for each lesson taught for a total of eight interviews: four with Abby and four with Kristen. The interviews occurred within one week of the lessons. The one week gap was needed in order for the researcher to review the lessons and reflection meetings, prepare the interview protocol, and meet with the researchers' graduate committee for feedback on the interview protocols. The interviews always occurred before the student teachers taught their next lesson, so that they would not confuse the different lessons they had taught. The interviews were audio recorded and transcribed by the researchers for later analysis. Table 1 shows the dates for the interviews of each task.

Analysis

The data analysis for this study consisted of four components: review of literature associated with the topics of the tasks, analysis of the cognitive demands of the tasks, analysis of the teacher knowledge, and analysis of the correlation between the cognitive demands of the tasks and the teachers' knowledge.

Review of literature about the mathematics of the tasks. In order to better understand the different components of knowledge the student teachers could have as they designed or modified the tasks, the researcher reviewed the mathematics education literature about the different topics of the tasks. The literature review highlighted knowledge that the student teachers both did and did not use to design and modify the tasks. The literature review was also helpful in the analysis of the cognitive demands of the tasks.

Analysis of the cognitive demands of the tasks. The cognitive demands of the four tasks used by the student teachers were analyzed using Stein et al.'s (1998) framework. Two of the tasks were designed by the student teachers. These tasks were analyzed both within and without the context of the lesson plan, i.e. the written and intended phases of a task. Two of the tasks were given to the student teachers by their cooperating teacher and then modified. These tasks were analyzed in their original and modified forms (written phase), and then within the context of the lesson plan (intended phase).

In order to increase the reliability of the study, the tasks in their original and modified forms or as they were designed by the student teachers were rated by four other graduate students in addition to the researcher. The raters attended an hour long training on how to use the framework to analyze the tasks. Three of the raters were consistent in their ratings. However, the fourth rater gave the tasks a rating that was different than the others, sometimes significantly different, on five of the six tasks they rated. Consequently, the codes from the fourth rater were discarded and not considered in the study. The other three rater's codes were consistent with the researcher and used in the results of the study to support the analysis of the tasks and provide additional reasons and justifications for the codes assigned to the tasks. Table 2 shows the codes for the tasks as coded by the raters and by the researcher.

Table 2

Comparison of Task Analysis Done by Graduate Raters and Researcher

Task	Rater 1	Rater 2	Rater 3	Rater 4	Researcher
Task 1					
Original	PwoC	PwoC	PwoC	PwC	PwC
Modified	PwoC	PwoC	PwoC	Math	PwoC
Intended	-	-	-	-	PwoC
Task 2					
Written	PwC	PwC	Math	PwoC	PwC
Intended	-	-	-	-	PwoC
Task 3					
Written	PwoC	PwoC	PwoC	PwC	PwoC
Intended	-	-	-	-	PwoC
Task 4					
Original	PwoC	PwoC	PwoC	PwoC	PwoC
Modified	Math	Math	Math	PwC	Math
Intended	-	-	-	-	PwC

Note. - = analysis not performed; PwoC = procedures without connections; PwC = procedures with connections; Math = doing mathematics.

Analysis of teacher knowledge. The analysis of the teacher knowledge was done in tandem with the analysis of the tasks and consisted of three stages. First, sentences from the lesson plans, reflection meetings, reflection papers, and interviews were coded using the modification of Hill et al.'s (2008) framework. The inferences of the student teachers' knowledge were based on the definitions described in Chapter 2. The first delineation inferred by the researcher was whether or not the statement contained evidence of pedagogical knowledge. If not, the researcher determined if the statement could be coded as CCK or SCK, depending on whether or not the knowledge use was in common with how others may use the knowledge. If there was pedagogical and mathematical knowledge in the statement, the researcher coded it as either KCS or KCT or curricular knowledge, depending on the type of pedagogical knowledge. Additionally, the knowledge inferred by the researcher could often be described as knowledge the student teachers' had while they were planning the task and knowledge the student teachers learned from their teaching experience. The student teachers often indicated that they either did not know something before they taught or they had not considered it before they taught. These instances were noted as times when the student teachers lacked knowledge.

The second stage of the analysis of the knowledge consisted of determining which knowledge was used to design or incorporate the task into the lesson plan. The researcher determined whether the knowledge was used by its presence in the lesson plan or an explicit indication by the student teachers that they had used that knowledge when planning the tasks.

In the third stage, the knowledge that was inferred from the coded sentences was synthesized and summarized into tables for each task. The researcher then inferred how the knowledge was used to plan the tasks. Finally, the researcher looked across the four tasks to see if there were any commonalities in how the student teachers used their knowledge to plan the lessons.

Analysis of the correlation between the knowledge and the tasks. The final part of the analysis consisted of looking for a correlation between the teacher knowledge used to design or modify a task and the cognitive demand of the task. This was done by looking at how the knowledge was used and determining whether that use either increased, decreased, or maintained the cognitive demand of the task. The researcher focused on changes in the phases of the tasks, i.e. from the original to modified form and from the written to intended phases. The researcher also considered times when the student teachers indicated that they did not use a particular knowledge when designing or modifying a task and how that may have impacted the cognitive demands of the tasks.

Results

The purpose of this chapter is to discuss the results of the analysis. The results are given separately for each of the four tasks. Included in the results is a description of the task, the cognitive demands of the task in its different forms, a description of teacher knowledge found in the literature, and a description of the knowledge the student teachers used as they prepared the task.

Task 1: Adding Ten Different Ways

Description of Task

Prior to teaching their first lesson, Kristen and Abby's cooperating teacher gave them an article from a newsletter (McAnallen, 2000) and asked them to create a three-day unit based on the ideas presented in the article. The second of the three lessons was recorded for this study. The task given for the recorded lesson was taken from a section of the article by McAnallen and included in the student teachers' lesson plan. Because the student teachers did not give the students a written task, the task had to be inferred from the lesson plan and from the article.

In the article, McAnallen (2000) presented an idea for a lesson plan with two objectives: to teach students to justify their work and to help students gain a conceptual understanding of addition. The teacher used what she called an *algebraic approach* to addition to accomplish the learning outcomes. In this algebraic approach, numbers are partitioned and recombined in different ways. For example, the numbers 27, 15, and 42 can be partitioned into tens and ones and recombined by adding the tens and ones (see Figure 3). Alternately, the numbers could be partitioned into fives and sevens and recombined by adding the fives and sevens separately (see Figure 4). For clarity in future examples, the actual numbers used in the article were 15, 25, and 37, while the numbers used by the student teachers in the lesson plan were 27, 15, and 42.

$27 = 20 + 7$
$15 = 10 + 5$
$42 = 40 + 2$
This means we add 20 + 10 + 40 = 70
and then we add 7 + 5 + 2 = 14,
so we have 70 + 14 = 70 + 10 + 4 = 84.

Figure 3. Sample student work from Task 1, partitioning into tens and ones (Lesson Plan 1).

$$
\begin{array}{rl}
27 = & 4+4+4+4+4+4+3 \\
15 = & 3+3+3+3+3 \\
+\ 42 = & 5+5+5+5+5+5+5+5+2 \\
\hline
= & 6(4)+6(3)+8(5)+2 \\
= & 24+18+40+2 \\
= & 20+10+40+4+8+0+2 \\
= & 70+14=84
\end{array}
$$

Figure ***4.*** Sample student work from Task 2, partitioning into fours, threes, and fives (Lesson Plan 1).

As a counter-example to justification, the author discussed the use of the “carrying” strategy in addition, stating that most students don’t understand that the “carried” one really represents a ten or a hundred, etc. In order to teach students to justify their work, the author suggested an algebraic approach to addition in which the students justify their addition through the use of “drafts”. The students partition the numbers for a first draft, combine some of them for a second draft, and possibly combine more if needed for a third draft. The author further emphasized the need for students to justify their work by giving the students the answer to a two-digit addition problem, 15 +

25 + 37, and then having the students focus on the process of adding the numbers rather than the answer itself.

According to McAnallen (2000), the "carrying" strategy of addition clouds the meaning of addition. "Addition is advanced counting . . . It doesn't matter where you start or end, just get 'em all" (p. 8). The article demonstrated this idea by showing the numbers in the addition problem as groups of ones that can be combined in any way. The teacher introduced the class to an axiom.

> An axiom is a proposition that we can't prove is true but we've never found a case that hasn't worked. Here is an axiom: If we take a set of equals and add them to a set of equals, the sum will be equal. (p. 8)

Through the use of the algebraic addition strategy, the teacher was trying to help the students understand the "algebraic assumption that left-hand side must always equal right-hand side" (p. 9). The article also suggested that, depending on the developmental level of the students, this addition strategy could be used to discuss the concept of similar terms in mathematics by having the students write the partitioning using multiplication and then discussing how you could combine the numbers. For example, in 3(5) + 5(5) + 6(6) + 1, only the three and five could be combined because they are similar terms.

The following paragraph from the article inspired the student teachers' task and lesson plan:

> Indeed, Rachel only assigns that one problem as homework, and requests that students find 10 different ways to add it. They must justify their work in the way that she has demonstrated. Sometimes a student will tell Rachel that they were taught a certain way to add. "That can be one of your ways," she'll tell them. (McAnallen, 2000, p. 9)

The student teachers modified the notion of adding one problem ten different ways and assigned the students in their class a similar task. Taken from the lesson plan itself, the student teachers wrote, “Teacher gives class 3 new numbers: 27 + 15 + 42. Teacher tells students to work in pairs with their neighbor to come up with as many different ways as they can to add these numbers together. Tell them they have 10 min” (Lesson Plan 1). Within the context of the lesson plan, the student teachers placed the task assignment following a brief review of the previous lesson where the students added 15, 25, and 37, “explaining again how to add each place value, and emphasizing that we can combine numbers in many different ways . . . as long as we make sure to count everything once” (Lesson Plan 1). After the students had had time to work on the task, the student teachers planned to ask several students with “really creative approaches” to present their work at the board, stating that the students will need to “explain their thinking” (Lesson Plan 1).

Cognitive Demand of the Task

The cognitive demand of this task was analyzed in three different stages: in its original form as given to the student teachers, in its modified form, and within the context of the lesson plan. The raters coded this task in its original and modified forms (see Table 2). In both forms the raters coded the task as procedures without connections. In its modified form I agreed with the raters; however, I coded the original form of the task as procedures with connections. The discrepancy is likely related to the selections of the articles read by the raters. In order to decrease the amount of necessary work for the raters, they were only given the excerpt from the article that explicated the task rather than the article in its entirety. However, given that the student teachers read the entire article and selected the task from the article, it is best to consider the task within the context of the article.

The lack of purpose in the task made determining the cognitive demands of the task difficult. It was difficult to determine the mathematical goal from the directions to add the problem ten different ways. This made it difficult to determine whether the students would be grappling with any mathematics or if they would descend into *unsystematic exploration* (Stein et al., 2000). Consequently, it was difficult to assign a cognitive demand to the task based only on the directions. Within the context of the article the mathematical goals of the task were fairly clear; it suggested several purposes: to help students conceptualize addition, to discuss the conventional use of the equal sign, to introduce like terms, and to discuss what it means to justify something mathematically. According to the article, the decision to pursue one of these purposes should depend on the developmental level of the students.

Using the framework from Stein et al. (1998), the task within the context of the article was coded as procedures with connections. The article emphasized that the algebraic approach to addition was intended to focus the students' attention on the meaning of addition. It also emphasized that students needed to justify their strategy, using the meaning of addition and the axiom discussed in class. The justification expectation meant that the task could not be "followed mindlessly" (Stein & Smith, 1998). The procedure of partitioning the numbers provided the students with a general approach to the problem, but the students still had the freedom to try a different partitioning. Additionally, the article focused on varying the task so that it was catered to the developmental level of the students in the class. This indicated that the author of the article wanted the task tailored to the students so that the task would be adequately challenging for the students. The use of multiple representations was the one point in the task analysis guide that was not met by the task.

The other raters determined that the task in the article was a procedures without connections task. They suggested two reasons for coding the task as they did: the placement of the task following the discussion simplified the task into an exercise and the lack of purpose and meaning in the task. One rater stated that "it could be rated higher if there was a good connection to a mathematical principle." The later reason for coding the task as procedures without connections would have been resolved had the raters read the article in its entirety since the article did contain some mathematical purposes. Although the former reason is valid, the examples given in the article did not limit the students' approaches the problem. On the contrary, the author encouraged the students to try different approaches to the problem as long as the students could justify the validity of their approach. In fact, the examples from the article were generated by the students. Although the raters brought up two good reasons to code the task as procedures without connections, their reasons are not as strong when the task is considered in the context of the article.

The student teachers modified the task when they put it into their lesson plan. Abby described how they changed the task by not emphasizing the axiom as much.

> I remember we changed it, but [pause]. Because in the article it emphasized more, it called it the Awesome Axiom. It was that we could add up all the numbers in a different way and as long as we had all the parts it would still be the same. But we didn't really talk about that it was an axiom. Like in the article, she had given her students a challenge that if they could disprove that she would give them a $50 bill. So it talked about how they all went home and tried to add the numbers in a way that wasn't going to give them the same answer. So we kind of skipped over that whole part. (Abby, Interview 1)

Although the student teachers understood the meaning of addition, as indicated by the statement “we can combine numbers in many different ways . . . as long as we make sure to count everything once”, Abby however indicated that they did not view having the students understand addition conceptually as the purpose of the task.

One the other hand, Kristen stated that she wanted her students to understand the axiom but that she did not want to use the term *axiom* with the students:

> Like there's the axiom. You know, like if you have two things that are equal and another set of two things that are equal, and if you add the two things together they remain equal. And I didn't want to say that in words; you know, that this is an axiom, but I wanted them to become comfortable with that mathematical idea. That even if I rewrite 15, 27 and 42, since these are all equal and since these still are equal, it's still going to be equal if I continue to recombine things. (Kristen, Interview 1)

From Kristen’s statement it seems that the student teachers wanted to convey the axiom to the students, but they did not want to discuss that it was an axiom or take time to have the students try to disprove the axiom.

In the directions the student teachers intended to give the students, they did not explicitly ask the students to justify or explain their solution strategy. By removing this component of the task from the directions, the student teachers implied to the students that justification was not a key purpose of the lesson. Although the student teachers titled the lesson “Proving Addition Facts” on the board, Kristen indicated that justification was not a mathematical purpose of the lesson:

> Well, that title was more for just hopefully giving the students a reason for doing it. I think that the day before we had called it “Adding Different Ways”, but they didn't care about adding different ways. So I thought, “Well in math it's kind of a big deal to be able to prove things, like prove it’s 82, so let's tell them that's why we're doing it. (Kristen, Interview 1)

The student teachers were trying to provide the students with a purpose for doing the task and they decided to use the notion of proving as the purpose. However, it was intended to motivate the students to work on the task rather than to get the students to grapple with the mathematics.

Given these two changes to the task, de-emphasizing the meaning of addition, and removing the justification, the task as modified by the student teachers was coded as procedures without connections by both the raters and the researcher. The raters' concerns about the unmodified task were the same in the modified version of the task: the students' prior experience made the task unproblematic and there was no purpose to the task. Additionally, one rater felt that the task was too easy for pre-algebra students.

Using the framework (Stein & Smith, 1998), the modified task fit better with the description of procedures without connections than with procedures with connections. The task itself required no explanation and consequently the students did not need to make connections to any mathematics. Additionally, the difficulty of the original task was in the required explanation; simply adding the numbers different ways required little cognitive demand to complete.

When considered in the context of the lesson plan, the intended task remained a procedures without connections task. If the student teachers had had an expectation that the justifications of approaches connected to a mathematical principle then the cognitive demand of the task may have changed. However, the anticipated student work given in the lesson plan indicated a focus on the procedure of adding rather than the meaning of addition. For example, in describing one possible solution, the student teachers partitioned 27 into 20 and 7, 15 into 10 and 5, and 42 into 40 and 2. They then said, "This means that we add 20 + 10 + 40 = 70 and then we add 7 + 5 + 2 = 14, so we have 70 + 14 = 70 + 10 + 4 = 84" (Lesson Plan 1). This example of an anticipated student response is typical of the anticipated student thinking in the lesson plan in that it walks through the arithmetic but does not describe why they could partition and add the way they did or even why one would want to do so. Additionally, in the formative assessment column, the student teachers read into the student response by saying that the "student understands how to add place value separately, [and] can correctly re-write each number in expanded form" (Lesson Plan 1). The student may or may not understand place value, but without more of an explanation beyond the arithmetic steps performed to solve the problem it is difficult to assess the students' understanding.

The interviews provided further evidence that the student teachers were looking for procedural explanations of the students' strategies. In the interviews, the student teachers were shown video clips of student explanations from their lessons and were asked whether the students justified their solutions. One of the students in Kristen's class partitioned the numbers but did not recombine them to confirm that both sides of the equal sign were still equal. When asked whether this student justified their answer Kristen said, "I was hoping, because he is a bright kid, so I was hoping, I was trying to get at how he recombined these. But it turns out that he hadn't anyway, he had just put 82" (Interview 1). Kristen did not accept this justification because the student had not explained the procedure he used to recombine the partitioned numbers.

In another video clip Kristen asked to elaborate on, the student stated what they did to solve the problem but did not explain why they chose the strategy they did or why their strategy was legitimate. When asked to evaluate this student's explanation Kristen said, “I think that this one is justifying because he's saying, adding the tens and adding the ones, in adding those together” (Interview 1). Even though in the video of the lesson the student does not talk about adding the tens and ones and why that is possible, the student teacher inferred that understanding from the students’ explanation and considered it an adequate explanation.

Abby also felt that simply explaining the steps taken to recombine the numbers was an adequate justification of the process. When shown one of her student’s explanations of their strategy where they explained their steps of recombining, Abby said that she thought it met their expectations:

> I like that he had, that he had taken the sevens that he took out from the ones. That he recognized that it was 14, and then he took out all the tens from each one and he made the 20 and then with that, he added the 24, which I guess is what he was showing here with the 14. I think I would have liked it better if he would have left the ten separate and written out the 14 as plus ten plus four. Then he could've put plus 14 and recognized that it was another ten and then recombined them. (Interview 1)

Abby focused on how the recombined the numbers by adding the ones and then adding the tens. By indicating that she was unsatisfied with the students’ strategy of not breaking up the 14, Abby indicated that it was the process that determined an acceptable explanation.

Although the student teachers indicated in the lesson plan that they expected the students to explain their strategy, the explanations were focused on the procedure and not connected to the mathematics. For this reason, the task remained at the procedures without connections level when considered in the context of the lesson plan.

Knowledge Associated with the Task

During the analysis of each task, a literature review of the topics of each task was performed. This literature review was done to gain a better understanding of the knowledge a teacher could have as the designed or modified the tasks; it is not meant to describe the knowledge pieces a teacher should have. The findings of the literature review are given in this chapter to justify areas where the student teachers may have lacked knowledge.

As indicated in the article (McAnallen, 2000), this task could have been used to accomplish several mathematical goals in the lesson. Consequently, there are several different components of knowledge that could be associated with this task. The list of knowledge compiled in this section is not meant to be exhaustive but to provide a glimpse of the knowledge the literature and task analysis suggested could have been used in designing the task.

Knowledge of whole numbers operations and early algebra as found in the literature. Task 1 dealt with whole numbers and primarily the operation of addition. The use of the multiplicative operation was also likely to occur in the task. Additionally, the article suggested that the algebraic approach to addition could lend itself to early algebra discussions. Consequently, a review of research on whole number operations and early algebra was performed to describe the possible knowledge the student teachers could have used to modify the task. In the reviewed research, knowledge could be inferred and categorized as KCS, KCT, and SCK. As the research was focused on mathematics teaching, it did not address CCK or pedagogical knowledge. Curricular knowledge was often beyond the scope of the reviewed literature because the literature, and the task for that matter, is dealing with one particular topic rather than how that topic fits in with the rest of the subject, although there existed some hints of local organization of the curriculum and prerequisite understanding.

The KCS in the literature covered common strategies, sequencing, and common misconceptions or difficulties. Children develop strategies for solving multidigit arithmetic problems without formal instruction, often inventing their own algorithms (Carpenter, Fennema, Franke, Levi, & Empson, 1999). One study identified two primary classes of strategies children used to solve multidigit addition and subtraction problems: *decompose-tens-and-ones* and *begin-with one-number* methods (Fuson et al., 1997). In the former strategy, the numbers were partitioned into tens and ones and then combined separately. In the later strategy, the child partitioned the second number into tens and ones and counted up or down from the first number. Other research projects have found similar strategies under slightly different names (Verschaffel, Greer, & Corte, 2007). A third strategy identified by some researchers is *compensating* or *varying*, where the child adjusts the numbers slightly to make the addition or subtraction easier (Verschaffel et al., 2007), although Fuson et al. (1997) considered these types of strategies to be incorporated into the two classes of strategies they identified. Carpenter et al. (1999) identified the following strategies of multidigit addition and subtraction invented algorithms: incrementing (begin-with-one-number), combining tens and ones, and compensating.

Common difficulties and misconception associated with multidigit arithmetic and transitioning into early algebra have also been identified. One misconception children commonly held was viewing multidigit numbers as adjacent single-digit numbers rather than considering the meanings of the different positions (Verschaffel et al., 2007). Many researchers have found the transition from arithmetic to algebra to be cognitively difficult for students (Verschaffel et al., 2007). The relational use of the equal sign, a common misconception found in middle-school students, was shown to influence students' ease transitioning from arithmetic to algebra (Knuth, Stephens, McNeil, & Alibali, 2006).

KCT found in the literature centered on the types of experiences teachers needed to provide students so that students could build their understanding of multidigit arithmetic and algebra. Verschaffel et al. (2007) stated that as students encountered experiences with counting and manipulating sets of objects students would begin to operate on those objects. Teachers can build on students' natural tendencies to use operations when counting sets. Another study found that teachers' use of the standard algorithm contributed to the development of more buggy algorithms (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998).

Careful selection of tasks is necessary if teachers want to build on students' arithmetic skills and push the students toward thinking algebraically. The teacher should select tasks that motivate a need for algebra by demonstrating its power (Verschaffel et al., 2007). A teacher wants to select arithmetic tasks that provide students with the opportunity to find patterns, generalize, conjecture, and justify relationships (Verschaffel et al., 2007). In the context of Task 1, if a teacher wants to begin to push students into thinking algebraically, they should select student work that demonstrates the power of algebra.

As already noted in the discussion of KCS, many students have a deficient understanding of the use of the equal sign, treating it as operational rather than relational

(Knuth et al., 2006). This deficit in understanding could be due the different treatments of the equal sign in arithmetic and algebra (Carpenter, Franke, & Levi, 2003). In an example akin to the adding ten different ways task, Falkner, Levi, and Carpenter (1999) illustrated how the teacher can develop students' algebraic reasoning by selecting tasks where the students explore the relational use of the equal sign:

> A child who has had many opportunities to express and reflect on such number sentences as 17 – 9 = 17 – 10 + 1 might be able to use the same mathematical principle to solve more difficult problems, such as 45 – 18, by expressing 45 – 18 = 45 – 20 + 2. This example shows the advantages of integrating the teaching of arithmetic with the teaching of algebra. By doing so, teachers can help children increase their understanding of arithmetic at the same time that they learn algebraic concepts. (Falkner et al., 1999, p. 233)

Discussion of SCK in the literature focused on representations and principled understanding of the mathematics. Verschaffel et al. (2007) discussed the different representations of numbers and operations on numbers, such as base-ten representations and a number line. Additionally, teachers could know the different ways algebra can be represented in arithmetic, such as pattern-finding and generalizations (Verschaffel et al., 2007). Teachers could also know that addition can be represented by the combining of sets or the joining of segments of different lengths (NRC, 2001).

Additionally, teachers need a principled understanding of the underlying mathematics in their students' strategies in order to determine their validity. For example, in the decompose-tens-and-ones strategy discussed earlier, teachers need to recognize the students' tacit use of the mathematical principle that numbers can be partitioned and then recombined (Verschaffel et al., 2007). Teachers could be aware of the use of the commutative and associative properties in arithmetic. These two properties allow for a great deal of freedom in arithmetic (NRC, 2001). The standard algorithms depend on these properties as well as the distributive property, yet their compact nature often hides the use of the properties (Verschaffel et al., 2007).

The literature contained some implicit references to curricular knowledge that dealt primarily with the prerequisite understanding students need in order to understand multidigit arithmetic. The use of a decimal number system requires a firm understanding of place value and different ways numbers can be represented, for example, standard and expanded forms (Verschaffel et al., 2007). Prior to instruction on multidigit arithmetic, children need ample experiences with the base-ten number system (NRC, 2001).

Knowledge the student teachers used to modify and incorporate the task. As the student teachers modified the task from the article and incorporated it into their lesson plan, they used knowledge from all six identified knowledge domains. For each category, there will be a discussion of the knowledge the student teachers used as well as a comparison of their knowledge to the literature.

Table 3 shows a list of the pedagogical knowledge used as well as sample statements. The pedagogical knowledge used in Task 1 included the principles of connecting to the students' prior learning, using contexts that interest students, encouraging the students to work in groups, managing the classroom with time limits, and observing students as they work on the task. The pedagogical knowledge demonstrated by the student teachers in this task was used with the purpose of motivating the students to want to do the task. The student teachers tried to motivate the students to work on the task by changing the title of the lesson to "Proving Addition Facts". The student teachers were planning to look for "creative" and "interesting" student work that would "excite" the students in the class and thus motivate the students to want to work on the task more. The student teachers were also hoping to motivate the students by having them work with partners. The student teachers used their pedagogical knowledge to manage the classroom by setting time limits, having the students work in groups, and walking around the classroom to increase proximity. They used their pedagogical knowledge to come up with strategies to help the students learn, for example reminding the students of the previous day's lesson and planning to have the students present their solutions to one another.

Table 3

Pedagogical Knowledge Used in Task 1

Pedagogical Knowledge	Example Statement
Try to connect new learning to previous student experiences.	Review of Day 1
An interesting context and creating ownership will motivate the students to want to do the task.	So we were trying to make it fun for them and that they could do because a lot of them when we started this, especially in Kristen's class on the first day, they were like, "We already know how to add, why do we have to do this?"
Have students work together so that they can help each other.	Students also help each other to understand how to add place values.
Manage the classroom by setting a time limit on the task.	Tell them they have 10 min.
The teacher should observe students as they work and have students present to facilitate the discussion.	Teacher walks around observing students working, and looks for creative ideas the students have.

Similar to the table for pedagogical knowledge, Table 4 shows a list of the curricular knowledge the student teachers used when planning the lesson and a sample statement. As the student teachers did not have much control over what was to be taught, they did not use their curricular knowledge often. The student teachers considered the state core to be a curricular resource that told them what they needed to teach. While this may not be true, the student teachers treated it as a fact and thus the researcher considered the notion that the state core could serve as a curricular resource to be knowledge. Additionally, the student teachers knew that the article they were given by their cooperating teacher could be used as a curricular resource. The student teachers were told that the task could be used to push the students toward algebraic thinking; however, whether the student teachers had the knowledge to actually do this will be discussed later. Finally, the student teachers knew that the rewriting of numbers was prerequisite to the students understanding and successfully completing Task 1.

Table 4

Curricular Knowledge Used in Task 1

Curricular Knowledge	Example Statement
The state core is a curricular resource.	We tried going through the State Core to see what objectives it met, but the only connection we could identify is with Standard I Objective 1-1, which requires students to be able to explain why addition works.
The article is a curricular resource.	In order to plan our lesson we were given an article addressing this addition activity.
The task could be used to bridge the gap between arithmetic and algebra.	And we talked to our cooperating teacher and we talked about how this could lead into combining like terms.
Place value and different ways of writing numbers are prerequisite to this lesson.	Because they've been learning tens, hundreds and they've also been learning the difference between a digit and a number, you know, that digits are zero to nine.

This task exemplifies some of the common misconceptions the student teachers had about the curriculum: First, the state core provides a curriculum (a list) of what should be taught as well as possible prerequisite knowledge of their students. Second, any article or activity that the cooperating teacher gave to the student teachers was worthy of being incorporated into the curriculum. (There is evidence that the student teachers begin to evaluate the different activities and articles given to them by their cooperating teacher later.) Although they were told by the cooperating teacher and the article that this task provided an opportunity to discuss like terms, the student teachers did not incorporate this idea into the lesson plan except in the title. In fact, Abby did not understand how the task could be used to discuss like terms because “these all are like terms already, because they're all whole numbers” (Interview 1).

Table 5 provides a summary of the student teachers’ KCT. The student teachers used this knowledge to plan how they would build on student thinking, to choose the numbers of the task to avoid certain false generalizations, and to determine how much information to give the students before they worked on the task. By giving the students the answer to the addition problem as was suggested in the article, they wanted the students to realize that the answer was not as important as the justification and thus encourage the students to justify their strategies. The student teacher’s planned move of questioning the students to help them understand the role of place value in the standard addition algorithm was a broad suggestion without specific questions. Even though the student teachers planned to ask questions, without the specific questions it is difficult to determine whether the student teachers’ questions actually would have helped the students understand.

Table 5

Knowledge of Content and Teaching Used in Task 1

KCT	Example Statement
Focus the students on the process rather than the answer.	Wednesday, when I began my lesson, I think it helped a lot that I put the three numbers up on the board and had the students quickly add them in any way they wanted to.
Look for particular student thinking to build on (connection between multiplication and addition, and place value).	But then also, I was looking for the connection between addition and multiplication.
Students tend to form false generalizations so vary the numbers to avoid this.	I think that he [The cooperating teacher] was worried that students would catch that pattern and only see that pattern and not do it if they see that pattern and that may have happened.
Misunderstandings regarding algorithm can be remedied by asking questions about place value.	Teacher asks students questions to try to help them understand that it is a 10 they are representing with that 1.

When compared to KCT found in the literature, the student teachers' KCT did not include how to build on students' arithmetic skills to push them into algebraic thinking. They also did not consider how to build on students' natural strategies nor possible buggy algorithms stemming from the method of addition taught in the lesson. Their concern to avoid false generalizations came primarily from their cooperating teacher. The article of the original task gave the pedagogical suggestion to provide the students with the answer to the addition problem.

Table 6 provides a list of the KCS the student teachers used when modifying Task 1 as well as an example statement. The student teachers knew that even though the students had already learned the addition algorithm, many students do not understand how the addition algorithm works and described how a student might interpret the “carried” one as having a value of one rather than representing a ten. Also, because the students already knew how to add, the student teachers knew that the task was too simple for junior high students. The student teachers also knew that some students did not have a conceptual understanding of addition nor did the students understand that multiplication was repeated addition. Finally, the student teachers took the students’ abilities to partition the numbers into tens and ones as evidence that the students understood the place value system.

Table 6

Knowledge of Content and Students Used in Task 1

KCS	Example Statement
If students partition the numbers using tens and ones then they understand place value.	Student understands how to re-write place value separately . . .
Students don’t understand the addition algorithm.	Student is using “carrying” correctly, but as we can see by his explanation, he doesn’t understand that the little “1” he writes over the tens column actually represents 10, not 1.
Students have difficulty connecting addition and multiplication.	Part of the reason was that I've noticed that some of the students have a problem understanding what multiplication is. But it really is, that the 15 is five plus five plus five, and that's three fives, three groups of five.
Students are already familiar with the addition algorithm.	I would say that every student is comfortable with adding using carrying.
Students don’t have a conceptual understanding of addition.	I don't think that as many of them really understanding that addition is just combining everything and the way you combine isn't as important as that you get everything exactly once, and so that's where they were lacking.
This task is too simple for junior high students.	We thought this was way too simple for junior high students.

The student teachers’ used their KCS to plan how they would evaluate student understanding, for example, by looking for evidence of understanding of place value in how the students partitioned and recombined the numbers. They also used KCS to anticipate student thinking. For example, they knew that they students were already familiar with the addition algorithm but may not understand why it works. They anticipated that some students would add this way, and using their KCT, they planned how they could help the students understand the algorithm better. Their knowledge of common student difficulties also motivated the student teachers’ purpose of the task and the kinds of student thinking they were looking for as the students worked.

The literature on KCS pointed to strategies of multidigit arithmetic and common difficulties. From the examples given in the lesson plan, the student teachers were only aware of one strategy aside from the standard algorithm that students might use to add multidigit numbers: decompose-tens-and-ones. This may be related to the article as it was the strategy promoted by the algebraic method. However, the student teachers seemed open to other methods given that they wanted to see “some really creative approaches”. One can assume that if the student teachers were aware of some of the other methods they would have included them as possible student solutions. Additionally, the student teachers showed no evidence that they knew of difficulties in the transition from arithmetic to algebra caused by a misconception of the equal sign even though this was implicit in the article.

SCK was not used much by the student teachers (see Table 7). Their SCK consisted of knowing why the addition algorithm worked and being able to conceptualize addition as the combining of sets. They used their conceptual understanding of addition to plan how they were going to explain the axiom to the students. The student teachers used their knowledge of how the addition algorithm worked to determine that this task may address this concept.

Table 7

Specialized Content Knowledge Used in Task 1

SCK	Example Statement
How the addition algorithm works.	Carrying is really more breaking it down into tens and ones.
How to conceptualize addition.	So to show that addition is just counting up the numbers and then putting each number once.

There were great differences in the SCK in the literature and the SCK the student teachers used to modify and plan Task 1, which was to be expected. Although the student teachers held one conceptual understanding of addition, that of combining sets, they showed no evidence that they knew that addition could also be represented by adjoining segments. This was not surprising given that the conceptualization of set combining aligned more with the addition strategy they wanted the students to learn.

The student teachers openly admitted that they had not considered the use of the associative and commutative properties in the strategy that they were teaching. Abby explained that it would have been a good idea to talk about the commutative and associative properties of addition and multiplication in conjunction with Task 1 because "these properties [commutative and associative] are things they need to learn about in this class because they are part of the core" (Reflection Paper 1). From this quote and the student teachers' use of the core as a list of things they needed to teach, the researcher inferred that if the student teachers had known that Task 1 could be used to discuss the commutative and associative properties, they would have included it in their lesson plan. Additionally, the student teachers were unable to describe the use of the distributive property in adding like terms.

Although the article containing the original task discussed the connection between the algebraic method of addition and combining like terms, the student teachers were unsure how this task related to combining like terms. When the cooperating teacher suggested that this task could be used to introduce like terms, the student teachers disregarded the suggestion. Abby indicated that:

> We had talked about giving them something more algebraic with different terms in it and having them deal with it, but having had them already thought about addition in a different way. Instead of just going through, and having them always add up numbers, but breaking them down and having something with variables in it. They could break it down and look for what was common in order to find like terms. I don't know that we would go from this [just using numbers] to talking about combining like terms because these are already like terms, because they're whole numbers. (Interview 1)

Abby's indication that like terms needed to be taught using variables and that all whole numbers are like terms pointed to Abby having a segregated view of algebra and arithmetic. She seemed to be unable to consider how algebra related to the arithmetic of the task.

After the reflection meeting where there had been a brief discussion about using the task to discuss combining like terms, Kristen indicated that she had thought more about the concept of combining like terms:

> And so I was trying to think about what he [the cooperating teacher] meant by combining like terms, and I think that the idea behind it is that we see all the sevens and so we can combine the sevens. And we see all these threes and so we can combine the threes, I think. (Interview 1)

From this statement, Kristen's view of algebra as generalized arithmetic does not seem to be as narrow as Abby's view. However, the notion of discussing like terms in the context of arithmetic appeared to be a novel idea to Kristen. The student teachers likely did not see the connection between the arithmetic of the task and algebra because they did not know what algebraic thinking to look for in their students' work, for example, justifying, generalizing, and pattern-finding.

Table 8 shows the CCK the student teachers used to plan and modify Task 1. The student teachers' CCK consisted of the knowledge that justification and proof are important mathematical processes, knowledge of the addition algorithm, and the ability to write numbers in different forms.

Table 8

Common Content Knowledge Used in Task 1

CCK	Example Statement
Mathematics involves justification.	Well in math, it's kind of a big deal to be able to prove things.
Addition algorithm and why it works.	I carry when I add, and that's fine.
Numbers can be written in many different forms.	Student . . . can correctly re-write each number in expanded form.

The student teachers' used CCK to determine the purpose of the task and to anticipate some student thinking. The student teachers knew that Reasoning and Proof was one of the Process Standards (NCTM, 2000), and so they "incorporated" proof into the task by titling the lesson "Proving Addition Facts". However, the expectation for student justification was more along the lines of an explanation of the students' thought process rather than a proof of why their process worked. The student teachers also used their knowledge of the addition algorithm and rewriting numbers to create some of the different ways that they predicted students would add the numbers.

In summary, the student teachers lacked knowledge of many common strategies students use to add multidigit numbers. Consequently, they relied on their own strategies for adding numbers in order to anticipate student thinking. The student teachers did not see a mathematical purpose for the task, and so they used the state core to identify the mathematics they should be teaching and relied on their knowledge of what mathematicians do in order to create a mathematical purpose. Additionally, the student teachers did not have adequate knowledge regarding the connection between arithmetic and algebra. They were "hoping to see students come up with different ways to add" (Lesson Plan 1), but were unsure as to what specific thinking they should be looking for in order to bridge the gap between arithmetic and algebra.

Task 2: Spatial Reasoning

Description of Task

The students had been making hexahedra in class. Kristen and Abby's cooperating teacher asked them to create a task for the students using the hexahedra. The student teachers consulted the state core to determine a mathematical topic that they could teach through the use of the hexahedra. The student teachers determined that the hexahedra could be used to teach the concept of surface area. They also decided that students first needed to have a conceptual understanding of area before they could understand surface area. The student teachers decided that these objectives aligned best with the core objective of deriving geometric formulas.

The student teachers did not provide the students with a written task, so the task had to be inferred from the lesson plan. In this case, the task consisted of a series of questions the student teachers planned to ask the students. On the previous class period, the student teachers gave the students six pieces of square paper and explained to the students how to fold the papers so that the six folded papers would fit together to form a hexahedron. At the start of the task, the students had folded the six sides of their hexahedron, but had not yet assembled the hexahedron. Figure 5 shows the relative size of a folded side (the shaded square) to the unfolded square paper and the fold marks that would result from folding the square paper. The questions in the task had the students use one of the folded sides. In the first question, the student teachers asked the students to "pull out one of [their] six squares [they] folded yesterday. If we let one edge of that square [referring to the shaded square in Figure 4] be one unit, what would be the area of the whole square?" Once the area of the square had been established, the student teachers handed another square of paper that was the same size as the unfolded hexahedron to the students. The student teachers then asked them to "work with their partner to come up with their best estimate of the area of the square of paper, using their smaller squares as one square unit." If the students oriented the folded square so that it was oriented the same direction as the unfolded paper, then the folded square did not cover the sheet of paper evenly and the students had to estimate the area measurement. The student teachers then asked the students to "unfold one of their smaller squares to see a way to get a more accurate estimate." When the square was unfolded the students could find squares congruent to the unit square which would improve the accuracy of their measurement.

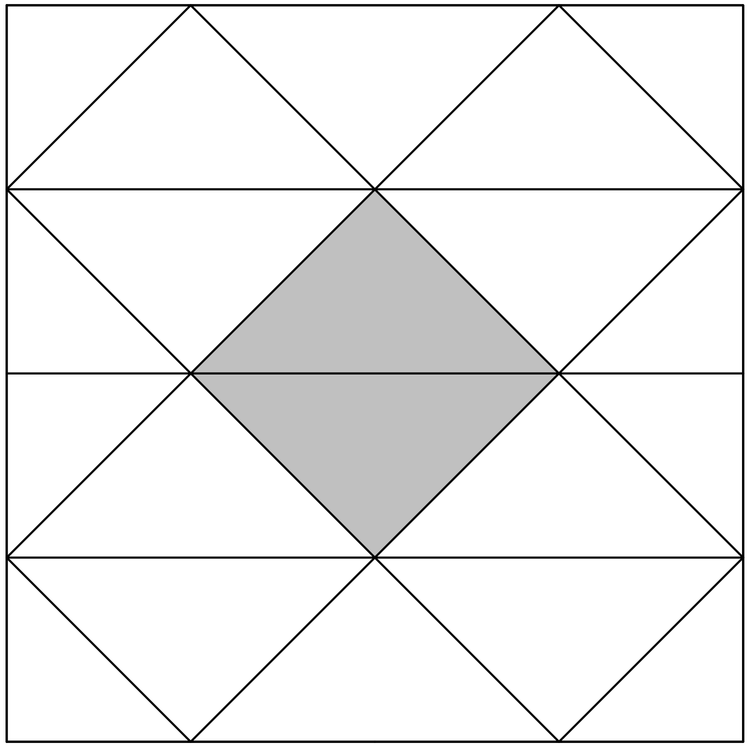

Figure 5. An unfolded side of a hexahedron used in Task 2**.**

Cognitive Demand of the Task

Two of the raters gave the written task a code of procedures with connections and one gave the written task a code of doing mathematics. The raters who coded the task as procedures with connections indicated that the task connected area to the meaning of counting square units, but felt that the students were guided too much by the suggestion that students unfold the paper to find a more accurate estimate. The rater who coded the task as doing mathematics did so because a pathway wasn't suggested until the end and the task required the students to access their own knowledge.

As a researcher, I also coded the task as procedures with connections, but with a reservation about the amount of anxiety it would cause the students. The task is connected to the concept of measurement as a count of a standard unit of the attribute to be measured. This fits well with the description that procedures with connections tasks "focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts" (Stein et al., 2000, p. 16). The pathway suggested also had a strong connection to the concept of measurement as counting. However, the cognitive demand of the task did not seem to fit with junior high school mathematics. The measurement standard expectations listed in *Principles and Standards for School Mathematics* places the expectations that students understand that measurements are estimates and how to best estimate a measurement as well as how to determine an appropriate unit of measure for a particular attribute in grades three through five and not in the junior high grades (NCTM, 2000). The expectations for junior high school students were much higher than this task would elicit.

In the context of the lesson plan, the cognitive demand of the task descended to procedures without connection. The anticipated thinking in the lesson plan did not contain much detail. They stated that the estimates would be "rough guesses" but did not plan to discuss how the students found the estimates. They also thought that the students might try to estimate the length and width and apply the area formula for a rectangle. There is no indication that they expected the students to explain their answers beyond "explain[ing] what they did" (Lesson Plan 2). Given the directions to unfold the square and the unproblematic nature of the task for junior high students, the task was already bordering on the procedures without connections cognitive level.   This insight into the expected student explanations made it so that the task fit better with the procedures without connections level than the procedures with connections.

Knowledge Associated with the Task

In order for the researcher to better describe and justify the knowledge a teacher could use when designing Task 2, the researcher included a review of the literature regarding area measurement. The first section contains some of the key knowledge components mentioned in the literature. The second section contains the knowledge the student teachers used to design Task 2 as inferred by the researcher.

Knowledge of measurement and area found in the literature. In order to understand area measurement, one must have an understanding of measurement in general. The literature reviewed dealt with measurement in general, common misconceptions of area, and the formula for finding the area of a rectangular polygon. Although this task did not deal directly with deriving formulas, the student teachers determined that the state core objective of deriving formulas aligned with the task.

First, the literature stated that most elementary students, secondary students, and even teachers do not have an adequate understanding of area (Ball et al., 2001; Bonotto, 2003; Stephan & Clements, 2003). This difficulty is compounded by the lack of experiences students have with measurement in general: lacking an understanding of the inverse relationship between size of unit and number of times it can be used to measure a region, that units can be partitioned into smaller regions to increase accuracy, and the use of different sized units in conversions (Lamon, 2007). For many students, measuring an area involves identifying numbers and applying an arithmetic operation without estimating the area first (Bonotto, 2003). Common difficulties students experience with area include: confusing area and perimeter, misapplying the area formula for rectangles to non-rectangular figures, using linear units, understanding the proportional relationship between the sides of a figure and its area, having gaps or overlaps, and varying the size of the unit (Outhred & Mitchelmore, 2000). Stephan and Clements (2003) attributed these difficulties to the complexity and sophistication of the array structure and multiplicative reasoning involved in the formula. Outhred and Mitchelmore (2000) stated that many of these errors could be traced to a non-conceptual understanding of area.

Baturo and Nason (1996) suggested two different strategies that children may use to find the area of a region using a unit of measure. First, students could iterate the unit by transposing and rotating the unit within the region. Second, the students could use as many units as necessary to completely cover the region without overlapping the units or leaving gaps. The student teachers intended the students to use the former strategy, although the students could have employed the latter if they had pooled all of their unit squares.

The literature also suggested various tools and pedagogical strategies teachers could employ to help students develop a better conceptual understanding of measurement and area (KCT). Stephan and Clements (2003) indicated that many of the traditional instructional tools used to measure and teach area hide the conceptual aspects of area. NCTM (1989) suggested that teachers have students use concrete materials to cover the region. However, Outhred and Mitchelmore (2000) indicated that research has shown that the use of such materials often conceals the conceptual underpinnings of area they are intended to teach and suggests the use of drawings as an alternative. In contrast, Stephan and Clements (2003) indicated that students' first experiences with measuring area should involving covering the region with a unit, incorporating classroom discussions that focus on the issues of overlapping, gaps, and precision. In order to get students to see the connection between rectangular area and formula, students need to be given experiences where they can explore arrays and their connection to the linear measurements of the rectangle (Battista, 2003).

The literature contained a great deal of information on the principles of measurement and area (SCK). Baturo and Nason (1996) described the process of finding a measurement: identifying the attribute to be measured, partitioning the attribute into units, and then counting the units. Stephan & Clements (2003) discussed assumptions involved in measuring an attribute by covering, for example, the chosen unit of measure is appropriate to measure the region, units do not overlap, and congruent units have equal areas.

Area has been defined as the "amount of two-dimensional surface that is contained within a boundary and that can be quantified in some manner" (Stephan & Clements, 2003, p. 10). Baturo and Nason (1996) suggested that area be considered from both the *static* and *dynamic* perspectives:

> The static perspective equates area with an amount of region that is enclosed within a boundary and the notion that this amount of region can be quantified. The dynamic perspective focuses on the relationship between the boundary of a shape and the amount of surface that it encloses so that, as the boundary approaches a line, area approaches zero. (p. 238)

Additionally, Baturo and Nason (1996) identified three types of knowledge associated with area: concrete, computational, and principled conceptual. Concrete knowledge included measuring an area with no gaps or overlaps for both regular and irregular shapes, using both standard and arbitrary units, being able to count parts of a unit, and an understanding of conservation of area. Computational knowledge incorporated the use of formulas and unit conversions. Principled conceptual knowledge included identifying the attribute that is to be measured and knowing why it is to be measured, that “area is a continuous attributed divided into discrete subunits” (Baturo & Nason, 1996, p. 244), area comes from squaring a linear attribute, the necessity of standard units for comparison, inverse proportionality of units, issues of precision, and the derivation of and connections among area formulas.

Knowledge the student teachers used to develop the task. Table 9 lists the student teachers’ KCS. The student teachers described their students’ tendencies to multiply any two numbers to find the area rather than multiplying the length and the width. These types of statements indicated that the student teachers knew that the students often misapplied the rectangular area formula and that students did not have a conceptual understanding of area. The student teachers anticipated that some of the students might not realize that the units could not overlap, but that most students would still be able to determine the area of the unfolded square by counting the units and fractions of units. The student teachers also knew that the students were uncomfortable with nonstandard units of measure and that the students may struggle because the folded square did not cover the unfolded square “evenly”, meaning in a grid pattern.

Table 9

Knowledge of Content and Students Used in Task 2

KCS	Example Statement
Students know that the area of a rectangle is length times width, but have no conceptual understanding of area.	I mean they know that area of a square is length times width, but maybe having that conceptual understanding of it . . .

KCS	Example Statement
Students have difficulty applying the area formula.	For example they did a problem in class where they had to know area, and the pool table problem—or the pool problem, almost every student knew that they had to multiply two numbers together, but very few students understood that it mattered which two you know.
Students are uncomfortable with a nonstandard unit of measure.	So you know, she just didn't understand that a unit is…an inch is just a type of a unit and we don't need to label it "inch".
Students may struggle with the task because the unit doesn't fit "evenly".	We thought that they might not realize that things can't overlap, and that they might have a hard time because the unit didn't fit evenly.
Students might not realize that the units cannot overlap.	We thought that they might not realize that things can't overlap, and that they might have a hard time because the unit didn't fit evenly.
Students will be able to count the unit and pieces of squares	Students will probably be able to use this suggestion to see that they need to rotate their smaller square and combine all the partial squares to get an accurate measure of the area.

For two of the items in the table, application of area formula and overlapping of units, Kristen indicated that they had discussed them while planning the lesson. However, neither of these ideas appeared in the lesson plan. The overlapping of units is a fundamental principle of measuring area and an idea that would likely present itself during the implementation of the task, but was not discussed in the lesson plan.

Another inconsistency between the lesson plan and what was mentioned in the interviews was the students counting partial units. The task required the students to count pieces of the whole unit and combine those pieces to make units. There was evidence in the lesson plan that the student teachers anticipated this thinking. However, Kristen stated in her interview that she was surprised when students counted partial units, "I intended them to use squares, because that was the unit. But then the student saw that the triangle is half of the unit." This inconsistency can likely be explained by the idea that Kristen anticipated the students counting all of the whole units and then the pieces of the whole unit, she did not anticipate that the students would partition the square unit into a smaller unit that they could then count to get a more accurate measurement.

The student teachers' used their KCS as a motivation to teach the lesson. They felt that the students did not have a conceptual understanding of area and that this task would help students develop that connection. It is unclear whether the student teachers intended the task to explain and clarify some of the misconceptions students had with the area formula for a rectangle. In the lesson plan and the reflection meeting, the student teachers discussed how students needed to be able to apply and derive formulas. However, Kristen acknowledged that the task "doesn't help them see very well why it's length times width" (Interview 1).

The student teachers used their KCS to anticipate student thinking. The anticipated student thinking in the lesson plan dealt with having to rotate the square to make it fit evenly, multiplying the length and width, and counting the units and partial units. The student teachers had observed students multiplying numbers to find area in the past and knew this was a strategy students might use to determine the area. Additionally, the student teachers knew that the students would be able to count the number of units and partial units because the students had used that strategy on a past task.

The student teachers knew that the students were uncomfortable with nonstandard units of measure. This knowledge was reflected in the student teachers choosing a nonstandard unit of measure for the class to use to measure the area of the paper.

However, there is no indication in the lesson plan that the student teachers intended to talk about issues of nonstandard units with the class.

Compared to the literature, the student teachers were aware of some student difficulties associated with area. The knowledge of these difficulties came from their experiences with their current students. They knew that most of their students did not have a conceptual understanding of area and that they often misapplied the area formula. They also knew that students would struggle with covering the area with no gaps or overlaps.

The student teachers' KCT (Table 10) centered on how to provide students with experiences that would build their conceptual understanding of area and using those experiences with area to lead into surface area. The student teachers knew that the action of covering the area with a unit would build a conceptual understanding of area. The student teachers had the knowledge that they needed to prepare the students for a discussion on surface area by providing the students with experiences that would develop their conceptual understanding of area first. The student teachers' strategy to have the students cover the area with a concrete unit was similar to what was suggested in the literature (NCTM, 1989; Stephan & Clements, 2003).

Table 10

Knowledge of Content and Teaching Used in Task 2

KCT	Example Statement
Help students conceptualize area by having them cover the area with a concrete unit.	And I think it helps them get a better conceptual idea of area to try to estimate in that way.

KCT	Example Statement
Help students conceptualize area by giving them a context where they have to visualize area.	So something that can help them understand, kind of picture, visualize area in some kind of context is very helpful to conceptual understanding.
Prepare students for surface area by reviewing area.	If they understood the surface area of their own cube, then that would help. But in order to do that then we decided to review what area was to see how it all ties together.

Table 11 lists the student teachers' SCK. There was evidence in the interviews that the student teachers knew that there could be no gaps or overlaps when measuring area. Additionally, the student teachers had knowledge of units, specifically that units must be congruent. Although there was some indication in Kristen's interview that she was aware of arrays and how they apply to the rectangular area formula, this knowledge was not included in the table because she did not use the knowledge to plan the task. She was aware that the rotation of the unit made it so that this task was not well suited for helping the students understand the multiplicative relationship in the area formula.

Table 11

Specialized Content Knowledge Used in Task 2

SCK	Example Statement
In finding area, there can be no overlaps or gaps.	Then we could have discussed that the squares couldn't overlap.
General understanding of units: units are what you count; congruency and consistency.	And so the reason is, if you are counting squares of all different sizes then you get to the end and you have 27 what?

The student teachers' had much of the SCK mentioned in the literature. They had some knowledge of units and measuring area with concrete units. The difference between the SCK of the student teachers and the SCK in the literature is in the breadth and depth of the knowledge (see Ma, 1999). For example, the student teachers knew some of the principles of area measurement such as covering the area completely and consistency of units. However, they did not show evidence that they understood how varying the size of the unit affects a measurement nor how unit conversions are connected to unit size. Their knowledge package of area was incomplete. Additionally, Kristen had a conceptual definition of area, but it was not articulated as well as it could have been.

Table 12 shows the student teachers' CCK, which consisted of the formula for finding the area of a rectangle, the notion that problem solving was a mathematical processes, the idea that there are many ways to estimate a measurement, and a conceptual definition of area. Their CCK was used to determine a purpose in the lesson (problem solving) and was also critical in the design of their task (having the students estimate the area as well as find the exact measurement).

Table 12

Common Content Knowledge Used in Task 2

CCK	Example Statement
Area (of a rectangle) is length times width.	Understanding of area as the length multiplied by the width.
Mathematics is problem solving.	Well, that was problem solving . . . That was a math idea that I kind of wanted to do.
Estimation in area measurement	I wanted them to see that there were many ways to estimate and there was also a way to get an exact answer.
Conceptual definition of area.	The area of the paper is the middle part, kind of.

The student teachers' curricular knowledge, which is listed in Table 13, included the core as a curricular resource and necessary prerequisite understanding for surface area. It is interesting that the student teachers did not view this task as the time to address issues related to units. This could have been because they wanted to use an arbitrary unit rather than a standard unit such as inches.

Table 13

Curricular Knowledge Used in Task 2

Curricular Knowledge	Example Statement
The state core is a curricular resource (deriving formulas).	And so we went through the core and really tried to decide what the big mathematical ideas were, and really it just was surface area and having a conceptual understanding of that.
Area is prerequisite to surface area.	I think you need to understand area before you get to surface area.
Units are also connected to area and surface area (but not necessarily prerequisite)	We kind of wanted a little bit to address units, oh yeah, we decided to do that later; we decided to really focus on units later.

The student teachers used their curricular knowledge to order the lesson. They knew that they needed to teach the students about area before they could adequately address surface area. Additionally, they decided that units were not necessary to understanding surface area and decided to address issues of units later.

They also used their curricular knowledge to find a purpose to the lesson by referring to the state core objectives. Although the purpose identified by the student teachers (deriving formulas) did not align well with the task, the student teachers knew that the state core was a tool they could use to determine the mathematics that should be taught in a pre-algebra course.

Table 14 lists the pedagogical knowledge the student teachers used to design Task 2. The pedagogical knowledge employed by the student teachers was used to manage the classroom and build on students' past experiences. The student teachers knew that it was good pedagogical practice to build off the students' past experiences. Earlier in the year, the students had made tetrahedrons and were asked to find the area of the folded paper. The student teachers wanted to connect this new task to their students' past experiences.

Table 14

Pedagogical Knowledge Used in Task 2

Pedagogical Knowledge	Example Statement
Try to connect new learning to previous student experiences.	. . . they did something very similar when they made tetrahedrons.
Have students work together.	I decided it might work better to try to have them working with each other more than with a whole class discussion.

Additionally, the student teachers had been experiencing management problems. The student teachers' used their pedagogical knowledge to try to address some of the management problems. In this case, the student teachers decided to have the students work in pairs and that she would help the individual pairs more rather than having a class discussion.

To summarize the student teachers' use of knowledge, they used KCS to anticipate student thinking and to provide the student teachers with a purpose for teaching the lesson. Their KCT allowed them to create experiences to help students conceptualize area. Their pedagogical knowledge was used to manage the classroom. Curricular knowledge was used to order the lesson and to identify the mathematics that should be taught in the classroom. While the student teachers had SCK, it was not well-connected or as deep as it could have been.

Task 3: Subtracting Integers

Description of Task

Prior to planning the third lesson, the student teachers had noticed that students were struggling with subtracting integers. They wanted to create a task in which students would have to subtract integers. They handed the students the following story problem:

> You have a lawn-mowing job and get paid every Friday. At the beginning of the week you had $32. By looking at what you did during the week, figure out how much money you had at the end of each day. (Make sure you show your work the way we discussed in class.)
>
> - **Monday**: You bought candy and balloons for your friend Joe's birthday. This cost you $8.
> - **Tuesday**: you went to Joe's birthday party at Seven Peaks and spent $17 total on your park pass and snack. On the way home, you stopped to get enough gas to mow your lawns tomorrow. The gas cost $22, and since you didn't have enough, you paid everything you had and borrowed the rest from Joe.
> - **Wednesday**: Your friend Susan paid you back the $4 she owed you. And you used this money to pay back part of your debt to Joe.
> - **Thursday**: The ice cream man came by and Joe let you borrow another $2 to buy ice cream.
> - **Friday**: Pay Day!! Your employers paid you $42 for the lawn mowing jobs you did this week. You paid off all your debt to Joe and kept the rest.

Within the context of the lesson plan, this task was placed after a discussion about different methods of subtraction, similar to the adding ten different ways task. The student teachers gave a two-digit subtraction problem, 23 – 18, to the students and discussed how they could partition the numbers so that they could subtract them in a useful way. For example, 23 = 10 + 13 and -18 = -10 – 8; the students could then combine 10 and -10 and 13 and 8 to get 0 + 5. The students were then expected to practice this method of subtraction in the story problem they were given.

Cognitive Demand of the Task

This task was coded as procedures without connections. There is no indication in the task that the expected explanation needed to extend beyond procedures. In fact the explanation needed to mimic what was shown during the discussion. Even though the task was set in a monetary context, the task itself did not elicit any connection to the underlying principles of integer subtraction. The students could easily solve the problem without having to understand how to subtract integers. In fact, the students could solve the problem without using negative numbers.

The other raters also coded this task as procedures without connections. Their explanations were related to the type of explanation required, describing it as a "rehearsed" explanation. One rater gave additional reasons, stating that "even though this task is long, it was not very demanding mathematically. Students would just have to add and subtract numbers. There is a context, but I would not say there are much connections being made while doing the procedure."

The intended phase of the task was coded as procedures without connections, possibly leaning toward memorization. Within the context of the lesson plan, the student teachers placed the task following a discussion of how to subtract multidigit numbers. The method for subtracting numbers further removed the meaning of subtraction. For example, in the method the students rewrote the multidigit subtraction problem 23 – 18. Connected to the meaning of subtraction, this operation involves subtracting 18 positive numbers. However, the student teachers had the students rewrite subtract 18 as equivalent to negative ten subtract eight. Conceptually the problem has changed from subtracting 18 positives to adding 10 negatives and subtracting 8 positives.

Mathematically the result will be the same, but they are conceptually different problems. By having the discussion about rewriting subtraction problems prior to giving the task, the student teachers removed the meaning of subtraction from the problem.

Knowledge Associated with the Task

Knowledge on the learning and teaching of integers found in the literature. This section contains knowledge that the student teachers could have possessed in the planning of Task 3. The domain of integer numbers is the result of a search for closure on the whole number system to the operation of subtraction (NRC, 2001; Verschaffel et al., 2007). Students often experience difficulties with arithmetic whenever their domain of numbers is extended to include larger domains (English & Halford, 1995; Verschaffel et al., 2007). Specifically, when the whole number system is extended to include negative whole numbers the relation of order and magnitude have to be reconceptualized (English & Halford, 1995). Although students may have had limited exposure to negative numbers, for example, temperature and debt (Ball, 1993), studies have found that children have a natural tendency to form an organized number line (English & Halford, 1995).

There are two primary approaches a teacher can take to the extension of a number system: appeal to the rules of arithmetic and intuition (NRC, 2001). Mathematically, "the extension of whole numbers to integers is an example of the axiomatic method in mathematics: basing a mathematical system on a short list of key properties" (NRC, 2001, p. 82). There are a number of representations that may appeal to students' intuition: money, frog on a number line, game-scoring, elevator (Ball, 1993), and hot coals and ice cubes.

There are positive and negative aspects to the money model of representing integers. The advantages are that the representation allows for all meanings of addition and subtraction and models relative quantities, for example negative five is less than negative two (Ball, 1993). However, students in Ball's class tended to separate the money they had and the money they owed, keeping track of both amounts separately rather than combining them into one amount. Additionally, the children were able to avoid the concept of negative numbers entirely by labeling the money owed as debt rather than as "-$8".

One pedagogical dilemma in the teaching of integers is the use of the "-" to refer to both subtraction and negative numbers. This notational choice often creates confusion between the operation and the negative number. Experienced students are able to rely on the context of the number sentence to determine the meaning of the symbol; however, novice students often struggle with the notation. Ball (1993) suggested the placement of the circumflex (^) above the number so that the students are able to focus on the negative number as a number rather than an operation.

Several mathematical principles underlie the teaching of integers. Ball (1993) described the two distinct representational uses of negative numbers: an opposite amount and a location relative to zero. Additionally, every number has a magnitude and direction; the issue of magnitude presents pedagogical difficulties:

> There is a sense in which -5 is more than -1 and equal to 5 even though, conventionally, the "right" answer is that -5 is less than both -1 and 5. This interpretation arises from perceiving -5 and 5 as both five units away from zero and -5 as more units away from zero than -1. Simultaneously understanding that -5 is, in one sense, more than -1 and, in another sense, less than -1 is at the heart of understanding negative numbers. (Ball, 1993, p. 379)

Knowledge of integers used to design the task. Table 15 lists the KCS the student teachers used to design Task 3. The student teachers knew that students experience difficulties performing operations on integers. Specifically, students struggle whenever they subtract a larger number from a smaller number, order two negative numbers, and subtract negative numbers. The student teachers thought that students conceptualize subtraction as “take away”. The student teachers also knew that the students would be able to solve simple subtraction problems that involved subtracting a smaller number from a larger number.

Table 15

Knowledge of Content and Students Used in Task 3

KCS	Example Statement
Students struggle with subtracting integers, specifically subtracting negative numbers and crossing zero.	We noticed in an assessment they had last week doing the order of operations that a lot of them when they have to subtract a negative, they knew the rules said, “you add that”. But we’re hoping to kind of bring that out, to have them see why it becomes addition; instead of just it's a rule.
Students struggle with ordering numbers, especially two negative numbers.	So he wasn't understanding that -69 is a bigger number than -75, so he added and got a bigger number.
Students have a conceptualization of the operation of subtraction as taking away, but it is difficult to transfer this understanding to negative numbers.	Well, they kind of know that subtraction is more or less, that it's taking away . . . But when they say, “Oh, I have seven things and then take away negative three things.” They have a hard time picturing that in context of how they understand subtraction.
Students can solve simple subtraction problems.	For example \$32 - \$8 = \$24 and \$24 - \$17 = \$7 are simple subtraction problems that we felt everyone should be able to solve without difficulty.

The student teachers' KCT centered on how the different representations of number line, money, and taking away could address certain difficulties students experience with negative numbers. For example, Kristen thought that the number line representation would be a helpful representation to help students order integers (i.e., the number that is the furthest right on the number line is the largest). Both Abby and Kristen thought that the familiarity of the context of money would help the students understand integer subtraction better. Additionally, their experiences with Task 1 indicated that having the students adding the numbers ten different ways did not produce the kinds of student thinking that they wanted, and so the student teachers felt that using a similar task would be ineffective. However, the cooperating teacher wanted the student teachers to teach the students a useful way of subtracting through the strategy of partitioning numbers. The student teachers decided to place a discussion on useful subtracting prior to having the students work on the task, but they did not consider how the placement of the task in relation to the discussion might impact student thinking. They later realized that they were restricting the possible student thinking by having the discussion before the task (see Table 16 for sample statements).

Table 16

Knowledge of Content and Teaching Used in Task 3

KCT	Example Statement
Money is a meaningful context for discussing negative numbers.	And I think we tried to, we did the idea using money because we thought it brought more conceptual understanding of it instead of just . . .
The number line may help students who are having difficulty ordering and subtracting numbers.	That helps them with their order because I think they have a hard time ordering integers that are negative.

KCT	Example Statement
The concept of removing negatives will help students understand the rule for subtracting negatives.	And so if we're taking away part of our negatives, that kind of helps them see why subtracting a negative is the same as adding. So instead of just saying that is suppose to be plus, but I don't know why, it just is.
Importance of careful use of terminology.	I think that either minus should not be used at all or it should only be used to mean subtract because when you use it either way it is confusing.
The ten different ways was not a productive task and the same idea shouldn't be used with subtraction as well.	He was having us go off of suggested the task of using two different numbers and showing 10 different ways to subtract those numbers, but I kind of felt that that sort of was silly, because the point of showing them the subtraction strategy is to partition in a way that makes it easy for you to do the subtraction.
Look for student thinking to reinforce the idea of partitioning the numbers to subtract.	Well, I was hoping that they would use some of the strategies that we had talked about, because I think that they help to illustrate what is going on better sometimes.

Another KCT listed in Table 16 is knowledge of important use of terminology in mathematics instruction. The use of proper mathematical terminology was addressed in an article about subtraction their cooperating teacher gave them. Although the student teachers knew that they needed to use proper terminology, there is evidence that the student teachers did not understand the terminology associated with subtracting integers (e.g., subtraction, negative, and minus). In the reflection meeting, the student teachers indicated that the article suggested they never use the word "minus". In her interview, Kristen expressed confusion about the definition and use of "minus". She knew that *subtraction* referred to an operation and that *negative* was a way to describe a number, but she had heard minus used for both *subtraction* and *negative*. This issue parallels the notation issue discussed in the literature review of the teaching of integers.

Table 17 lists four different components of SCK the student teachers used to design Task 3. First, the student teachers knew how to create situations where the students would have to subtract a larger number from a smaller number and subtract negative numbers. These were types of integer problems that the student teachers noticed caused difficulties for the students. The student teachers wrote the Tuesday portion of the task to force the students to deal with subtracting a larger number, and they wrote the Wednesday portion of the task to prompt students to subtract a negative number. These portions of the task could accomplish the goals of the student teachers; however, they did not consider how the context of money would allow students to avoid using negative numbers by simple labeling the money owed as debt. Consequently, the tasks the student teachers wrote did not necessary force the students to wrestle with the issues the student teachers had hoped.

Table 17

Specialized Content Knowledge Used in Task 3

SCK	Example Statement
Writing the task so that the students have to cross zero and subtract negative numbers.	On this one on Tuesday, we ended up after, so here after they go to seven peaks, they ended up having seven dollars left, but then he goes to get gas, which was 22, and so they are subtracting a bigger number from a smaller number.
The difference between adding a negative and subtraction.	I didn't want to say subtracting a negative is adding because it's not adding; it results in the same thing is adding.
The subtraction method of partitioning the numbers explains borrowing.	And it helps them see why borrowing works because you’re just rewriting it as 20 plus 12 instead of 32. And so you’re taking the 8 from the 12 and that’s essentially what you do when you borrow.
Different representations: number line and zero pairs.	Because for me, that's how I make sense of it all . . . I picture a number line.

Second, one of the student teachers acknowledged the conceptual difference between subtracting a negative number and adding a positive number. Kristen described subtracting a negative numbers as removing negatives from the set and adding a positive number as adding positives to the set. Kristen then explained that the result of the two operations would be the same. Abby, on the other hand, felt that the students just used the rule “add the opposite”, but when asked to explain how the rule worked, Abby just stated that they were the same thing (Interview 3).

Third, the student teachers’ had two of the representations mentioned in the literature: number line and zero pairs. Kristen was more comfortable with the number line representation, while Abby preferred the idea of zero pairs. Kristen was explicit in her preference and Abby implicit.

Finally, the student teachers knew that the strategy of partitioning numbers could help explain to the students why the method of “borrowing” worked in subtraction. For the subtraction problem 32-8, the students could partition 32 into 20 plus 12. This is what happens when you “borrow” to subtract. Ma (1999) termed this *regrouping* (p. 12).

After teaching the lesson, the student teachers discussed how their purpose of the task did not align with the task. First, Abby found that the students in her class avoided using negative numbers and decided that the context of money had not forced the students to use negative numbers. Additionally, they intended the task to be used as a way to practice the subtraction strategies that they had taught during the discussion, not to elicit student thinking of integers. The student teachers wanted the students to practice the partitioning strategy as they worked on Task 3. The task was meant to supplement the discussion that occurred before the lesson rather than to provide student thinking that could be discussed later. If they had planned to elicit student thinking from the task then they would have altered the task. Finally, Kristen acknowledged that the subtraction problems in the task were too simple and did not motivate the students to need to use the subtraction method they had taught.

The student teachers used SCK to create problems in the task that would elicit certain student thinking (e.g. subtracting a negative number) and to decide which representations they would use while teaching. They did not consider how their task aligned with their teaching objectives. This may be because they lacked key components of SCK for integers.

Their CCK fell primarily into two categories: how they solved subtraction problems through partitioning and principles of integers (see Table 18). The student teachers found partitioning the subtraction to be a useful way to mentally subtract numbers and wanted to teach this to their students. Abby in particular found the method enlightening as she had never considered the concept of partitioning the numbers; she had always used the algorithm or her fingers. The student teachers also knew that the partitioning strategy for subtracting integers was similar to the partitioning strategy for Task 1. The student teachers also had knowledge of integers, specifically the rule for subtracting integers as well as how the integers fit into the number system.

Table 18

Common Content Knowledge Used in Task 3

CCK	Example Statement
The strategy for subtraction is similar to the strategy for addition.	We are actually kind of doing the same thing that we did with addition. But we are doing it with subtraction.

CCK	Example Statement
The rule for subtracting numbers.	Kristen: well, then, because I know, minus the negative is plus.
How integers fit into the number systems.	This led to a good conversation in which whole and rational numbers were also mentioned.
Partitioning the subtraction allows us to do it in our head.	It helps me to do it in my head. And when I was trying to figure out a pattern, 77 and whatever else, to get the differences. I partitioned the different ones. It just helped me to do it in my head a lot easier.

The student teachers used their CCK to determine the topic of what they wanted to teach. They wanted to teach this useful way to subtract numbers because they had personally found it to be helpful. They recognized that combining integers was an important aspect of the partitioning strategy, so they decided that they needed to teach integers as well, specifically how subtracting a negative number gives the same result as adding a positive number. They had knowledge of the number systems, but did not explicitly plan to talk about the concept beyond the idea that integers were an extension of the whole numbers.

There was only one indication of how the student teachers used their pedagogical knowledge to plan the lesson. In Kristen's third interview, she stated, "Well, if they can do one problem with you helping them that hopefully would increase their understanding, but that isn't enough to necessarily think that they could apply those principles in any situation." This is similar to the pedagogical practice of leading the students from solving the problems with teacher assistance to solving the problems on their own. This aligns with a behaviorist perspective of how students learn.

There was evidence of the teachers using pedagogical knowledge in their lesson plan to manage the classroom. For example, in the lesson plan the student teachers have the students clear their desks from distractions. The student teachers did not discuss management issues in the other data gathered for this task.

There was no evidence that the student teachers used any curricular knowledge when planning the lesson. This could have been because they were assigned the topic by their cooperating teacher. They did use the state core to determine that the students needed to learn how to add and subtract integers and rational numbers.

In summary, the student teachers used KCT and CCK to determine an objective. They also used CCK to identify that integers were used in their method of subtraction. The student teachers wanted the students to encounter certain types of problems and used SCK to write problems that addressed specific issues. They used KCT to decide how to remedy student misconceptions.

Task 4: Proportional Reasoning Problems

Description of the Task

The final task the student teachers used in the collected data was a set of problems where the students could have used proportional reasoning to solve the problems. The cooperating teacher gave the student teachers a worksheet containing nine proportion problems with different contexts. The nine proportion problems can be found in Appendix I; a sample problem from the worksheet read as follows:

> If You Hopped Like a Frog . . .
> Frogs are champion jumpers. A 3-inch frog can hop 60 inches. That means the frog is jumping 20 times its body's length. How tall are you? If you could jump 20 times your body length, how far could you go?

The student teachers modified the task by selecting four of the nine problems and giving one problem to every pair of students in the classroom rather than having every student work on all nine problems. They also changed the individual problems by omitting the scale factors and adding additional questions. The students were assigned to work on their problem and to create a poster explaining how they found the solution. The problems were given to the students prior to any discussion of how to set up proportions to solve the problem; however, the students had been solving similar problems where they could have used proportional reasoning for several days. The students did not use the strategy of setting up proportions; most students found a scalar multiplier to help them solve the problem. The student teachers wanted the students to develop the strategy of setting up proportions to solve these problems. The student teachers gave the students one of four problems. The modified problems can be found in Appendix J; a sample problem is as follows:

> If You Hopped Like a Frog . . . Frogs are champion jumpers. A 3-inch frog can hop 60 inches. If you could jump like a frog, how far could you hop in one jump? How many jumps would it take you to jump down a football field (100 yards)? How far would you go if you hopped 30 times?

Cognitive Demands of the Task

The written phase of this task was coded twice: in its original form as it was given to the student teachers and in its modified form. The original task was coded as procedures without connections. By providing the students with information about the multiplicative relationship, the task becomes easier for the students and strongly implies the pathway that should be taken to solve the problem. Additionally, there was no explanation required.

The modified task was coded as doing mathematics by both the researcher and the raters. The raters felt that the coding of this task depended largely on previous experiences of the students. The raters felt that if the students had not been taught how to solve these proportional problems previously, then the students would experience a great deal of anxiety and would have to access relevant knowledge. However, if the task were to be given to the students following a lesson on how to solve problems by setting up proportions, then the raters would have given the task a coding of procedures without connections. While the students in the class had previous experience with solving problems that involved proportional reasoning, they had not been explicitly taught to set up a proportion and did not use that strategy.

The other concern the raters expressed in the coding of this task was the lack of justification that was required in the task. However, that concern was not strong enough for the task to be coded as something different from doing mathematics. The raters felt that not giving the students a pathway to follow balanced the need for a justification.

Within the context of the lesson plan, the intended task declined to procedures with connections. First, the purpose of the task for the student teachers was to get the students to develop the procedure of setting up proportions. They hinted that the students should set up proportions by including proportions on the warm-up and problem-of-the-day at the beginning of the lesson. However, the student teachers decided not to suggest any method for solving the task as the students worked on it. Additionally, the justification required by the student teachers in the lesson plan was for the students to explain their thought process rather than to justify their work mathematically.

Knowledge Associated with the Task

Knowledge of proportional reasoning found in the literature. There exists a large body of research dedicated to the development of proportional reasoning. This list of knowledge was generated from a small sampling of the research available. The research reviewed included research that assimilated much of the literature.

The literature provided examples of SCK in defining proportional reasoning, identifying different types of proportional problems, and describing the foundational understanding involved in being able to reason proportionally. As proportional reasoning and proportionality have often been ill'defined in the literature, Lamon (2007) provided the following definition for *proportional reasoning:*

> I propose that *proportional reasoning* means supplying reasons in support of claims made about the structural relationships among four quantities in a context simultaneously involving covariance of quantities and invariance of ratio or products; this would consist of the ability to discern a multiplicative relationship between two quantities as well as the ability to extend the same relationship to other pair of quantities. (pp. 637-638)

Additionally, proportional reasoning involves the recognition that there are two scalar multipliers that are inverses of one another (Lamon, 2007). A proportion is two equal ratios. Other important definitions involve the difference between ratios and rates. Mathematically, a ratio compares two quantities of the same unit; whereas a rate compares two quantities with different units. For example, 4 miles:5 miles is a ratio and 4 miles/3 hours is a rate.

Three types of proportional problems were identified in the literature: missing value, numerical comparison, and qualitative comparison (Kaput & West, 1994; Lamon, 2007; NRC, 2001). In missing value problems, the student is given a complete ratio or rate and a second ratio or rate (either implicitly or explicitly) with one of its quantities missing, and the student is expected to solve for the missing value (Kaput & West, 1994; Lamon, 2007; NRC, 2001). Numerical comparison problems involve determining which of two ratios or rates is greater (Lamon, 2007; NRC, 2001). Finally, in qualitative comparison problems one of the quantities in the ratio or rate has been altered and the student is expected to determine how that would change the ratio. For example, "What happens to the price of a balloon if you get more balloons for the same amount of money" (NRC, 2001, p. 243).

Lamon (2007) described what is involved in understanding proportionality. Such understanding includes knowing the difference between addition and subtraction, knowing when to apply (and when not to apply) proportions to model a situation, and understanding the difference between direct and inverse proportionality in the context of a function, graph, or constant of proportionality.

Kaput and West (1994) discussed features of multiplicative word problems that make the problems easier or harder to solve. Problems were easier to solve if the given ratio is in reduced form, there is an integer scalar multiple, the quantities in the unit are contained is some way, the wording of the problem contains the phrases "for each" or "for every" or if the rate is something students would be familiar with, such as speed or price. Features associated with difficult word problems included quantities where one was not a factor of the other and quantities that were relatively close, resulting in the students using an additive rather than multiplicative approach.

In terms of KCS, the literature discussed difficulties, strategies, and learning development sequences of proportions and proportional reasoning. Both children and adults struggle with proportional problems (Weinberg, 2002). Ball et al. (2001) stated that less than half of the middle school teachers in one study were able to solve a simple proportion problem, and the authors attributed the difficulty to an overgeneralization of whole number arithmetic. Furthermore, students' ability to give correct answers to proportion problems does not necessarily guarantee that the students are able to reason proportionally; for example, cross multiplication allows students to avoid reasoning about the constant of proportionality (Lamon, 2007; Weinberg, 2002).

Common student errors involved in solving missing value proportion problems included using only two of the three quantities given in the problem or using all three quantities with incorrect operations (Weinberg, 2002). For example, students often use an additive rather than multiplicative approach when solving proportion problems (NRC, 2001).

Children have several strategies for solving proportional situations and solving proportions themselves. Some of the different strategies children use to solve problems with proportional situations include dividing to find a unit rate, repeatedly subtracting the unit rate from the whole, using a series of operations that would result in equivalent fractions, and setting up proportions (Weinberg, 2002). When students set up a proportion with a missing value, Weinberg (2002) identified three ways that they solve for the missing value: cross multiplication, isolating the variable using multiplication, and finding equivalent fractions.

Researchers agreed that the process of developing proportional reasoning is long-term (Lamon, 2007). Children begin by recognizing the relationships as multiplicative rather than additive. Once this has been established, children need to be able to determine the invariants in the situation. Understanding is further solidified as children are able to conceptualize rates (NRC, 2001).

Literature on the KCT of proportional reasoning was not as prolific as SCK or KCS, consisting of suggestions for teaching proportions. Lamon (2007) claimed that students need to be pushed to look beyond obvious observations and that direct instruction was necessary for the development of proportional reasoning. Students needed to be taught when to apply and when not to apply proportional reasoning to a problem (Lamon, 2007; Weinberg, 2002), as well at the strengths and weaknesses of different strategies and their use in particular proportion problems (Weinberg, 2002).

Finally, the literature emphasized that proportional reasoning problems should be an integral part of the middle school curriculum. All of NCTM documents indicated that proportions were an integral part of the curriculum connecting the mathematics taught in the middle grades (NCTM, 1989; 2000; 2006). Proportions and proportional reasoning were identified as focal points in the grades six and seven curricula (NCTM, 2006).

Knowledge used by the student teachers to modify and incorporate the task. The literature highlighted some of the knowledge the student teachers could have had as they designed Task 4; it was not expected that the student teachers possess all of the knowledge described in the literature. Occasionally, there was evidence that the student teachers did not possess some of the knowledge they could have used. First, they did not have a clear definition of what proportional reasoning entailed beyond being able solve proportional problems in context. Furthermore, the student teachers did not indicate that they knew of the existence of different types of proportional reasoning problems nor did they know how different features of proportional reasoning problems made the problems easier or more difficult. And although they knew that they wanted their students to develop the strategy of using proportions to solve problems, they were unsure how explicit they needed to be.

Table 19 shows the CCK the student teachers used to modify Task 4 as well as accompanying example statements. The students knew that proportions could be used to solve the task because they had used proportions to solve the task. Additionally, the student teachers knew that first finding a unit rate, and then multiplying by the unit rate could solve the task because Abby had used that strategy to solve the original task. They knew how to determine if two proportions were equivalent. They also knew of different applications of proportions, such as the proportionality of the human body.

Table 19

Common Content Knowledge Used in Task 4

CCK	Example Statement
Proportions can be used to solve the task.	We each took this home and solved the problems separately, and I used proportions.
The task can be solved by finding a unit rate and then multiplying.	I did these problems without setting up a proportion.
Proportions are equivalent ratios and can be set up many different ways.	It just needed to be the same proportion.
Different applications of proportions.	So knowing someone's measurement from their waste to the floor knowing that the proportion is the same. You should be able to figure out their height, if you have your height and measurement from waste to the floor.

By solving the problems given to them by their cooperating teacher, the student teachers' used CCK to identify the mathematics involved in the task. The student teachers used proportions to solve the problems and decided that proportions were the fundamental mathematics of the task. The equivalence of different proportions was also part of the fundamental mathematics of the task.

Additionally, the student teachers repeatedly indicated that they thought that the students were using proportions to solve the problems, but that they were “doing it in their head" (Abby, Interview 4). The student teachers were projecting their solution strategy of setting up a proportion into what the students were doing to solve the problem. Abby also solved the problems by multiplying by the unit rate. This CCK was used to anticipate student thinking.

The student teachers' SCK is listed in Table 20. The student teachers knew that the task as it appeared in its original form would not force the students to use proportions to solve the task. They also knew that there were advantages to the strategy of setting up a proportion. Kristen identified organization as one advantage of setting up proportions. She felt that proportions, especially when used to do unit conversions, helped to keep the work organized. The student teachers also knew how to identify the scalar multiplication strategy by looking at student work. However, the student teachers continued to express the idea that the students were setting up the proportions in their heads.

Table 20

Specialized Content Knowledge Used in Task 4

SCK	Example Statement
The task in its original form does not require the use of proportions.	So the original problem gave them that scale factor, and we deleted that. So they had to figure out 20.
Advantages to using proportions to solve problems.	And the spider one is more useful to have proportions, because there is more to keep track of. It involves dividing and then multiplying and all these things.
Features of student work where they used the strategy of finding a scalar multiplier.	So it looks like all of them just divided by three to get the 66.6 and then you jump that much times your height.

As mentioned in Table 20, the student teachers knew that the task in its original form did not require the students to set up a proportion to solve the task. The student teachers used SCK to modify the task. Originally, the task gave the students the unit rate. The student teachers knew that with this unit rate the students would not have to set up a proportion to solve the problem; they could just multiply given quantity by the unit rate to find the answer. As a result of this knowledge, the student teachers modified the task by removing the unit rate and making the given rate more complex. Additionally, the student teachers knew that one of the benefits of using proportions is that it helps to keep everything organized, especially when dealing with a lot of unit conversions. Consequently, the student teachers added more questions to the task to try to make the task more complex.

The student teachers' KCS came primarily from their student teaching experiences (see Table 21). The student teachers had observed that the students did not set up proportions to solve proportional reasoning problems. Consequently, they knew that it was unlikely that students would set up a proportion to solve Task 4. The student teachers also knew that their students would be able to solve Task 4 because they had observed their students solving similar problems in the past. The most common strategy that the student teachers observed was that of finding the unit rate. Additionally, the student teachers had heard the students talk about cross multiplication on some of the warm-up quizzes at the beginning of class. The student teachers observed that the students likely did not know why cross multiplication worked. The student teachers also knew that the students had never seen a proportion in their warm-up quizzes where the variable was in the denominator. The student teachers anticipated that if the students set up a proportion to solve Task 4 with the variable in denominator, then the students may not be able to solve the proportion.

Table 21

Knowledge of Content and Students Used in Task 4

KCS	Example Statement
Students unlikely to set up proportions to solve problems.	However, I also knew by previous exposure to proportion problems, that my students were not comfortable with setting up proportions to solve these problems.
Students have good proportional reasoning.	I knew, by previous problems and class work, that my students for the most part had quite good proportional reasoning.
Most common strategy is to find a scalar multiplier.	I guess once they figure out part of it, most of them were just multiplying instead of actually setting up those proportions.
Common difficulties and misunderstandings include when the variable is in the denominator, equivalent proportions, definitions of terms, and a conceptual understanding of cross multiplication.	There were many students actually that knew how to cross multiply, and so this was no problem for them because they just across multiplied and then it was out of the denominator.

The student teachers' used KCS to anticipate student thinking and provide the teachers with a purpose for the task. Knowledge of past strategies they had seen students employ was used to anticipate student thinking in the lesson plan. Additionally, they believed that their students used their proportional reasoning skills to solve task, but were reluctant to set up proportions. This was the student teachers' purpose in teaching the lesson: to get the students to develop the strategy of setting up proportions to solve the task.

Table 22 lists two components of KCT the student teachers used to plan Task 4: strategies the student teachers used to get the students to set up a proportion, and how to help students solve the proportion once it had been set up. First, how to get the students to use proportions presented a problem for the student teachers. They used their KCT to decide how they could promote the strategy. They did not want to direct the students too much, so they did not imply a particular strategy in their directions. Instead, they decided to give the students problems where they could use proportions and look for that type of student thinking to have presented at the board. They also included examples of proportions in the problem-of-the-day and the warm-up given at the beginning of class. They thought that if the students saw proportions being used to solve unit conversions then students might use the strategy in the task as well.

Table 22

Knowledge of Content and Teaching Used in Task 4

KCT	Example Statement
Get students to set up proportions by giving them more opportunities to solve problems where they could use proportions and have them do presentations.	But we were hoping that at least some, we wanted the presentations to address that, that some people would say that they just need to multiply the numbers, so they would set up proportions and solve them.
How to help students solve the problem once they have set up a proportion.	I asked him what we would do if x was in the numerator and there was a number in the denominator.

Second, once the students had set up a proportion to solve the problem, the student teachers anticipated that the students might have difficulty solving some of the proportions, particularly when the variable was in the denominator of one of the ratios. The student teachers planned questions they could ask to help the students through these difficulties. They also planned to use the thinking to discuss the equivalency of different proportions.

There was one piece of curricular knowledge that was used by the student teachers: the state core can be a curricular resource. They used this knowledge to determine the underlying mathematics. They used the state core to find occasions when proportions could be used (e.g., similar triangles and unit conversions. They also used *Principles and Standards for School Mathematics* (NCTM, 2000) to justify the inclusion of proportions in the curriculum.

The student teachers used their pedagogical knowledge to encourage participation in the task and to manage the classroom (see Table 23). The student teachers selected problems with an interesting context and avoided problems that would require the students to measure something (e.g. head circumference) or would make students uncomfortable (weight). This selection was done to encourage students to participate in the task; they did not want the task to discourage the students from engaging. They also planned to have the students create posters to present their thinking. The rationale was to have the students teach one another and to create ownership of the task.

Table 23

Pedagogical Knowledge Used in Task 4

Pedagogical Knowledge	Example Statement
Motivate students to work by giving them an interesting context, having them work in pairs, and giving each pair a different problem.	And then maybe we thought the context was more interesting.
Strategies for improving presentations	First, I needed to convince the presenters that they needed to prepare with the intent to be able to teach their peers. . .
Problems will take too much time if the students have to measure.	This when we get into, because then they would have to take that measurement and we didn't want to take the time.

The student teachers also tried to avoid management issues by having the students work in pairs and giving the students different problems to work. From their student teaching experience, the student teachers knew that the students were more willing to work in class if they were able to work with their peers. Additionally, the student teachers indicated that they thoughts students learned better when they were able to discuss their ideas with their peers. The student teachers had also learned that their students were more likely to work on the task if everyone in the class was not working on the same task. If everyone in the class had the exact same task to work on, then some of the students would let other students in the class find the solution. They thought that more students would participate if they had slightly different problems in the task.

Discussion and Conclusions

This chapter includes discussion regarding both research questions as well as answers to those questions. The question of how student teachers used their pedagogical and mathematical knowledge for teaching to design or modify tasks will be discussed first. A discussion on the affects the student teachers' knowledge had on the cognitive demands of the tasks will follow. The chapter also explores the limitations of the study as well as implications for teacher education and future research.

Evidence of Knowledge Used by the Student Teachers

Prior to discussing how the student teachers used their knowledge to design and modify tasks, it is necessary to examine what knowledge was used in relation to the Hill et al. (2008) framework. There was evidence that some of the knowledge domains were used quite frequently by the student teachers in the preparation of their tasks; these knowledge domains were KCS, CCK, pedagogical knowledge, and KCT. On the other hand, there was little evidence that the student teachers used curricular knowledge or SCK in the way the research described (see Ball & Bass, 2000; Ball et al., 2005; Hill et al., 2008; Ma, 1999) to prepare the tasks.

There was ample evidence that the student teachers used KCS in the preparation of their tasks. The student teachers often discussed student strategies they had seen students employ as well as their students' misconceptions. In the case of all four tasks, knowledge of students' misconceptions or lack of particular mathematical knowledge motivated the student teachers' purposes behind the tasks: students did not have a relational understanding of the addition algorithm, students did not have a conceptual understanding of area, students struggled with subtracting integers, and students did not know how to use proportions to solve problems. The student teachers identified many examples of student thinking in relation to the mathematical topics they hoped to address through the use of the selected tasks.

There was evidence that the student teachers used their CCK in the planning of each task. The student teachers used CCK to solve the problems in their tasks and to determine the mathematical topic of the task. For example, the student teachers determined that proportions could be used to solve Task 4 because they themselves had used proportions to solve the task. As will be discussed in more detail later, the student teachers also applied CCK in order to accomplish teacher moves that Hill et al. (2008) would claim require different knowledge than CCK, such as interpreting student thinking, anticipating student thinking, and determining the fundamental mathematics of the task.

The student teachers also discussed pedagogical issues and employed their pedagogical knowledge in the preparation of the tasks. When planning their tasks, the student teachers would consider the size of student groups, how to get students to participate, and how to manage the classroom. Although mathematics could influence such decisions, these considerations were never in relation to the mathematics of the lesson and were based purely on the student teachers' pedagogical knowledge.

The student teachers used KCT to a lesser extent than CCK. Although they knew that they were supposed to build on student thinking, their plans for such teacher moves were often vague, such as "facilitate the discussion by asking questions" (Lesson Plan 3). This description of how the student teachers planned to build on student thinking was so far removed from the mathematics that it was considered pedagogical knowledge rather than KCT. Most often, KCT was used in the formative assessment of the lesson plan to plan how the student teachers could help the students realize their misconceptions. For example, in Task 1 the student teachers planned to ask specific questions about a student's addition strategy to help the student realize their misconceptions about the addition algorithm.

The little use of curricular knowledge can most likely be attributed to the lack of control the student teachers had over the curriculum. The student teachers were assigned to teach particular topics, and in some cases particular tasks, by their cooperating teacher. Thus the student teachers were not required to make curricular decisions and did not use much curricular knowledge.

Understanding how the particular topic the student teachers were teaching with a given task fit into the unit, into the subject, and into mathematics as a whole was curricular knowledge over which the student teachers did have control. Although the student teachers used the state core and the NCTM website as curricular resources to identify topics the students had learned in previous years, they did not use the resources to help them determine how the topic fit into mathematics in general.

Although there is evidence that the student teachers used some SCK to design or modify the tasks, their SCK rarely aligned with SCK identified or discussed in the research. Furthermore, if you apply Ma's (1999) theory of profound understanding of fundamental mathematics to the student teachers' SCK, the student teachers' knowledge lacked the depth and breadth described by Ma. For example, in planning Task 1 (adding ten different ways) the student teachers did not have the depth of knowledge to understand the fundamental use of the properties in the strategies they were teaching.

Nor did they have the breadth of knowledge to see the connection between algebra and arithmetic.

Ball et al. (2005) included knowledge of different representations of a mathematical concept as part of SCK. The student teachers often had different representations for the mathematics in the task. In Task 1 the student teachers had one representation of addition: that of combining sets. The student teachers had two ways of representing operations on integers for Task 3: zero pairs and number lines.

Connection between Teacher Knowledge and Teacher Moves

Hill et al. (2008) illustrated the knowledge domains in their framework through of teacher moves. For example, teachers used KCS to anticipate student thinking and to sequence instruction to align with ways students learn a mathematical topic. They used KCT to build on student thinking and to remedy student errors. Teachers used SCK to interpret student thinking and to explain and represent mathematical ideas.

Often, how the student teachers used their knowledge to plan or modify tasks aligned with the teacher moves identified by Hill et al (2008). The student teachers used KCS to anticipate and evaluate student thinking. Their knowledge of students' common misconceptions provided the student teachers with motivation for the purposes of their tasks. This motivation led to the use of KCT as the student teachers considered what experiences they should provide their students in order to remedy the misconceptions. The student teachers also used KCT to plan how to build on student thinking and to determine how much guidance or direction they should give the students in the directions of the task. The student teachers used SCK to create problems in their tasks that would elicit particular mathematics and to create mathematical explanations.

Naturally, there were times when the student teachers needed to perform a particular teacher move but lacked the knowledge to do so. For example, in Task 1 the student teachers may have needed to anticipate student thinking but the only strategy they knew the students would use was “carrying”, so they used their CCK that they had gained from the article to anticipate additional student thinking. These moments provided key insight into the knowledge on which the student teachers relied in order to perform teaching moves when they lacked the most valuable knowledge.

Reliance on Common Content Knowledge

Whenever the student teachers needed to perform a teacher move that required knowledge they lacked, the student teachers employed their CCK. This way of using CCK was most often done in the case where SCK was lacking. The student teachers used CCK to perform a myriad of teaching moves that Hill et al. (2008) included as part of other knowledge domains. There was evidence that the student teachers used CCK to anticipate student thinking, construct a mathematical explanation, interpret student solutions, and identify the mathematical topic of the task rather than delving into the fundamental mathematics of the task.

Anticipating student thinking. On two occasions, the student teachers relied on CCK rather than KCS to anticipate student thinking. The student teachers used CCK to anticipate different methods the students would use to approach the adding ten different ways task. They also used CCK to predict how students would interpret debt in the subtracting integers task.

In the adding ten different ways task, the student teachers anticipated two methods that the students would likely use to solve the problem. They were aware that students already knew the algorithm for adding multidigit numbers and would likely use the algorithm as one of their ways. Additionally, the algorithm was the method the student teachers usually used to add multidigit numbers. The other method the student teachers anticipated came from the article their cooperating teacher gave them to read and was characterized as an algebraic method. This method involved partitioning the addends and recombining the partitioned pieces in different ways. The student teachers did not know whether the students would partition and recombine the numbers when performing multidigit addition. The student teachers used CCK rather than KCS to anticipate this particular student solution.

In the subtracting integers task, the student teachers assumed that the students would interpret debt in terms of negative numbers. This assumption was based on how the student teachers interpreted debt as well as how debt is interpreted by people in general. Consequently, the student teachers designed an integer task in the context of money and debt. The student teachers used their CCK rather than their KCS to predict how the students would interpret debt. In fact, there is evidence in the literature that students will avoid the interpretation of debt as a negative integer (Ball, 1993).

Constructing a mathematical explanation. According to the framework, the construction of an explanation should involve SCK (Ball et al., 2005; Hill et al., 2008). In the case of the area task, the student teachers used their CCK to describe area and surface area. When asked to construct a conceptual definition of area for students, Kristen used the word “inside” as part of the definition. Compare the use of the word “inside” to the word “outside” used in Kristen’s conceptual definition of surface area. Kristen admitted that these definitions were her own conceptualizations of area and surface area. These conceptions were in common with how people of other professions view the terms. Although Kristen was attempting to construct an explanation for students, she used CCK in order to form the explanation rather than SCK.

Interpreting student solutions. The interpretation of student solutions often requires the use of SCK (Ball et al., 2005; Hill et al., 2008). However, in the case of Task 4, the student teachers used CCK to interpret how students were solving the problems. Prior to giving the students Task 4, the student teachers had given the students other tasks that required the students to reason proportionally in order to arrive at a solution. The student teachers noticed that the students were not setting up proportions to solve the problems, but were using other strategies instead. Not setting up proportions to solve the problems was of great concern to the student teachers as they were preparing the task. The student teachers wanted the students to develop the strategy of setting up proportions to solve problems. They were unsure how to accomplish this and decided that if they had the students create a poster to show their work, the students would set up proportions. The student teachers interpreted the previous student solutions they had seen as the students setting up proportions in their head to solve the problem. The student teachers were projecting their own solution strategy for solving the task onto what they observed students doing to solve the problems. This is similar to what A. G. Thompson and Thompson (1996) found in their study of a middle school teacher interpreting the student's conception of rate. The student teachers were using CCK to interpret student thinking rather than SCK.

Identifying the mathematical topic of the task. Although not mentioned explicitly in the literature, the teacher move of determining the fundamental mathematics of the task would involve the use of SCK. Teachers have to decompose their mathematical knowledge in order to teach mathematics (Ball & Bass, 2000); this is something that is not in common with other professionals. However, the student teachers were often unable to clearly articulate the fundamental mathematics involved in the tasks and instead relied on the mathematics they used to solve the problem to determine the mathematical topic of the task. For example, the student teachers described the fundamental mathematics concepts of the proportional reasoning problems in their lesson plan as follows: “Understand the concept of proportions and how they can be used to solve for missing values” (Lesson Plan 4). The student teachers’ description of the fundamental mathematics concepts was more of an identification of the mathematical topic they wanted to address with the task rather than the fundamental mathematics involved in proportional reasoning and setting up proportions. Similar identifications of the mathematics topics can be found in the fundamental mathematics concept section of the lesson plans for the area and subtracting integers tasks.

This conception of the fundamental mathematics as the topic the student teachers wanted the students to learn was echoed in the interviews with the student teachers. Both student teachers described the fundamental mathematics of a task in general terms as “just kind of the core principles that come out through the task” (Kristen, Interview 4) and “the big mathematical ideas, the key concepts” (Abby, Interview 4). The student teachers viewed the fundamental mathematics as the mathematical topic they wanted the students to learn by doing the task.

This conception of the fundamental mathematics may be related to the student teachers’ lack of SCK. In the case of all four tasks, the student teachers determined the mathematical topic of the task by the mathematics the student teachers used to solve the task; they used their CCK.

In planning the adding ten different ways task, the student teachers were unsure of the mathematical purpose of the task. From the various data resources used, it seemed that the student teachers failed to identify the mathematical purpose for a variety of reasons. First, they had not considered the importance of a clear, sound mathematical objective for the teacher and students to be a critical component of a worthwhile mathematical task (NCTM, 2007). Although the student teachers did have a vague purpose motivated by the fact that most students do not understand the addition algorithm (KCS), they were uncertain how to use the student thinking to bring out the notion of place value in the algorithm (KCT). In fact, their sole consideration for selecting student thinking to be presented was “creative” ideas, dependent upon the use of a variety of operations. This indicated that they did not consider how their purpose could be achieved through the use of student thinking (KCT). They also did not think deeply about the mathematics present in the task itself. They were told by both their cooperating teacher and the article used to develop the task that the task could be used to lead into combining like terms. However, it is evident from the interviews that they did not understand how the task related to combining like terms. Consequently, they settled on a purpose of having the students prove the addition facts because “it's kind of a big deal to be able to prove things” (Kristen, Interview 1) in mathematics. The student teachers used their CCK that reasoning and proof were things mathematicians do in order to determine the purpose of the task.

The student teachers had two purposes for the subtracting integers task: to develop a way to subtract that “made sense” and to help the students understand subtracting integers better. Their purpose was motivated by the student teachers’ experiential CCK and KCS; their own experiences with subtraction were nonsensical and their students seemed be struggling with subtracting integers, especially when subtracting a larger number from a smaller number or subtracting a negative number. The student teachers were assigned by their cooperating teacher to teach useful ways for subtracting, referring to partitioning the numbers in the subtraction problem in such a way that makes the subtraction easier. Following the lesson and reflection meeting for Task 1, the student teachers realized that the students did not engage in any particular mathematics because the task itself did not elicit any particular thinking. The student teachers learned that they, as teachers, need to have a clear purpose for using a task, and that the students also need to have a mathematical purpose for doing the task. From their experiences with Task 1, they felt that they needed more of a purpose to Task 3 and its accompanying lesson. Knowledge of subtracting integers was used by the student teachers when they partitioned and recombined the numbers in the subtraction problem and so the student teachers felt that integers would be a good purpose for the day’s lesson.

The placement of the useful subtraction discussion before Task 3 confused the distinction between subtraction and negative numbers. Conceptually, subtraction can be defined as “taking away” (which is how the student teachers described subtraction), so subtracting a negative number involves removing some of the negatives. The student teachers relied on this conception to construct their task; the student teachers created a task where the students could “take away” some of their debt. However, their discussion of subtracting in useful ways did not use this conceptualization of subtraction. In fact, it seemed to be counter to the conceptualization of subtraction as “taking away”. For example, in order to perform the subtraction 63 – 29 which conceptually involves removing 29 positives from the set, the student teachers planned to rewrite the negative 29, conceptually changing the problem to adding 29 negatives, as -20 – 6 – 3, once again changing the conceptualization of the problem. Failure to consider how the conceptualization of the useful ways to subtract was counter to developing a conceptual understanding of integer subtraction indicated a failure to use aspects of their SCK. They did not have adequate SCK to evaluate the alignment of their purpose and task because they had relied on CCK to identify the mathematical topic of the task.

As the students had already been using proportional reasoning to solve problems in their classes, the purpose of the proportional reasoning task for the student teachers was to get the students to use the strategy of setting up proportions to solve problems involving multiplicative relationships. Using their CCK, the student teachers determined that setting up proportions was the mathematical topic of the task because the student teachers had used proportions to solve the problems themselves. However, they did not give the students problems that necessarily motivated the use of proportions; in fact, they included problems where a scalar multiplier could be identified easily, with one problem even involving fractions which would lead to the use of complex fractions when setting up a proportion. The student teachers’ use of CCK to identify the mathematical topic of the task resulted in a misalignment between the task and the student teachers’ purpose in using the task.

The student teachers identified the mathematics of the tasks by the mathematics they used to solve the problems, or their CCK. The student teachers did not have adequate SCK to determine the fundamental mathematics of the task and had to rely on their CCK. They also used the state core to motivate, identify, and justify the mathematical purposes of the tasks. Their reliance on the core could also be related to their lack of SCK. There was no evidence in any of the tasks that they considered how their purpose aligned with the mathematics of the task beyond superficial consideration of the mathematical topic of the task.

The Effect of Knowledge on the Cognitive Demand of the Task

Factors Associated with a Decrease in Cognitive Demand

In the previous chapter, the results of the cognitive demand of the tasks were given. Synthesizing the results of each task, common factors were associated with decreases in the cognitive demands. These factors were expecting an explanation of what was done rather than a mathematical justification when the students presented their work, removing the connections to meaning from the task, providing students with ready access to relevant knowledge, and not tailoring the task to the level of the students.

In the case of all four tasks, the student teachers expected an explanation of what was done rather than a mathematical justification. In the case of Task 1, the explicit request for a justification in the original task was removed when the student teachers modified the task. None of the other tasks in their written phase ever explicitly asked for a justification, although the lesson plan for Task 4 has the students create posters to demonstrate their solution. Still, the student teachers expected an explanation of the students' processes rather than a mathematical reason for each step in the process.

In Task 1 and Task 3, the student teachers removed possible connections to the meaning of the mathematics as the task passed through different phases. In the former task, the student teachers removed meaning as the task passed from it original form to its modified form by not emphasizing the axiom, that if you add a set of equals to another set of equals then their sums will also be equal, and its connection to the meaning of addition. In the latter task, the student teachers removed the connection to subtraction from the task as it passed from the written phase to the intended phase by placing a discussion on different ways of subtracting prior to giving the students the task. In the discussion, the student teachers used subtracting positive numbers and adding negative numbers interchangeably; however, the two operations are conceptually different.

The students' access to relevant knowledge played a role in the decrease in cognitive demands of Task 1 and Task 4. In the former task, the student teachers made the knowledge needed in order to successfully complete the task easily accessible to the students by teaching the students the method of partitioning prior to having them add the numbers and expected the students to continue to rehearse the procedure in ten different ways. In the latter task, the student teachers hinted that the students should use proportions to solve the problem by incorporating proportion problems into the quiz at the beginning of the period. The student teachers planned to discuss how the students solved the proportion problems in the quiz as a class prior to having them work on the task. Thus, the student teachers made the knowledge necessary for successfully completing the task easily accessible to the students.

Finally, in Task 1 and Task 2, the student teachers decreased the cognitive demand by not considering the cognitive level of the students in the design of the task. In

Task 1, the students were expected to do arithmetic in different ways. However, junior high students should be expected to generalize arithmetic and should be pushed to think more algebraically. The student teachers did not adjust the task in this way to meet the needs of their students. The latter task was designed to help the students gain a better understanding of area measurement. However, according to the NCTM (2000) students this type of task would be more appropriate in the upper elementary grades rather than in the junior high grades. Junior high students should be expected and prodded to develop and justify area formulas, which is more than was expected in the task.

Impact of Knowledge on the Cognitive Demand

Given that the cognitive demands of the tasks changed as the tasks passed from one phase to another, it was natural to wonder whether the student teachers' knowledge contributed to the decreases. The analysis revealed that the decreases in cognitive demand could be attributed to a lack in the student teachers' knowledge. Additionally, the presence of a particular knowledge (e.g. CCK, KCS, KCT, etc.) did not necessarily lead to an increase in the cognitive demands of the tasks.

The student teachers' expectations of the kinds of justifications they anticipated their students would give for the problems were along the lines of an explanation rather than a mathematical justification. This was connected to the student teachers' lack of mathematical content knowledge (which encompasses both CCK and SCK) as well as KCS. Although the student teachers expected the students to give presentations of their thinking, the sample student presentations given in the lesson plans indicated that the student teachers did not have a clear picture of the type of thinking they wanted presented. They were hoping for particular ways of solving the tasks, for example, looking for students who set up proportions to solve the proportional reasoning task or looking for students who used the different operations creatively to add ten different ways. The student teachers were more concerned with the process than the justification. In the interviews, the student teachers were asked to give examples of ideal justifications. The student teachers responses indicated that an explanation of the process was sufficient. The expectation of an explanation rather than justification could be related to the student teachers' knowledge in two ways: their own knowledge of mathematical justification was deficient, whether in general or in the particular cases of the tasks; or the student teachers were unsure the level of justification they could expect from the students (KCS).

Task 3 highlighted how a teacher could inadvertently remove meaning from a procedure due to a lack of teacher knowledge. The task was disconnected from the underlying mathematics in two ways: the juxtaposition of the intended discussion of subtraction and the task in the lesson plan, and the task itself did not require the students to engage in the fundamental mathematics of operations on integers. In Interviews 2 and 3, both student teachers indicated that they understood the rules of subtracting integers; Kristen even had a number line representation for explaining how to subtract integers. The student teachers also understood how they could partition numbers in different ways in order to subtract numbers more easily. Both of these pieces of knowledge are CCK. However, the decision to place a discussion on subtraction that confused subtracting positive numbers with adding negative numbers before a task intended to help the students conceptualize subtraction indicated that the student teachers were lacking SCK and KCT. The student teachers needed to use SCK to realize that the discussion they had planned removed the meaning of subtraction from the procedure they wanted to teach in the discussion by ignoring the conceptual changes from subtracting positive numbers to adding negative numbers. The student teachers could have used SCK to modify the discussion so that it was consistent with the meaning of subtraction or they could have used KCT to not put the discussion and the task in the same lesson plan. Even though Kristen indicated in her third interview that the task did not require the students to partition the numbers to subtract, she did not recognize that the strategy of partitioning the numbers was counter to the mathematical meaning that subtracting negative numbers involves removing negatives from the set she was trying to create through the use of the task.

Ironically, the student teachers tried to create Task 3 so that students would have to grapple with the issues of subtracting negative numbers. They chose the context of money because it offered a situation to discuss negative numbers (i.e. debt) and what it means to subtract a negative number (remove some of your debt). I argue that the student teachers used CCK in order to select the context for the task. If they had used SCK, they may have recognized that the use of negative numbers was not necessary to solve the task. Even though the student teachers had CCK, they needed SCK in order to increase or maintain the cognitive demand of the task.

Another reason for a drop in cognitive demand was the student teachers' inability to tailor tasks to the cognitive level of their students. This inability was best exemplified in Task 2. The student teachers had created a task that held connections to underlying mathematical principles. However, the student teachers did not have the KCS in order to determine whether the task was appropriately challenging for the students.

A similar situation occurred in Task 1. Although the student teachers had the KCS to realize the task would likely not be very challenging for the students, they lacked the SCK to make the task more challenging. Specifically, the student teachers lacked knowledge of how the task could be used to generalize arithmetic and lead into a discussion on like terms. The student teachers needed SCK in order to increase the cognitive demand of the task.

The accessibility of relevant knowledge for the students was another issue contributing to a decrease in cognitive demand. In Task 4, the student teachers wanted the students to develop the strategy of setting up proportions to solve problems. In order to get the students to use proportions, the student teachers made the knowledge of proportions more accessibly to the students by placing a problem involving proportions in the quiz. This placement decreased the cognitive demand of the task. In some ways this decrease can be traced to a lack of KCS. The student teachers did not know that students are often reluctant to set up proportions. More importantly, the student teachers lacked SCK. The numbers in the task did not promote setting up proportions as a strategy, and in some ways it actually discouraged the use of the strategy. Rather than using their SCK to consider how the numbers of the problems in the task could elicit a particular strategy, the student teachers decreased the cognitive demand of the task by making the knowledge of proportions less time-removed for the students.

Most of the time, the tasks experienced either a decrease in cognitive demand or a maintenance of an already lower-level cognitive demand. The decrease in cognitive demand in the tasks could be linked to the student teachers’ lack of a particular domain of knowledge. Even though the student teachers used other domains of knowledge, they often lacked the SCK needed to prevent a decrease in the cognitive demands of the tasks.

Negative case analysis. Mertens (2005) suggested that a negative case analysis adds credibility to a study because it provides additional support to the hypothesis. While most of the tasks decreased in cognitive demand, there was one time when the cognitive demand of the task increased. The student teachers were initially given a set of problems to serve as Task 4. This initial form of the task was considered a procedures without connections task. However, the student teachers felt that the task in that form would not elicit the student thinking they desired. Consequently, the student teachers omitted the excessive information in the problems that made the task too easy. This omission increased the cognitive demands of the task.

As this is a case when the cognitive demand of the task increased rather than decreased, it is important to consider the student teachers' knowledge that contributed to the increase. Even though the student teachers first used CCK to solve the problems themselves, doing so helped them recognize that the problems were relatively easy and there was no need to set up a proportion to solve them. However, when Kristen suggested that they remove the information that made the task too easy she was no longer using CCK. I submit that by identifying the components of the problem that made it so that the students would not use proportions, the student teachers were using SCK. The knowledge needed to identify what made the task easy is not knowledge one would expect other professionals besides teachers to possess. Thus, SCK led to an increase in the cognitive demand of the task.

Conclusions

The discussion earlier in the chapter described how the student teachers used their knowledge to design, modify, and plan how to incorporate the tasks into their lessons. They often used KCS, pedagogical knowledge, and CCK. They occasionally used KCT and rarely used SCK and curricular knowledge. Interestingly, the student teachers used their CCK to anticipate student thinking even though KCS would have been a more valuable knowledge domain to use. Similarly, the student teachers used CCK to construct mathematical explanations, interpret student solutions, and identify the mathematical topic of the task. Ball et al. (2005) indicated that such teaching activities required the use of SCK to effectively accomplish.

The first research question asked how student teachers use their knowledge to design or modify mathematical tasks. The analysis showed that the student teachers usually used their knowledge in ways that were common with how Hill et al. (2008) described (e.g., using KCS to anticipate student thinking, using KCT to remedy student misconceptions, using SCK to represent mathematics). However, the student teachers used CCK to perform many teaching jobs that Hill et al. claimed required a knowledge domain different from CCK.

The cognitive demands of the tasks often changed as the student teachers modified or incorporated the task into their lesson plan. In most cases, there was a drop in the cognitive demands of the tasks. Factors explaining these changes included expecting a explanation of the students' thinking processes, removing connections to the mathematics from the task, providing the students with the knowledge needed to solve the task, and not considering the developmental level of junior high students. These factors were associated with a lack of knowledge, usually SCK, in the student teachers.

The second research question asked how the student teachers' knowledge impacted the cognitive demands of the task. The analysis indicated that the presence of the student teachers' SCK had the greatest impact on the cognitive demand of the task. When the student teachers were missing SCK and had to rely on CCK instead, the cognitive demand of the task descended. However, when the students had SCK, as was the case in Task 4, the cognitive demands of the task increased.

Limitations

This study was limited in the sampling of the participants in that only two student teachers were used. It was limited in the data collection by only video recording the first day of each task, the timing of the interviews, and the due date of the reflection papers. The analysis in the study was in the use of only one researcher to code for knowledge and the need to infer the knowledge of the student teachers.

Initially, the study planned to sample two different pairs of student teachers. However, as mentioned in Chapter 3, the data from one pair of students was not useable. This meant that the results of this study came from one pair of student teachers. Although there was a plenty of data gathered on the two student teachers, the use of more student teachers would have provided a broader data set.

The researcher did not anticipate that the tasks planned by the student teachers would last for multiple days. However, the length of the class periods often made it difficult for the student teachers to close a task in one day. The researcher had only arranged to video record the first day of the task and consequently did not video record all of the enacted phase of the task. Originally, the researcher had intended to study the enacted phase of the task, but was limited to studying only the written and intended phases of the task.

The interviews occurred after the student teachers had taught the lesson and discussed the lesson in the reflection meeting. There was a lag of about five or six days between the lesson and the interview. Consequently, the student teachers may have been influenced by the reflection meeting and lesson, and misrepresented some of the knowledge that they used in the task. It would have been nice to interview the student teachers within a day of their lesson. It also would have improved the study if the interviews could have occurred prior to the enacted phase of the task so that the researcher could have collected more accurate data about what the student teachers had intended rather than what had happened.

There was also a lag in time between the reflection meeting and the reflection paper. The student teachers had approximately one week to write the reflection paper. The reflection paper was not always focused on the lesson the student teachers taught, and consequently were not as valuable to this study. Additionally, the student teachers sometimes wrote the reflection papers following their interviews with the researcher. Their papers then reflected some of the ideas that were discussed in the interviews. It is difficult to know how much the interviews impacted reflection papers.

The study was also limited by the use of only one researcher to code the data. Although multiple raters were used to code the tasks, only one researcher coded the data for the different domains of knowledge. The use of only one researcher impacts the reliability of the study. As the knowledge had to be inferred from the data, other researchers could have coded the data differently. Additionally, as the knowledge had to be inferred from the data, it is impossible to know for sure exactly what knowledge the student teachers used to design, modify, and plan their tasks.

Implications

Implications for Future Research

This research project was a study of novice rather than experienced teachers. Although this study answers the question of how student or novice teachers use their knowledge when designing, modifying, and planning tasks, it does not answer how teachers in general use their knowledge. Beginning teachers may be more likely to lack the knowledge needed to accomplish certain aspects of their job and consequently rely on CCK. However, the literature has suggested that even experienced teachers do not have sufficient knowledge for teaching, which could mean that experienced teachers continue to rely on their CCK (see Ball et al., 2001; Fennema & Franke, 1992; Mewborn, 2003). Future research should investigate whether experienced teachers continue to rely on CCK.

Future research should also investigate how teachers who rely on CCK differ from teachers who do not. One research article indicated that a student teacher failed to learn from her teaching experiences because she did not take the time to reflect on the experience (Borko et al., 1992). If the student teacher had reflected on the teaching experience and the meaning of fraction division, would she have gained some SCK that she could have used later in her teaching? Is reflection a characteristic that differentiates the teachers who have SCK from those who rely on CCK?

Additionally, could there be a difference in the type of knowledge elementary teachers rely on to supplement their SCK? Research has indicated that elementary teacher are often more insecure about mathematics than secondary mathematics teachers. Would an elementary teacher rely on CCK or is there a different knowledge domain on which elementary teachers would rely? Personal experience indicates that elementary teachers may rely on pedagogical knowledge when they lack SCK.

The data gathered in this study and scope of the project prevented the investigation of how the student teachers used their knowledge during the enacted phases of the tasks and how that knowledge impacted the cognitive demands of the task. Stein et al. (2000) found that there was often a significant decrease in the cognitive demand of a task during the enacted phased of the task. This change was likely impacted by the knowledge of the student teachers (Stein et al., 2007); precisely how the teacher knowledge impacts the transformation is still unknown.

This research project also did not study how student teachers select and evaluate tasks. The student teachers were often given a task to teach the class, and at a minimum given a situation from which to create a task. Thus, the student teachers did not fully use their knowledge to select worthwhile mathematical tasks. Instead, the student teachers were forced to try to adapt the given tasks. It would be interesting to study how student teachers and teachers in general use their knowledge to evaluate and select tasks for the classroom. Osana et al. (2006) already suggested that there was a correlation between teacher knowledge and their ability to correctly evaluate tasks, but Osana et al. did not look at the kinds of knowledge the preservice teachers used to make their evaluations.

Finally, future research should investigate how SCK is developed. Is it something that can be taught to preservice teachers or can it only be learned through teaching experiences? Answers to these questions will improve teacher education programs.

Implications for Teacher Education

This was a study of how student teachers used their knowledge to plan and design tasks. The findings indicated that the student teachers had not developed the SCK they needed to design and modify tasks and had to rely on CCK. This has several implications for teacher education, the most important being how to get preservice teachers to develop SCK.

It would be impossible to teach preservice teachers all the SCK they would need in order to teach mathematics. For one, there is not enough time in the teacher preparation program to cover all the material in depth. Additionally, the SCK needed to teach has not been fleshed out in its entirety for all of mathematics, nor could it ever be. Therefore, simply adding more methods classes or content classes where the preservice teachers learn more about the mathematics they will be teaching is not enough. The preservice teachers need to be given opportunities to develop SCK. However, as research has not yet determined how SCK develops, it is difficult to know the types of experiences that would develop SCK.

Some possible experiences could include having preservice teachers evaluate and sort tasks. Preservice teachers could also be asked to modify or change tasks so that they would be at a higher-level of cognitive demand. Preservice teachers could also look at videos of teaching moments and discuss how the enacted phase of the task differed from the written or intended phases.

Although not part of the scope of this project, there was some indication that the student teachers gained SCK during their student teaching experience. The experiences that promoted the development of SCK were unique to the student teacher program of the university. In the interviews with the student teachers, they indicated that they had gained SCK after teaching the lesson and reflecting on it in their reflection meetings, reflections papers, and discussions between themselves as well as discussions with their cooperating teacher. This indicates that reflection may be key to the development of SCK and points to the adoption of student teaching programs and teacher education programs that promote reflection.

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Appendix A: Lesson Plan Template

Department of Mathematics Education Lesson Plan Template
Cover Sheet

Name	Date:
Lesson Title	
Fundamental Mathematics Concepts	
Requisite Mathematics	

Lesson Sequence

Unit Plan Sequence	**Time**	**Student Thinking and Responses**	**Formative Assessment**
Launching Student Inquiry			
Supporting Productive Student Exploration of the Task			
Facilitating Discourse and Public Performances			
Unpacking and Analyzing Students’ Mathematics			

Appendix B: Abby Interview Protocol 1

Interview Protocol
Student Teacher: Abby
Lesson Date: September 20, 2007

1. Who originally designed this task? Did you modify it in any way? Why did you choose the numbers 15, 27, and 42?
2. Can you evaluate this task in terms of the students' mathematical abilities?
3. What were your main mathematical objectives in using this task?
4. What types of student thinking did you want to see students come up with during the task?
5. In the lesson plan, you titled the lesson Proving Addition Facts. Could you give an example of an adequate justification?
6. I'm going to show you some of the students' presentations of their work. After listening to their presentation, I want you to describe whether or not you think the student adequately proved their addition strategy.
   Segment 1
   Segment 2
   Segment 3
   (Segment 4)
7. In the lesson plan, it states that another purpose of this task was to lead into combining like terms. How could you use this student's work to introduce combining like terms? Could you create an example of the optimal student work you could use to introduce combining like terms?
8. You showed a subtraction example to review ideas from the previous day. (Review the teaching segment.) What do you think the student meant when he said, "plus 70"? Could there be a mathematical reason?
9. In the warm-up, there was a question about the divisibility rule for four. In the reflection meeting, you said that you think you understand why four works. Could you describe that for me?
10. In the warm-up, why didn't it matter whether or not we performed the parenthesis or exponential operation first?

Appendix C: Kristen Interview Protocol 1

Interview Protocol
Student Teacher: Kristen
Lesson Date: September 20, 2007

1. In the reflection meeting, you said that your primary goal was to decrease the amount of complaining you had witnessed the day before. Why do you think that so many students were complaining?
2. Who originally designed this task?
   Did you modify it in any way?
   Why did you choose the numbers 15, 27, and 42?
3. Can you evaluate this task in terms of the students' mathematical abilities?
4. What were your main mathematical objectives in using this task?
5. What types of student thinking did you want to see students come up with during the task?
6. In the lesson plan, you titled the lesson Proving Addition Facts. Could you give an example of an adequate justification?
7. I'm going to show you some of the students' presentations of their work. After listening to their presentation, I want you to describe whether or not you think the student adequately proved their addition strategy.
   Segment 1
   Segment 2
   (Segment 3)
8. In the lesson plan, it states that another purpose of this task was to lead into combining like terms. How could you use this student's work to introduce combining like terms?
9. Could you create an example of the optimal student work that you could use to introduce combining like terms?
10. In the warm-up, there was a question about the divisibility rule for four. In the reflection meeting, you said that you think you understand why four works. Could you describe that for me?

Appendix D: Abby Interview Protocol 2

Interview Protocol
Student Teacher: Abby
Lesson Date: September 26, 2007

1. Preparation
    a. In your lesson on "Proving addition facts", you seemed to indicate that you were unsure of the purpose or the big mathematical ideas of the task you were teaching. Is that correct?
    b. Prior to teaching the surface area lesson, did you feel that you understood the big mathematical ideas better than with the addition facts lesson? Why? How did you prepare more?
    c. After having taught the lesson, do you (still) feel that you understood the big mathematical ideas of the lesson?
    d. How are you feeling about your preparation on the big mathematical ideas for tomorrow's lesson?
2. Discussion of Task A (24:30)
    a. When you were gathering students' estimates of the area, did you think it was important to find out how the students found their estimates? Why?
    b. When you first heard the estimate 3 1/8, what did you think about that answer?
    c. At the time, why did you decide not to ask the 3 1/8 group how they found their estimate?
3. Discussion of Task B (33:00-37:00)
    a. What are some of the big ideas that students need to understand about area? What does it mean to have a conceptual understanding of area?
    b. Do you think Mosiah understands area? What does he understand, what doesn't he understand?
4. Warm-Up Problem 2 (8:00-8:40)
    a. What kinds of student thinking were you expecting students to use to solve this equation?
    b. Did you notice any of these strategies when you were wandering around the classroom?
    c. What is the value of the strategy the students used ("substituting different numbers")?
5. Warm-Up Problem 3 (8:40-10:00)
    a. How well do you think these students understood how to graph coordinate points?
    b. Do you think they could identify which quadrant the point is in without graphing the point?
    c. Do you think they could give the ordered pair of a graphed point (especially a point on one of the axes)?

Appendix E: Kristen Interview Protocol 2

Interview Protocol
Student Teacher: Kristen
Lesson Date: September 26, 2007

1. Preparation
    a. In your lesson on "Proving addition facts", you seemed to indicate that you were unsure of the purpose or the big mathematical ideas of the task you were teaching. Is that correct?
    b. Prior to teaching the surface area lesson, did you feel that you understood the big mathematical ideas better than with the addition facts lesson? Why? How did you prepare more?
    c. After having taught the lesson, do you (still) feel that you understood the big mathematical ideas of the lesson?
    d. How are you feeling about your preparation on the big mathematical ideas for tomorrow's lesson?
2. Task A and Task B
    a. What were the big mathematical ideas of the area tasks?
    b. What does it mean to have a conceptual understanding of area?
    c. In task one, you asked the students to estimate the area using their unit square while in task two you asked them to find the exact area by unfolding their unit square and counting triangles. Why did you suggest they use triangles to find the exact area?
    d. What does it mean to find an exact answer to area?
3. Warm-Up Problem 1 (6:00-7:15)
    a. When explaining to the student why -11 and +7 didn't work, you said "because we went 11 negative and only came back 7 positive." How does this describe why the sum of -11 and 7 is not 4?
    b. Is there another way you could have described adding integers?
    c. Why did you choose to use a number line representation?
    d. Why is the product of -11 and 7 the same as the product of -7 and 11?
4. Warm-Up Problem 2 (7:15-7:55)
    a. What kinds of student thinking were you expecting students to use to solve this equation?
    b. Did you notice any of these strategies when you were wandering the classroom?
    c. Why were you surprised when the student used a "working backwards" strategy?
    d. Although you had her explain her thinking, you didn't discuss it anymore with the class. Why did you decide not to discuss this idea more?

Appendix F: Abby Interview Protocol 3

Interview Protocol
Student Teacher: Abby
Lesson Date: October 3, 2007

1. Task
    a. What were your student learning objectives of this task?
    b. How did this task help accomplish ________ objective?
    c. How well did the students understand ____________ objective of the task?
2. How did you feel about changing the order of the discussion and the task? How do you feel the task and discussion went during the class?
3. Why did you ask the question, "Could there be different solutions to the problem?" (26:20) Do you think the students understood why the answer is the same?
4. Did Mosiah and Amber approach Tuesday the same way? How were their approaches different?
5. Why did you ask the question, "What does this 15 represent?"

Appendix G: Kristen Interview Protocol 3

Interview Protocol
Student Teacher: Kristen
Lesson Date: October 3, 2007

1. Task
    a. What were your student learning objectives of this task?
    b. How did this task help accomplish _________ objective?
    c. How well did the students understand _____________ objective of the task?
2. Why did you decide to have the discussion first and then give the students the task?
3. Partitioning Subtraction (18:00)
    a. Why did you use multiplication when you were repartitioning the problem $23-$18?
    b. Why would it be negative 10?
    c. What is the difference between subtraction and a negative number?

Appendix H: Abby and Kristen Interview Protocol 4

Interview Protocol
Student Teacher: Abby and Kristen
Lesson Date: December 6, 2007

1. How do you determine the fundamental mathematics in a task?
    a. What types of outside resources do you use to determine the fundamental mathematics in a task?
    b. How does the CORE help you determine the fundamental mathematics?
2. What did you talk about when you were planning this task?
3. How did you find the proportional reasoning problems you used for your task on Thursday?
    a. Why did you choose those particular problems over some of the other problems that were in the book?
4. Are all these problems of equal difficulty?
    a. Which do you think are harder/easier?
    b. Why?
5. How did you expect students to solve these problems?
    a. What are some other strategies the students could have used?
6. What was your primary student learning outcome?
    a. How could problem 1 help students learn how to __________________?
    b. How could problem 2 help students learn how to __________________?
    c. How could problem 3 help students learn how to __________________?
    d. How could problem 4 help students learn how to __________________?
    e. Is one of these problems more likely than the others to lead students to set up a proportion to solve? Why?
7. How could you use these students' work to set up a proportion?
8. Why did you decide to give the groups different problems to solve?

Appendix I: Original Phase of Task 4

If You Hopped Like a Frog . . .

Frogs are champion jumpers. A 3-inch frog can hop 60 inches. That means the frog is jumping 20 times its body's length. How tall are you? If you could jump 20 times your body length, how far could you go?

If You Were as Strong as an Ant . . .

Ants may be ting, but they are great weight lifters. An ant weighing 1/250 of an ounce can easily lift a bread crumb weighing 1/5 of an ounce. That means the ant is lifting 50 times its own weight. How much do you weigh? If you could lift 50 times your weight, could you lift a 3,000-pound car?

If You Had the Brain of a Brachiosaurus . . .

Large dinosaurs had tiny brains. Brachiosaurus weighed about 80,000 kilograms, but its brain weighed only about 200 grams (0.2 kilograms). So its body was about 400,000 times as heavy as its brain. What is your weight? What would your brain weigh if it weighed 1/400,000 as much as your body?

If You Scurried Like a Spider . . .

Considering its length, a female house spider is faster than any other animal, even a cheetah. It can move 33 times the length of its own body in 1 second. If you could run 33 times your body length in a second, how many feet per second could you run? How long would it take you to run the length of a 100-yard football field? How far could you run in one minute if you could keep up that pace?

If You Swallowed Like a Snake . . .

Your lower jaw is hinged to your upper jaw, but a snake's jaw is not. If a snake wants to eat something big, it can simply drop its entire lower jaw to get its mound w-i-d-e open. A western diamondback rattlesnake with a head just 1 inch wide can swallow a whole gopher measuring 2 inches across. That means the snake is eating something twice as wide as its head. How wide is your head? What could you swallow in the classroom that is twice as wide as your head?

If You Ate Like a Shrew . . .

Shrews are among the smallest of mammals, but their appetites are huge! A shrew that weighs just 1/5 of an ounce eats about 3/5 of an ounce of yummy insects and worms each day. That means it eats 3 times its own weight daily! How much would you be eating each day if you could eat like a shrew?

If You High-Jumped Like a Flea . . .

A flea just 3 millimeters high can spring more than 200 millimeters into the air---almost 70 times its own height. The Statue of Liberty is 93 meters above the ground. Measure your height in centimeters and figure out how high you'd go if you could jump 70 times your height. How high could you jump? Could you land on the top of the Statue of Liberty?

If You Flicked Your Tongue Like a Chameleon . . .

Chameleons are experts at standing still, unnoticed by unlucky insects. When one flies by, . . . Zzzzzzzap! Out goes a very long tongue. The fly is now food. A 1-foot chameleon may have a 6-inch tongue. Its tongue is half as long as its body. How long would your tongue be if you had a tongue like a chameleon's? What could you touch with your tongue from your seat?

If You Craned Your Neck Like a Crane . . .

A whooping crane that's 4 feet tall (48 inches) has a 16-inch neck. That means its neck is 1/3 the height of its body. How tall are you? How long would your neck be if it were 1/3 your height?

Appendix J: Modified Phase of Task 4

If You Hopped Like a Frog . . . Frogs are champion jumpers. A 3-inch frog can hop 60 inches. If you could jump like a frog, how far could you hop in one jump? How many jumps would it take you to jump down a football field (100 yards)? How far would you go if you hopped 30 times?

If You Scurried Like a Spider . . . Considering its length, a female house spider is faster than any other animal, even a cheetah. It can move 165 times the length of its own body in just 5 seconds. If you could run like a spider, how many feet per second could you run? How long would it take you to run the length of a football field (100 yards)? How far could you run in one minute if you could keep up your pace?

If You Ate Like a Shrew . . . Shrews are among the smallest of mammals, but their appetites are huge! A shrew that weighs just 1/5 of an ounce eats about 3/5 of an ounce of yummy insects and worms each day. How much would you be able to eat each day if you could eat like a shrew? How many pounds of chocolate would you be able to eat in a week?

If You High-Jumped Like a Flea . . . A flea just 3 millimeters high can spring more than 200 millimeters into the air. If you jumped like a flea, how high could you jump? The Statue of Liberty is 93 meters above the ground. If you could jump like a flea, could you jump over the Statue of Liberty?