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Fusion of Character Tables and Schur Rings of Dihedral Groups

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FUSIONS OF CHARACTER TABLES AND SCHUR RINGS OF DIHEDRAL GROUPS

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ABSTRACT

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A finite group H is said to *fuse to* a finite group G if the class algebra of G is isomorphic to an *S-ring* over H which is a subalgebra of the class algebra of H. We will also say that G *fuses from* H. In this case, the classes and characters of H can fuse to give the character table of G. We investigate the case where H is the dihedral group. In many cases, G can be completely determined. In general, G can be proven to have many interesting properties. The theory is developed in terms of S-ring of Schur and Wielandt.

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Contents

Chapter 1

Introduction

We investigate S-rings as developed by Schur and Wielandt and the fusion of groups. We begin Chapter Two by stating the definitions of S-rings and fusion of groups. A finite group H is said to *fuse to* a finite group G if the class algebra of G is isomorphic to an S-ring over H which is a subalgebra of the class algebra of H. We will also say that G *fuses from* H. In this case, the classes and characters of H can fuse to give the character table of G. We investigate the case where H is the dihedral group. We cover some necessary preliminaries, which include a discussion of the group algebra, the Magic Rectangle Condition and some relevant basic results in Chapter Three. In Chapter Four, we discuss the main problem of which groups fuse from D_{2n} and look at both the odd and even cases for n. We end the chapter by solving the $2n = 8p$ case. Finally, in the Chapter Five, we dicuss the classification of Schur rings over cyclic groups and apply it to our problem in the $2n = 8p^2$ case.

Chapter 2

Main Results

Let G be a finite group and $\mathbb{C}G$ be its group ring over $\mathbb{C}$. Suppose $W \subset G$, define $\overline{W} := \sum_{w \in W} w \in \mathbb{C}G$ and $W^{(-1)} := \{w^{-1} | w \in W\}$. In general, for $n \in \mathbb{Z}$, define $W^{(n)} := \{w^n | w \in W\}$.

Definition 2.0.1. *Suppose $\{S_i\}_{i=1}^m$ is a partition of a finite group G satisfying three conditions:*

(1) $S_1 = \{1\}$.

(2) If $S_i = \{g_1, \cdots, g_s\}$ then $S_i^{(-1)} = \{g_1^{-1}, \cdots, g_s^{-1}\} = S_j$ for some j.

(3) If $i, j \leq m$, then $\bar{S}_i \bar{S}_j = \sum_k a_{ijk} \bar{S}_k$ where a_{ijk} is a non-negative integer for all i, j, k. These a_{ijk}'s are called the structure constants of the group G.

An S-ring S over G is the subalgebra of $\mathbb{C}G$ generated by $\bar{S}_1, ..., \bar{S_m}$.

We call each S_i a *S-principal subset* of G. The set of all S-principal subsets will be denoted by *D(S)*. A subgroup H of G is an *S-subgroup* if $\overline{H} \in S$.

Example 2.0.2. *Suppose G is a finite group. Let $S_1 = \{1\}$ and $S_2 = G \setminus \{1\}$. We can easily check that this forms an S-ring over G. This is called the trivial S-ring.*

Example 2.0.3. *Let the S-principal subsets be the conjugacy classes of G. It can* *easily be checked that this partition also satisfies the S-ring conditions. This S-ring is the class algebra of G.*

Example 2.0.4. *If N is a normal subgroup of G, then the conjugacy classes of G contained in N form an S-ring over N. Denote this S-ring $F_G(N)$.*

Definition 2.0.5. *A finite group H is said to fuse to a finite group G if the class algebra of G is isomorphic to an S-ring over H which is a subalgebra of the class algebra of H. We will also say that G fuses from H. If G fuses from H, then under the isomorphism, each class C of G corresponds to a subset $C^* \subset H$, where C^* is a union of conjugacy classes of H. In this situation, we will say that C and C^* correspond.*

If X is a set, then $|X|$ will denote the cardinality of X. Suppose $n = 2m$. Let

$$Dic_{2n} = \langle a, b | a^m = b^2, b^4 = 1, b^{-1}ab = a^{-1} \rangle.$$

We call Dic_{2n} the *dicyclic group* of order $2n$.

Our main theorem is:

Theorem 2.0.6. *Suppose G fuses from D_{2n}.*

i) If n is odd, then $G \cong D_{2n}$.

ii) If $n = 2m$, and m is odd, then $G \cong D_{2n}$.

iii) If $2n = 8p$, then $G \cong D_{2n}$ or $G \cong Dic_{2n}$.

iv) If $2n = 8p^2$, p an odd prime, then we have either, i) G is dihedral or dicyclic; or ii) p=3 and G is the Frobenius group of order 72.

In the case where $|G| = 4m$, m is even with $Z(G) = \{1\}$, we can deduce the following important properties for G:

1. G has at least two classes of size m.

2. $G/G' \cong \mathbb{Z}_2^2$.

3. G has a unique non-trivial class of odd size, which is a class of involutions.

4. Each non-linear irreducible complex character of G is of even degree.

5. G has a degree 2 irreducible character.

6. Each character of G is real;

7. Each element of G is conjugate to its inverse.

8. G is 2-nilpotent, that is, the Sylow 2-subgroup of G has a normal complement in G.

9. G is solvable.

10. The Sylow 2-subgroup of G is dihedral, dicyclic or quasidihedral.

11. Let M be the normal subgroup of G which corresponds to $\langle x \rangle \leq D_{2n}$. Then (G, M, G') is a Camina triple, that is, conjugacy classes of G in $G \setminus M$ are union of cosets of G'.

12. G is not a direct product.

13. G has at most $d(n) + 3$ normal subgroups, where $d(n)$ is the number of divisors of n. Furthermore, for each divisor $d < n$ of n, G has unique normal subgroup of order d.

We will hereafter refer to these properties as *Property* n, $1 \leq n \leq 13$.

Chapter 3

Preliminaries

We introduce the group algebra, the Magic Rectangle Condition and state some basic results.

3.1 The Group Algebra

Suppose G is a finite group with elements $\{g_1, ..., g_n\}$. We define a vector space over $\mathbb{C}$ with basis elements $\{g_1, ..., g_n\}$ and denote it by $\mathbb{C}G$. The elements of $\mathbb{C}G$ are formal sums of the form,

$$\lambda_1 g_1 + ... + \lambda_n g_n \ (\lambda_i \in \mathbb{C}).$$

The rules for addition and scalar multiplication are the usual ones, that is, if

$$x = \sum_{i=1}^{n} \lambda_i g_i \text{ and } y = \sum_{i=1}^{n} \mu_i g_i,$$

are elements of $\mathbb{C}G$, $\lambda \in \mathbb{C}$, then,

$$x + y = \sum_{i=1}^{n} (\lambda_i + \mu_i) g_i \text{ and } \lambda x = \sum_{i=1}^{n} (\lambda \lambda_i) g_i.$$

We now define a product on $\mathbb{C}G$ to make it into an algebra:

$$\left(\sum_{g\in G}\lambda_g g\right)\left(\sum_{h\in H}\mu_h h\right)=\sum_{g,h\in G}\lambda_g\mu_h gh,$$

where $\lambda_g, \mu_h \in \mathbb{C}$. With these operations, $\mathbb{C}G$ is an algebra over $\mathbb{C}$ of dimension n. We call this algebra the *group algebra*. An important vector subspace of $\mathbb{C}G$ is the *center of the group algebra*, denoted $Z(\mathbb{C}G)$, which is defined as,

$$Z(\mathbb{C}G)=\{z\in\mathbb{C}G : zr=rz \text{ for all } r\in\mathbb{C}G\}.$$

It is a standard result that $Z(\mathbb{C}G)$ is generated by $\overline{C_1},...,\overline{C_s}$, where the C_i's are the conjugacy classes of G. We also call $Z(\mathbb{C}G)$ the *class algebra* of G.

3.2 Magic Rectangle Condition

Suppose H fuses to G. Let $B_1 = \{e\}, B_2 = \{C_{21}, ..., C_{2r2}\}, ..., B_f = \{C_{f1}, ..., C_{frf}\}$ denote the partition of the classes of H determined by the fusion. Then the character table of H satisfies the Magic Rectangle Condition[Smith], that is, there exists a partition of the irreducible characters of H,

$$\Phi_1 = \{\chi_1\}, \Phi_2 = \{\chi_{21}, ..., \chi_{2s2}\}, ..., \Phi_f = \{\chi_{f1}, ..., \chi_{fsf}\},$$

so that for each Φ_i, B_j the number

$$\tau_{ij} = \frac{\sum_{m=1}^{r_j} |C_{j_m}| \chi_{it}(C_{j_m})}{(\sum_{m=1}^{r_j} |C_{j_m}|) d_i} \tag{3.1}$$

is independent of $\chi_{it} \in \Phi_i$, for $1 \le t \le s_i$, and is equal to the value

$$\frac{\sum_{m=1}^{s_i} d_{im} \chi_{im}(C_{j_k})}{\sum_{m=1}^{s_i} d_{im}^2}, \tag{3.2}$$

which is independent of $C_{j_k} \in B_j$ for $1 \le k \le r_j$. Thus we have

$$\tau_{ij} = \frac{\sum_{m=1}^{r_j} |C_{j_m}| \chi_{it}(C_{j_m})}{(\sum_{m=1}^{r_j} |C_{j_m}|) d_i} = \frac{\sum_{m=1}^{s_i} d_{im} \chi_{im}(C_{j_k})}{\sum_{m=1}^{s_i} d_{im}^2}. \tag{3.3}$$

Because of the well defined value of τ_{ij} above, we obtain an $f \times f$ fused table with the rows indexed by the Φ_i's and columns indexed by the B_j's where the value of the ijth entry is $\eta_i \tau_{ij}$ where η_i is

$$\eta_i = \sqrt{\sum_{m=1}^{s_i} d_{i_m}^2}. \tag{3.4}$$

This $f \times f$ fused table is the character table for the fused group G. The irreducible characters in each Φ_i is said to *fuse* to give an irreducible character of G.

We show that $\mathbb{Z}_6 = \langle x \rangle$ fuses to D_6. The conjugacy classes of D_6 are

$$\{\{1\}, \{(12), (13), (23)\}, \{(123), (132)\}\}.$$

We can easily check that the following correspondence gives the required fusion:

$$\{1\} \mapsto \{1\},$$

$$\{x^2, x^4\} \mapsto \{(123), (132)\},$$

$$\{x, x^3, x^5\} \mapsto \{(12), (13), (23)\}.$$

The character table for the cyclic group C_6 is:

	e	x^2	x^4	x^3	x	x^5
χ_0	1	1	1	1	1	1
χ_3	1	1	1	-1	-1	-1
χ_2	1	ρ^4	ρ^2	1	ρ^2	ρ^4
χ_4	1	ρ^2	ρ^4	1	ρ^4	ρ^2
χ_1	1	ρ^2	ρ^4	-1	ρ	ρ^{-1}
χ_5	1	ρ^4	ρ^2	-1	ρ^{-1}	ρ

where $\rho = e^{2\pi i/6}$ and $\chi_i(x) = \rho^i$. If we partition the columns according to the fusion as $\{e\}, \{x^2, x^4\}, \{x^3, x, x^5\}$ and the rows as $\{\chi_0\}, \{\chi_3\}, \{\chi_2, \chi_4, \chi_1, \chi_5\}$, we can easily check that the magic rectangle condition is satisfied. The fused table is:

	e	$\{(1,2,3)\}$	$\{(1,2)\}$
χ_0	1	1	1
χ_1	1	1	-1
χ_2	2	-1	0

The fused table is the character table for D_6.

3.3 Basic Results

We now prove some basic results concerning fusions of groups.

Lemma 3.3.1. *Suppose H fuses to G, where the classes of G are $C_1, .., C_t$ and the corresponding subsets of H are $H_1, ..., H_t$. Then $|C_i| = |H_i|$ for all $i \leq t$. In particular, we have $|H| = |G|$.*

Proof. Consider C_i and H_i for some i. We have

$$\bar{C}_i \bar{G} = |C_i| \bar{G},$$

and

$$\bar{H}_i \bar{H} = |H_i| \bar{H}.$$

Since C_i and H_i correspond, we must have $|C_i| = |H_i|$ as required. ☐

We then have this easy corollary.

Corollary 3.3.2. *If B is an abelian group and if A fuses to B, then A and B are isomorphic.*

Proof. By Lemma 3.3.1, we see that $|A| = |B|$ and all the principal subsets of A are of size 1. Since B is an abelian group, the correspondence under the isomorphism of *S-rings* is a one-to-one map of singletons. Moreover, the multiplication tables of the two groups are the same since the structure constants are the same under the fusion. ☐

The following result will be useful in proving the subsequent three lemmas, of which we will make constant use.

Lemma 3.3.3. *[LMa] Let S be an S-ring over the group G. Suppose there is a normal subgroup H of G such that $\overline{H} \in S$. Let $\rho : G \rightarrow G/H$ be the natural epimorphism.*

i) If $D = g_1A_1 \cup \cdots \cup g_kA_k$ is an S-principal subset of G where the $A_i's$ are nonempty subsets of H and $g_1H, ..., g_kH$ are distinct cosets of H, then $|A_1| = \cdots = |A_k|$.

ii) If D_1, D_2 are S-principal subsets of G, then either $\rho D_1 \cap \rho D_2 = \emptyset$ or $\rho D_1 = \rho D_2$.

Proof. i) Since, $A_i \subset H$ for all i, we have,

$$\begin{aligned} \bar{D}\bar{H} &= |A_1|g_1\bar{H} + \cdots + |A_k|g_k\bar{H} \\ &= \lambda\bar{D} + \sum \lambda_i \bar{D}_i \\ &= \lambda(g_1\bar{A}_1 + g_2\bar{A}_2 + \cdots + g_k\bar{A}_k) + \sum \lambda_i \bar{D}_i, \end{aligned}$$

for integers λ, λ_i and where the D_i's are principal subsets. But clearly, by comparing coefficients, we must have that $\lambda = |A_1| = \cdots = |A_k|$.

ii) Suppose $D_1 = g_1B_1 \cup \cdots \cup g_kB_k$ and $D_2 = g_1C_1 \cup \cdots \cup g_kC_k$ where $B_i, C_i \subset H$ with B_i, C_i not necessarily nonempty and $g_1H, ..., g_kH$ distinct cosets of H. Assume, $\rho D_1 \cap \rho D_2 \neq \emptyset$. Then there is some j such that $B_j \neq \emptyset$ and $C_j \neq \emptyset$ and we have,

$$\bar{D_1}\bar{H} = \lambda_1\bar{D_1} + \lambda_2\bar{D_2} + \cdots$$

where λ_1, λ_2 are positive integers since $g_j\bar{H}$ must be in the product above. Suppose $g_iC_i \neq \emptyset$. But this can happen only if the corresponding g_iB_i isn't empty in the decomposition of D_1 else g_iH is an empty set and $g_i\bar{C_i}$ can't appear on the right side of the equation. Similarly, if $B_i \neq \emptyset$, we consider $\bar{D_2}\bar{H}$ and conclude that $C_i \neq \emptyset$. Thus the map either sent principal subsets to disjoint sets of cosets or to the same set. $\square$

Now we state a useful result due to Humphries and Johnson[HumphriesJohnson].

Definition 3.3.4. *Let G, H be groups and let A be an S-ring on G. We say that H fuses to A if there is a subalgebra of $Z(\mathbb{C}H)$ which is isomorphic to A.*

Lemma 3.3.5. *[HumphriesJohnson] Suppose that H fuses to G. Let N be a normal subgroup of G. Then there is a normal subgroup M of H such that M fuses to $F_G(N)$ and H/M fuses to G/N.*

Proof. Since N is normal in G, $N = C_1 \cup \cdots \cup C_t$, where the C_i's are conjugacy classes of G. Let $H_1, ..., H_t$ be the corresponding subsets of H under the fusion and $M = H_1 \cup \cdots \cup H_t$. We first prove that M is a subgroup. Since the structure constants of products of the $\bar{H}_i$'s and $\bar{C}_i$'s are the same, we see that M is closed under multiplication and thus is a subgroup. It is normal, being a union of conjugacy classes.

To prove the second part, let $\pi : G \to G/N$ be the quotient map and $G/N = \{w_1, ..., w_u\}$. Let $D_i = \{w_{i_1}, ..., w_{i_{k_i}}\}, i \leq s$, be the conjugacy classes of G/N. Let $E_i = \pi^{-1}(D_i), i \leq s$, which is a union of cosets of N. If $\pi(v_i) = w_i$, for $v_i \in G$, then $E_i = \{v_{i_1}N, ..., v_{i_{k_i}}N\}$. If $w_i w_j = w_k$, then we have $(v_iN)(v_jN) = v_kN$ so that $\overline{v_iNv_jN} = |N|\overline{v_kN}$. Thus if $\bar{D}_i\bar{D}_j = \sum_k \lambda_{ijk}\bar{D}_k$, then $\bar{E}_i\bar{E}_j = |N| \sum_k \lambda_{ijk}\bar{E}_k$ and the structural constants for the D_i's and E_i's differ by a factor of $|N|$.

Now to simplify notation, we write the conjugacy class D_i, a set of cosets of N, as $D_i = \{N_1, ..., N_t\}$. Then by Lemma 3.3.3, $E_i = \pi^{-1}(D_i)$ is a union of conjugacy classes of G, say $J_1, ..., J_n$ where each J_j is evenly spread out over the N_k's as defined by that lemma. More precisely, $J_j = g_1A_1 \cup \cdots \cup g_tA_t$, where the A_i's are nonempty subsets of N of equal size and $g_iA_i \subset N_i$, $1 \leq i \leq t$. In other words, the J_j's partition E_i's. Consider $J_1 = g_1A_1 \cup \cdots \cup g_tA_t$. Since H fuses to G, let K_1 denotes the corresponding set in H. We note that K_1 is a union of classes of H. By the same lemma, K_1 is evenly spread out over the t cosets of M, say, $M_1, ..., M_t$. To see this, consider $\overline{J_1N} = |A_1| \sum_{i=1}^{t} \overline{N_i}$. Since H fuses to G, this implies $\overline{K_1M} = |A_1| \sum_{i=1}^{t} \overline{M_i}$, where the M_i's are cosets of M. Let B_i denote this set of cosets.

We claim that the correspondence between such B_i's and D_i's will be our required fusion of H/M to G/N. Now, $F_i = (\pi')^{-1}(B_i)$ is a union of conjugacy classes of H. We now show that just as E_i was partitioned into J_j's above, F_i can be partitioned into $K_1, ..., K_n$ such that K_j correspond to J_j under the fusion of H to G. Consider J_k, for some k. Suppose J_k corresponds to J_k' in H. Then $\bar{J}_k\bar{N}$ is a linear combination of $N_1, ..., N_s$, a set of cosets which contain J_1. By looking at structure constants, we see that $\bar{J}_k'\bar{M}$ must be a linear combination of cosets which contain K_1, i.e., the set $M_1, ..., M_t$. Thus J_k' must be one of $K_1, ..., K_n$. Similarly we can switch the role of H and G to get the correspondence $K_j \to J_j, j \leq n$. Similar to the above analysis, the structure constants for the B_i's and the F_i's differ by a factor of $|N|$ and thus the structure constants for the B_i's and D_i's are the same as required. □

Corollary 3.3.6. *If H fuses to G, then there is a subgroup N containing H' such that $H/N \cong G/G'$.*

Proof. By Lemma 3.3.5, there is a normal subgroup $N \subset H$ corresponding to $G' \subset G$. That same results shows that H/N fuses to G/G'. But the latter is an abelian group and by Corollary 3.3.1, we have $H/N \cong G/G'$. That N contains H' follows since H/N is abelian. □

Lemma 3.3.7. *Suppose that H fuses to G and let N be a normal subgroup of H such that $\bar{N}$ is in the S-ring over H. Then there is a normal subgroup M of G such that H/N fuses to G/M.*

Proof. Let the conjugacy classes of G be $C_1, ..., C_t$ and the corresponding subsets in H be $H_1, ..., H_t$. By hypothesis, we have $\bar{N} = \sum_k \mu_k \bar{H}_k, \mu_k \in \{0, 1\}$. Let M be the union of the C_k's where $\mu_k \neq 0$. Then M is closed under multiplication and is normal as it is a union of conjugacy classes. By 3.3.5, we have that H/N fuses to G/M. □

Chapter 4

Groups which fuse from D_{2n}

In this chapter, we begin our investigation of groups G which fuse from a dihedral group.

4.1 The Dihedral Group

Recall that the dihedral group, D_{2n}, has the following presentation:

$$D_{2n} = \langle x, y : x^n = 1, y^2 = 1, y^{-1}xy = x^{-1}\rangle.$$

Here are some properties of this group that will prove useful:

The subgroup $\langle x\rangle$ is cyclic of size n and index 2, and thus is normal in D_{2n}.

If $n = 2m + 1$, then the $\frac{n+3}{2} = m + 2$ conjugacy classes of D_{2n} are:

$$C_i = \{x^i, x^{-i}\}(0 \le i \le m), C_{m+1} = \{x^j y | 1 \le j \le n\}.$$

Moreover, $Z(D_{2n}) = \{1\}$.

If $n = 2m$, the $\frac{n}{2} + 3 = m + 3$ classes of D_{2n} are:

$$C_i = \{x^i, x^{-i}\}(0 \leq i \leq m), C_{m+1} = \{x^{2j}y|1 \leq j \leq m\}, C_{m+2} = \{x^{2j-1}y|1 \leq j \leq m\}.$$

In this case, $Z(D_{2n}) = \langle x^{n/2} \rangle \cong \mathbb{Z}_2$.

The question that we want to investigate is: which groups fuse from dihedral groups?

4.2 D_{2n} fuses to G where n is odd

In this section, we investigate the case when n is odd and prove that G must be the dihedral group.

Lemma 4.2.1. *Let G be a finite group. Suppose G fuses from the dihedral group D_{2n}, where n is odd. Then G has a class C with n elements and all elements of C have order 2.*

Proof. Since D_{2n} has a class of size n, G has a class C of size at least n. And since $|C|$ divides $|G|$ and C clearly doesn't contain the identity, we have that $|C| = n$. Let $x \in C$. Since G acts transitively on C by conjugation, we have

$$|G| = 2n = |C||Stab_G(x)| = n|Stab_G(x)|.$$

So the subgroup $Stab_G(x)$ has size 2. But x and e are in $Stab_G(x)$ and so we must have $x^2 = e$. Thus all elements of C have order 2. □

Next, we prove a result of Mann[Mann] which we will need later. If G is a finite group, we denote by $k = k(G)$ the number of conjugacy classes of G. Let $d_1, \cdots, d_k$ be the degrees of the irreducible characters of G, and write $T(G)$ for the sum of these degrees, that is,

$$T(G) = \sum_{i=1}^{k} d_i.$$

Then we also have

$$|G| = \sum_{i=1}^{k} d_i^2.$$

Lemma 4.2.2. *[Mann] For k and $T(G)$ defined above, we have*

$$\sqrt{|G|} \leq T(G) \leq \sqrt{|G|k}.$$

Proof. The left-hand inequality is clear as

$$\sqrt{d_1^2 + \cdots + d_k^2} \leq \sqrt{(d_1 + \cdots + d_k)^2} \leq d_1 + \cdots + d_k.$$

Let $d = (d_1, ..., d_k)$ and $v = (1, ..., 1)$. By the Cauchy Schwartz Inequality, we have:

$$|\langle d, v\rangle|^2 \leq \langle d, d\rangle \cdot \langle v, v\rangle,$$

which gives

$$T(G)^2 \leq |G|k,$$

and so

$$T(G) \leq \sqrt{|G|k},$$

as required. □

Lemma 4.2.3. *[Mann] Let t(G) be the number of elements of G whose square is 1. Then there exist numbers $v_i = 1, 0, or -1$ for $i = 1, \cdots, k,$ such that*

$$t(G) = \sum_{i=1}^{k} v_i d_i.$$

Proof. See [Mann]. □

Corollary 4.2.4. *[Mann] $t(G) \leq T(G)$.*

Proof. From Lemma 4.2.3, we have

$$t(G) = \sum_{i=1}^{k} v_i d_i \leq \sum_{i=1}^{k} d_i = T(G),$$

as required. □

We apply these results to G. Since the class C of G has n involutions, we see that $t(G) \geq n+1$ and using Lemma 4.2.2, we have

$$n+1 \leq t(G) \leq T(G) \leq \sqrt{2nk}.$$

This gives

$$(n+1)^2 \leq 2nk,$$

and so

$$(n^2+2n+1)/(2n) \leq k,$$

and finally,

$$[n/2+1+1/(2n)] \leq k. \tag{4.1}$$

By (4.1) and the fact that k is bounded above by the number of conjugacy classes of D_{2n} we have

$$n/2+1+1/(2n) \leq k \leq n/2+3/2.$$

As n is odd, $k = n/2+3/2$, which is the number of conjugacy classes for D_{2n}. Thus G has the same number of classes as D_{2n}. We have just proved:

Proposition 4.2.5. *Suppose G fuses from the dihedral group, D_{2n}, where n is odd. Then G has a class, C, of size n and every other conjugacy class of G has size 1 or 2. Further, the fusion of D_{2n} to G determines an isomorphism of the class algebras of D_{2n} and G.*

We will take advantage of the following three equivalent statements:

1) H fuses to G and G fuses to H.

2) $Z(\mathbb{C}G) \cong Z(\mathbb{C}H)$.

3) G and H have the same character table.

The equivalence of 1) and 2) is trivial. The equivalence of 2) and 3) can be found in [Feit, pg 42]. Now we prove Theorem 2.0.6 part i).

Theorem 4.2.6. *With the same hypotheses as Proposition 4.2.5, G is isomorphic to D_{2n}.*

Proof: By Proposition 4.2.5, we have $Z(\mathbb{C}G) \cong Z(\mathbb{C}D_{2n})$. Thus the character tables for the two groups are the same. By a result of [Cel], stated below, G is isomorphic to D_{2n}.

Proposition 4.2.7. *[Cel] If G is a finite group with the same character table as D_{2n} or Dic_{2n} then G is isomorphic to D_{2n} or Dic_{2n}. Moreoever, D_{2n} and Dic_{2n} have the same character table if and only if $n \equiv 0(4)$.*

Proof. See [Cel]. □

4.3 D_{2n} fuses to G where n is even

Proposition 4.3.1. *Suppose D_{2n} fuses to G where n is even. Then G does not have a class of size n.*

Proof. Suppose G does have a class C of size n. Let k be the number of classes of G. As in the case when n is odd, each element in C is an involution. By Lemma 4.2.2 and 4.2.4, we have

$$n + 1 \leq t(G) \leq T(G) \leq \sqrt{(2nk)}.$$

By the same argument, we conclude that

$$k \geq [n/2 + 1 + 1/(2n)].$$

Since n is even, we have

$$k \geq n/2 + 2. \tag{4.2}$$

If n is even, the center of D_{2n} has size 2 and thus by the fusion, the center of G must have size at most 2. Suppose first that $Z(G)$ is trivial. Since G has a class of size n, it also has $k - 2$ classes each with at least two elements so that

$$1 + 2(k - 2) \leq n.$$

Solving for k and using (4.2), we obtain

$$\frac{n}{2} + 2 \leq k \leq \frac{n}{2} + \frac{3}{2},$$

which is a contradiction. Thus $Z(G)$ has order 2, that is, $Z(G) = \{1, z\}, z^2 = 1, z \neq 1$. But this implies that for any $g \in C$, $Stab_G(g)$ contains $\{1, g, z, zg\}$, which is also a contradiction since this implies by the Orbit-Stabilizer Theorem that $|G| = 4n$. $\square$

As a consequence of this proposition, we have Property 1.

We now prove Property 2.

Lemma 4.3.2. *If D_{2n} fuses to G, $n = 2m$ and $Z(G) = \{1\}$, then the normal subgroups D'_{2n} and G' correspond under the fusion. Thus, G/G' is isomorphic to $\mathbb{Z}_2^2$.*

Proof. The dihedral group D_{2n} has two classes C_1, C_2 of involutions, each of size m. By Proposition 4.3.1, these two classes do not fuse. Thus G has two classes of size m. Let E_1, E_2 be classes of G corresponding to C_1, C_2 and $h_i \in C_i, g_i \in E_i, i = 1, 2$. In this case, we have $\bar{C_i} = h_i\overline{D'_{2n}}, i = 1, 2$, which implies $\bar{C_i}^2 = m\overline{D'_{2n}}, i = 1, 2$. Thus, because of the fusion, there is a normal subgroup $N \subset G$ corresponding to D'_{2n} such that $\bar{E_i}^2 = m\bar{N}, i = 1, 2$. By Lemma 3.3.5, D_{2n}/D'_{2n} fuses to G/N. Since $[D_{2n} : D'_{2n}] = 4$, we have $[G : N] = 4$.

By Lemma 3.3.6, there is a subgroup M of D_{2n} containing D'_{2n} which corresponds to G' with $D_{2n}/M \cong G/G'$. We claim that $M = D'_{2n}$ and $N = G'$. The quotient G/N has size 4. Thus G/N is abelian and therefore N contains G'. We then have the natural onto homomorphism $G/G' \to G/N$. And because $D'_{2n} \subset M$, we also have the natural homomorphism $D_{2n}/D'_{2n} \to D_{2n}/M$. Since these homomorphisms are onto, we must have $|G/G'| \geq 4$ and $|D_{2n}/M| \leq 4$. We have

$$4 = |D_{2n}/D'_{2n}| \geq |D_{2n}/M| = |G/G'| \geq |G/N| = 4.$$

Thus, $M = D'_{2n}$ and $N = G'$. $\square$

Lemma 4.3.3. *Suppose $n = 2m, m \geq 2$. Then the dihedral group $D_{2n} = \langle x, y | x^n, y^2, y^{-1}xy = x^{-1}\rangle$ has three subgroups of index 2. They are:*

$$M_1 = \langle x\rangle; \quad M_2 = \langle x^2, y\rangle; \quad M_3 = \langle x^2, xy\rangle.$$

Here, M_1 is cyclic and M_2, M_3 are dihedral groups of order $n = 2m$. In addition, for $1 \leq i \neq j \leq 3$, $M_i \cap M_j = \langle x^2 \rangle = D'_{2n}$. If C_1, C_2 are classes of D_{2n} of size m, we can reindex M_2, M_3 so that C_1 is a class in M_2 and C_2 is a class in M_3.

Proof. Clearly, $M_i, i = 1, 2, 3$ has index 2. Conversely, any subgroup of index 2 is normal and contains the commutator subgroup, D'_{2n} with $D_{2n}/D'_{2n} \cong \mathbb{Z}_2^2$. Since $\mathbb{Z}_2^2$ has three subgroups of index 2, by the Correspondence Isomorphism Theorem, there are three subgroups of D_{2n} of index 2 containing the commutator subgroup. The rest is obvious. □

Lemma 4.3.4. *Let N be a subgroup of G of index 2. Let $x \notin N$ such that $G = \langle N, x \rangle$. Suppose $g \in N$ and denote by C the class of g in N and C' the class of g in G. Then, either $C' = C$ or $C' = C \cup C^x$ with $C \cap C^x = \emptyset$. Thus either $|C'| = |C|$ or $|C'| = 2|C|$.*

Proof. Since N has index 2 in G, N is normal in G. Let $\{1, a\}$ be the transversal for N. Thus, $G = \langle N, a \rangle$. For each $h \in G \setminus N$, we can write $h = h_0 a$, with $h_0 \in N$. Then, $C^h = C^{h_0 a} = C^a$. We now show that either $C = C^a$ or $C \cap C^a = \emptyset$, or equivalently, if their intersection is nonempty, then $C = C^a$. Suppose $u \in C \cap C^a$. Then $u \in C$ and $u = w^a$, with $w \in C$. We pick any $v \in C$. Since C contains both u, v, there is a $y \in N$ such that $v = u^y$. We have,

$$v = u^y = (w^a)^y = w^{ay} = w^{yy^{-1}ay}.$$

But $y^{-1}ay = za$ for some $z \in N$. Thus,

$$v = w^{y(za)} = (w^{yz})^a \in C^a.$$

Thus, $C \subset C^a$ and $C = C^a$ since they have the same size. □

Theorem 4.3.5. *Suppose D_{2n} fuses to G, with $n = 2m$, where m is odd. Then $G \cong D_{2n}$.*

Proof. By Lemma 4.3.2, $G/G' \cong \mathbb{Z}_2^2$. Now, $\mathbb{Z}_2^2$ has three subgroups of index 2, all of which are normal. We denote their complete preimages in G by N_1, N_2, N_3. By Lemma 3.3.5, let M_1, M_2, M_3 be the corresponding subgroups in D_{2n}. We can rename these groups to coincide with those in Lemma 4.3.3. Now consider the two dihedral subgroups of D_{2n}, M_2 and M_3. As m is odd, M_2, M_3 each has a class of size m. Thus C_1 and C_2 are also classes in M_2 and M_3, respectively. Under the fusion, N_2, N_3 must each have a class of size at least m. Denote these classes by E_2, E_3. But since N_i has order $2m$ and $E_i \subset N_i$, we have that $|E_i| = m, i = 2, 3$. It is then clear that E_2, E_3 must correspond to $C_1, C_2 \subset D_{2n}$. Consider any element $g_i \in E_i$. By Lemma 4.3.4, the conjugacy class of g_i in N_i has size m or $m/2$. As m is odd, we see that E_i must also be a class in N_i, $i = 2, 3$. This implies that each element in E_i must have order 2. Hence, the E_i's are classes of involutions.

Let $E = G \backslash (N_2 \cup N_3)$. Then E is a union of classes of G. Let E' be the corresponding union of classes of D_{2n}. Since E' is the complement of $M_2 \cup M_3$ in D_{2n}, we have

$$E' = \{x, x^3, ..., x^{2m-1}\}.$$

E' has m elements and is its own inverse. Since x has order $2m$ and m is odd, E' has an involution. Thus, E also has an involution. We then have $t(G) \geq 2m+2$. As before, we have,

$$2m + 2 \leq t(G) \leq \sqrt{4mk},$$

where k is the number of classes of G. Thus, $k \geq m+2+1/m$. But k, being an integer, implies $k \geq m + 3$. But for $n = 2m$, D_{2n} has $m + 3$ classes. Thus the centers of the two group algebras are isomorphic. By Proposition 4.2.7, $G \cong D_{2n}$ or $G \cong Dic_{2n}$.

Now we prove that G can't be isomorphic to the dicyclic group. Since m is odd, by Proposition 4.2.7, the character tables of the dihedral and dicyclic groups are different and hence $Z(\mathbb{C}D_{2n}) \ncong Z(\mathbb{C}Dic_{2n})$. Thus, $G \cong D_{2n}$. □

We prove Property 3.

Proposition 4.3.6. *Suppose that D_{2n} fuses to G, with $n = 2m$, m is even and $Z(G) = \{1\}$. Then G has a unique class C of odd size. Moreoever, C is contained in G' and is a class of involutions.*

Proof. Since n is even, D_{2n} has nontrival center of size 2, say $Z(D_{2n}) = \{1, z\}$. Let C_1, C_2 be the two classes of size m. By assumption, we have $Z(G) = \{1\}$, so that the class $\{z\}$ of D_{2n} must fuse with some of the other classes of D_{2n}. By order consideration, since classes have orders dividing the group, it can't fuse with C_1 or C_2. Thus the class $\{z\}$ must fuse with a nonzero number of classes of D_{2n} all of which have size 2. The other fusions will consist of classes of order 2 and thus have even size. Moreover, C is contained in the subgroup, N_1, of G corresponding to the full rotational subgroup of D_{2n}. But since $\{1\}$ and C are the only classes of odd size contained in N_1, we must have that $C \subset G'$. Since $|C|$ is the only nontrivial class of odd size, we have that $C^{-1} = C$ and thus there must be at least one element in C which is its own inverse, i.e an involution. Since all elements in C have the same order, C is a class of involutions. □

Now we prove Properties 4-7.

Proposition 4.3.7. *Suppose that D_{2n} fuses to G, where n is even. Then the character degrees of G are*

$$1, 1, 1, 1, d_5, ..., d_k,$$

where $d_5, ..., d_k$ are even. Also, r, the number of non-linear characters of D_{2n} which fuse to give a character of G, is a square.

Proof. Since G/G' has size 4, G has 4 linear characters. By the Magic Rectangle Condition, the non-linear characters of G are formed by fusing r of the degree 2 characters of D_{2n}. By using the first column, i.e. the column of the identity class, the degree of any non-linear character of G is

$$\frac{\sum_{m=1}^{s_i} d_{im}\chi_{im}(C_{j_k})}{\sum_{m=1}^{s_i} d_{im}^2} \cdot \sqrt{\sum_{m=1}^{s_i} d_{im}^2} = \frac{\sum_{i=1}^{r} 2(2)}{\sum_{i=1}^{r} 2^2} \cdot \sqrt{\sum_{i=1}^{r} 2^2} = 2\sqrt{r},$$

which is even as required. ☐

Lemma 4.3.8. *Suppose that D_{4m} fuses to G where $Z(G) = \{1\}$. Then G' is not abelian.*

Proof. Suppose that G' is abelian. Let $x \in G'$. Then $G' \subset C_G(x)$. Since $G/G' \cong \mathbb{Z}_2^2$, the conjugacy class containing x must have 2 or 4 elements. But this is a contradiction as we have proven that there is a class of odd size in G'. ☐

Lemma 4.3.9. *Suppose that D_{4m} fuses to G where $Z(G) = \{1\}$. Then $f = min\{\chi(1) | \chi \in Irr(G), \chi(1) > 1\} = 2$.*

Proof. By Proposition 4.3.7, f is even. Suppose $f \geq 4$. Then $4 = |G/G'| \leq f$ and thus by [Isaacs(5.14)(b); pg 75], we have G' is abelian, which is a contradiction to Lemma 4.3.8. ☐

Lemma 4.3.10. *If D_{2n} fuses to G, then each character of G is real.*

Proof. Since each element of D_{2n} is conjugate to its inverse, the entries in the character table for D_{2n} are real. But the character table of G can be obtained from the character table of D_{2n} by fusing using the Magic Rectangle Condition. Thus each character of G is real. □

Proposition 4.3.11. *Suppose that D_{4m} fuses to G. Then every $g \in G$ is conjugate to its inverse.*

Proof. Since each character of G is real by Lemma 4.3.10, each element of G is conjugate to its inverse. □

We prove Properties 8-9.

Proposition 4.3.12. *Suppose that D_{4m} fuses to G. Then G is 2-nilpotent and solvable.*

Proof. Since all the non-linear characters of G are even, then by [Huppert, page 313], G is 2-nilpotent, that is, the Sylow 2-group of G has a normal complement. Let N be the normal subgroup of odd order and index a power of 2. By Feit-Thompson, N is solvable. And since G/N is a 2-group, it is solvable. Thus G is solvable. □

We call G a p-group of maximal class if G has order p^n and nilpotent class $n-1$. We state two theorems from Finite Groups by Huppert[Hup]. First, a *quasidihedral group* is the group with the following presentation,

$$\langle r, s \mid r^{2^{n-1}} = s^2 = 1, srs = r^{2^{n-2}-1} \rangle.$$

Theorem 4.3.13. *If $G = 2^n$ and of maximal class, then G is dihedral, dicyclic or quasidihedral.*

Theorem 4.3.14. *Let G be a p-group of order p^n, $n \geq 3$. G is a p-group of maximal class iff there exists an element in G whose conjugacy class size is p^{n-2}.*

Lemma 4.3.15. *If G possesses an element x with $|C_G(x)| = 4$, then the Sylow 2-group, P, of G is dihedral, dicyclic or quasidihedral. In particular, $|P/P'| = 4$ and P has a cyclic group of order $|P|/2$.*

Proof. Let P be a Sylow 2-group of G of order 2^n. Since $|C_G(x)| = 4$, $C_G(x)$ is a 2-group. Without loss of generality, we may assume $C_G(x) \subset P$. By definition, $C_P(x) = P \cap C_G(x)$, thus $|C_P(x)| = 4$. The class of x in P then has size 2^{n-2} and thus by Theorem 4.3.14, P is a 2-group of maximal class and P is a dihedral, dicyclic or quasidihedral by Theorem 4.3.13. The rest follows. □

Now, we prove Properties 10-13.

Corollary 4.3.16. *Suppose that D_{4m} fuses to G, where m is even and $Z(G) = \{1\}$. The Sylow 2-group of G is dihedral, dicyclic or quasidihedral.*

Proof. Consider C_1, the class of size m in G. Pick any $x \in C_1$. Since $|C_G(x)| = 4$, the result follows from Lemma 4.3.15. □

Definition 4.3.17. *Suppose G is a finite group and M, N are proper normal subgroups with $N \leq M$. We say that (G,M,N) is a Camina triple if the conjugacy classes in $G \backslash M$ are union of cosets of N. If $M = N$, (G,N) is a Camina pair.*

Proposition 4.3.18. *Suppose H fuses to G. Suppose that (H,M,N) is a Camina triple and suppose that there are normal subgroups M', N' of G which correspond to M and N, respectively. Then (G, M', N') is a Camina triple.*

Proof. Consider a conjugacy class $C' \subset G \setminus M'$. We prove that C' is a union of cosets of N'. Let C' corresponds to $C \subset H \setminus M$ under the fusion. Notice that C is a union of cosets of N. By Lemma 3.3.5, H/N fuses to G/N'. By the proof of the same lemma, this fusion of quotient groups is determined by the fusion of H to G and since C is a union of cosets of N, C' must be also. □

Corollary 4.3.19. *Suppose that D_{4m} fuses to G, where m is even and $Z(G) = \{1\}$. Let M be the normal subgroup of G corresponding to $\langle x \rangle \leq D_{4m}$ under the fusion. Then (G, M, G') is a Camina triple.*

Proof. Since $(D_{4m}, \langle x \rangle, D'_{4m})$ is a Camina triple, by Proposition 4.3.18, (G, M, G') is a Camina triple as required. □

Proposition 4.3.20. *Suppose that D_{4m} fuses to G, where m is even. G is not a direct product.*

Proof. Suppose that G is a direct product of nontrivial subgroups, H and K. Then D_{2n} is also a direct product of nontrivial subgroups. But this is only true if m is odd, contradicting our hypothesis. □

Proposition 4.3.21. *Suppose that D_{2n} fuses to G where $n = 2m$. Then G has at most $d(n) + 3$ normal subgroups, where $d(n)$ is the number of divisors of n. Furthermore, for each divisor $d < n$ of n, G has unique normal subgroup of order d.*

Proof. By Lemma 3.3.5, each normal subgroup N of G corresponds to a normal subgroup in D_{2n}. Thus the number of normal subgroups of G is no more than the number of normal subgroups of D_{2n}, which is $d(n)+3$. The rest follows since with the exception of M_2 and M_3 of Lemma 4.3.3, the only other normal subgroups of D_{2n} are the cyclic subgroups. □

4.4 D_{2n} fuses to G where $|G| = 8p$

Now we prove our main theorem part iii).

Theorem 4.4.1. *Suppose that D_{2n} fuses to G where $|G| = 8p$. Then $G \cong D_{2n}$ or $G \cong Dic_{2n}$.*

Proof. We use the classification of groups of order $8p$ by [Western]. There are 15 isomorphism classes. We can eliminate types 1)-3) since they are abelian. By Proposition 4.3.20, we can also eliminate types 4), 5),7)-9) and 14). The commutator subgroup of types 6), 13) and 15) has index at least 8 and hence can also be eliminated by Property 2, Proposition 4.3.2. Type 11) violates Properties 6 and 7 and hence can also be eliminated. We have eliminated all but types 10) and 12) which are the dihedral and dicyclic groups. □

Chapter 5

Schur Rings over Cyclic Groups

5.1 Classification of S-rings over Cyclic Groups

In this section, we introduce three types of S-ring and note a result of [LMan] which says that an S-ring over $\mathbb{Z}_n$ is one of these three types.

Suppose S is an S-ring over a cyclic group $\mathbb{Z}_m$ for some m. Then S is one of the following three types:

Type 1: A given *S-ring* S over G is of this type if we can find $H \leq Aut(\mathbb{Z}_m)$ such that principal subsets of S are orbits of H.

Type 2: Suppose a cyclic group $\mathbb{Z}_m$ is a direct product of two subgroups, H and K. Let H' and K' be S-ring over H and K, respectively. Let $\{D_1, ..., D_t\}$ and $\{E_1, ..., E_s\}$ be the principal subsets of these S-ring. Then the S-ring of this type is generated by the following principal subsets: $\{\overline{D_i \times E_j} : 1 \leq i \leq t, 1 \leq j \leq s\}$. We call S the *dot product* of H' and K'.

Lemma 5.1.1. *[LMan] Let H, K be subgroups of G with H a normal subgroup and $H \subset K$. Let S be an S-ring over K. Suppose H is an S-subgroup and $\rho : G \to G/H$ is* *the natural mapping. Then $D(\rho^*(S)) = \{\rho(D) : D \in D(S)\}$ generates an S-ring over K/H.*

Proof. This is a direct consequence of parts i) and ii) of Lemma 3.3.3. $\square$

Reversing this process, suppose $S_{G/H}$ is a S-ring over G/H with $D(S_{G/H}) = \{E_1, ..., E_r\}$ where $E_1 = \{H\}$. We define $D_0 = \{e\}$, $D_1 = H \setminus \{e\}$ and $D_i = \rho^{-1}(E_i)$ for $i = 2, ..., r$. Then $\{D_0, D_1, ..., D_r\}$ generates an S-ring over G. Moreover, $\rho^*(\rho^{-1}(S_{G/H})) = S_{G/H}$.

Type 3: To describe our next construction, we'll restate Lemma 3.3.3.

Lemma 5.1.2. *[LMa] Suppose there is a normal subgroup H of G such that $\overline{H} \in S$. Let $\rho : G \to G/H$ be the natural epimorphism.*

i) If $D = g_1A_1 \cup \cdots \cup g_kA_k$ is an S-principal subset of G where the $A_i's$ are nonempty subsets of H and $g_1H, ..., g_kH$ are distinct cosets of H, then $|A_1| = \cdots = |A_k|$.

ii) If D_1, D_2 are S-principal subsets of G, then either $\rho D_1 \cap \rho D_2 = \emptyset$ or $\rho D_1 = \rho D_2$.

Proposition 5.1.3. *[LMan] Using the notation defined above, suppose $K \supset H$ is a subgroup in G with K/H being an $S_{G/H}$-subgroup. For any S-ring S_K of K with $\overline{H} \in S_K$ and $\rho^* S_K = (\mathbb{Z}[K/H]) \cap S_{G/H}$, there is an S-ring S over G such that*

$$D(S) = D(S_K) \cup \{\rho^{-1}(E) : E \in D(S_{G/H}) \text{ with } E \not\subseteq K/H\}.$$

Furthermore, $S \cap \mathbb{Z}[K] = S_K$ and $\rho^* S = S_{G/H}$.

Proof. See [LMan, Proposition 1.4]. $\square$

The S-ring S constructed above is the *wedge product* of S_K and $S_{G/H}$ and will be denoted as $S_K \wedge S_{G/H}$.

Thus for any given S-ring S over $G = \mathbb{Z}_m$ of this type, there exist cyclic subgroups H and K satisfying the above proposition with $S = S_K \wedge S_{G/H}$.

Example 5.1.4.

Suppose $G = A_4$. Let

$$H = K = A_4' = \{1, (12)(34), (13)(24), (14)(23)\}$$

be subgroups of G. Let

$$D(S_{G/H}) = \{\{A_4'\}, \{(123)A_4'\}, \{(132)A_4'\}\}.$$

Trivially, K/H is an $S_{G/H}$-subgroup. Let

$$D(S_K) = \{\{1\}, \{(12)(34), (13)(24), (14)(23)\}\}$$

be the principal subsets which generate the *S-ring* over K. Clearly, we have that $\overline{H} \in S_K$ and $\rho^* S_K = (\mathbb{Z}[K/H]) \cap S_{G/H}$. Then the principal subsets which generate the wedge product over G are

$$\{\{1\}, \{(12)(34), (13)(24), (14)(23)\}, \{(123)A_4'\}, \{(132)A_4'\}\}.$$

This is the class algebra of A_4 which we have shown to be a wedge product. We now state some important results from [LMan] about S-ring over cyclic groups.

Throughout this section, we assume G is cyclic and $|G|$ is not prime. Let S be the S-ring over G. Assume S is nontrivial and imprimitive. Since S is nontrivial, we can consider all the proper S-subgroups of G, partially order them and pick H the minimal, nontrivial subgroup. Then $S \cap \mathbb{Z}[H]$ is a primitive *S-ring* over H by choice of H. Suppose further that $|H|$ is not prime. Then $S \cap \mathbb{Z}[H]$ must be trivial by Corollary 5.2.4. Thus $H \setminus \{e\}$ is a principal subset. Let p be a divisor of $|H|$ and the unique subgroup of H of order p be denoted P_1.

With this setup, we have the following consequences.

Theorem 5.1.5. *[LMan, Thm 3.4] Suppose $H \setminus \{e\}$ is a principal subset of S and the order of H is not prime. For any principal subset D which is not a union of H-cosets, $D = E$ or $D = E \cdot (H \setminus \{e\})$, where E is a principal subset such that $\langle E \rangle \cap H = \{e\}$.*

Proof. See [LMan, Theorem 3.4]. □

We now prove a theorem which describes S using smaller subgroups. Consider

$$\{K | K \text{ is an } S\text{-subgroup with } K \cap H = \{e\}\}.$$

This set is nonempty as $\{e\}$ is in it. Pick K with the property that $|K|$ has the largest order.

Theorem 5.1.6. *[LMan, Thm 3.5] Suppose $|H|$ is not a prime and $H \setminus \{e\}$ is a principal subset. Let K be as above. Then $S(KH) = S(K) \cdot S(H)$. In particular, if $G = KH$, then $S = S(K) \cdot S(H)$. And if $HK \neq G$, and $\rho : G \to G/H$ is the natural homomorphism, then $S = S(HK) \wedge \rho^* S$.*

Proof. Clearly, $|H|, |K|$ are relatively prime. If $D \subset K$, we're done since $D = D \cdot \{e\}$. If D is any principal subset in $KH \setminus K$, then D can't be a union of H-cosets since KH is precisely the union of kH-cosets. By Theorem 5.1.5, $D = E \cdot (H \setminus \{e\})$ or E, where E is a principal subset whose generated group intersects H trivially. Since $E \subset HK$, $\langle E \rangle \subset K$. Thus $S(KH) = S(K) \cdot S(H)$.

To show the last part of the theorem, we need only show that D is a union of H-cosets if D is not in HK. Suppose not, then by 5.1.5, $D = E \cdot (H \setminus \{e\})$ or E. Since $D \nsubseteq HK$, we have that $E \nsubseteq K$. But this implies $\langle E \rangle K$ is a S-subgroup properly containing K, which also intersects H trivially, contradicting our choice of K. Thus D must be a union of H-cosets. □

Example 5.1.7.

Suppose $G = \mathbb{Z}_{12} = \langle x \rangle$. Suppose the S-ring S over G is generated by the following principal subsets:

$$\{\{e\}, \{x^4, x^8\}, \{x^3, x^6, x^9\}, \{x, x^2, x^5, x^7, x^{10}, x^{11}\}\}.$$

Let $H = \langle x^3 \rangle$. Clearly, H satisfies the conditions of the theorem. In this case, $K = \langle x^4 \rangle$. We then see that $S = S(K) \cdot S(H)$, where $S(K) = \{\{e\}, \{x^4, x^8\}\}$ and $S(H) = \{\{e\}, \{x^3, x^6, x^9\}\}$.

Example 5.1.8.

Let G be as above. Let S be the S-ring generated by

$$\{\{e\}, \{x^3, x^6, x^9\}, \{x, x^4, x^7, x^{10}\}, \{x^2, x^5, x^8, x^{11}\}\}.$$

Let $H = \langle x^3 \rangle$ so that H satisfies the conditions of the theorem. In this case, K is the trivial subgroup. Clearly, $HK \neq G$. Let $\rho : G \to G/H$ be the natural homomorphism. We see then that $S = S(HK) \wedge \rho^* S$, where $S(HK) = S(H) = \{\{e\}, \{x^3, x^6, x^9\}\}$.

We now state the two main results of [LMan]. As before, G is cyclic and S is the nontrivial S-ring over G. The first result [LMan, Cor. 4.3 and Thm 4.5] states that if G has at least one Sylow subgroup which is not an S-subgroup then S is a dot or wedge product of smaller subgroups.

The second and main result of [LMan] is that if there is a subgroup H which is not an S-subgroup, then S is a dot product or a wedge product of smaller subgroups. It is the second result which we wish to be precise.

Suppose S is a nontrivial S-ring over a cyclic group G. Suppose there exists a subgroup H which is not an S-subgroup. By Corollary 4.3 and Theorem 4.5 of [LMan], we assume H is not a p-Sylow subgroup and that every p-Sylow subgroup is an S-subgroup. Suppose $|H| = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$, where each p_i is prime and $r_i \geq 1$. Let H_i

denote the unique subgroup of order $p_i^{r_i}$. Since H is not an S-subgroup, not all H_i are S-subgroups. Without loss of generality, assume H_1 is not an S-subgroup. To simplify our notation, we write p for p_1, r for r_1 and P_r for the p-Sylow subgroup of G. Suppose $\{e\} \leq P_{a_1} \leq \cdots \leq P_{a_k} = P_r$ are all S-subgroups in P_r, where $0 = a_0 < a_1 < a_2 \cdots < a_k = r$. Since H_1 is a p-group and H_1 is not an S-subgroup, $a_{i+1} - a_i \neq 1$ for some i. Let i be the smallest integer such that $a_{i+1} - a_i \neq 1$. Set $a_i = s$ and $a_{i+1} = t$. We then have the following lemma.

Lemma 5.1.9. *[LMan, Lemma 5.1] Suppose all Sylow-subgroups of G are S-subgroups and there exists a nontrivial subgroup H which is not an S-subgroup. Then there exists a prime divisor of G, integers s, t with $s + 2 \leq t$ such that $P_t \setminus P_s$ is a principal subset.*

We assume the conditions of the previous lemma. Let $|G| = p^r m$, where p doesn't divide m. Let M be the subgroup of order m. Since every Sylow subgroup of G is an S-subgroup and M is a product of Sylow subgroups, M is also an S-subgroup.

Theorem 5.1.10. *[LMan, Thm 5.2] With the same assumptions and notation of the previous lemma, we have,*

i)If $s = 0, t = r$, then $S = S(P_r) \cdot S(M)$.

ii)If $r > t, s = 0$, then $S = S(P_t M) \wedge \rho^* S$, where $\rho : G \to G/P_t$ is the natural homomorphism.

iii)If $s \geq 1$, then $S = S(P_s M) \wedge \rho^* S$, where $\rho : G \to G/P_s$ is the natural homomorphism.

Proof. Part i) and ii) follows directly from 5.1.6. □

5.2 Application to Fusion

Now we apply the results from the previous section to our problem.

Proposition 5.2.1. *Suppose that* $D_{4m}, m = 2k$ *fuses to* G *where* $Z(G) = \{1\}$*. Let* S *denote the* S*-ring on* $\mathbb{Z}_{2m} = \langle x \rangle \leq D_{4m}$ *which fuses to* $F_G(M_1)$ *and* T *the* S*-ring on* $\mathbb{Z}_m = \langle x^2 \rangle = D'_{4m}$ *that fuses to* $F_G(G')$*. Then* $\mathbb{Z}_m$ *has a nontrivial subgroup* H *such that* $\bar{H} \notin S$ *and* $\bar{H} \notin T$*.*

Proof. Since m is even, there is an element in $\mathbb{Z}_m$ of order 2. Let H be the subgroup generated by this element. Suppose $\bar{H} \in T$, then $\bar{H} - 1$ is an element of T, implying that the S-ring on D_{4m} has a non-identity class with one element. As $Z(G)$ is trivial, this is a contradiction. □

Proposition 5.2.2. *Suppose that* $D_{4m}, m = 2k$ *fuses to* G *where* $Z(G) = \{1\}$*. Let* S *denote the* S*-ring on* $\mathbb{Z}_{2m} = \langle x \rangle \leq D_{4m}$ *which fuses to* $F_G(M_1)$ *and* T *the* S*-ring on* $\mathbb{Z}_m = \langle x^2 \rangle = D'_{4m}$ *that fuses to* $F_G(G')$*. Then* S *and* T *are either dot products or wedge products of strictly smaller rings.*

Proof. By Theorem 5.2.4, since $2m$ and m are not primes and $2m - 1$ and $m - 1$ do not divide $4m$, S and T must be imprimitive. By Proposition 5.2.1, Lemma 5.1.9, and Theorem 5.1.10, the result follows. □

An *S-ring* S on G is *primitive* if for every S-principal set D, $\langle D \rangle = G$ or $\langle D \rangle = \{1\}$. If S is not primitive, S is *imprimitive*. The followimg lemma will lead us to a fundamental result of Wielandt.

Lemma 5.2.3. *Let* H *be an abelian group and let* p *be a prime dividing* $|H|$*. Then* $G = \langle x^p | x \in H \rangle$ *is a proper subgroup of* H*.*

Proof. Define $\rho : H \to H$ by $\rho(x) = x^p$. Since H is abelian, ρ is a homomorphism and $G = Im\ \rho$. By Cauchy's theorem, there is at least one element of order p and thus $Ker\rho \neq \{e\}$. Then $|G| = |Im\rho| < |H|$. ☐

Theorem 5.2.4. *[Scott] If H is a finite Abelian group not of prime order, and $P \neq E$ is a cyclic Sylow p-subgroup, then there is no nontrivial, primitive S-ring R over H.*

Proof. By Cauchy's theorem, there exists a subgroup K of H of order p. By Sylow's Theorem, K lies in some conjugate of P and hence in P as P is the unique Sylow p-subgroup. Since P is cyclic, K is unique. Suppose R is a nontrivial, primitive S-ring over H, and let $\bar{t} = \sum x_i \neq 1, x_i \in H$, be a basis element of R. By the Binomial Theorem, $\bar{t}^p \equiv \sum x_i^p (\text{mod } p)$. Suppose $\bar{t}^p \not\equiv je(\text{mod } p)$, for some integer j and e the identity of H. But this implies $Q = \{x_i^p\}$ generates a nontrivial subgroup of H, a contradiction since $\bar{Q} \in R$.

Thus, $\bar{t}^p \equiv \sum x_i^p \equiv je(\text{mod } p)$. For all $h \in H$, either h has no pth roots in H or an entire xK coset of them. To see this, suppose $h = x^p$ some $x \in H$. Then $(xk)^p = x^p k^p = h$, for any $k \in K$. Conversely, if $x^p = y^p = h$, then $(xy^{-1})^p = 1$ and $xy^{-1} \in K$. Thus, $xK = yK$.

For all i, either $x_i^p = e$ or $x_i^p = g$ for some $g \neq e$. In the first case, $x_i \in K$, else K is not the unique subgroup of order p. In the latter case, there must be $p - 1$ other x_i's with $x_i^p = g$ else $\sum x_i^p \neq je(\text{mod } p)$. Thus we can divide up the elements of t into two groups: elements in K and xK cosets of pth roots. Hence,

$$\bar{t} = a + b\bar{K}, a \in \mathbb{Z}K, b \in \mathbb{Z}H.$$

And since $< t >= H$ and $H \neq K$, we see that $b \neq 0$.

Claim: There is a basis element of the form $\bar{t} = b\bar{K}$. Suppose not, pick a basis element $\bar{t_1} = a_1 + b_1\bar{K}$ such that $L(b_1)/L(a_1)$ is a maximum and another basis element

$\bar{t_2} = a_2 + b_2\bar{K} \neq t_1^*$, possibly $t_2 = t_1$. We then have

$$\bar{t_1}\bar{t_2} = a_1a_2 + (a_1b_2 + b_1a_2 + b_1b_2|K|)\bar{K},$$

where the identity is not in the right term (since $t_2 \neq t_1^*$) and thus

$$\frac{L(a_1b_2 + b_1a_2 + b_1b_2|K|)}{L(a_1a_2)} > \frac{L(b_1a_2)}{L(a_1a_2)} = \frac{L(b_1)}{L(a_1)}.$$

But this is a contradiction since $\bar{t_1}\bar{t_2}$, being a linear combination of $t_j \neq e$, means

$$\begin{aligned}\bar{t_1}\bar{t_2} &= \lambda_3\bar{t_3} + \cdots + \lambda_k\bar{t_k}\\ &= \lambda_3(a_3 + b_3\bar{K}) + \cdots + \lambda_k(a_k + b_k\bar{K})\\ &= (\lambda_3a_3 + \cdots + \lambda_ka_k) + (\lambda_3b_3 + \cdots + \lambda_kb_k)\bar{K}\end{aligned}$$

And,

$$\frac{L(\lambda_3b_3 + \cdots + \lambda_kb_k)}{L(\lambda_3a_3 + \cdots + \lambda_ka_k)} \leq \frac{\frac{L(\lambda_3b_1a_3)}{L(a_1)} + \cdots + \frac{L(\lambda_ka_kb_1)}{L(a_1)}}{\lambda_3L(a_3) + \cdots + \lambda_kL(a_k)} = \frac{L(b_1)}{L(a_1)},$$

since $\frac{L(b_n)}{L(a_n)} \leq \frac{L(b_1)}{L(a_1)}$ for all n. Thus, $\bar{t} = b\bar{K}$ is a basis element.

Let

$$M = \{h | h \in H, th = t\}$$

and

$$\text{N}=\{h \in H | h \text{ occurs with coefficient } L(t) \text{ in } t^*t\}.$$

Then, as proven in Lemma 5.2.5 , $M = N$. Clearly, $K \subset M$, and M is a proper subgroup of H, else $H = K$. But $\bar{M} \in R$, contradicting the assumption that R is primitive. □

Lemma 5.2.5. *Suppose*

$$M = \{h | h \in H, th = t\}$$

and

$$N = \{h \in H | h \text{ occurs with coefficient } L(t) \text{ in } t^*t\}$$

as above. Then $M = N$.

Proof. Suppose $t = \sum_{i=1}^{r} x_i$, so that $L(t) = r$. Let $n \in N$. Then,

$$t^*t = L(t)n + \cdots = rn + \cdots$$

So for all $x_i^{-1} \in t^*$, there exists a unique $x_{j_i} \in t$ such that $x_i^{-1}x_{j_i} = n$. Hence,

$$\begin{aligned} tn &= (x_1 + \cdots + x_r)n \\ &= x_1 n + \cdots + x_r n \\ &= x_1 x_1^{-1} x_{j_1} + \cdots + x_r x_r^{-1} x_{j_r} \\ &= x_{j_1} + \cdots + x_{j_r} \\ &= t. \end{aligned}$$

Thus $n \in M$ and $N \subset M$. Conversely, let $m \in M$. Then

$$\begin{aligned} mt &= m(x_1 + \cdots + x_r) \\ &= (x_1 + \cdots + x_r). \end{aligned}$$

Thus, for all j, there exists a unique i_j, such that $mx_{i_j} = x_j$, that is, $m = x_j x_{i_j}^{-1}$. And so,

$$\begin{aligned} t^*t &= (x_1^{-1} + \cdots + x_r^{-1})(x_1 + \cdots + x_r) \\ &= x_1 x_{j_1}^{-1} + \cdots + x_r x_{j_r}^{-1} + \cdots \\ &= m + \cdots + m + \cdots \\ &= rm + \cdots . \end{aligned}$$

Hence, $M \subset N$. □

The following corollary is thus trivial.

Corollary 5.2.6. *Any S-ring S on a cyclic group satisfies one of:*

1. *$dim(S) = 1{,}2$;*
2. *S is imprimitive;*
3. *$|G|$ is prime.*

Proposition 5.2.7. *Suppose that D_{4m} fuses to G. Then there is a non-trivial normal subgroup H of G which corresponds to a subgroup $N \cong \mathbb{Z}_h$ of $\mathbb{Z}_n \subset D_{2n}$ such that $D_{2n}/N \cong D_{2n/h}$ fuses to G/H.*

Proof. By Lemma 4.3.2, $D'_{2n} \cong \mathbb{Z}_m$ fuses to $F_G(G')$. Let S be the *S-ring* on D'_{2n} determined by this fusion. We claim S is imprimitive. Suppose not, then by Corollary 5.2.6, our only case is (1). Clearly, $dim(S) \neq 1$. If $dim(S) = 2$, we have two classes of sizes 1 and $m-1$. This means G has a class of size $m-1$. But $(m-1)||G|$ implies $m - 1 \leq 4$, contrary to hypothesis. Thus S is imprimitive as claimed.

Hence there is some principal set $E \subset \mathbb{Z}_m$, such that $N = \langle E \rangle \neq \{1\}, \mathbb{Z}_m$. By [Wielandt, Proposition 23.6], $\overline{N} \in S$. Let $H \leq G'$ be the corresponding subgroup promised by Lemma 3.3.7. Thus we have $N = \mathbb{Z}_h$ cyclic and

$$D_{4m}/N = D_{4m}/\mathbb{Z}_h \cong D_{4(m/h)}$$

is the dihedral group which fuses to G/H by Lemma 3.3.5. □

Proposition 5.2.8. *Suppose that D_{4m} fuses to G with m even and $Z(G) = \{1\}$. Let $S = F_G(D_1)$ be the S-ring for the dihedral group $D_1 \cong D_{2m}$. Then S is not a direct product of S-rings.*

Proof. If S is a dot product of S-rings, $S = S_H S_K$, then $D_{2m} = HK$, for nontrivial subgroups H, K, where $m = 2k$, with k odd. In fact, H and K must be $\mathbb{Z}_2$ and D_{2k}, respectively. But this implies $e \neq x \in H$ must be a principal subset, contradicting the fact that $Z(G)$ is trivial. □

5.3 D_{2n} fuses to G where $|G| = 8p^2$

Proposition 5.3.1. *Suppose that D_{4m}, m even, fuses to G and let C' be a class in G with corresponding subset $C \subset \mathbb{Z}_m \subset D_{4m}$. Suppose that $|C|$ is odd and that C is invariant under the action of $Aut(\mathbb{Z}_m)$ so that C is a union $C = O_1 \cup ... \cup O_t$ of $Aut(\mathbb{Z}_m)$ orbits. Since C is odd, we may assume that $O_1 = \{x^m\}$. Let ζ be a primitive $2m$th root of unity and let $\chi^{(h)}$ be the character of D_{4m} such that,*

$$\chi^{(h)}(x^u) = \zeta^{hu} + \zeta^{-hu}, \chi^{(h)}(yx^u) = 0.$$

Let n_v be the order of an element in O_v and let μ denote the Mobius function, ϕ the Euler function. Let the index i correspond to the partition of the characters containing $\chi^{(h)}$ and the index j to C. Then for τ_{ij} defined in the Magic Rectangle Section, we have,

$$\tau_{ij} = \frac{1}{|C|}((-1)^h + \sum_{v=2}^{t} \frac{\phi(n_v)}{\phi(\frac{n_v}{gcd(h,n_v)})}\mu(\frac{n_v}{gcd(h,n_v)})). \tag{5.1}$$

The character value of G is $\tau_{ij}(2\sqrt{w})$ where w is the number of characters in that part of the partition which contains $\chi^{(h)}$, in this case w is a square and is an integer.

Proof. If $Y = Y_n$ is the set of all primitive nth roots of unity, then $\bar{Y} = \mu(n)$.[NivenZuckerman, page 197] If $m \in \mathbb{N}$, we have,

$$\bar{Y}^{(m)} = \frac{\phi(n)}{\phi(\frac{n}{gcd(n,m)})}\bar{Y}_{n/gcd(n,m)}.$$

Since $C \subset \mathbb{Z}_m$, each conjugacy class not equal to O_1 has 2 elements. From the Magic

Rectangle Condition, Equation 3.1,

$$\tau_{ij} = \frac{\sum_{m=t_1}^{t_{rj}} |C_m| X_{w_k}(C_m)}{[\sum_{m=t_1}^{t_{rj}} |C_m|] d_i},$$

we see that the denominator is $2|C|$. From the character table of the dihedral group, we have $\chi^{(h)}(x^m) = 2(-1)^h$. Each orbit O_v, $v > 1$, is a union of elements of order n_v. It is also a union of classes of D_{4m}. The character value for each class $\{x^u, x^{-u}\}$ is $\zeta^{hu} + \zeta^{-hu}, 1 \leq h \leq m-1$. Since O_v is a union of elements of order n_v, the value of Equation 3.1 on O_v is,

$$2 \sum (\zeta^{hu} + \zeta^{-uh}) = 2\bar{Y}_{n_v}^{(h)}.$$

Thus the numerator of τ_{ij} is,

$$\begin{aligned} 2(-1)^h + 2\sum_{v=2}^{t} \bar{Y}_{n_v}^{(h)} &= 2(-1)^h + 2\sum_{v=2}^{t} \frac{\phi(n_v)}{\phi(\frac{n_v}{gcd(n_v,h)})} \bar{Y}_{\frac{n_v}{gcd(n_v,h)}} \\ &= 2(-1)^h + 2\sum_{v=2}^{t} \frac{\phi(n_v)}{\phi(\frac{n_v}{gcd(n_v,h)})} \mu(\frac{n_v}{gcd(n_v,h)}). \end{aligned}$$

Finally, since the column of the character table of D_{4m} corresponding to the non-trivial class of size one is $(1, 1, 1, 1, \pm 2, ..., \pm 2)$, the value of τ_{ij} corresponding to that class is a sum of 2's and -2's. Thus the corresponding entry in the character table of G from the Magic Rectangle Condition is,

$$\begin{aligned} \eta_i \tau_{ij} &= \tau_{ij} \sqrt{\sum_{m=1}^{s_i} d_{im}^2} \\ &= \tau_{ij} \sqrt{4w} = 2\sqrt{w}\tau_{ij}, \end{aligned}$$

as required. ∎

Theorem 5.3.2. *Suppose that G fuses from a dihedral group and that $|G| = 8p^2$ where p is an odd prime. Then we have either, i) G is dihedral or dicyclic; or ii) p=3 and G is the group of order* 72.

Proof. Let $C \subset \mathbb{Z}_{2p^2}$ corresponds to the class C' of G which has odd size given by Property 3, Proposition 4.3.6. Then $|C|$ is p or p^2 as these are the only odd divisors of $|G|$. We first show that C must be a union of automorphism classes of Z_{2p^2}, that is, we show that if C contains an element of a certain order, it must contain all of the elements of that order in $\langle x^2 \rangle = D'_{8p^2} \cong \mathbb{Z}_{2p^2}$.

For every principal subset T, we define $T^{(n)} = \{g^n | g \in T\}$. By [Wielandt, Theorem 23.9], a subset $T \subset D'_{8p^2} \cong \mathbb{Z}_{2p^2}$ is a principal subset if and only if $T^{(n)}$ is a principal subset for n prime to $2p^2$. Suppose $g \in C$ and $h \in \mathbb{Z}_{2p^2}$ with $|g| = |h| = r$. We show $h \in C$. Since $|g| = |h| = r$, there exists a t with $gcd(t, 2p^2) = 1$ such that $g^t = h$. By [Wielandt, Thm 23.9], $C^{(t)}$ is also a principal subset. Since t is odd and $a = x^{2p^2} \in C$, $a^t = x^{2p^2} \in C^{(t)}$. But $C^{(t)}$ is a principal subset implies $C^{(t)} = C$. Thus $g^t = h \in C$.

We first claim that $C = C_2 \cup C_{2p}$. There are two cases to consider: $|C| = p$ or $|C| = p^2$.

If $|C| = p$, then C is either $C_2 \cup C_p$ or $C_2 \cup C_{2p}$ where C_i denotes the set of elements in D'_{2n} of order i. We show that the first case yields a contradiction.

Clearly, C is invariant under the action of $Aut(\mathbb{Z}_m)$ and $|C|$ is odd. We can thus apply Proposition 5.3.1. The character value of G for the column corresponding to C and the character corresponding to the fusion of characters containing χ^h is

$$\frac{1}{p}\left((-1)^h + \frac{\phi(p)}{\phi(p/gcd(h,p))}\mu(p/gcd(h,p))\right)2\sqrt{w}.$$

If we consider the partition containing χ^h, where h is odd and $gcd(h, p^2) = p$, in the character table for D_{2n}, then in the fusion the character table value for G for the rows of the partition of characters which contain χ^h and the column of C' is $2(p-2)\sqrt{w}/p$. By Proposition 5.3.1, this is an integer, hence $w \geq p^2$. But $1 \leq h < 2p^2$, thus h can take on exactly p values. Then $w \leq p$, which is a contradiction.

If $|C| = p^2$, then we have four cases: i)$C = C_2 \cup C_p \cup C_{p^2}$, ii)$C = C_2 \cup C_p \cup C_{2p^2}$, iii)$C = C_2 \cup C_{2p} \cup C_{p^2}$, iv)$C = C_2 \cup C_{2p} \cup C_{2p^2}$.

Case (i): $C = C_2 \cup C_p \cup C_{p^2}$; as before the character value is

$$\frac{1}{p^2}\left((-1)^h + \frac{\phi(p)}{\phi(p/gcd(h,p))}\mu(p/gcd(h,p)) + \frac{\phi(p^2)}{\phi(p^2/gcd(h,p^2))}\mu(p^2/gcd(h,p^2))\right)2\sqrt{w}.$$

If h is odd and $gcd(p^2, h) = 1$, this equates to $-4\sqrt{w}/p^2$. As this value must be an integer, we have $w \geq p^4$ but there can only be at most $2p^2 - 1$ such values of h, which again is a contradiction.

Case ii): $C = C_2 \cup C_p \cup C_{2p^2}$; the character value for this case is,

$$\frac{1}{p^2}\left((-1)^h + \frac{\phi(p)}{\phi(p/gcd(h,p))}\mu(p/gcd(h,p)) + \frac{\phi(2p^2)}{\phi(2p^2/gcd(h,2p^2))}\mu(2p^2/gcd(h,2p^2))\right)2\sqrt{w}.$$

If h is odd and $gcd(p^2, h) = 1$, this equals $-4\sqrt{w}/p^2$. Since this value is an integer, $w \geq p^4$ but only p such values of h satisfy this condition, which is a contradiction.

Case iii): $C = C_2 \cup C_{2p} \cup C_{p^2}$; the character value is

$$\frac{1}{p^2}\left((-1)^h + \frac{\phi(2p)}{\phi(2p/gcd(h,2p))}\mu(2p/gcd(h,2p)) + \frac{\phi(p^2)}{\phi(p^2/gcd(h,p^2))}\mu(p^2/gcd(h,p^2))\right)2\sqrt{w}.$$

If h is odd and $gcd(p^2, h) = p$, the above value is $-4\sqrt{w}/p^2$. But this gives $w \geq p^4$, a contradiction.

Case iv): $C = C_2 \cup C_{2p} \cup C_{2p^2}$; the character value is

$$\frac{1}{p^2}\left((-1)^h + \frac{\phi(2p)}{\phi(2p/gcd(h,2p))}\mu(2p/gcd(h,2p)) + \frac{\phi(2p^2)}{\phi(2p^2/gcd(h,2p^2))}\mu(2p^2/gcd(h,2p^2))\right)2\sqrt{w}.$$

If $gcd(h,p^2)$ is not p^2, our value is 0 and if $gcd(h,p^2) = p^2$, that is, when $h = p^2$, our value is -2. Thus we see that case iv) is the only possible case. And in this case, the common value of χ^h for the columns of the classes making up C by the Magic Rectangle Condition is 0 for all h except when $h = p^2$ in which case we get -2. Thus the character χ^h of D_{8p^2} does not fuse with any other character and is a character of degree 2. Notice we can also check by using the Magic Rectangle Equation 3.2 directly to get -2. Moreover, for any $h \neq p^2$, since χ^h gives the value 0, χ^h must fuse with at least one other degree 2 character. Thus G only has one irreducible character of degree 2.

By Property 8, Proposition 4.3.12, the Sylow 2-subgroup of G has a normal complement, that is, $G = NS$. Consider the *S-ring* $F_G(N)$ on N determined by the fusion. If this *S-ring* is imprimitive, we can find a nontrivial subgroup J of N with $\bar{J} \in F_G(N)$. Then $|J| = p$ and by Lemma 3.3.5, D_{8p} fuses to G/J. Thus G/J is dihedral or dicyclic by Theorem 4.4.1. But, assuming $p > 2$, G/J has more than one irreducible representation of degree 2. By lifting these irreducible characters to those of G, we have that G also has more than one irreducible character of degree 2 which is a contradiction.

Thus the S-ring on N is primitive and fuses from a cyclic group of order p^2. And by [Wielandt Theorem 25.4], since p^2 isn't prime, $F_G(N)$ is trivial. Thus G must have a class of size $p^2 - 1$. Then we must have that $(p^2 - 1)|8$ and the only prime which satisfies this divisibility is 3. Thus G is the group of order 72 in this case.

We can now assume that $C = C_2 \cup C_{2p}$. In [Yuanda, pg. 77-93], Yuanda classified groups of order $8p^2$ using 6 lemmas. By Property 10, the Sylow-2 group of G must be dihedral, dicyclic or abelian of type [4,2]. We can thus eliminate all the groups from Lemmas 2 and 4. We can also eliminate groups from Lemma 3 since the index of the commutator subgroup of these groups is at least 8. Groups from Lemma 1 can also be eliminated by Property 10 and the requirement that G has trivial center. From the remaining lemmas, there are only 2 groups with trivial center and $|G/G'| = 4$. These two groups can be eliminated since they each have a class of size p^2 contradicting Property 3 in which we proved that G has a unique nontrivial class of odd size.

Now, suppose G has nontrivial center. Then $Z(G) \cong \mathbb{Z}_2$. As before, Property 10 will help us eliminate all the groups from Lemmas 2 and 4. There are three groups from Lemma 1 with $Z(G) \cong \mathbb{Z}_2$. Two of these groups are dihedral and dicyclic. The last one has presentation:

$$G = \langle a, x, y | a^{p^2} = x^4 = y^2 = 1, x^y = x^{-1}, a^x = a^{-1} = a^y \rangle.$$

But this group violates Property 6 and 7. The element axy, for example, is not conjugate to its inverse. Finally, there are 5 groups from Lemmas 5 and 6 with $Z(G) \cong \mathbb{Z}_2$. But all five violates Property 13 since each has at least 2 distinct normal subgroups of size p.

We have eliminated all groups in the classification except for the two groups listed on page 79 of [Yuanda], which are the dihedral and dicyclic groups. □

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