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What Are Some of the Common Traits in the Thought Processes of Undergraduate Students Capable of Creating Proof?

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WHAT ARE SOME OF THE COMMON TRAITS IN THE THOUGHT PROCESSES OF UNDERGRADUATE STUDENTS CAPABLE OF CREATING PROOF?

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ABSTRACT

WHAT ARE SOME OF THE COMMON TRAITS IN THE THOUGHT

PROCESSES OF UNDERGRADUATE STUDENTS

CAPABLE OF CREATING PROOF?

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Master of Arts

Mathematical proof is an important topic in mathematics education research. Many researchers have addressed various aspects of proof. One aspect that has not been addressed is what common traits are shared by those who are successful at creating proof. This research investigates the common traits in the thought processes of undergraduate students who are considered successful by their professors at creating mathematical proof.

A successful proof is defined as a proof that successfully accomplishes at least one of DeVilliers (2003) six roles of proof and demonstrates adequate mathematical content, knowledge, deduction and logical reasoning abilities. This will typically be present in a proof that fits Weber’s (2004) semantic proof category, though some syntactic proofs may also qualify. Proof creation can be considered a type of problem,

and Schoenfeld’s (1985) categories of resources, heuristics, control and ability are used as a framework for reporting the results.

The research involved a) finding volunteers based on professorial recommendations; b) administering a proof questionnaire and conducting a video recorded interview about the results; and then c) holding a second video recorded interview where new proofs were introduced to the subjects during the interviews. The researcher used Goldin’s (2000) recommendations for making task based research scientific and made interview protocols in the style of Galbraith (1981). The interviews were transcribed and analyzed using Strauss and Corbin’s (1990) methods. The resulting codes corresponded with Schoenfeld’s four categories, so his category names were used.

Resources involved the mathematical content knowledge available to the subject. Heuristics involved strategies and techniques used by the subject in creating the proof. Control involved choices in implementing resources and heuristics, planning and using time wisely. Beliefs involved the subjects’ beliefs about mathematics, proof, and their own skills. These categories are seen in other research involving proof but not all put together.

The research has implications for further research possibilities in how the categories all work together and develop in successful proof creators. It also has implications for what should be taught in proofs courses to help students become successful provers.

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Chapter One: Introduction

One of the most discussed topics in mathematics education research is that of proof. The role of proof in mathematics education, the importance of proof in mathematics and the difficulties of proof for students of mathematics have all been discussed a great deal. The National Council for Teachers of Mathematics [NCTM] Principles and Standards for School Mathematics (2000) emphasize the importance of proof in mathematics education, saying:

> Being able to reason is essential to understanding mathematics…By the end of secondary school, students should be able to understand and produce mathematical proofs – from hypotheses – and should appreciate the value of such arguments…Reasoning and proof should be a consistent part of students' mathematical experience in pre-kindergarten through grade 12. (p. 56)

Communicating the products of discovered and constructed mathematical ideas is necessary in the education of students, and proof can be a natural vehicle.

Yackel and Hanna (2003) argue for the importance of proof because education is currently shifting away from behaviorist theories toward an emphasis on reasoning in learning. This emphasis on reasoning has brought more awareness to mathematical argumentation and justification, and since proof is the end result of argumentation for mathematicians, it is seen as central to mathematical thinking. Hanna (1996) states that "proof deserves a prominent place in the curriculum because it continues to be a central feature of mathematics itself … [and] it is a valuable tool for promoting mathematical understanding" (p. 2).

Personal Interest

My personal interest in proof comes in part from my own struggle with it in my schooling. When I took my first “proofs” course at Brigham Young University, called “An Introduction to Proof”, I remember feeling there were 3 kinds of proof: those that used induction, those that used contradiction, or those that could only be patterned after other proofs I had seen. My struggle with proof continued through Abstract Algebra and Basic Analysis. My main strategy in these classes was to memorize proofs, whether or not I understood them. My geometry class was a different experience, as I was able to appeal to other known facts and definitions as I examined relationships in geometric drawings. In this class I also had a strong support structure of friends and much of the work was done as a group. Proof was not easy in that class, but it made more sense.

When I returned to Brigham Young University to pursue my master’s degree in mathematics education, my first class was Theory of Analysis. I learned much in this class. The structure was very different from my experience in its prerequisite. The professor did not lecture, he asked questions. He demanded that we know our definitions, telling us that we could not prove basic facts if we did not know the definitions on which they relied. While I may have known this unconsciously beforehand, as I had used the idea in geometric proofs, this was the first time it had been explicitly stated in an analysis proof setting. It was a real breakthrough in proof for me. Because the professor did not lecture, students were expected to prove theorems as homework and present them to the class. What developed in that classroom was a wonderful place for debate and discussion of proof. We analyzed each other’s methods, determining if there were gaps in the logic, or if there were other ways to do the same proof. Often more than one proof of a theorem was presented. The professor truly was a guide, keeping us on track and at the same time being a member of the community analyzing proposed proofs.

My next course was Galois Theory. It was taught in a more standard lecture format, but I believe that my experience in my Theory of Analysis class changed the way I understood proof. I asked many questions about proofs presented in class, trying to understand the logic and reasoning necessary as well as what definitions and previous results were being used where. In working on the homework with classmates, I found myself saying, "Well, let's look at the definitions that apply to this statement. Maybe there is something there that will help us." In exploring our definitions and previous results, we often found the proof, and just as often did not. But we learned a lot in the process about the subject at hand, exploring the topic more than I had ever thought such a simple homework question would support.

While I was taking these courses, I was also taking my core mathematics education classes and a discussion seminar on mathematics education research. Among the things we talked about in these classes were discussions and readings on the philosophies of mathematics education and philosophies of mathematics. This often led to pondering how these philosophies applied in the mathematics classes we were taking concurrently. I learned much as I struggled to think about and qualify my own beliefs about mathematics and mathematics education. As I thought about what mathematics truly consisted of and what mathematics education was meant to accomplish, I often came back to thinking about the role of proof in the mathematics classroom, why it was difficult for so many, and what it meant to create proof.

Topic Development

While taking all these classes, I realized that I enjoyed reading and thinking about proof. Specifically, three papers stood out: Harel and Sowder's (1998) Students' Proof Schemes: Results from Exploratory Studies, Moore's (1994) Making the Transition to Formal Proof, and Recio and Godino's (2001) Institutional and Personal Meanings of Mathematical Proof. This led me to really ponder the relationship between these ideas, that is, between proof schemas, meanings of proof and difficulties students have making the transition to formal proof. After presenting in the practicum, it became clear to me that what I was interested in was the process of developing proof - not the end results, not the difficulties of creation, but what led to the successful development of proof.

Understanding the proof development process can be interesting and useful to the mathematics education research community. By understanding the thought processes that lead to the creation of proof, we can learn much about how to teach proof. Part of the difficulty of learning to create proof is that there are processes in proof creation that are unconscious or hidden from plain view. These processes will reveal much about how proof is understood and viewed by the those doing proofs as well as how instruction on proof might be modified.

After my initial gathering of data, it was apparent that a framework was needed to help describe the successful practice of proof creation. From how I defined successful proof creation, it was suggested that I consider Schoenfeld's 1985 book that discusses successful problem solving. His framework for what is necessary to be successful at problem solving complemented my definition of successful proof creation as a problem solving process and so I adopted his framework for structuring my analysis, allowing for additional possibilities.

This leads to my research question: What are some of the common traits in the thought processes of students capable of creating proof? In the next chapter I will address the literature on proof that applies to my study, showing why my study will be of interest and bring new information to the research community, and build my theoretical framework. Then in chapter three I will outline in detail my research methods. In chapter four are my results and analysis and chapter five is the conclusion.

Chapter 2: Conceptual Context

The conceptual context of this paper is the lens through which my research findings will be interpreted. In this chapter I develop that lens and make clear the assumptions that ground my research. First I discuss the definition of proof. By carefully determining what a proof is and when a proof is created successfully, it will be possible to analyze the process of creation, which is my focus. In addition, I consider the previous research that guided the development of my research focus.

What is proof?

The question of what constitutes a proof seems innocuous, yet it requires much thought and careful analysis to answer adequately. A successful proof can generally be thought of as having the appropriate mathematical conceptual knowledge and adequate deductive reasoning skills applied with some correct language and formatting. However, the formal language of proof is not necessary for a successful proof to occur. A successful informal proof will have the presence of appropriate logical deductions and adequate mathematical knowledge. Weber (2004) defines three major categories of proof creation. He defines semantic proof production as occurring

> …when the prover uses instantiations of relevant mathematical objects to suggest and guide the formal inferences that he or she draws. By instantiations, [he means] a systematically repeatable way of thinking about a mathematical object that is internally meaningful to that individual. (p. 9)

Weber (2004) considers semantic proof to be a proof that is created when the individual understands the mathematical situation and then uses his understanding to create a successful proof. In contrast, Weber defines syntactic proof production as occurring when an individual produces a proof solely by manipulating correctly stated definitions and other facts and theorems in a logically permissible way. "Such a proof production might be colloquially defined in the mathematical community as a proof that is written by 'unpacking the definitions' and 'pushing symbols'." (p. 8) Weber defined procedural proof to be a proof that is created when a person follows a set of rules they believe will yield a successful proof. When a procedural proof is created, the person "may or may not be aware of how their resulting work establishes the statement to be proven." (p. 5) Weber's categories of syntactic, procedural and semantic proof are the three types of proof production he observed in undergraduates.

Weber (2004) points out that procedural proof production is not entirely useless. The procedurally created proof gives the creator practice at applying and becoming effective at using a new technique for proof creation. This becomes another method of proof creation that the prover is capable of using. Thus it can be useful in giving the creator practice in following logical rules and techniques. Syntactic proof differs from procedural proof because strong guidance from an authority (book or person) on how to proceed is lacking for the prover when creating a syntactic proof. The prover must decide how the assertions are logically connected together in syntactic proof production, while in a procedural proof, the prover is following rules given to them by an authority. In comparison, a semantic proof creator constructs personal meaning about relevant mathematical objects and reasons about them in his proof, gaining understanding about the concepts being explored in the proof.

Weber's (2004) three categories of proof creation are interesting for several reasons. Semantic proof does not necessarily employ the formal language of proof. It may be a complete proof that has little or no formal language in it. It may be mostly pictorial with verbal explanation. There are many possibilities. Procedural proof can appear to be a very nice proof but generally the creator will have no understanding of what has been created. This shows a lack of mathematical content knowledge and possibly a lack of reasoning skills. Thus while a procedural proof may appear to be successful, in questioning the creator, it can be determined that it was created without understanding and should not be considered a successful proof creation. Syntactic proof creation will be considered successful if the creator can explain the proof to some extent. Often, a syntactic proof will not evidence the mathematical content knowledge required to create a meaningful and successful proof.

Now consider what a proof provides to the mathematical community. According to DeVilliers (2003) there are six main roles of proof: verification, explanation, discovery, communication, systemization, and intellectual challenge. Verification is a checking process in determining the truthfulness of a statement. Explanation is a process where the proof explains aspects of the concepts inside the proof. Discovery is a process of discovering more about a topic or concept by attempting a proof. Communication is explaining and sharing knowledge with others in a clear manner. Systemization involves creating and examining the system of operation in which the proof exists. Intellectual challenge proofs are undertaken just for the sake of a challenge. Some of these roles are used more by students of mathematics and some are used more by mathematicians. A proof may not accomplish all of these roles at once yet these are all important functions of proof at different times and in different contexts. Moreover, different proofs of the same statement will accomplish different roles. For example, in proving Pythagoras' theorem, a visual proof like Baudhayana's will provide verification and explanation better than Euclid's proof for many math students. However, Euclid's proof can be more intellectually challenging and is systemizing in nature, being the culmination of his Book one of Elements. It could be said that a proof must accomplish one of DeVilliers six roles of proof in order to be considered a proof. However a proof that is explanatory for some individuals may not be for others. Thus it becomes important to consider for whom must the proof accomplish one of these functions, e.g. for the individual creating the proof or for their peers. The role a proof plays will depend upon why the proof is being created, and for what audience. If it is being created by an individual to personally determine the truthfulness of a statement, the proof will likely be different from the proof that is created by an individual to persuade others of the truth of that same statement. In summary, the purposes, context, and audience of a proof are important aspects to consider as the success or adequacy of a proof is judged.

For this study, when I considered whether a person had truly created a proof or not, I considered first which of Weber's (2004) categories was most accurate in describing the proof. A successful proof evidences mathematical content knowledge and deduction and logical reasoning knowledge. Typically, successful proofs will fit in the semantic proof category, though a few syntactic proofs may also qualify. The audience, context and purposes of the proof will affect how these two types of knowledge are evidenced, so I also considered these aspects. A successful proof would have to accomplish one of the purposes discussed by DeVilliers (2003).

Proof as Problem Solving

Proofs can be viewed as a type of problem solving. A successful proof has both mathematical content knowledge and deductive and logical reasoning skills evidenced. These two types of knowledge interact to help a prover work their way through a proof, identifying problems and then solving them to create the successful proof.

Schoenfeld (1985) outlines four categories of knowledge and behavior that he considers necessary to adequately describe a person's problem solving abilities in mathematics. According to his definition, "it is a particular relationship between the individual and the task that makes the task a problem for that person. The word *problem* is used here in this relative sense, as a task that is difficult for the individual who is trying to solve it." (p. 74) He also uses the following definition from the *Oxford English Dictionary*, "Problem: A doubtful or difficult question; a matter of inquiry, discussion, or thought; a question that exercises the mind." (Simpson & Weiner, 1989) Some types of proof creation can be viewed as a type of problem solving. This is because the creation of the proof is often a problem for the prover, challenging them to think, inquire, discuss and exercise their mind in determining their course of action. So it is likely that characteristics of successful provers will be similar to successful problem solving abilities.

Though Schoenfeld (1985) uses his entire book to outline and explain his categories, I will attempt to summarize and explain the categories in a more abbreviated manner. Briefly, the categories are *resources*, *heuristics*, *control* and *belief systems*.

Resources can be defined as the "mathematical knowledge possessed by the individual that can be brought to bear on the problem at hand." (Schoenfeld, 1985, p. 15) This category includes more than facts about mathematical ideas. It also includes algorithmic procedures, non-algorithmic procedures that are “routine” to the individual, intuitions, informal knowledge about the domain, and understandings or propositional knowledge about the rules of working in the domain. Resources may also include incorrect knowledge. Students make consistent error patterns in research and this is indicative of them having incorrect understandings about concepts. Representations can provide support for resources and allow students to access problems that may be inaccessible in different representations. When an individual demonstrates appropriate mathematical content knowledge in a proof creation, it will come from accessing the resources available to the individual.

Heuristics is a category that involves “strategies and techniques for making progress on unfamiliar or nonstandard problems or rules of thumb for effective problem solving.” (Schoenfeld, 1985, p.15) These include drawing figures and using suitable notation, exploiting related problems, working backwards or reformulating a problem into a similar, but easier solved problem, and testing and verification procedures. The origins of this category can be traced to Polya (1945) and his book *How to Solve It*. Schoenfeld writes that numerous heuristics studies have been done but heuristics have proven more complex than expected. Heuristic strategies also rely on the available resources in the domain of the problem. Shaky knowledge of subject matter cannot be overcome by good heuristics. Heuristic strategies are not necessarily easily developed. This is because they are often vaguely taught or hinted towards, and having good strategies doesn’t mean one has the ability to use the strategies appropriately to solve a problem. An individual who demonstrates adequate deductive and logical reasoning skills in proof creation will have accessed and used heuristic strategies and techniques they have available and recognize as relevant.

Control is defined as "global decisions regarding the selection and implementation of resources and strategies." (Schoenfeld, 1985, p. 15) This includes planning, monitoring, assessing, decision making skills, and conscious metacognitive acts. Control is resource allocation in problem solving. It is a major factor in the success or failure of the individual. Research shows that student problem solving performance can be improved by teaching heuristic strategies if they are taught within the framework of a prescriptive control strategy. It is also thought that participation in cooperative problem-solving experience and the internalization of aspects of that experience can lead to better control abilities. Good control abilities are essential in proof creation. Individuals who go on wild goose chases of tangentially related information are not exercising appropriate control and will become frustrated. Individuals who have good control abilities will use their resource and heuristic abilities wisely and apply them to the proof problem to demonstrate their deductive and logical reasoning skills as well as their mathematical content knowledge.

Belief Systems is a category that involves one's "mathematical world view," (Schoenfeld, 1985, p. 15) which is the set of (not necessarily conscious) determinants of an individuals' behavior about self, the environment, the topic and about mathematics in general. It is a critical category which has an effect on the individual's learning of material in the other categories as well as affecting whether they will access that learning in a problem solving situation. Students with skewed belief systems will not be successful provers because they will not see the proof creation process as relevant or useful to them.

Proofs that are created will be in a procedural and possibly syntactical method, and will access their resources and heuristics abilities very differently from those who have belief systems that recognize proof as useful and helpful to them as individuals. Additionally, different purposes of proof will be accomplished based on the belief systems held by the individual. Those who create procedural and some syntactical proofs will do so possibly because they believe the purpose of proof is verification of information already believed to be true.

These four categories and the characteristics of individuals within these categories will be a major part of my research lens. I believe that belief systems, control abilities, heuristics, and resources will describe all of the common characteristics I will see in my subjects.

Proof research literature

In this section I discuss a number of studies on proof that are relevant to my research, and discuss how they relate to Schoenfeld's (1985) categories.

Recio and Godino (2001) researched how various meanings of proof in different contexts affect student's beliefs about proof. They used a questionnaire to collect data about general university students' capability to build simple deductive proofs as freshmen. The questionnaire had two proof problems, one arithmetic based and the other geometry based. Student responses were put into five categories.

1. The answer is very deficient (confused, incoherent).
2. The student checks the proposition with examples, without serious mistakes.
3. The student checks the proposition with examples, and asserts its general validity.

4. The student justifies the validity of the proposition, by using other well-known theorems or propositions, by means of partially correct procedures.
5. The student gives a substantially correct proof, which includes an appropriate symbolization (p. 86)

The data were first analyzed in quantitative tables, to determine whether the type of problem and response category were unrelated variables; this suggested that students used similar methods, independent of the question posed. This was a validity check on their results. Their main conclusion was that most freshmen had difficulty with deductive proof. Next, they proposed an interpretation of why this was.

The five categories in Recio and Godino's (2001) model were interpreted as personal proof schemes that the subjects held and used in response to proof problems with elementary content and structure. Those with a type 2 response were confirming the truth of the propositions using particular examples, thus it was seen as an *explanatory argumentative scheme*. Type 3 responses were using examples to verify the propositions, using empirical-inductive procedures, thus they are viewed as *empirical-inductive proof schemes.* Type 4 answers used informal logic approaches, so they are *informal deductive proof schemes.* Type 5 answers are elementary deductive proofs, thus they are seen as *formal deductive schemes.*

Recio and Godino (2001) then point out that there are other contexts in which proof is used which can affect student understanding of proof. For example, in daily life people use intuitive arguments, which are situationally based. These arguments do not require deductive logic. In the experimental sciences, hypotheses are validated by the results of experiments, and theories are established based on these results, without confirming that it is true in all situations. New theories will refine existing theories by the addition of further research results. This type of justification is empirical-inductive in nature. Even in the mathematical classroom, the teacher often uses informal reasoning to argue the truth of things. So based on these ideas that proof can have different social contexts, the research argues that because students are simultaneously members of different institutions where proof has different meanings, it is often difficult to discriminate between the respective types of argumentation. Thus with this framework, student proof schemes are seen as related to institutional meanings of proof. Those using an explanatory scheme are reflecting their elementary intuitive argumentation styles; those with empirical-inductive schemes are using a more scientific view of proof. Those using informal deductive schemes are using the type of reasoning often used in the classroom by the teacher, with a strong intuitive component, and those with the formal deductive schemes are closes to the usual ways rigorous proof is done.

Recio and Godino's (2001) paper discusses how personal and social meanings of proof can affect students' proving abilities. The idea that a student may not view proof strictly in the way that mathematicians view proof is important. Their beliefs about proof are part of their belief systems about mathematics. As Schoenfeld (1985) discovered in his research, the beliefs that an individual has about mathematics affects their problem solving performance.

> "In other cases, much of the mathematical knowledge that the students had at their disposal, and that they should have been able to use, went unused in problem solving. This was not because they had forgotten it (a matter of resources) or because they ran out of time to use it (a matter of control) but

because they did not perceive their mathematical knowledge as being useful to them , and consequently did not call upon it." (p. 13)

I hypothesized that in my research, successful provers would view proof similarly to mathematicians. This view of proof would aid my research subjects in understanding what is expected of them when they are presented with a proof problem. Their beliefs about proof would influence how long they spend on a proof problem, how they approach the proof and what they view as success in the creation of a proof. Those with a different view of proof would create a different final product than their peers because of their beliefs about proof.

Harel and Sowder's (1998) research was to learn more about college students' proof understanding, production and appreciation. Their focus was on students' schemes of mathematical proof. Their data comes from six separate sources, five one-semester teaching experiments in various mathematics classes, and one case study of a precocious junior-high student. Data were collected from classroom observations in field notes, retrospective notes, clinical interviews, homework and tests. Two of the classes had more extensive data collected, including video-taping and transcribing of all class sessions and observations from graduate students as well as the previous forms of data mentioned above. Their validity checks include interviews with other math majors at a different institution and numerous revisions of their model until it reached a stable stage after analysis. The goal of their report is to explain and demonstrate the different categories of proof schemes that they observed.

In their research, Harel and Sowder (1998) found three major categories of proof schemes for students, each of which had several subcategories. It is pointed out that these schemes are not viewed as mutually exclusive; it is possible for people to hold more than one proof scheme at a time. The three main categories are *external conviction*, *empirical* and *analytical*.

External conviction is characterized by the belief that mathematical proof must come from an outside source other than the individual. Three subcategories in this external conviction scheme are the *ritual* scheme, where the appearance of the argument means more than the actual argument, the *authoritarian* scheme, where a book or a teacher is the source of conviction, and the *symbolic* scheme, where the usage of mathematical symbols means that a proof is being established, without understanding the meaning of the problem situation.

Empirical schemes use physical or sensory experiences to find conviction. Students have two subcategories here, *inductive* and *perceptual*. Those with an inductive proof scheme use specific cases to evaluate a conjecture. They believe the statement to be true after observing its truth in several cases. Perceptual schemes use mental images and pictures to make observations, but the subject is unable to change or transform the images. For example, the subject may think only of an isosceles trapezoid when thinking of a trapezoid, and not consider other possibilities.

Analytical proof is broken into two major subcategories: *transformational* and *axiomatic*. Those possessing analytical schemes are using some form of deductive logic to reason about conjecture. Those using one of the transformational proof schemes are using pictures and mental images similar to a perceptual empirical scheme. However, in these schemes, the individual is capable of manipulating the images in order to reason about them logically. The axiomatic scheme exists when an individual understands to some degree that justification starts with statements that are accepted without proof. Within both the transformational and axiomatic schemes, Harel and Sowder (1998) identify more subcategories; however it is not necessary for me to describe them here.

Harel and Sowder's (1998) research is very interesting and applicable to my own. They have developed a comprehensive structure about proof schemas and what proofs can be created based on the schema that an individual holds. For this study, I hypothesized that students would move back and forth between schemas while developing their proofs (although this is not how Harel and Sowder viewed their schemas) I believe that proof schemas are mainly part of the belief system category of proof as problem solving that we have from Schoenfeld (1985). However, this category is very influential and affects student success in proof creation.

The purpose of Galbraith's (1981) research was to investigate student perception and understanding of some processes involved in mathematical argumentation. His method of investigation was clinical interviews with several hundred children ages twelve to fifteen. He used three tasks in the interviews, working through all tasks with each student. Since the research was on a large scale, it was not feasible for one individual to conduct all the interviews, thus a set of protocols was designed to help the interviewers be consistent. The protocols are like flow charts, telling the interviewer what to do based on student success or failure at the previous step. The purpose of this was to probe student reasoning at different levels and points of item discussion. The data was tape-recorded for retrospective analysis. Validity of data is claimed because of consistency in responses across interviews, even those with variable age, gender, school and interviewer, and because results agreed with pilot study data.

Eight clusters of response patterns were found in the students' reasoning. These are considered essential parts of successful thought processes in mathematical reasoning for students. Students need these process skills as well as an agreement with the class instructor on the power and purpose of these skills. The skills that Galbraith (1981) found are summarized below, directly from Galbraith's paper.

A. Variety/Completeness in checking
   (a) Variety in choice of special cases
   (b) Thoroughness of checks
   (c) Avoidance of conjectures on insufficient evidence
B. Proof/Explanation related to external principle
   (a) Recognition that a principle is present
   (b) Identification of the principle
   (c) Application of the principle
C. Linking of inferences
   (a) Identification of chains
   (b) Acceptance of Lack of Closure within chains
D. Domain of validity of generalizations
   (a) Need for system in generating/examining special cases implied by definitions and statements
   (b) Significance of a counter-example
   (c) Mechanism of refutation by counter-example
E. Literal interpretation of the data
   (a) Local interpretation of statements

(b) Global conservation of contexts

F. Evaluating statements/distinguishing implication and equivalence

(a) Avoidance of Centration (evaluation of whole based on consideration of only part of statement)

(b) Separation of conditions and conclusions

(c) Awareness of distinction between conjecture and defined knowledge

G. Meaning of definitions

(a) Properties of definitions

(b) Awareness of need to restructure basic schemas

H. Proof structure

(a) Analysis of a proof into components

(b) Evaluation of the components (p. 26-27)

I see these eight clusters of thought as being part of problem solving resources, control and heuristics. Their variety and completeness in checking is a heuristic strategy that they may have developed or still need to develop more. Relation of proof to an external principle and the recognition, identification and application of that principle is an aspect of control and their ability to monitor progress. Their ability to exploit related problems is a heuristic, and their knowledge about the problem they are working with is their resources. Linking of Inferences is a heuristics strategy. Domain of validity of generalizations is also a heuristic strategy, with some control elements. Use of counter-examples in this category is a heuristic and the need for systematical testing is an element of control. Literal interpretation of data is heuristics and being able to use the data in aiding the proof creation. Evaluating statements is a heuristic skill of being able to see implication and equivalence. Definitions are a resource available to a prover, and using them is a heuristical skill. Proof structure will come thru control and heuristics, and be affected by their belief systems about proof. So these eight clusters of reasoning responses that Galbraith (1981) found are found inside Schoenfeld's (1985) framework of categories, showing how interactive the categores are with one another.

Moore's (1994) research is concerned with the cognitive difficulties university students experience in learning formal mathematical proof. His research used grounded theory methodology and involved non-participant observation in a class devoted to helping students in their transition to formal proof as well as interviews with the professor and several members of the class. His major findings were that students have difficulty with concept understanding, mathematical language and notation and getting started in their proofs. Within concept understanding were five subcategories of difficulties that he focused on. These subcategories were the inability to state the definition, no intuitive understanding of the concept, inability to use concept images, inability to generate and use examples and not knowing how to structure the proof from a definition. These findings are all examples of student deficits in heuristics and resources.

Although this research does a good job of focusing on the process of proof, it focuses on the negative, or the difficulties that students have with proof. The question of whether students who are successful in proof are strong in the areas he found others having difficulty with remains unanswered. Though the research shows observed interactions among the difficulties, a negative model does not imply an opposite positive model. Moore's (1994) paper provides the difficulties that students had when learning to create proof. Although it cannot be assumed that the only strengths held by successful provers are in the categories that his subjects had difficulties in, I hypothesized that these would show up as some of the strengths I would see in my subjects. The difficulties that he found can be classified in Schoenfeld's (1985) categories as lacking the resources, heuristics, and control abilities to create the proof. Their concept understanding, definition difficulties and mathematical language and notation were all resource problems. Their difficulties with knowing where to start were control and heuristics problems. While I do not see explicit problems with belief systems, it is postulated that some of their resource and heuristics difficulties are due to their belief systems and not believing that their resources and heuristics will be useful to them in their proof creation.

Conclusion

Weber (2004) outlines three basic types of proof created by college students. These are procedural, syntactical and semantic proofs. According to DeVilliers (2003), proof has six main roles in mathematics. These possible roles are verification, explanation, discovery, communication, systemization and intellectual challenge. A successful proof will accomplish at least one of these six roles for its intended audience as well as demonstrate appropriate mathematical content knowledge and deductive or logical reasoning skills. This will generally occur only in semantic proofs, but may occur in some syntactical proofs. Proof can be considered a type of problem solving. Schoenfeld (1985) describes the four categories necessary to adequately characterize problem solving performance. These categories are resources, heuristics, control and belief systems.

Much of the research on proof that exists currently can be viewed through this lens of proof as a problem solving process. Recio and Godino (2001), Galbraith (1981), Harel and Sowder (1998), and Moore (1994) all wrote influential papers on proof that can be viewed in this manner and support the view of proof as a problem solving process. Table 1 below summarizes the categories in Schoenfeld's (1985) problem solving scheme that are addressed by each of the proof studies discussed above.

Table 1

Aspects of Problem Solving Discussed in Proof Literature

Aspect of Problem Solving	Proof Literature Discussing this Aspect
Beliefs	DeVilliers, Harel & Sowder, Recio & Godino, Moore, Weber
Resources	Galbraith, Moore, Weber
Heuristics	Galbraith, Moore, Weber
Control	Galbraith, Moore, Weber

It is expected that my research will sustain and corroborate what others have found but by using this lens of proof as a problem solving process we can learn more about proof and what it means to be a successful proof creator.

Chapter 3: Research Methods

As has been shown in the last chapter, much research has been done about proof, but little of it has been primarily focused on students capable of creating proof. Thus, my study investigated some common traits of thought processes in students capable of creating proof, with the goal of providing new information about the learning of proof that can inform better instruction. Using my theoretical framework as a guide I examined the mental processes used by students as they worked on creating proofs. In this chapter I discuss research methodology.

Theoretical Background

Goldin (2000) argues that by carefully considering and adjusting for various aspects of task-based interviews, it is possible to use them in a rigorous, scientific manner to conduct research. Among those aspects specifically addressed by Goldin are the environment, the structure of the interview, the questions, and the tasks. Goldin goes on to discuss five issues that will affect the scientific validity of interview based research. They are control and design, replicability & generalizability, importance of mathematical content, role of cognitive theory, and interplay among task and contextual variables. When these five issues are sufficiently addressed, validity in results can be argued. Furthermore, these issues can be addressed by using the ten broad methodologically-based principles he proposes and explains in his chapter.

Scientific research involves the careful description of all methods employed in the observation of subjects. This includes the researcher distinguishing between what is completely controlled, what is partially controlled, and what was not controlled. This is done for several reasons. First, it is an essential step so that future researchers will be able to recreate the experiment and obtain similar results. Second, it is necessary so that when inferences and results are developed from the collected data, the researcher can logically reason about the effect of the varying controls on the results. Goldin (2000) points out that it is the presented tasks that are subject to control, not the interpreted tasks. He says: "Quality research not only addresses the variables that are controlled, but also includes explicit consideration of known variables that are uncontrolled, seeking to understand and allow for their possible effects." (p. 527) Without careful consideration of all variables, the validity of results becomes more questionable. Such lack of consideration gives way to the report being characterized as an anecdote.

It is not easy to quantify results in task based interviews. However, generalization is part of scientific research and necessary for progress. Goldin (2000) discusses Piaget as an example of using interviews in his research. Piaget's results have been verified by many different researchers in a variety of settings and cultures. The corroboration of his results with others is part of what made his findings so influential. They were generalizable and replicable. Goldin argues that a large body of anecdotal accounts does not provide grounds for generalizability, noting that astrology, medical quacks and other pseudoscientific beliefs have numerous anecdotes to support their claims.

The content and structure of the mathematical task-based interview is subject to the researcher's control and one that must be addressed more than superficially. Deeper semantic and mathematical structures that may occur in various task domains should also be considered. Goldin (2000) states that the analysis of the structures and possible interactions among mathematical topics is an important part of the research.

All scientific investigation is based in theory and guided by theory, thus mathematical task-based interview research should also be based in theory. Our definitions are essential to what we will observe and thus our theories which guide are definitions must be carefully considered. All researchers will have preconceived ideas, and it is better to make them explicit and part of the theoretical model than tacit with their effect on results unknown. All assumptions about subject's competencies, cognitions, attitudes, pathways, beliefs, strategies, etc., should be carefully considered in the interview design. This careful consideration of our cognitive theories will help us in the design of contingencies and in drawing our inferences about the observations.

Additionally, the interplay between variables must be considered. The environment of a task based interview will have social and cultural aspects which may have consequences and influence on the subjects' behavior. Goldin (2000) points out that these interactions mean that some observations will be traded for others. Also, researchers must be open-minded and ready to deal with unanticipated events.

In order to address and provide a solid foundation for research that utilizes task-based interviews, Goldin (2000) proposes ten principles. These principles help establish quality standards in mathematics education research and allow for good progress. They are:

- Design task based interviews to address advance research questions
- Choose tasks that are accessible to the subjects
- Choose tasks that embody rich representational structures
- Develop explicitly described interviews and establish criteria for major contingencies

- Encourage free problem solving
- Maximize interaction with the external learning environment
- Decide what will be recorded and record as much of it as possible
- Train the clinicians and pilot-test the interview
- Design to be alert to new or unforeseen possibilities
- Compromise when appropriate

Goldin has carefully considered aspects of task based interview research, including control and design, replicability & generalizability, importance of mathematical content, role of cognitive theory, and interplay among task and contextual variables and clearly laid out a plan for quality research to be conducted. His proposed principles help researchers to achieve more validity and also to design effective and quality research.

In designing my study, I addressed applicable principles from Goldin (2000) in several ways. The principle of designing task based interviews that addressed advanced research questions was appropriately addressed because my research questions were developed first, and my methodology was appropriately chosen so that it addressed them. I wanted to see what successful provers did, so I gave subjects proofs to do and observed them in action. I also carefully defined what constituted a proof, which allowed me to know if my research subjects were successful in creating proofs when I observed them. I chose my subjects to be able to do proofs, by seeking advice from professors. I also sought advice from professors in choosing the proof problems. This addresses the principle of choosing good tasks that are accessible to the subjects. I developed explicitly described interviews and established criteria for major contingencies by following Galbraith (1981) and I created my interview protocols as an outline of questions based on situations in the interview so that the interviews would be standardized as much as possible. These protocols can be viewed in the appendices. I purposely did not offer suggestions or hints, and allowed students to do what they wanted in order to solve the problems, which encouraged free problem solving. The principle of deciding what will be recorded and recording as much of it as possible was followed because I recorded everything in video and collected all papers used by subjects in working on tasks as well. The principle of training the clinicians and pilot-testing the interview was simple to keep because I had a pilot test and was the only clinician in the data collection. The last two principles of designing to be alert to new or unforeseen possibilities and compromising when appropriate where followed because I structured my interviews to allow for individual differences in knowledge and ability in proof creation. While I tried to use many of the same problems with each subject, each subject had different strengths and weaknesses.

Two of Goldin's principles do not apply, namely maximizing interaction with the external learning environment and choosing tasks that embody rich representational structures. These did not deal with what my study was investigating and what I wanted to observe. I was not interested in whether subjects used representations and how they affected proof creation or how interaction with the learning environment affected proof. These principles were not useful to me in structuring my study but the other principles were very helpful.

Research Methods

I first conducted a pilot study with two volunteers, having them complete my questionnaire and completing the first interview with them, so that I could practice my questioning and my videoing skills. The pilot study was a success. It indicated I had some interesting problems on the questionnaire as I got very different answers from my two capable volunteers and I gained some interview experience which was very helpful in knowing more of what I would be doing in my data collection.

When it was time to begin the study, I had twenty-seven students from Brigham Young University's mathematics and mathematics education junior and senior undergraduates recommended to me by faculty in these departments. Students were recommended based on three criteria. These criteria were being a junior or senior undergraduate majoring in mathematics or mathematics education at Brigham Young University, being a capable prover, and being able to communicate clearly about proof. I chose to select subjects from junior and senior undergraduates because they have some experience with proof creation, having taken several upper level proof based courses by this time in their schooling, but they are still relatively new at the process overall. Thus, the process of proof creation is less automatic for them and they were able to communicate about their thoughts and processes more clearly than someone who has been creating proof for many years (e.g. a professor of mathematics).

I asked for volunteers through e-mail. Seven volunteered to participate, but two withdrew early and two withdrew later due to time constraints on their schedule, leaving three subjects total. I chose to work with one student at a time. This was so that I could prepare and focus on each interview individually, and could also test hypotheses developed with one student in my work with the next.

My three subjects were Carl, Candace and Matt (all names are pseudonyms). I interviewed them in that order. Carl was a senior double majoring in mathematics and statistics. Carl was in the beginning of his basic analysis and abstract algebra courses when our interviews occurred. Candace was a junior majoring in mathematics education. She was also a tutor in the math lab. Matt was a senior majoring in mathematics and minoring in philosophy. He used some of his logic skills from philosophy in his proofs as well.

To begin, I asked each student to fill out a questionnaire that contained four proof problems (see Appendix A). I chose these proof problems for several reasons. They cover a broad range of mathematical topics, so that I investigated my subjects' proving abilities overall in mathematics and not just in a specific branch of mathematics. In addition, they are varied in possible approach strategies. Thus, a student cannot take the same approach on each problem, and the approaches taken on each problem should vary from student to student. This was so many different thought processes and strategies that students use in proof creation would occur. I intentionally avoided problems that would suggest proof by induction, a special type of proof that does not typically involve semantic proof creation as it is taught as a syntactical or procedural proof creation.

The questionnaire asked the student to provide their best proofs. It also asked them to not erase anything that they wrote down, but merely cross it out so that it was still legible. After they returned this questionnaire, we scheduled the first interview to occur within a week of them returning the questionnaire. Before that interview I reviewed their proofs. In reviewing the proofs ahead of time, I was able to know what the student had done to create the proof as much as possible so that the first interview could focus on their processes and strategies instead of why the proof was correct. In the first interview, I asked them to verbally explain what they had written and how they thought about the problem, thoroughly explaining their proof and what they tried and did while working on the proof. The interview was video recorded. I asked questions to prompt them to explain their thoughts more.

This interview provided useful information on several fronts. It allowed me to verify that they were capable with proof and that they were good at communicating about their proofs and allowed me to start to understand how they approached and worked on proof. All three volunteers were capable of discussing their thoughts clearly, so a second interview was scheduled to take place a week or two after the first, based on their schedules. This task-based interview on proof asked the student to create proofs in front of the interviewer while being video-recorded and with occasional prompts from the interviewer to explain their thoughts verbally. At the end of this second interview, they were also questioned about their beliefs about mathematics and proof. These responses were to aid in understanding and interpreting the data, as a kind of member check. These questions about belief were added in consideration of Recio and Godino's (2001) paper on the possible meanings of proof to individuals. I wanted to be able to know if the students' beliefs were strongly affecting their proof creation.

After the process of interviewing Carl was complete, some analysis was performed. I transcribed the interviews and did an initial read through and marked possible codes. I then did a second read through the interviews to see if there were any other codes that appeared to me at that time. When I felt I had identified all of the codes I could at that time, Candace was asked to go through the same interview process. After Candace, Matt went through the process. While designing the study I had felt that four or five subjects would have been desirable, but there were no more volunteers and I did have a plentiful amount of data available. All three subjects were capable provers and good communicators. They each had differences and similarities, so there were plenty of rich data on which to perform analysis.

I chose task-based interviews to help me study students' thought processes. Because we do not have mind readers and interpreters, the next best thing is students who are capable of talking about their thoughts and ideas while working on tasks. Written questionnaires alone would not have generated enough information about student thought processes. Task-based interviews allowed me to be present, observing and recording student thought processes while they created proof. The questionnaires allowed me to have subjects reflect back on their proof creation, and then I was able to compare that with their proof creation in the task-based interview. This was a validity check, as it allowed me to check the inferences of what they say they do with what they did do in the task-based interview. I used the responses to these questions more than I had anticipated because they helped me understand how mathematics was viewed by my subjects and how they viewed what they were doing.

My list of possible tasks is in appendix D, and was created from problems used in studies from my literature review and bibliography, textbook problems from introduction to proof courses and with the aid of professors at Brigham Young University. They were chosen as tasks that the students were capable of solving and that they were good discussion tasks, as determined by myself and some professors that I consulted. Good discussion tasks were determined by considering whether the proof had multiple approach strategies that would cause the students to think carefully about how they would create a proof. I chose which tasks each student would complete after the first interview. I reviewed what had occurred in the first interview, and what appeared to be the strengths and weaknesses of that person's proving abilities and then tested this with the proofs I chose to ask them to complete in the second interview. I also chose tasks from a broad range of mathematics in the second interview to allow for different proof strategies on different types of problems to be evident. Because the proofs in the first interview were created before the interview, it also allowed me to get a different view of their proof creation from this second interview where the proofs were created during the interview. As an example of how I used the first interview to plan the second, during the first interview, Carl said he had struggled with the geometric proof and that he was not as capable at creating these types of proof. I chose to present him with two different geometric proofs at the beginning of interview two, the first being to show that the bisectors of supplementary angles were always complementary. This proof was completed very quickly by Carl and did not cause him to think deeply. He said "Seems like there's not a lot to prove. By definition, the bisector of A will have half measure A and bisector of B will have have measure of B, add together, they will be half the sum. There sum is 180, so it will be 90".

The interviews were video-recorded, and transcribed for analysis. I used Galbraith's (1981) research in structuring my question protocols. Galbraith outlined his questioning protocols ahead of time so that the interviews were standardized as much as possible and easily compared. My questions were written in a similar fashion, as an outline of questioning protocols so that the interviews would be standardized as much as possible. This was done so that I would ask each subject similar questions about each proof. I was concerned that otherwise, I might forget to ask about difficulties on one subjects' proofs and not know what that individual viewed as the difficulties in their proofs. By standardizing my interviews, I ensured that such a thing did not occur. These question protocols are located in appendices B and C, for interviews 1 and 2 respectively. The questions are general, so that they can be used for all types of proof problems. I wrote these questions to help prompt the students to talk about their proof creation when they were having trouble expressing themselves, and also to make sure that many aspects of proof creation were discussed on each proof that was created in the interviews. The belief questions at the end of the second interview were added to aid in validity checks, allowing me to determine the subjects' beliefs about proof and aid in determining that my interpretation of their proof creation tools was correct.

Analysis

Initial analysis was done by reading the transcripts and creating open codes in the style of Strauss and Corbin's (1990) *Basics of Qualitative Research*. After open coding, I had over thirty possible codes. To begin the axial coding, I started reviewing my codes and refining them. I worked to create solid definitions of each code by examining the text clips and determining what had stood out to me and made me think they had commonalities. Some of the codes were too general, and had to be clarified and made more specific. Some of them were too specific and not supported by enough data.

From this initial axial coding, I developed nine major codes. By repeating the refining process for selective coding, five of these codes collapsed into other codes for four major codes which corresponded closely to Schoenfeld's (1985) four categories. Because of the close correspondence, I chose to use his category names. The four categories were resources, heuristics, control and belief systems. The way these categories were demonstrated in my data will be discussed in detail in the next chapter.

Chapter 4: Results

As mentioned in chapter 3, I chose Schoenfeld's (1985) categories to encompass my codes and findings. I believe this to be the clearest and most accurate interpretation of my data. I have not interpreted my data to fit the categories of Schoenfeld, but have found my data fits the observations he made concerning mathematical problem solving and can be applied to proof creation. This is true because I coded the data before realizing how well it fit his categories. Additionally, I am able to apply Schoenfeld's categories without losing my lens of how I defined proof and success in proof creation. This is because Schoenfeld was originally dealing with problem solving more than proof, and he does not explicitly define the things that I have defined.

Schoenfeld's (1985) categories of Resources, Heuristics, Control & Beliefs encompass my findings which answer my research question. The things that make a student a successful creator of proof are in the types of resources, heuristics and control they have available to them, and the beliefs that they carry about mathematics and mathematical proof. The specific types of each category that help a student to be successful are discussed in detail in this chapter.

Resources

Resources is a broad cognitive category that includes different types of knowledge available to subjects in their proof creation. It includes informal and intuitive knowledge regarding the domain of the proof, facts and definitions that are relevant to the proof, algorithmic and "routine" non-algorithmic procedures that relate to the proof and understandings about the rules of the domain of the proof (Schoenfeld 1985). Thus resources are the facts, definitions, procedures, concepts and knowledge available to an individual that they are consciously able to use in proof creation.

The resources used by my subjects were of several different types. For example, all of my subjects used *definitions* as resources in their proof creations. When explaining his set theory proof from the questionnaire, Carl said "Just the proposition, then you just have to basically change the set notation into the notation of propositions so that the *ands* and *ors* can be… so that you can distribute the *or* across the *and* and then switch it back into sets." Carl clearly considered this a proof where the definition was easily used to create the proof. When discussing her proving abilities, Candace remarked "Um, if I'm familiar with the definitions that I need, um… then I usually feel like I can [create a proof]." When discussing the last proof on the questionnaire, Matt remarked "So I got out those definitions. That seems to be the running theme here, is definitions that I looked up first."

Specific mathematical knowledge was also a resource often used by my subjects. When working on his proof of the divisibility by nines rule, Matt used modular arithmetic. He saw the different powers of 10 as all being equivalent to 1 mod 9, and that played a powerful role in his proof. "And then, um, I realized that each one of those 10-to-the-whatevers would be congruent to 1 mod 9. And so, if we multiply them, it would, they just disappear. And so, then the digits themselves would have to be congruent to 0 mod 9, themselves." The concepts and definitions of modular arithmetic are an important resource available to Matt in this proof creation.

The definitions, facts, resources and specific mathematical knowledge available to my subjects determined much about their success in creating a proof. Without appropriate definitions or knowledge, and an understanding of how to use these in proof creation, my subjects would not have been successful at proof creation. This was clearly demonstrated when I proposed the following task to Carl: Prove that the intersection of any finite collection of open sets is open. . Carl was able to prove the statement for sets on a number line, which he was familiar with. But he did not have a definition for a more open set available to him and the proof was unapproachable as a result. Matt approached the same task, and as he had taken a class involving the subject material, had the definitions available to him and was easily able to structure a proof.

Heuristics

Heuristics is a cognitive category involving the strategies and techniques used to make progress on a proof problem. It includes rules of thumb, usage of notation, representations, pulling from previous similar proofs, testing and verifying statements, and reformulating proof problems into something more solvable (among others). Heuristics and resources are continually interacting with one another. Without the appropriate resources, heuristics cannot necessarily help someone in proof creation.

An example of using the heuristic strategy of *considering similar proofs* is found with Carl. He explained that considering similar proofs he had already done was helpful for him because his brain had already built connections between concepts. This was clearly evidenced in the interviews when Carl uses his proof of the nine’s divisibility rule to begin the structure of his eleven’s divisibility rule proof. In working on his proof of the nines divisibility rule it took him a while to develop how to represent what happened with a generic number of digits in the number. He struggled with what notation would be effective and pondered how to make sure all cases were considered. His starting point was induction, because he felt he needed a general form for the number and he could add 9 to 9*k*. Then he had to consider how 9*k* was the sum of the digits and consider how adding 9 would affect those digits:

> The last digit and then in the ones place and then the tens will increase by 1 unless its 9 and it'll go to 0 and then that'll keep going for as many consecutive 9's as you have and then you'll add one at the end. So you're subtracting one on one end and adding one on the other end and any 9's in between are 0'd out so your new sum is divisible by nine.

However, in the eleven's divisibility proof, he quickly remembered what he had done in the proof of nine's divisibility and used it to help him structure his proof, explaining to the interviewer the similarities between the proofs.

Carl also provides an example of the limitations placed on heuristics by resources. When working on an analysis proof about the intersection of an infinite number of open sets, Carl was unable to go beyond a finite group of one-dimensional sets. He was aware that open sets could exist in more dimensions, but did not have any other knowledge about them. Additionally, he felt he had shown it was true for any finite number of one-dimensional open sets but was unable to extend it to an infinite number. Characteristics of infinite sets were not part of his available resources. His use of heuristics allowed him to break the proof down into a simpler, similar proof about a collection of finite one-dimensional open sets, which he was able to prove. However, his lack of resources prevented him from extending this proof toward the originally posed proof.

The issue of *notation* arose with my subjects, in several instances. When working on the quadrilateral inside a square problem, Matt began labeling segments with various letters. What the letters would stand for was not immediately obvious so I questioned him about it. When asked why he used labels, he replied that it was a kind of shorthand: “I think of them more as just a shorthand way of writing the whole description, other than the variable.” Matt had chosen variables to represent things in the picture. He implicitly knew what they all meant though he had not written down the labels for the variables anywhere. When working on the divisibility by nines proof on the questionnaire, Carl explained that he had encountered issues with how to represent the generic number being divided by 9. He said, “it’s easy to explain this verbally, but then trying to write it down so that its clear and anybody can follow it is a little bit different, you have to write, you have to think of new symbols and things that will hold true throughout.” Ultimately, Carl chose to represent the generic number $9k$ as d_1 thru d_n, while those of $9k + 9$ were denoted as D_1 thru D_n . He then examined cases of what the sums of d_k and D_k would be to show that his induction proof was true. Candace took a similar induction approach to this problem, and encountered difficulty with how to represent the sum of the digits. She chose similar notation, but reasoned about the cases differently and used summation notation. In contrast, Matt did not have the notational issues that plagued Candace and Carl on this problem, as he chose to use modular arithmetic and his notation was all based on learned conventions.

The use of notation can be a hindrance in proof creation, when students are faced with a new situation and are unsure about what notation to use. This is demonstrated by Carl and Candace’s struggles in representing the sum of digits. However, notation, when it is known and used with confidence, can be a powerful tool to aid a student explaining their proof and in arriving at the solution. Matt’s abilities with modular notation and in creating his own shorthand in the square and quadrilateral problem were aids to him in arriving at the solutions quicker. In the square and quadrilateral problem, he was able to create simple equations with the notation that allowed him to prove the statement relatively quickly. In the divisibility by nines problem, modular notation allowed him to skip the difficulty of generically representing all the digits of a number and then adding nine to the number and deal with the messy abstract changes and instead look at the whole number mod 9and then each digit's place value mod 9. In this manner, he quickly saw the number *k* which was equivalent to 0 mod 9 could also be written as a_0*10^0 + a_1*10^1 +. . . +a_n*10^n . where each a_i was a digit of *k*. He then took the mod 9 of this entire expression and considered each piece mod 9. Since each 10^i was equal to 1 mod 9 however, they could be collected and written as 1 mod 9 * (a_1 + a_2 + . . . + a_n) mod 9 = *k* = 0 mod 9. When questioned about the rules of modular arithmetic, he answered he had used the rules appropriately and understood them from use.

Making sketches or drawings of figures, sets, functions, and so forth is a useful heuristic strategy in the context of creating a proof. When working on a proof that involves geometrical figures, making sketches of the figure and properties held by that figure can help determine whether a statement is true or not, and also what path to take to write the proof. When working on the trapezoidal proof in the questionnaire, Matt made many sketches of isosceles trapezoids and their properties before even starting the proof. He used the sketches to determine if the statement was true or false and how he would prove it once this was known. In his sketches he examined triangles, which he explained were trapezoids with the top base equal to zero, and isosceles trapezoids of varying lengths in the top and bottom sides. He made calculations about angles and congruencies in the trapezoids, and these considerations made it into his proof which included some sketching of trapezoids. When working on the variation of the triangle inequality, Candace recognized that it was similar to the triangle inequality and decided to use vectors to represent what the statement said. She spent a few minutes making sure her drawing represented the statement correctly and used it to reason about the truth of the statement. This only took her so far in the proof though because she could not see how it would handle negative numbers. Her sketch was of two vectors, *a* and *b*, originating from the same point, in different directions, and then the vector that could be drawn between their endpoints was labeled *a* – *b*. She could not see how this could represent a negative distance, and moved on to using a number line. The vector representation was only as useful to her as her understanding of vectors and magnitudes allowed it to be. While it did help her in the proof, her resource limitations prevented it from helping her more.

Another heuristic strategy is to *reformulate* a proof problem into other similar problems. This allows the prover to examine the problem from different angles to help find a solution. When working on the isosceles trapezoid proof on the questionnaire, (By connecting the midpoints of an isosceles trapezoid, you obtain a rhombus) Matt used drawings to help him think about the proof in a simpler form. He said:

> I was thinking that because you could make this top one as small as you wanted, you could make it arbitrarily small. And so I thought, I could make it almost just like a triangle … 'Cuz if it was true, then it would have to be true for, like, an epsilon like, up there, really small. And so I thought, I mean, it might not necessarily prove it, but it would be, if it was false for a triangle or true

> for a triangle, then I would think that it would probably be the same for the other one.

By thinking about the top base of the trapezoid as arbitrarily small, Matt reduced the proof question to a similar question about triangles, namely, by connecting the midpoints of an isosceles triangle and the top vertex, do you obtain a rhombus? This helped him think about the proof problem and determine if the original statement was true.

These Heuristic strategies of considering similar proofs, reformulating proofs into different problems, notational choices and using sketches, drawings or graphs to aid in proof creation were essential to the success of the proof creators. Without these techniques, they would not have known where to start or what to do when they were stuck. Having various strategies for considering and thinking about the proof and making it more approachable for them aided them in their proof creation. This was demonstrated by Moore's 1994 study on student difficulties in proof. They were not able to use concept images or generate and use examples. They also had difficulty with mathematical notation and language. These deficits relate directly to the skills my successful proof creators had in their arsenal.

Control

Control is a category that includes cognitive and meta-cognitive acts by the individual. These acts are about implementing resources and heuristics in proof creation. Planning how to create a proof, monitoring progress in the proof, making decisions about what to do in the proof, thinking about the proof, are all typical processes included in this category.

During the first interview, which was about the proofs created based on the questionnaire, Matt indicated that the length of his written proofs was an indicator of its difficulty level to him. He said that more could be assumed in the shorter proofs because they were trivial and basic. Longer proofs were those were more was needed to be shown. He said "you can make a fairly simple proof extremely long and detailed without really having much, um, it really wouldn't be enlightening at all, it would just be really detailed and that's it." Matt's acts of determining what was assumed in a proof and what was not assumed are acts of control. He determined what resources and heuristics he would need based on what he would assume and require to be proved as background to the proof problems. As an example, in his work on the divisibility by nines proof discussed previously, his written proof does not prove that $10^i = 1 \bmod 9$. He states that as a fact. When questioned, he explained it as being an obvious step that was unnecessary to prove individually.

Thinking about a proof during proof creation as a means of control in the implementation of resources and heuristics was used often by my subjects. Candace indicated that she thought about the divisibility by nines proof in the back of her mind for more than a day, which finally resulted in her progressing in her proof. She said "this was the one that I would think about even while I was doing other things." It is considered a conscious control decision, as she also states that "I think this was the last one that actually finished writing up but probably the first one that I wrote something down about." She chose to stop work on it and think about how she would represent the place value for a while before finishing the proof. After one proof that Matt worked on in the second interview, he told me "It took me forever. Um… I think it's 'cuz, um, I think the main thing was, I just didn't stop to gather my thoughts because I was talking so much. … That's a lot of it. 'Cuz, when I get stuck to a point like that, usually I, I just need to stop and say, Okay, this is what I'm gonna do." During interviews, all my subjects' decision to spend some time pondering of the proof and planning what to do was evidenced by their tendency to trail off from talking and stare off into space or at the paper intently for a period of time, though several times it was also explicitly stated that they needed to spend some time thinking. This is a control mechanism they had that allowed them to step back and consider what they had done and what still needed to be done and how they should accomplish their goals. It is a conscious decision to step back and ponder what their next step should be.

When planning how to create a proof, individuals, based on their available heuristics and resources and the type of proof problem they are encountering, will employ different strategies. When discussing the first proof on the questionnaire, which was a set theory proof, Candace indicated that her initial thoughts were that it reminded her of her abstract algebra class and a proof she had helped another individual with in her job at the math lab. She said that she remembered that "we just need to show subsets." This was where she got her idea about how she was going to prove it. However, she couldn't remember what exactly she needed to show, so she decided to show all combinations. (This refers to x being in each subset and all combinations of subsets). She is using the heuristic of considering similar proofs in helping her plan her proof. She is conscious of her choice and so this is evidence of control. Candace also stated the other way to prove it would be with Venn diagrams, but that those don't count as proofs, "because they're almost … an example." She proceeded to work out all the cases she could see. Her proof contained seven individual cases (which was not all combinations) and their verifications. This is an example of her employing her cognitive control over implementing her available resources and heuristics. She chose which heuristic strategy she would use, which was affected by some of her beliefs. Her choice is where her control is evidenced. She felt that a proof of all combinations was the best available strategy for creating a proof available to her.

In contrast, Carl's first thoughts when looking at the same proof problem were "Oh, I did that in [my introduction to proofs course]. As well as it was equivalent to changing notation into *and* and *or* symbols". In his words: "you just have to basically change the set notation into the notation of propositions so that the *ands* and *ors* can be… so that you can distribute the *or* across the *and* and then switch it back into sets." Because his resources (i.e. the definitions he remembered) here were better than Candace's, and he was confident in his notational heuristics, the planning of how to create the proof was a short period, as it seemed obvious to him what needed to be done and simple for him to do.

Monitoring progress during a proof creation is a control method that helps an individual make sure they are making progress towards a viable proof and not pursuing a dead end. Such monitoring may involve a conscious decision to step back and examine what has been done and if it is helpful, and it may be subconscious, just seeming natural to re-examine the path and determine if continuing on the same path is the best course. When working on the "divisibility by 120" proof problem in the second interview, Matt's initial strategy was to factor the polynomial n^5-5n^3+4n. After factoring it completely, however, he declared that it seemed like the statement he was proving in fact wasn't going to be true, that he should have checked some examples first, and he was going to do so now. He proceeded to check several examples, determined that it did seem like it was true and then indicated he needed to think about why for a few minutes. After reflecting on his factored polynomial, he proceeded to use modular arithmetic to explain why it was a true statement. In this proof creation, we see Matt exercising cognitive control of his proof creation process – based on his beliefs about the truth of the problem.

My subjects' control abilities, both cognitive and meta-cognitive, were vital to their successful proving abilities. The abilities to determine what can be assumed and what can not, to monitor progress in the proof creation and to plan how the proof will be created were all essential to their success. Galbraith (1981) saw these skills as necessary in his study. Moore (1994) also reported that one of the main difficulties student had with creating proof was where to start. My subjects could plan how they were going to create the proof and what they were allowed to assume in the proof creation.

Beliefs

Beliefs is an affective category that includes a person's beliefs about mathematics and proof, including their proof schemas in the sense that Harel and Sowder (1998) used the term. It also includes beliefs about a person's own mathematical capabilities. Beliefs determine much about a person's behavior in proof creation. When I was designing my research, I chose to include questions about beliefs concerning mathematics in the second interview to aid in the accurate interpretation of subjects' proof heuristics. However, the answers to these questions are not the only indicators of subjects' beliefs about mathematics and proof. Some commentary in the proof creation process also gave insight about subjects' beliefs.

When Candace was discussing the set proof in the questionnaire, she indicated that she had created Venn diagrams to help her in understanding the problem. However, she said that the diagrams did not count as a proof, but just an example of one way of looking at the sets. She explains it as “I have to think of them more as an example rather than proof.” So she dropped that line of reasoning and chose to make a (faulty) list of all the possible set combinations and check each individually. This is an example of her beliefs affecting her control acts and consequently the resources she used in this proof creation.

On the bijection proof on the questionnaire, Candace wanted to use matrices to demonstrate the injective and surjective properties. She believed that it should be a valid method of proof, but did not have the experience to be certain. In discussing the proof, she said:

> Where things are bijections with algebra, I’ve never done it using matrices. And I didn’t know if bijection meant the same thing. I, I knew that if I could do it with a matrix, I could do it algebraically, but I… I really, I think that you can do it using linear algebra, that that is valid. I just, I don’t feel confident doing it. Didn’t have the experience to do that… No, I didn’t have experience with linear algebra and so I, I think I doubted using that.

Thus her beliefs about her capabilities and available resources affected the type of proof that she created.

My subjects also discussed the idea of one proof being more elegant or better than another proof. They had beliefs about whether their proof was a “good” proof or an awkwardly constructed one. When talking about it, Matt said:

> Um, I think I could make a more… I don't know what a better proof means, but uh, probably a prettier proof or a more, I don't even know what that means, either. It's kind of subjective. But I, I think I could make a proof that I think would be a lot more impressed with, maybe.

I also found that what it meant to create a proof varied among my subjects. Their beliefs about what proof was were evidenced in their proof creations. Candace told me: "And so…and I guess that's what proof is, …Whether or not you can connect definitions to the other… to make statements equal to one another." This was evidenced in her proof creations, as she relied heavily on her definitions as a resource in structuring most of her proofs. She placed definitions in a main role in all of her proofs on the questionnaire and in the second interview.

It was apparent that belief played a significant role in the proof creation process. For example, Candace explained why she wasn't sure if her proof was valid. "But I don't, I just don't have experience to back it, I haven't done it before and seen that it works, so that's kind of why I doubt it." Carl explained why he felt his informal proof was valid. "I think even if somebody…that has more experience with this kind of thing, even if you can see holes in the proof or places where it could be better, you understood the idea that I was saying and could tell that I'd covered the possibilities." Matt asserted that he understood the concepts in the proof. " 'Cuz I know how to prove this! I'm just trying to think of how to explain this a little bit, and I'm just floundering here."

Positive beliefs' about mathematics and mathematical proof have a strong influence on how students go about proof creation. My subjects beliefs' about what were valid methods of proof creation, about what made a proof valid and complete and what made a proof more elegant or better than another proof were more in line with the beliefs held by mathematicians. This is shown in the papers discussed in chapter two by Recio & Godino, Harel & Sowder and also DeVilliers.

Summary

Table 2 below provides a summary of the specific behaviors of my students in creating proof, discussed above and categorized according to Schoenfeld's problem-solving framework. This summary provides a compact answer to my research questions. In the next chapter, I discuss the contributions of my study, and provide some suggestions for future research and implications for instruction.

Table 2

Results of Study Organized by Aspects of Problem Solving Discussed in Proof Literature

Aspect of Problem Solving	Specific Use by Students in Study
Beliefs	What constitutes valid methods of creating proof
	What constitutes an elegant proof or makes a proof better than another
	How to know when a proof is valid and complete
Resources	Facts and definitions related to the proof
	Specific mathematical knowledge in the domain of the proof
Heuristics	Considering similar proofs as a guide to proof creation
	Reformulating a proof into similar but easier problems
	Choices of notation to aid the proof
	Making sketches, drawings, graphs
Control	Determining what can be assumed in a proof situation
	Monitoring progress towards the proof
	Planning how to create a proof

Chapter 5: Conclusion

Answer to the Research Question

The question that drove this research project was: What are some of the common traits in the thought processes of students capable of creating proof? This question arose from my personal interest and struggle with proof in mathematics and was inspired by some interesting research papers I read in mathematics education classes. These papers included Harel and Sowder (1998), Moore (1994), Galbraith (1981) and Recio and Godino (2001). My review of these papers and others led me to conclude that no one had addressed what was happening in successful proof creation at the university level. Much of the research on proof had classified students according to the final product. People had investigated difficulties that students were having with proof, but few had considered or studied the process of what was happening when proof was being created successfully.

I carefully considered and defined what a successful proof creation would be in my research and began with a small pilot study. I refined my data collection process from the results of my pilot study and collected my data. Galbraith's (1981) research influenced my structure, as did Goldin's (2000) writings about how to make task based research scientific and rigorous. I used interview protocols of questions to keep the questions asked of each subject similar and had think-aloud protocols to prompt and aid subjects in sharing their thoughts during the proof creation process. Tasks from the interview were compiled from books and professors and selected according to subject abilities, while keeping task selection similar overall. I video-taped the interviews and collected all papers that were written on by subjects. After the interviews were transcribed, I analyzed my data, following Strauss and Corbin's (1990) guidelines. After analyzing the data and refining my categories, my final categories closely aligned with

Schoenfeld's (1985) categories. Notably, I found that my results could be clearly organized using the categories already created by Schoenfeld to describe successful problem solving. Proof can be considered a type of problem solving, and the process of proof creation is similar to the process of problem solving.

I found that there were many factors affecting successful provers as they participated in the proof creation process. These are that successful provers are good problems solvers. They have sufficient resources and heuristics available to them. They can use facts and definitions related to the proof problem and specific mathematical knowledge in the domain of the proof to their advantage. They are able to consider similar proofs and if necessary, reformulate the proof question into an easier one to answer. They are capable of making good notational choices to aid their proof creation, and make use of sketches, drawing and graphs to learn more about the proof problem they are solving. They have good cognitive and meta-cognitive control over the allocation of these resources and heuristics. They are able to determine what they are able to assume about a proof situation and what must be verified. They make plans for proof creation and monitor their progress on the proof so their resources and heuristics are used more effectively and efficiently. Their beliefs about mathematics and problem solving are positive. They have beliefs about what makes a proof valid and complete, and what methods are valid for proof creation. They have ideas and beliefs about what makes one proof more elegant or better than another proof.

Contributions of the Study

As noted, above, most studies of proof have taken a deficit viewpoint, attempting to explain unsuccessful attempts at proof. By focusing on successful provers, this study takes a step toward understanding the aspects of student understanding that allow for success. This is supported by coming to see writing proofs as a problem solving process, and arguing for bringing to bear the literature on problem solving to understanding proof production.

Suggestions for Further Research

There are a number of possible ways to extend this study. A larger scale project with more participants could find more cognitive and meta-cognitive tools that students use in the successful proof creation process and verify my results. More can be learned about the affective processes that have an effect on proof creation.

Additionally, the findings of this study could be incorporated into a study on teaching students to be successful proof creators. An investigation into which aspects of successful proof creation are key points in the development of successful proof creators would be useful in seeing which elements seem to change an average student into a successful student.

Implications for Instruction

The use of these results to aid in the improvement of instruction in mathematical proof needs to be considered, discussed and analyzed, implemented, researched and refined. These tools and processes can be an aid to teachers of proof, as they are things that could be taught explicitly to students as skills they need to work at developing. Teachers can also help develop the affective processes students need by cultivating a classroom atmosphere that breeds these types of beliefs and behaviors. It is important that teachers to be aware of and understand these proof tools that need developing among students to create capable provers, so they can try to aid this development, while means of implementation are researched further.

After completing this research, if I was called to teach an introduction to proofs class, there are several things I would do in my classroom. I would have students read about the different purposes of proof, perhaps as discussed by DeVilliers (2003). I would make sure they were exposed to some of the more “classic” proof problems from which proof strategies can be obtained, and explicitly discuss these strategies. I feel I have found many of these in my informal discussions with professors to collect proof problems for my interviews. An example is the proof that the square root of 2 is irrational, a classic proof for learning proof by contradiction. After allowing students to attempt to prove this fact, and discussing it and showing the traditional proof, I would lead a discussion about why this proves that the square root of 2 is irrational. Then I would have them attempt similar proofs. Similarly, we would discuss the importance of definitions, and the various ways definitions can be used in a proof. I would set a timer for 20 minutes, give them a proof problem they’ve never seen and have them work on it in groups. Then when the timer rang, we would first discuss the proof creation process, not the answer, and look at the different strategies attempted, the time allotted to the strategies, and how or why students decided to abandon (or not abandon) strategies for other ideas. I would discuss how reformulating a proof problem into smaller proof problems can aid in finding how to solve the larger proof problem. Currently, I believe that by explicitly discussing with students many of the attributes I saw in my research, I might make them more aware of the traits they need to be successful provers.

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Appendix A

Questionnaire

Name____
Phone number________________________________
Alternate means of contact______________________

Please place your name and contact information at the top of this paper. Fill out this questionnaire to the best of your ability and return it to my (K. Duff) mailbox in 260 TMCB. You will then be contacted to set up an interview time.

Prove or disprove the following statements. Please use additional sheets of paper which you attach to this sheet to show all your work. Do not erase, just place a line through or an X over anything that is not part of your final proof, so that it is still legible. Please write clearly.

1. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

2. If a whole number is divisible by nine, then the sum of its digits is divisible by nine.

3. If you connect the midpoints of the sides of an isosceles trapezoid, it always forms a rhombus.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (ax - by, bx + ay)$, where *a, b* are numbers with $a^2 + b^2 \neq 0$. Prove that *f* is a bijection.

Appendix B

Interview 1 (Follow up to Questionnaire)

The following is a list of questions that the interviewer will draw from to probe the subjects' proofs on the questionnaire. In the course of the interview, some questions may be used more than once, and some may not be used at all.

What were your initial thoughts about this problem?

Can you elaborate on that?

Why did you choose to use that approach?

Can you tell me more?

How did you accomplish that?

What were you considering at that point?

How was that helpful?

When were you convinced of the truthfulness (or falsity) of the statement you were asked to prove?

What role did that belief play in your thought process?

Did you become stuck at any point?

What did you do then?

How did you decide that you were done with the proof?

Do you believe that this is the best proof you could create?

Why?

Appendix C

Interview 2

Part 1: The following is a list of questions that the interviewer will draw from to probe the subjects' thoughts while working on the proofs presented. In the course of the interview, some questions may be used more than once, and some may not be used at all.

What are your initial thoughts about this problem?

Can you elaborate on that?

Why are you choosing to use that approach?

Can you tell me more?

How will you accomplish that?

What are you considering at that point?

How is that helpful?

Are you convinced of the truthfulness (or falsity) of the statement you have been asked to prove?

How is that influencing your thoughts?

Are you stuck?

What will you do to move forward in the proof?

Are you done with this proof?

How do you know?

Do you believe that this is the best proof you can create?

Why?

Part 2: The following is a list of questions that the interviewer will then use after the proof tasks to probe the students beliefs about proof. In the course of the interview, some questions may not be used.

What is mathematics?

What is proof?

What role do proof exercises play in your mathematics learning?

How do you feel about proofs in general?

Can you elaborate there on what you mean?

How do you feel about your ability to create proof?

Can you say more?

Do you work best alone or with others? Why?

Do you think you have a general approach to proof problems? What is it?

How do you think your mathematical abilities compare with other students in your major? Why do you feel that way?

How are your proving abilities compared with other students in your major? Why do you think that?

Appendix D

Bank of tasks for interviews (excluding tasks on questionnaire for first interview)

There are $180(n-2)$ degrees in an n-gon.

The bisectors of adjacent, supplementary angles form a right angle.

$n^3 - n$ is divisible by 6 for all $n \in \mathbb{Z}$

A point p is a limit point of a set M if and only if every neighborhood of p contains a point of M distinct from p.

The union of any finite collection of closed sets is closed.

The intersection of any finite collection of open sets is open.

You have a square of side length one. Inside this square you are going to place five points. Prove that at least two of the points will be within $\sqrt{2}$ or $\frac{1}{\sqrt{2}}$ of each other.

Prove that $\mathbb{Z} X \mathbb{Z}$ is not cyclic.

Prove that if A and B are subsets of G with $A \subseteq B$ then $C_G(B)$ is a subgroup of $C_G(A)$.

If n is not prime, then $\mathbb{Z}/n\mathbb{Z}$ is not a field.

Show that [0,1] is uncountable.

Let G be a group of order pq, where p and q are prime numbers. Prove that every proper subgroup of G is cyclic.

Let F be the splitting field of $x^4 - 2$ over $\mathbb{Q}$. Find, with proof, all intermediate fields between F and $\mathbb{Q}$.

Let r and s be algebraic over F. Prove that $r+s$ is algebraic over F.

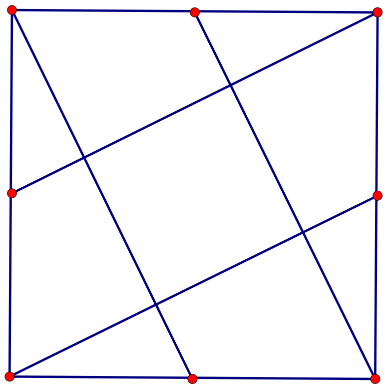
Prove $120/(n^5 - 5n^3 + 4n) \forall n \in \mathbb{Z}$

Suppose that gcd(a,b) = 1 and that a/n and b/n. Prove that ab/n.

Prove that $(2n)!/(2^n n!)$ is an odd number.

Prove that if 2^n-1 is prime, then n is prime.

Given a square with the midpoints connected in such a fashion as shown, determine, with proof the area of the middle quadrilateral.



Why is every palindrome number with an even number of digits divisible by 11? What can be said about palindromes with an odd number of digits?

Prove that $A \subseteq B \equiv \overline{A} \cup B = U$

Let n be an integer. Prove that $(n\mathbb{Z}, +, *)$ is a commutative ring.

Prove that $\forall a, b \in \mathbb{R}, |a| - |b| \leq |a - b|$

Prove that $1/2^n \to 0$ as $n \to \infty$

Define a binary operation * on Z by a*b=a+b+1. Prove that Z with * forms an abelian group.