2008-03-13

How Eighth-Grade Students Estimate with Fractions

Audrey Linford Hanks
Brigham Young University - Provo

HOW EIGHTH-GRADE STUDENTS ESTIMATE WITH FRACTIONS

by

Audrey Linford Hanks

A thesis submitted to the faculty of

Brigham Young University

In partial fulfillment of the requirements for the degree of

Master of Arts

Department of Mathematics Education

Brigham Young University

April 2008

BRIGHAM YOUNG UNIVERSITY

GRADUATE COMMITTEE APPROVAL

of a thesis submitted by

Audrey Linford Hanks

This thesis has been read by each member of the following graduate committee and by majority vote has been found satisfactory.

Date	Daniel K. Siebert, Chair
Date	Steven R. Williams
Date	Keith R. Leatham

BRIGHAM YOUNG UNIVERSITY

As chair of the candidate's graduate committee, I have read the thesis of Audrey Linford Hanks in its final form and have found that (1) its format, citations, and bibliographical style are consistent and acceptable and fulfill university and department style requirements; (2) its illustrative materials including figures, tables, and charts are in place; and (3) the final manuscript is satisfactory to the graduate committee and is ready for submission to the university library.

____
Date

____
Daniel K. Siebert
Chair, Graduate Committee

Accepted for the Department

____
Keith R. Leatham
Graduate Coordinator

Accepted for the College

____
Thomas W. Sederberg
Associate Dean, College of Physical and Mathematical Sciences

ABSTRACT

HOW EIGHTH-GRADE STUDENTS ESTIMATE WITH FRACTIONS

Audrey Linford Hanks

Department of Mathematics Education

Master of Arts

This study looked at what components are in student solutions to computational estimation problems involving fractions. Past computational estimation research has focused on strategies used for estimating with whole numbers and decimals while neglecting those used for fractions. An extensive literature review revealed one study specifically directed toward estimating with fractions (Hanson & Hogan, 2000) that researched adult estimation strategies and not children's strategies. Given the lack of research on estimation strategies that children use to estimate with fractions, this study used qualitative research methods to find which estimation components were in 10 eighth-grade students' solutions to estimation problems involving fractions. Analysis of this data differs from previous estimation studies in that it considers actions as the unit of analysis, providing a smaller grain size that reveals the components used in each estimation solution. The analysis revealed new estimation components as well as a new structure for categorizing the components. The new categories are whole number and

decimal estimation components, fraction estimation components, and components used with either fractions or whole numbers and decimals. The results from this study contribute to the field of mathematics education by identifying new components to consider when conducting future studies in computational estimation. The findings also suggest that future research on estimation should use a smaller unit of analysis than a solution response to a task, the typical unit of analysis in previous research. Additionally, these results contribute to mathematics teaching by suggesting that all components of an estimation solution be considered when teaching computational estimation, not just the overarching strategy.

ACKNOWLEDGMENTS

"…With men this is impossible; but with God all things are possible" (Matthew 19:26).

I would like to thank my family for their love, patience, support, and sacrifice as I completed this thesis. Without their many hours of service in my behalf, this accomplishment would still only be a dream. Additionally, I would like to thank my graduate committee for their help, guidance, and support. I would specifically like to thank my committee chair, Dan Siebert, for his seemingly endless patience with me. Without his belief in my abilities, I could not have pushed through the long hours necessary to complete this endeavor.

TABLE OF CONTENTS

LIST OF TABLES

Chapter 1: Introduction

Estimation is perhaps one of the most important mathematical processes, both inside and outside of schools. Children and adults alike use estimation on a daily basis when engaging in activities such as evaluating quantities, finding approximations, and considering the reasonableness of solutions presented to them. From finding the better buy at the grocery store to determining how long it will take to reach a destination, estimation permeates our lives (R. E. Reys, 1984). In fact, in everyday situations, estimation can often be more useful than precise calculations (Threadgill-Sowder, 1984).

The definition of estimation depends on which type of estimation that is being referred to. There are two types of estimation - measurement estimation and computational estimation. Measurement estimation is gauging approximately how many or how much of something there is or there needs to be (R. E. Reys, 1984). Computational estimation is approximating the answer to a numeric computation. Examples of measurement estimation include estimating how much cereal to pour in a bowl, how many jellybeans are in a jar, or the distance between two points. On the other hand, estimating 39.82×51.351 is an example of computational estimation. The distinction between the two is vital because measurement and computational estimation engage students in very different mental processes, leading to different strategies in solving problems.

I am investigating computational estimation in this study, instead of measurement estimation, because computational estimation has more practical application in the mathematics classroom in at least two ways. First, though often neglected, computational estimation comprises part of secondary mathematics curricula, while measurement estimation does not. Since it is in the curricula, it would be helpful to learn more about it so it can be taught more effectively. Second, aside from being part of the curriculum, computational estimation can be used every day in the classroom to check the appropriateness of an answer to a computation or to decide on a reasonable approximation to one when appropriate (Threadgill-Sowder, 1984), whereas measurement estimation, since it does not involve computations, cannot be used in this manner. Thus, a study of computational estimation has more value than a study of measurement estimation would for informing mathematics instruction in the classroom.

There are two kinds of research that have been done on computational estimation: Studies that researched strategies used for computational estimation and studies that examined the development of computational estimation skills. Studies on estimation strategies generally used middle school students, high school students, and adults as subjects (R. E. Reys, Rybolt, Bestgen, & Wyatt, 1982; Threadgill-Sowder, 1984). These studies also focused on students whose mathematical ability was either high or unknown. The tasks used for the interviews involved a balance of whole number and decimal problems almost to the complete exclusion of problems with fraction operations. Those who interviewed the students in the studies asked for explanations of their reasoning and then deduced from these explanations general strategies for computational estimation.

The second type of computational estimation research, studies that have focused on the development of computational estimation skills, surfaced after estimation strategies were identified. This kind of research primarily included studies that involved teaching estimation strategies to students and observing the development of the previously identified targeted skills (Case & Sowder, 1990; LeFevre, Greenham, &

Waheed, 1993; Sowder & Wheeler, 1989). The skills that were observed are those same skills and strategies that were found in the previous research on estimation strategies. Some of these studies suggested how to teach computational estimation skills and incorporated these teaching methods into their studies of how students develop these skills.

The aforementioned body of literature on computational estimation is problematic in two ways. First, the previous literature on computational estimation almost completely ignores estimation with fractions. Past research has focused instead on estimation with whole numbers and decimals. With the understanding of fractions being so different conceptually from the understanding of whole numbers and decimals, it seems highly likely, although we do not know, that estimation strategies for fractions would be different than, or modified versions of, those used for whole numbers or decimals. However, it is difficult to tell since virtually no research has specifically focused on fraction estimation. The second problem with previous literature is that the limited estimation research that actually does consider fractions (Hanson & Hogan, 2000) looks at adult estimation strategies rather than those of children. Although this study can give us some insight into fraction estimation strategies adults might use, we do not know if children would employ similar strategies. Since children and adults have different approaches to doing mathematics (Carpenter, Franke, & Levi, 2003; Mack, 1990), it seems reasonable that their strategies for estimating with fractions could also differ, even if in subtle ways.

Because not very much research has been conducted involving computational estimation in recent years, and considering the lack of attention to children's estimation strategies used with fractions, there is more to be learned from expanding our understanding of computational estimation strategies. By doing this, we can be better equipped as teachers and mathematics educators to assist students in increasing their abilities in the now poorly attended field of computational estimation with fractions. Thus, the purpose of this study is to examine the way students use computational estimation to estimate with fractions.

The remaining chapters in this thesis describe the study I conducted to answer this research question. In Chapter 2, I present a critical review of the literature to construct a theoretical framework for my study. In Chapter 3, I describe the methodology for the study. In Chapter 4, I present the fraction estimation components I found in the data. Finally, in Chapter 5, I conclude by presenting the implications and contributions of this study as well as its limitations and directions future research could take as a result.

Chapter 2: Literature Review

In this chapter I discuss some of the literature that influenced this study. First, I discuss some of the differences between whole number and fraction reasoning. I do this to highlight the need for computational estimation research that specifically examines children's estimation with fractions. Second, I discuss computational estimation strategies already identified in past research. Although I am focusing on computational estimation with fractions, it is essential to understand the strategies used for whole numbers and decimals, as well as the few fraction strategies already identified in prior literature, to see if they surface in my data of students estimating with fractions. Third, I discuss the methodologies presented in previous estimation literature. I argue for the potential benefits of adopting a smaller grain size to analyze students' estimation solutions.

Differences Between Whole Number and Fraction Reasoning

To understand why it is important to look at fraction estimation strategies, we must see that whole number reasoning differs greatly from fraction reasoning. The fundamental principles behind children's thinking in these two areas have many differences, but I am going to focus on those differences between identifying quantities in each type of number and differences in how each number type is grouped. Again, this is not an attempt to show all the differences between the two, but rather just to show that there is enough of a difference to warrant a study of estimation with fractions.

Identifying quantities with the two different number types involves two different sets of processes. With whole numbers, children name or count objects in a sequence corresponding with an action like pointing or keeping track of them in their mind, eventually arriving at the last object (not double-counting), and using the number name of that last object to quantify the set of objects (Verschaffel, Greer, & De Corte, 2007). In contrast, fractions are quantified by iterating parts of a whole or partitioning a whole into parts, and not merely by counting separate objects (Siebert & Gaskin, 2006). The similarity here is that both types of number can be iterated (for whole numbers 1 is iterated, for fractions a part of the whole is iterated). However, the biggest difference here is that with whole numbers, the whole is not partitioned to find a quantity.

Another big difference between reasoning with whole numbers and fractions is the way that they are grouped. Whole numbers and decimals are grouped by powers of 10, reflecting the Base 10 system that we use on a day-to-day basis. Fractions, however, are parts of a whole and when reasoning with fractions, they have no meaning unless they are related to the whole (Siebert & Gaskin, 2006); thus, they are grouped as parts of the whole. These are very distinct ways of grouping these two types of numbers.

Again, this inventory of the differences between reasoning with whole numbers and fractions is very limited. However, since there is such a difference in reasoning between whole numbers and fractions, it would seem that the strategies for estimation with whole numbers and fractions might be different. This is one of the important reasons for conducting this study.

Computational Estimation Strategies

In order to create a framework with which to view the data that I find in this study, I first will examine the estimation strategies for whole numbers and decimals as well as those few found for fractions. This helps in understanding the computational estimation strategies already found as well as ensures that I will be able to recognize whether the strategies students use to estimate with fractions fit within the previously identified strategies, are modifications of previously identified strategies, or if they are new strategies altogether.

The three primary categories of strategies found in research on whole number and decimal estimation are reformulation, translation, and compensation (see Table 1). Within each of these categories are more specific estimation strategies that are used by children as they engage in computational estimation. These strategies were found by R. E. Reys et al. (1982) in their study on strategies that good computational estimators use. Subsequent studies on computational estimation used the results of this study as a framework of processes and strategies or found similar results (LeFevre et al., 1993; B. J. Reys, R. E. Reys, & Penafiel, 1991; R. E. Reys, 1984; Sowder & Wheeler, 1989). I describe each of the strategies that fall into these categories and provide an example for each strategy. I will first discuss the whole number and decimal estimation strategies followed by the few fraction estimation strategies that have been found. Following this discussion I will show how this research is inadequate to describe students' estimation strategies for fractions.

Table 1

Whole Number and Decimal Computational Estimation Strategies (based on the findings in R. E. Reys et al., 1982)

Process and strategy	Definition of strategy
Reformulation	
Front-end use of numbers	Using traditional rounding or truncating to change a number to a nearby multiple of 5,

	10, 100, etc.
Substitution	Changing the form of a number or rounding a number to a compatible number that is close and makes it easier to operate in the problem
Translation	
Processes equivalent to but different than the original problem	Processing the numerical values in the problem in a different order than is stated, but is still equivalent mathematically
Changes operation	Changing the operation given in the problem to one that is equivalent (e.g. averaging)
Compensation	
Intermediate compensation	Adjusting the values of numbers in a problem based on the inaccuracies created by previous rounding
Final compensation	Adjusting the value of a solution based on the inaccuracies created by rounding in the problem

Reformulation is a process in which students use various methods to change the numbers involved in a computational estimation problem without changing the structure of the problem. There are two main types of reformulation strategies: front-end use of numbers and substitution. When employing the front-end use of numbers strategy, a student truncates the numbers or rounds them to the nearest 5, 10, 100, or other appropriate number for the problem. In substitution, students substitute the existing number with one that is close but makes the problem easier to deal with. This can be done in three different ways. First, in a problem such as 252 ÷ 3, a student might substitute 252 with 240 or 270 because 3 divides 240 and 270 easily. Second, finding a compatible number also includes the use of benchmarks – numbers that are more commonly used and with which the student is more familiar. The third way substitution is used is to change the form of a number, like a fraction to a decimal or a decimal to a percent.

The process of translation changes the structure of the problem to be more manageable. This category was not clearly described in the literature but seemed to consist of two different strategies: using processes that are different, but equivalent to, the problem and changing the operation. This first strategy consists of changing the order in which you might perform operations but still keeping the answer equivalent. An example of this would be if a student was asked to estimate a 15% tip from a bill at a restaurant and they chose to find 10% and then add half of that to find their answer. The second strategy, changing the operation, is used when changing a problem from one operation to another. For example, changing an additive problem to a multiplicative problem or vice versa. Averaging is an example of changing the operation in this way because students change an addition problem by averaging the numbers and then multiplying by how many numbers there are.

Compensation in a computational estimation problem is a process by which adjustments are made to either the numbers or operations in the problem to account for error introduced in a previously employed estimation strategy. Compensation can be done either intermediately or at the end of a problem (called final compensation). When the intermediate strategy is used, numbers or operations are adjusted to compensate for all the changes made to other numbers in the problem thus far. For example, if given three numbers to add that are all less than the numbers a student might normally round to, in compensation the student might round one number down, rather than up, to compensate for the fact that the other numbers are getting larger by rounding. Compensation can be done at the end as well. For example, a student may recognize that his or her estimated solution may be higher or lower than the actual answer because of rounding that occurred to numbers in the problem, and then adjust the answer up or down accordingly.

In addition to these estimation strategies found in general computational estimation literature, researchers have also identified a few estimation strategies that are specific to fractions (see Table 2). There were two fraction estimation strategies found in the study by Hanson and Hogan (2000), with one of those strategies found in the fraction literature by B. J. Reys, Kim, and Bay (1999), and one found in the study by Cramer, Post, and delMas (2002). These strategies are rounding using benchmarks, creating common denominators, and mentally modeling the fractions.

Table 2

Fraction Estimation Strategies from Previous Literature

Name of Strategy	Definition of Strategy
Rounding using benchmarks (Hanson & Hogan, 2000; B. J. Reys et al., 1999)	Changing a fraction to a fraction that is more frequently used and more familiar, such as 1, 1/2, 3/4, etc.
Creating common denominators (Hanson & Hogan, 2000)	Changing one or both fractions to have common denominators by some procedure
Mentally modeling the fraction (Cramer et al., 2002)	Mentally visualizing the fraction by comparing the numerator and denominator

Rounding the fractions to a benchmark consists of changing one fraction to a different fraction that is more frequently used and with which they have more familiarity. This was one method that adults used when presented with problems involving estimation with fractions. Benchmarks are numbers like 1, 1/2, 3/4, or other common fractions (B. J. Reys et al., 1999). There are several settings in which students could implement this strategy. One such setting might be for a student to recognize that 23/25 is close to 1 and then to operate using 1 instead of 23/25.

Another strategy used by adults in estimating with fractions was creating a common denominator. There were two different ways that this was done, but each ended with the same result, a common denominator. The first way is using the traditional method of finding a common denominator. The second way is ignoring small differences in the denominators and making them the same. In ignoring the small differences in the denominators, a student working with the denominators 19 and 21 might change one of the denominators to be the same as the other or else change both to have a denominator of 20, without changing the numerators.

The third fraction estimation strategy mentioned above is mentally modeling the fraction. This strategy was found in a study involving children, and consists of visualizing the fraction mentally by comparing the numerator with denominator. It is unclear from the literature whether this is an additive or multiplicative comparison or something else entirely. This strategy surfaced in a study comparing two methods of instruction. One of the methods of instruction included mentally modeling fractions by considering the relationship between the numerator and denominator. Only students who received this instruction used this strategy on the estimation problem given in an interview. Because students in the control group did not exhibit this strategy, it seems likely that this strategy is detectable only if students have been explicitly instructed in using this method.

There are two reasons that the above strategies may not be adequate for studying children's computational estimation with fractions. The first is that most of these strategies result from computational estimation involving whole numbers and decimals it is likely that whole number and decimal estimation strategies differ from strategies for fraction estimation. The second important problem to take into consideration is that of children's strategies versus adults' strategies. Although the studies done on whole number and decimal computational estimation have looked at children's strategies extensively, the study on fraction estimation by Hanson and Hogan (2000) only looked at strategies that adults used. The one estimation strategy found in fraction literature, mentally visualizing fractions, was one used by students, but it is difficult to tell how this strategy fits in with other fraction estimation strategies. Furthermore, it is unclear if it can be generalized beyond the specific instructional setting in which the strategy was exhibited.

Methodologies

In past estimation research, it is unclear exactly what unit of analysis was used by the researchers, because the unit of analysis was rarely clearly stated. However, the nature of the results suggest that the unit of analysis was the complete solution to an estimation problem. For example, in R. E. Reys et al. (1982), entire solutions were offered in the results sections as examples of estimation strategies. This suggests that the authors were using an entire solution as the unit of analysis. Another specific case of this is when Hanson and Hogan (2000) indicated that one of the difficulties they had in coding came because students changed strategies in the middle of the problem. Their difficulty in classifying this solution suggests that they were also using the entire solution as the unit of analysis. All in all, I was unable to find any evidence in the computational estimation literature that researchers had used a unit of analysis other than the entire solution as they analyzed their data.

One of the difficulties presented by using the entire solution as the unit of analysis has already been hinted at by Hanson and Hogan (2000) above, and that is that sometimes it seems as if an entire solution involves more than one estimation strategy. For example, in a paper by B. J. Reys et al. (1991), an explanation was categorized as a front-end strategy, but the authors did not address the compensation that occurred in the same explanation. In that same article, a solution contained four estimation strategies but is only noted as having one. The student responded to the problem of finding 30% of 54,215 by “changing 30% to 1/3 then dividing 54,000 by 3” (p.370). This explanation was identified as translation (although the specific strategy within the category of translation was not noted) but it appears as though the student was using substitution (changing 30% to 1/3), front-end use of numbers (rounding or truncating 54,215 to 54,000), and translation (dividing by 3 instead of multiplying by 1/3) in this explanation. Because estimation solutions can contain more than one estimation strategy, a smaller grain size might be beneficial in analyzing students’ responses to estimation problems.

A smaller grain size might also be useful in identifying aspects of a solution that are important to estimation but previously unidentified as a part of estimating. For example, the student in the solution above would not have been able to arrive at an answer if he or she had not divided by 3. Although division by 3 is typically seen as computation rather than estimation, in this solution it is an essential component of successful estimation. Past research has attended only to components of the solutions which appear unique to estimation, ignoring other aspects of the solution that are also a common part of solutions to tasks not involving estimation. While this focus on unique aspects of estimating has certainly uncovered many strategies that play an integral part in estimating, it may have exaggerated the importance of these strategies and neglected other strategies that are prevalent in the solutions to a variety of quantitative tasks and problem situations. Without using a smaller grain size to identify all of the different parts of the solution, it is impossible to know which strategies, whether they be unique to estimation or not, play a significant role in estimating.

When using a smaller grain size to analyze estimation solution, it does not make sense to call the subparts strategies, because in the past, strategies have been equated with entire problem solutions. Thus, I will call subparts of solutions components rather than strategies, whether or not those subparts are unique to estimation solutions.

No one else has looked at all components a student uses in computational estimation as the grain size for analysis. We do not know how useful this grain size might be since it has not been tried before. This grain size will not only identify clear estimation components for fraction estimation, but also possibly uncover the other important parts or components of estimation solutions. Thus, in this study I will address the following research question: What components are in student solutions to computational estimation problems involving fractions?

Chapter 3: Methods

In this chapter I discuss the design of the qualitative study that I conducted to help answer my research question. This consists of a description of the subjects and settings, tasks, collected data, and data analysis.

Subjects and Settings

The students chosen for this study were selected from two schools in suburban areas in the Western United States. I taught mathematics at both schools and at each school worked extensively with the other mathematics teachers in the classroom, at department and faculty meetings, and at in-service training. As a result, I became very familiar with the ways the teachers at the two schools view teaching and mathematics. I selected these two schools, which I will call Dover Middle School and Sutherland Junior High, for my study mainly because the teachers of the two schools have very different philosophies and orientations towards teaching and mathematics, not particularly because of the convenience. These differences are evidenced both by the teachers' perceived role in the classroom and the standards for acceptable performance and passing grades. Dover Middle School teachers view their role as more of a guide to help students gain mathematical understanding, with the assumption that it may take a variety of methods to achieve such understanding. Additionally, the percentage for a passing grade in all of the math classes at Dover Middle School is 70%. At Sutherland Junior High, the teachers feel it is their responsibility to "tell" as best as they can with the assumption that many of the students will lack the ability to understand. The mathematics teachers at this school are free to choose their own percentage for a passing grade in their classes, and this percentage varies between about 40% and 50%. Because the estimation procedures and understanding of fractions that students exhibit are likely influenced by the mathematical orientation of the school they attend, using these two schools gives variation to the data that will be analyzed.

At each school, five students from the eighth grade were selected, representing various levels of mathematical achievement. I chose eighth-grade students because one of the schools, Sutherland, consisted of the eighth and ninth grade and the other school, Dover, consisted of the seventh and eighth grade and I wanted to keep the grade level the same for all the students involved. At Sutherland, two were chosen from the lower level of mathematical achievement, selected from a Pre-Algebra class; one from the middle level, an Algebra class; and two from the upper level, a Geometry class. At Dover, one was chosen from the lower level of mathematical achievement, selected from a Pre-Algebra class; two from the middle level, an Algebra class; and two from the upper level, a Geometry class. In each of these classes, I went in and explained the project I was working on and told them that if they were interested they should let their teacher know. Once a list of willing students was compiled, I worked with the teacher to find which of those students fell into the desired grade range for each class. In Geometry and Algebra I was looking for students in the A to B range and in Pre-Algebra I was looking for students in the B to C range. This was to ensure a sufficient separation between achievement levels. Of the 10 students interviewed, 7 were males, 6 who were white and 1 who was Hispanic, and 3 were females, who were all white.

I purposely tried to recruit students with a wide range of mathematical abilities to compensate for shortcomings I perceived in the previous estimation research. In past studies, many researchers focused only on those students who were at higher levels of mathematical ability or those who had been pre-screened as good estimators. This does not seem likely to give a complete picture of the components that students use in estimation. Likewise, it does not take into consideration how a student's ability to estimate is affected by their knowledge of the material being considered (in this case their understanding of fractions). I anticipated that lower-level students would have a weaker understanding of fractions than higher-level students, and thus I expected to see a difference in their estimation performance, providing a wider range of estimation components.

My decision to interview eighth graders was also purposeful. I did not want to interview elementary age students because, although the interviews would surely yield interesting results, the students would not necessarily have a deep enough understanding of fractions or number sense to give data that would be helpful in answering my research question. However, I also did not want to interview older (or more advanced) students, because they often find it easier to compute rather than estimate when presented with a mathematical problem as I have noticed in fraction estimation interviews I have conducted with older students in the past. I thought that eighth graders might be ideal for this study because they are likely to possess a better understanding of fractions than elementary students and perhaps less likely than high school students to move immediately toward computation when faced with an estimation task.

Tasks

When initially developing tasks for the interviews, I began with 10 symbolic fraction problems. These problems involved the operations of addition, subtraction, multiplication, and division. The numbers used were proper fractions, mixed numbers, and whole numbers. At the end of the interview, I included tasks to elicit the student's understanding of fractions. I interviewed a tenth-grade student with these prepared tasks and found out two important things. First, for these tasks, it was much easier for the student to compute than it was to estimate, which was part of the reason that I selected younger students for the remainder of this study. The second was that the student's responses seemed to be affected by the fact that the problems were all symbolic. Thus, I realized I could not answer my research question with just these tasks. So I began a process of task development that took place in several steps.

I began my task modification by first examining which problems led the student to estimate rather than directly calculate the answer. Of the 10 estimation problems, the student engaged in estimation on only 1 problem. I examined this problem to see why it elicited an estimated response and the others did not. I discovered that it was the only problem that had fractions complicated enough to make estimating easier than computing the answer with the known algorithms. Consequently, I changed the other estimation problems to include more complicated fractions.

As I considered this interview as well as the computational estimation literature, I wanted to find out if the student would use different components if the problems were word problems rather than symbolic problems. Thus, I added word problem tasks that involved estimation with fractions. These problems also involved the operations of addition, subtraction, multiplication, or division of fractions.

After modifying the tasks of the interview significantly, I went through a process of further refining these tasks to ensure that they would elicit useful data. I repeated this process twice, although the subsequent changes were much less significant after the first changes that I made, involving the changing of a numerator or denominator or wording to make the problems more clear.. Each cycle involved interviewing a student, transcribing or taking field notes from the audiotape of the interview, analyzing the data to see whether or not the tasks were eliciting estimation components, and then modifying the tasks to better elicit students' estimation components. After the second iteration of this process I found that the interview protocol needed only one small modification and so was ready to be used in the interviews I conducted at the two schools.

The final set of tasks, resulting from the above process, had two sections (see Appendix). The first section contained four symbolic problems, one for each of the four arithmetic operations. The second section contained four word problems, also involving the four different operations. The questions for the interviews were asked in the same order for every interview. Although asking the questions in a different order for each student may have resulted in different responses, with so few participants, it was a variable I chose to keep constant.

Data Collection

I personally conducted a single interview in the fall of 2006 with each of the 10 eighth-grade students from the two schools mentioned above. During each interview, the estimation tasks were read out loud and repeated if the student wanted to hear it again. I did this so that students would not be hindered in their responses by any struggles with reading. However, all the students were informed that if they would like to see the question in written form they could ask to see it. If they asked to see the question, I showed them a paper with only that problem typed in large font. I read it as many times as they wanted and provided it in written form to ensure that each student was able to give a response that was not hindered by not remembering all of the information from hearing it once. Each student was allowed as much time as needed to answer each estimation task to elicit as much of the thinking going into his or her estimation solutions. None of the students were allowed to use scratch paper during the interview so that it would be less likely that they would attempt to compute rather than estimate. Each interview was video- and audiotaped, and lasted between 25 to 45 minutes. Following the interviews, each tape was transcribed.

Analysis

The first decision that I had to make regarding my coding and analysis was what unit of analysis I would use for my coding. I decided to use an action or implied action as my unit of analysis. An action or an implied action is a segment of transcript that either alludes to or directly describes a single operation on a quantity or quantities, where operation is loosely defined to mean an arithmetic operation, a comparison, rounding, etc. For instance, I considered the following to be examples of actions: “I multiplied 5 times 12” (arithmetic operation), or “I knew that five-ninths is close to one half” (implied comparison). When I tried a unit larger than that, such as a task or even a single response to any question asked during the interview, I found that there was often more than one component being used within the question or response, and I did not feel that one code for the entire unit captured what was happening in the student’s estimation as discussed in the literature review. When I tried a unit smaller than an action or implied action, such as a word, it was too small to have any meaning by itself. Thus, using an action or implied action allowed for multiple components to emerge in each solution without being eclipsed by a single dominating strategy that classified an entire answer. These coded actions represented the components found the in the students' estimation solutions.

I initially began coding by searching for the components identified as strategies in previous literature (those described in Tables 1 and 2), but quickly encountered two problems. The first problem, which I had anticipated, was that there were actions or implied actions that the students engaged in to estimate that did not match any of the previous strategies. The second problem was that the unit of analysis differed so greatly from any past research that some of the components that I observed in the interviews did not necessarily fit into any of the previous categories due to the incongruence of unit size, necessitating a reorganization of the overarching framework of the estimation strategies.

As I encountered these problems, I proceeded in a somewhat cyclical process. Every time I found an action or implied action that did not fit into a code I already had, I created a new code to capture the nature of the action. I did this an interview or two at a time, creating new codes for actions or implied actions that did not fit into my previous coding scheme, and then returning to previously coded data to see whether or not that component was also present there. I occasionally encountered actions or implied actions that caused me to split, combine, or redefine categories.

During the process of coding and creating new components, I began to try and organize them under an overarching framework. I first tried to see if the new categories could be grouped into categories that were presented in prior research. This did not seem to work because the unit of analysis varied so much from the previous research to my own analysis, as previously mentioned. I tried to see if I could make two generic categories of whole number and decimal estimation components and fraction estimation components, disregarding the previous categories of reformulation, translation, and compensation. This was a better fit for the estimation components I was finding, but components such as performing an operation did not quite fit into these categories since it was independent of which number type was used. That was when I created a third category for estimation components used with either fractions or whole numbers and decimals. As I finished coding, each component fit into one of these three categories. Thus, this new framework was what I used to organize the estimation components I found.

Chapter 4: Results

One thing that became very clear as I analyzed my data was that the categories found in previous literature were not an effective way to organize the components that I found. This is principally because, as mentioned earlier, the unit of analysis was so much broader in the literature that I reviewed than it was in my own research. Another reason for this is that this study focused on estimating with fractions rather than just with whole numbers and decimals. As such, although there were many components that surfaced in my data that matched previous strategies, there were also many new components that did not fit into the structure that has generally been accepted for estimation strategies. As I added new categories to accommodate the new components, I found that components described by preexisting strategies fit into some of the new categories as well, causing me to restructure the overall framework.

The restructured framework consisted of three categories. These categories are whole number and decimal estimation components, fraction estimation components, and estimation components used for either fractions or whole numbers and decimals. This new organization acknowledges the role that the type of number in the estimation problem plays in influencing what components are invoked in the student. For each estimation component I provide a definition of the component and an example from the data of a student using the component. Then I provide three different counts of the use of each component in the transcripts: first, how many times it was used by all students, out of a total of 365 components used; second, how many students used this component, out of a total of 10 students; and lastly, how many problem solutions the component was used in, out of a total of 77 problem solutions given by students. There were 77 problems solutions, rather than 80, because there were 3 problems in which the student gave no solution at all.

There are two important caveats that need to be mentioned concerning these counts of the components. First, because of the limited number of subjects in this study and the way these subjects were selected, these counts should not be used to make generalizations concerning the frequency of these components among eighth-grade students in general I have included these counts only because I anticipate that the reader may be interested in how often these components appeared in my data. Second, each action or implied action was coded for a component regardless of whether that component yielded a correct answer. Thus, the counts should not be interpreted as the number of times a component was used successfully or appropriately.

Whole Number and Decimal Estimation Components

Some of the components that I found were categorized as whole number and decimal estimation components (see Table 3). By this I mean components that were used to operate on whole numbers or decimals only. The context in which the components were used often involved fractions, but the particular action was limited to operating on whole numbers and decimals, or operating on numerators or denominators as if they were whole numbers (without considering how these changes influenced the numeric value of the fraction). Most of these components were found in the previous literature, but because they were defined so broadly in those studies, I found that I needed to split these general strategies into more specific components so as to better fit how students were estimating in the fraction tasks.

Table 3

Whole Number and Decimal Estimation Components

Name of component	Definition of component
Front-end use of numbers[a]	Rounding or truncating a whole number or the numerator and denominator of a fraction separately to 10s (not necessarily the closest 10)
Substitution[a]	Rounding a whole number or the numerator and denominator of a fraction separately to a number other than 10s
Changing to a decimal[a]	Changing a fraction to a decimal
Rounding decimals to a benchmark[a]	Rounding a decimal to a nearby decimal or fraction that may be easier to work with
Forming and iterating a composite number	Fusing two numbers together for the purpose of iterating

[a]R. E. Reys et al. (1982), albeit not necessarily mentioned by the same name.

Front-end use of number. The front-end use of number component consists of rounding or truncating a whole number or the numerator and/or denominator of a fraction separately to 10s (not necessarily the closest 10). An example of this component in my data is when one student was working with the fraction 39/72 and said, “I rounded 39 up to 40 and 72 up to 80” to get the fraction 40/80. The student treats the numerator and denominator as separate whole numbers. This is why this component is a whole number estimation component. This component was used a total of 35 times by 6 students in 21 problem solutions.

This component has been identified as a common strategy for estimating with whole numbers and decimals by other researchers (Hanson & Hogan, 2000; LeFevre et al., 1993; B. J. Reys et al., 1991; R. E. Reys et al., 1982; Threadgill-Sowder, 1984). Rounding and truncating are both included in this component because it was often difficult to tell whether the student was following rounding rules or truncating. For instance, if a student changed 52 to 50, it is unclear if they did it because they were rounding to the nearest 10 or because they were simply truncating the number.

Substitution. Substitution is the process of rounding a whole number or the numerator and/or denominator of a fraction separately to a number other than 10s. It is used to make computation in estimation easier. An example of this component in my data is when a student was working with 6/25 and said, “I rounded down the six twenty-fifths to make it five twenty-fifths.” Five twenty-fifths is a convenient fraction to work with. This component was used a total of 10 times by 4 students in seven problem solutions.

Since the use of this component, as well as front-end use of number, is for the purpose of having the computations or simplifications work out nicely, in many cases rounding to 10s may actually be substitution, but from the student explanations it is not apparent what the intention behind some of their actions were. Historically, the difference between front-end use and substitution has not been clear. Thus, since it was too difficult to always distinguish which component the students were using, for the sake of clarity in my analysis I chose front-end use of number to consist of rounding or truncating part of a fraction to a nearby multiple of 10 and substitution to consist of rounding a whole number or the numerator and/or denominator of a fraction separately to a number other than 10s.

This component has been identified in previous literature (LeFevre et al., 1993; B. J. Reys et al., 1991; R. E. Reys, 1984; R. E. Reys et al., 1982) similarly to the way the students used it in the interviews conducted in this study. However, although rounding to benchmarks and changing the form of the number were all grouped under the category of substitution in much of the prior estimation literature, I separated those components so as to eliminate some of the confusion existing in this other literature as to which component a student was actually using.

Changing to a decimal. This component is simply changing a fraction to a decimal. This may be done using memorized facts or using a rote procedure. An example of this component in my data is when one of the students changed 3 1/8 to a decimal: “three and one eighths is three point one-two-five.” In this example, like many instances in the transcripts, it was unclear whether the student was using a memorized fact or a rote procedure. However, due to the speed with which he changed the fraction to a decimal, it seems that he used a memorized fact. This component was used a total of five times by 3 students in four problem solutions.

Past literature (LeFevre et al., 1993; B. J. Reys et al., 1991; R. E. Reys et al., 1982; Sowder & Wheeler, 1989) only briefly mentioned this component (often associated with substitution) and refers to changing numbers from one form to another. However I modified this category in two ways. First, in this study this category only includes exact conversions. In the past, both exact conversions and approximations were included in this category. I made this change because I only encountered exact conversions in the transcripts. Second, only conversions that involved switching from fractions to decimals were included in this category. In prior research, this category included all conversions from one number type to another. I restricted this category to conversions from fractions to decimals to capture what seemed to be attempts by students to avoid reasoning about fractions by converting fractions to decimal form.

Rounding decimals to a benchmark. Rounding decimals to a benchmark is a component that consists of rounding a decimal to a nearby decimal or fraction that may be more familiar or easier to work with (a benchmark). An example of this in my data is when a student had changed a fraction to a decimal and then took that decimal, .44, and said, "point four-four, that can round up to a half." This component was used a total of one time by 1 student in one problem solution.

As noted in the literature review, using benchmarks is a strategy that has been documented in past literature (Hanson & Hogan, 2000; B. J. Reys et al., 1999; B. J. Reys et al., 1991). However, none of the estimation literature has indicated how benchmarks are used except to identify the use of benchmarks as a strategy. Refining this strategy into more specific components may allow us to see more clearly how benchmarks can be used in estimation.

Forming and iterating a composite number. The forming and iterating of a composite number component involves fusing two numbers together for the purpose of iterating. An example in my data is when a student was estimating how many boxes of chocolates he would need to give to his diabetic friends (see Appendix, problem 5) and figured that "it's two for every three, then I times that [the two and the three] by two because for every three diabetics it took two boxes." The student formed a composite number ("it's two for every three"), and then iterated it ("I times that by two"). This component was used a total of two times by 1 student in one problem solution.

This component has not been found as a strategy found in previous research on estimation, although composite numbers do arise in the literature on fractions and proportional reasoning (Lamon, 1999). Part of the reason for this could be the lack of word problems in past research. Word problems would be a more likely place for a student to create composite numbers because in a word problem a student may encounter two numbers that might be divided to create a fraction, and instead of dividing them to get a single quantity, he or she might continue to view them as two numbers linked together but still refer to them as separate objects.

Fraction Estimation Components

Many of the components that surfaced in my data were fraction estimation components (see Table 4). I refer to these components as fraction estimation components because they could not be directly used on whole numbers and decimals. Even though some of these components have been identified as strategies before, several of them have not appeared in any of the previous estimation literature.

Table 4

Fraction Estimation Components

Name of component	*Definition of component*
Comparing with benchmarks[a]	Comparing the fraction with a fraction that is more frequently used and with which they have more familiarity, such as 1, 1/2,

	3/4, etc. and then possibly rounding the original fraction to the benchmark
Comparing two separate fractions	Comparing the size of two fractions, either fractions from the original problem or fractions created as the student reasons about the problem (not necessarily a familiar fraction)
Increasing the numerator and denominator proportionally (multiplicative reasoning)	Changing one fraction to have a different denominator and changing the numerator proportionally
Creating common denominators[b]	Changing one or both fractions to have common denominators
Converting a mixed number to an improper fraction	Taking a mixed number and changing it to an improper fraction by some procedure
Iterating a fraction	Repeated addition of a fraction to reach a new number
Operating with mixed numbers	Performing an arithmetic operation with mixed numbers without first converting the mixed numbers to improper fractions
How many unit fractions in a whole	Using the knowledge that there are x 1/x's in a unit whole to find how many 1/x's in a particular whole number

Simplifying fractions	Making the quantities in the numerator and denominator smaller while keeping the fraction equivalent

[a]Hanson and Hogan, 2000; B. J. Reys et al., 1999.
[b]Hanson and Hogan, 2000.

Comparing with benchmarks. The component of comparing with benchmarks consists of comparing the fraction with a fraction that is more frequently used and more familiar, such as 1, 1/2, 3/4, etc. and then possibly rounding the original fraction to the benchmark. An example of this from my data is when a student rounded 14/19 to 2/3 because, "I just figured that fourteen nineteenths is pretty close to two thirds." This component was used a total of 35 times by 7 students in 17 problem solutions.

Using benchmarks is a component that is identified in previous research (Hanson & Hogan, 2000; B. J. Reys et al., 1999; B. J. Reys et al., 1991). However, these studies did not indicate whether benchmarks were commonly used as an estimation strategy. In contrast, the students in my interviews used it extensively. This seems to be a powerful tool in fraction estimation, perhaps more powerful than resorting to components that fall under the previous category of whole number and decimal estimation. This is because a student with a strong understanding of fractions might have a sense of how different fractions compare to benchmark fractions and use this component instead of rounding the numerator and denominator separately. By comparing and rounding to a nearby benchmark, the student may be able to produce a more meaningful and accurate estimate than if he or she first treated the numerator and denominator separately by using front-end components of rounding or truncating.

Comparing two separate fractions. Comparing two separate fractions consists of comparing the relative sizes of two fractions, where the fractions either came from the original problem or were generated by the student while reasoning about the problem (not necessarily a familiar fraction). An example of this from my data is when a student made a comparison between two fractions in the problem (see Appendix, problem 6), 1/8 and 1/3, and concluded, “One eighth is smaller than one third.” The data in this instance, and in many instances when this component was observed, were insufficient to determine exactly how the student arrived at their decision as to which fraction was larger. All the evidence indicates is that they made the comparison. This component was used a total of five times by 3 students in four problem solutions.

This estimation component has not been identified in past estimation literature. This could be, in part, because the unit of analysis used in previous research was not sufficient to bring to the surface some of the interim components not necessarily producing a direct result. For instance, in the example above, the student is subtracting 1 1/3 from 3 1/8 and after making the comparison of 1/8 and 1/3, the student continues on the problem by saying that three minus one is two and then the final answer is a little less than two because “I know I’m gonna need to take some out of this two to be able to, um, take, um, one-third away.” If the unit of analysis were just the problem solution, as in previous research, one of those researchers may have identified this solution as compensation and disregarded the component of comparing two fractions, even though the comparison led to the adjustment at the end.

Increasing the numerator and denominator proportionally. The component of increasing the numerator and denominator proportionally involves changing one fraction to have a different denominator and changing the numerator proportionally (i.e., multiplicatively, not additively). An example of this is when a student changed one fraction to another, saying, "I'll say six twenty-fifths is around eight thirtieths." This is multiplicative reasoning because if it were additive reasoning, he would have added five to the numerator, making it 11, because he had increased the denominator by five. This component was used a total of nine times by 2 students in two problem solutions.

Increasing the numerator and denominator proportionally is a newly identified component. There is nothing like it in the estimation literature. This is an important method because it reflects a sophisticated application of proportional reasoning in a multiplicative situation.

Creating common denominators. The component of creating common denominators consists of using a procedure to change one or more fractions to equivalent fractions with a common denominator. Unfortunately, I have no evidence of what processes that students used as they engaged in this component; they seemed to be doing nothing more than following a rote procedure. The students may have been thinking about common denominators more conceptually, rather than procedurally, but none of their explanations were thorough enough to provide evidence of that. An example of this component from my data is when one of the students was faced with the problem regarding how much chocolate the people in Germany and the United States eat together (see Appendix, problem 8). Given the two quantities 7/30 and 6/25, she found that "the common denominator of twenty-five and thirty is one-fifty." She then proceeded to incorrectly calculate the numerators that would accompany the new denominator of one hundred fifty. This component was used a total of 12 times by 2 students in seven problem solutions.

This component is one of the few documented in the literature about fraction estimation strategies. Hanson and Hogan (2000) not only observed this strategy, but also emphasized that when students used this strategy, they often used it incorrectly. I also noticed this same pattern in my data as I coded and analyzed the interviews. One instance is noted in the example above. Although this student found a common denominator that was correct, she was unable to produce the correct numerators corresponding to the common denominators.

Converting a mixed number to an improper fraction. Converting a mixed number to an improper fraction consists of taking a mixed number and changing it to an improper fraction by some procedure. An example of this from my data is when one student was working with the mixed number 2 1/3, "Two and one third is [pause] that would be six, seven thirds." This component was used a total of 17 times by 5 students in 12 problem solutions. Previous literature on estimation has not identified this component. It is unique to fraction estimation.

Iterating a fraction. Iterating a fraction is repeatedly adding a fraction multiple times to reach a new number. An example of this from my data is when a student iterated 2/3 three times, "So two thirds, and then another two thirds, and another two thirds – that adds up to two." This component was used a total of 13 times by six students in seven problem solutions. This is another new component not found in past literature on estimation.

Operating with mixed numbers. The component of operating with mixed numbers consists of performing an operation on mixed numbers without first converting the mixed number into an improper fraction. An example from my data is when a student got 1 2/3 as his answer when he started with three and "minused one and a third." This component was used a total of six times by 6 students in six problem solutions. Operating with mixed numbers is another component that has not been identified before.

How many unit fractions in a whole. Students use the component of how many fractions in a whole when using their knowledge that there are x 1/x's in a unit whole to find how many 1/x's in a particular whole number. Although the student did not directly say this, it was the most plausible component for the explanations the students gave for the items coded this way. In almost all of the parts that were coded this way, a student gave at least a partial solution based on how many of a unit fraction were in a whole number. It seemed as though this could be because they knew how many of those unit fractions were in one and then used this to find how many were in another whole number. An example of this in my data is when a student was trying to figure out how many slabs of cement he could pour with 1 17/19 tons of sand (see Appendix, problem 7). He first rounded this mixed number to two and then declared the answer to be four because, "there's four halves in two." This component was used a total of five times by 4 students in four problem solutions. Since this is strictly a fraction estimation component, and was not identified in the small amount of research on fraction estimation, it is a new component that surfaced in my data.

Simplifying fractions. This component can be described as making the quantities in the numerator and denominator smaller while keeping the fraction equivalent.

However, I was usually unable to determine from the data what specific reasoning the students were using when simplifying. An example of this component is when a student had already used substitution to change 6/25 to 5/25 and then simplified to 1/5, explaining, “five twenty-fifths, which is just a fifth.” This component was used a total of 17 times by 5 students in 11 problem solutions. Simplifying fractions is another new component that has not been documented before.

Estimation Components Used for Either Fractions or Whole Numbers and Decimals

Some of the components that I found could be used with either fractions or whole numbers and decimals (see Table 5). To be considered for this category, a component is not based upon or does not draw from any powerful knowledge from either number type. This is why they were not split into separate components and grouped in the first two categories. I found three components from my data that fall into this category, only one of which has been documented (R. E. Reys et al., 1982).

Table 5

Estimation Components Used for Either Fractions or Whole Numbers and Decimals

Name of component	*Definition of component*
Compensation[a]	Adjusting a quantity in the problem or solution to account for the error introduced in the solution by the previous rounding of a different quantity
Performing an operation	Performing an arithmetic operation on two quantities
Form of the answer	Finding a solution by considering what the

answer might look like

---

[a]R. E. Reys et al. (1982).

Compensation. The compensation component consists of adjusting a quantity in the problem or solution to account for the error introduced in the solution by the previous rounding of a different quantity. An example of this component from my data is when a student was estimating how much a fifth plus a third is. This student said it would be "a little less than a half." This is compensating because he did not say that it was a half, indicating that it was less than a half because, "I know a third's less than a half and a fifth's like a lot less than a half" so it would not be quite one half. In the past this component has been separated into intermediate and final compensation, but in my coding I grouped them together because there were not important distinctions between the two in my data. This component was used a total of 18 times by 5 students in eight problem solutions.

This is a component that could easily be used for fractions (as the example above illustrates) or whole numbers and decimals. It was used for both number types in my data. The reason it is in this category is because, although it could be argued that some of these components could use powerful reasoning, none of the students that I interviewed did this. For example, they may have noted that they made a quantity bigger or smaller, but did not seem to consider how much bigger or how much smaller they had adjusted it when compensating. There were many examples in the literature of how this was used with whole numbers and decimals, but no examples of where it was used with fractions.

Performing an operation. Performing an operation is a component where students perform an arithmetic operation on two quantities. An example of this from my data is when a student explained, “I multiplied two-thirds by two-thirds.” This component falls into this category because of the procedural nature of the operations performed. This component was used very often because most of the solutions required it to reach an estimated answer. This component was used a total of 168 times by 10 students in 67 problem solutions.

This component has not been specified in previous literature, but not because students were not performing operations. This is an example where the difference in the unit of analysis creates important distinctions in the components found. Students were most certainly performing operations in previous estimation research, but researchers did not document it as a strategy, presumably because the unit of analysis they were using caused them to overlook some components students used in estimation.

Form of the answer. This component can be described as finding a solution by considering what the answer might look like. An example of this from my data is when a student explained, in partial explanation for why his answer would be “two and some-odd fraction, … Lots of the time when you add fractions together, they won’t be whole numbers…when you have a fraction like twenty-eight over forty-one plus a third, that’s not going to be a whole number.” This student thought that the answer would consist of a whole number and a fraction because the sum of two fractions is seldom a whole number. The student’s estimated answer was at least partially determined by the form of what he thought it should look like. This component is in this category, rather than that of fraction estimation components, because although the student may be using some knowledge of fractions, knowing that fractions often add to be another fraction would not be considered powerful knowledge of fractions. This component was used a total of seven times by 1 student in four problem solutions.

Form of the answer is not a component that has been found in past literature. Sowder and Wheeler (1989) briefly mentioned it once when a student's answer was 10 1/2 and it was noted that maybe the student changed it from 11 to 10 1/2 because he or she thought it should have a fraction in the answer since the problem had fractions in it to begin with. However, that was the extent that it was mentioned, and it was definitely not named as a strategy used in estimation.

Estimation Solutions

To give a sense of the complex way that estimation components are combined to create a solution to an estimation problem involving fractions, I now present a solution taken directly from the transcripts of my data. The italicized sections are the actions that were coded. In brackets following each italicized section are the component names for the actions. The student is represented by A2 and I represents the interviewer.

> Task: The people in Germany eat $\frac{7}{30}$ of the chocolate in the world and the people in the United States eat $\frac{6}{25}$ of it. Estimate how much of the world's chocolate the people in the two countries eat together.
>
> A2: *Six twenty-fifths would be, that's about one fifth* [comparing with benchmarks]. And so, one fifth, we eat one, one fifth of the world's chocolate and then *Germany eats like a third* [comparing with benchmarks]. *And a fifth plus a third would be* [performing an operation] *like a little less than half* [compensation].
>
> I: Ok. And how did you get a fifth and a third?
>
> A2: Because *I rounded down the six twenty fifths to make it five twenty fifths* [substitution] *which is just a fifth* [simplifying fractions] and then *I rounded up Germany's for a third* [comparing with benchmarks].

I: Ok. And so then when you added a third and a fifth how did you get a little bit less than half?

A2: Um, because I just, *I know a third's less than a half* [comparing two separate fractions] and *a fifth's like a lot less than a half* [comparing two separate fractions], it's like fifth, oh wait, and *they like add up* [performing an operation] *and they're close, somewhere close to a half* [compensation].

In this section of transcript we notice how many components are in a computational estimation solution. Six different estimation components are used in this solution. Three of these components are fraction estimation components, one of them is a whole number estimation component, and two of them are components used for either fractions or whole numbers and decimals. Three of these components are clearly estimation techniques (comparing with benchmarks, substitution, and compensation), while the other three components seem to support the estimation occurring here to arrive at the solution. However, without the supporting components, this student would not have arrived with an estimation solution. This complexity draws attention to why it is essential to look at all of the components that go into estimation.

If I had taken the traditional stance of only classifying the solution with one estimation strategy, I would have overlooked a large part of the work the student engaged in to solve the estimation problem. Clearly looking at this smaller grain size was effective because I was able to identify more estimation components being used in a single estimation solution. Additionally, there are supporting components to the estimation that may not be considered estimation techniques but are nonetheless a fundamental part of an estimation solution.

Using this knowledge, we can answer how students estimate with fractions. They estimate with components unique to fractions, mingled with other components (whole number and decimal), as well as with supporting components.

Summary

As shown above, after analyzing my data, I found three categories of estimation components. These are whole number and decimal estimation components, fraction estimation components, and components used for either fractions or whole numbers and decimals. Some of these components are computational, some are estimates, and some are similar to the whole number and decimal or fraction estimation strategies that were found in previous estimation literature.

Chapter 5: Conclusion

This study was motivated by the importance of estimation in everyday life and in the mathematics classroom as well as by the observation of limitations in previous estimation literature. Most of the previous research did not address estimation with fractions, and the one study that did focus on this was done with adults rather than with children. This is problematic because whole number and decimal reasoning is so different from fraction reasoning (as discussed in the literature review) and because adult reasoning is so different from that of children. Additionally, the unit of analysis used by researchers in previous computational estimation studies was not only not defined, but the inferred grain size was so large (an entire solution to a task) that individual components could easily be overlooked when more than one were used to find an answer. The methodology in this study was developed by taking these shortcomings into account, and resulted in the identification of many new components used in fraction estimation. In the remainder of this chapter, I discuss the implications and contributions for research and teaching, as well as the limitations of this study and possible future directions research may take as a result of these findings.

Implications and Contributions for Research

One implication for research comes from the unit of analysis used in my methodology. Unlike previous estimation studies, my study was based on a unit of analysis of much smaller size. Using an action or implied action as the grain size to code the data, I was able to find many components that may not have surfaced otherwise. This suggests that by following this precedence and using a smaller unit of analysis, future research could be more successful at gaining a clearer picture of what students do when they use computational estimation.

Likewise, contributions to the field of mathematics education arise from the new estimation components identified here as well as those components identified in past research as strategies that were clarified and developed further. These contribute to the field in two ways. First, they highlight the fact that past research in computational estimation is not sufficient for understanding how students estimate, particularly with fractions. Second, they give new components to consider when conducting future research on estimation.

Implications and Contributions for Teaching

Through this study I have shown that some components of estimation solutions can give a clearer picture of how students estimate with fractions. Based on these findings, it is clear that we cannot simply limit the teaching of estimation to general strategies, but must also teach the individual components of computational estimation identified in this study so that students can successfully estimate with fractions.

Limitations and Future Directions

The first limitation of this study is the small sample. Although I have been able to identify components that are used in computational estimation involving fractions, the small number of students prevents me from making general statements concerning how common these components are. Thus, future research studies that include larger sample sizes are needed in order to see how widespread these components are among the general population of students and to see if there are other estimation components that these students did not use.

The second limitation is that this analysis was limited solely to identifying components that students used when there are potentially other important findings in the data, besides just the components themselves. For example, it may be useful to look for patterns among components to see if specific groups of components are used together. As such, finding patterns among components in future research could be a fruitful endeavor.

The third limitation of this study is that the correctness of each solution was not considered as part of my analysis. It could be very helpful in the development of teaching methods for estimation to know whether a given component typically leads to correct solutions and which types of errors students tend to make while using certain components. Future research should, therefore, include an analysis of the relationship between components and correctness of answers.

With the importance that computational estimation plays in our every day lives as well as in the classroom, it would be valuable to learn as much as we can about estimation in order to inform classroom instruction. The above potential directions for research, in addition to the findings from this study, are a step toward that goal.

References

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.

Case, R., & Sowder, J. T. (1990). The development of computational estimation: A neo-Piagetian analysis. *Cognition and Instruction, 7*, 79-104.

Cramer, K. A., Post, T. R., & delMas, R. C. (2002). Initial fraction learning by fourth- and fifth-grade students: A comparison of the effects of using commercial curricula with the effects of using the rational number project curriculum. *Journal for Research in Mathematics Education, 33,* 111-144.

Hanson, S. A., & Hogan, T. P. (2000). Computational estimation skill of college students. *Journal for Research in Mathematics Education, 31,* 438-499.

Lamon, S. J. (1999). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers.* Mahwah, NJ: Lawrence Erlbaum Associates.

LeFevre, J., Greenham, S. L., & Waheed, N. (1993). The development of procedural and conceptual knowledge in computational estimation. *Cognition and Instruction, 11*, 95-132.

Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education, 21*, 16-32.

Reys, B. J., Kim, O., & Bay, J. M. (1999). Establishing fraction benchmarks. *Mathematics Teaching in the Middle School, 4*, 530-532.

Reys, B. J., Reys, R. E., & Penafiel, A. F. (1991). Estimation performance and strategy use of Mexican 5^{th} and 8^{th} grade student sample. *Educational Studies in Mathematics, 22*, 353-375.

Reys, R. E. (1984). Mental computation and estimation: Past, present, and future. *The Elementary School Journal*, *84*, 546-557.

Reys, R. E., Rybolt, J. F., Bestgen, B. J., & Wyatt, J. W. (1982). Processes used by good computational estimators. *Journal for Research in Mathematics Education, 13*, 183-201.

Siebert, D., & Gaskin, N. (2006). Creating, naming, and justifying fractions. *Teaching Children Mathematics, 12,* 394-400.

Sowder, J. T., & Wheeler, M. M. (1989). The development of concepts and strategies used in computational estimation. *Journal for Research in Mathematics Education*, *20*, 130-146.

Threadgill-Sowder, J. (1984). Computational estimation procedures of school children. *Journal of Educational Research*, *77*, 332-336.

Verschaffel, L., Greer, B., & De Corte, E. (2007). Whole number concepts and operations. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 557-628). Reston, VA: NCTM.

Appendix: Interview Tasks

Symbolic problems:

1. $\frac{49}{52} - \frac{1}{4}$

2. $\frac{14}{19} \times \frac{2}{3}$

3. $2\frac{1}{3} + \frac{28}{41}$

4. $5 \div \frac{39}{72}$

Word problems:

5. Suppose $\frac{21}{29}$ of a box of chocolates contains the limit of how much sugar a diabetic can eat in one day. You want to give that much chocolate to each of 6 diabetic friends. Estimate how many boxes of chocolates you would need in order to do that.

6. You are running in a 5k run (which is $3\frac{1}{8}$ miles). You have just passed the $1\frac{1}{3}$ mile marker. Estimate how many more miles you have to run to the finish line.

7. You have $1\frac{17}{19}$ tons of sand and it takes $\frac{1}{2}$ ton of sand to pour 1 square slab of cement. Estimate how many slabs you can pour with the amount of sand you have.

8. The people in Germany eat $\frac{7}{30}$ of the chocolate in the world and the people in the United States eat $\frac{6}{25}$ of it. Estimate how much of the world's chocolate the people in the two countries eat together.