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Expanded Mathematical Treatment for "Spectral Bias in Adaptive Beamforming with Narrowband Interference"

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Expanded Mathematical Treatment for “Spectral Bias in Adaptive Beamforming with Narrowband Interference”

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Abstract

This technical note presents extended versions of some mathematical derivations found in the paper “Spectral Bias in Adaptive Beamforming with Narrowband Interference” which is under review for publication in IEEE Signal Processing Letters. Though the SP Letters paper is self contained and complete, this note provides intermediate steps for some of the equation derivations to assist the interested readers who would like to re-create the results.

I. Introduction

A. Background

The following mathematical developments include some intermediate steps which due to space limitations were not included in “Spectral Bias in Adaptive Beamforming with Narrowband Interference.” Though in that paper, the presentation is complete and self contained, it may not be obvious without some effort how some results were arrived at. This note serves as supplementary material provided to facilitate the interested reader who would like to re-derive the results. It is not intended as a stand-alone paper and as such does not include adequate introduction, problem definition, experimental results, or analysis of results. “Spectral Bias in Adaptive Beamforming with Narrowband Interference,” as submitted for review in IEEE Signal Processing Letters is considered the original source for publishing these ideas.

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A demonstration program to illustrate scooping behavior is available in [1]. This MATLAB (R) script shows how scooping occurs with both LCMV and subspace projection beamforming, and how the proposed algorithm corrects scooping.

Section and equation numbering in this note matches that of the original SP Letters paper.

B. Signal Model

The P element sensor array which produces a length $P \times 1$ complex baseband sample data vector

$$\mathbf{x}[n] = \mathbf{a}s[n] + \sum_{q=1}^{Q} \mathbf{v}_q[n] d_q[n] + \boldsymbol{\eta}[n] \tag{1}$$

where $s[n]$ is the SOI, $\boldsymbol{\eta}[n]$ is noise, and $d_q[n]$ is one of Q "detrimental" interfering sources. Vectors $\mathbf{a}$ and $\mathbf{v}_q[n]$ are normalized array spatial responses to $s[n]$ and $d_q[n]$ respectively. Assume that $s[n]$, $d_q[n]$, and $\boldsymbol{\eta}[n]$ are wide sense stationary random processes and $s[n]$ is spatially fixed. Due to interferer motion, $\mathbf{v}_q[n]$ varies, but over L time samples called the "short term integration (STI)" window, $\mathbf{v}_q[n]$ is approximately constant. The $d_q[n]$ are narrow band.

The beamformer output is

$$y[n] = \mathbf{w}_j^H \mathbf{x}[n], \;\; j = \lfloor n/L \rfloor \tag{2}$$

where $\mathbf{w}_j$ is an adaptive weight vector computed for the jth STI and $\lfloor \cdot \rfloor$ denotes rounding toward $-\infty$.

The PSD estimate, $\hat{S}_y[k]$ is formed from $y[n]$ using Welch's overlapping windowed averaged periodogram, which for a fixed beamformer with $\mathbf{w}_j = \mathbf{w}$ can be expressed in matrix-vector form

$$\begin{aligned}
\hat{S}_y[k] &= \frac{\gamma}{M} \sum_{m=0}^{M-1} \left| \sum_{n=0}^{N-1} g[n] y[n + m(N-O)] e^{-i2\pi kn/N} \right|^2 \\
\hat{\mathbf{S}}_y^T &= [\hat{S}_y[0], \cdots, \hat{S}_y[N-1]] \\
&= \frac{\gamma}{M} \sum_{m=0}^{M-1} \left| \mathrm{DFT}_N \{ \mathbf{y}_m^T \bullet \mathbf{g}^T \} \right|^{\odot 2} \\
&= \frac{\gamma}{M} \sum_{m=0}^{M-1} \left| \mathrm{DFT}_N \{ \mathbf{w}^H \mathbf{X}_m \mathbf{G} \} \right|^{\odot 2} \\
&= \frac{\gamma}{M} \sum_{m=0}^{M-1} \left| \mathbf{w}^H \mathrm{DFT}_N \{ \mathbf{X}_m \mathbf{G} \} \right|^{\odot 2}, \qquad (3) \\
\mathbf{y}_m^T &= \big[y[m(N-O)], \cdots, y[m(N-O)+N-1] \big] \\
\mathbf{X}_m &= \big[\mathbf{x}[m(N-O)], \cdots, \mathbf{x}[m(N-O)+N-1] \big]
\end{aligned}$$

where $i = \sqrt{-1}$, M is the number of length N DFT averaging windows used, O is the overlap between successive windows, $\bullet$ is the Hadamard element-wise matrix product, $\mathbf{g} = [g[0], \cdots, g[N-1]]^T = \text{diag}\{\mathbf{G}\}$ is the spectral shaping window (e.g. Hamming). The DFT (or equivalently FFT) operates separately along all matrix rows of its argument, and $|\cdot|^{\odot 2}$ denotes element-wise magnitude squared and '$\hat{\cdot}$' indicates an estimated quantity. In the final step to obtain (3), $\mathbf{w}^H$ is factored out using linearity of the DFT. Setting $\gamma = 1/\text{tr}\{\mathbf{GG}\}$ properly scales the PSD estimate.

Exploiting stationarity over an STI, define the time dependent array autocorrelation matrix as

$$\begin{aligned} \mathbf{R}_x[n] &= E\{\mathbf{x}[n]\mathbf{x}^H[n]\} \approx \mathbf{R}_{x,j}, \quad j = \lfloor n/L \rfloor \\ \mathbf{R}_{x,j} &= \sigma_s^2 \mathbf{a}\mathbf{a}^H + \sum_{q=1}^{Q} \sigma_{d_q}^2 \mathbf{v}_{q,j}\mathbf{v}_{q,j}^H + \sigma_e ta^2 \mathbf{I} \end{aligned}$$

where $\mathbf{v}_{q,j} = \mathbf{v}_q[jL]$. The STI sample estimator of $\mathbf{R}_{x,j}$ is

$$\hat{\mathbf{R}}_{x,j} = \frac{1}{L} \sum_{n=jL}^{(j+1)L-1} \mathbf{x}[n]\mathbf{x}^H[n] \tag{4}$$

II. Spectral Scooping

A. Analysis

Assuming i.i.d. noise and no SOI, the observed signal is

$$\mathbf{x}[n] = \mathbf{v}_j \, \sigma_d \, e^{i\omega_d n} + \boldsymbol{\eta}[n]. \tag{7}$$

The true covariance is then

$$\mathbf{R}_{x,j} = \sigma_d^2 \, \mathbf{v}_j \mathbf{v}_j^H + \mathbf{R}_z = \sigma_d^2 \, \mathbf{v}_j \mathbf{v}_j^H + \sigma_s^2 \mathbf{a}\mathbf{a}^H + \sigma_\eta^2 \, \mathbf{I}. \tag{8}$$

Using (7) and (8) in (4), the sample covariance for STI j is

$$\begin{aligned}
\hat{\mathbf{R}}_{x,j} &= \frac{1}{L}\sum_{n=jL}^{(j+1)L-1}\Big(\sigma_d^2\,\mathbf{v}_j\mathbf{v}_j^H + \sigma_d\,\mathbf{v}_j\,e^{i\omega_d n}\boldsymbol{\eta}^H[n] \\
&\quad + \sigma_d\,\boldsymbol{\eta}[n]\mathbf{v}_j^H e^{-i\omega_d n} + \boldsymbol{\eta}[n]\boldsymbol{\eta}^H[n]\Big) \\
&= \sigma_d^2\,\mathbf{v}_j\mathbf{v}_j^H + \frac{\sigma_d}{L}\,\mathbf{v}_j\left(\sum_{n=jL}^{(j+1)L-1}\boldsymbol{\eta}^H[n]\,e^{i\omega_d n}\right) \\
&\quad + \frac{\sigma_d}{L}\left(\sum_{n=jL}^{(j+1)L-1}\boldsymbol{\eta}[n]\,e^{-i\omega_d n}\right)\mathbf{v}_j^H + \sigma_\eta^2\mathbf{I} + \mathbf{E}_j \\
&= \sigma_d^2\,\mathbf{v}_j\mathbf{v}_j^H + \frac{\sigma_d}{L}\left(\mathbf{v}_j\,\mathbf{n}_j^H + \mathbf{n}_j\mathbf{v}_j^H\right) \\
&\quad + \sigma_\eta^2\mathbf{I} + \sigma_s^2\mathbf{a}\mathbf{a}^H + \mathbf{E}_j, \qquad (9)\\
\mathbf{n}_j &= e^{-i\omega_d jL}\sum_{n=0}^{L-1}\boldsymbol{\eta}[n+jL]\,e^{-i\omega_d n} \qquad (10)
\end{aligned}$$

Sample error in the noise covariance estimate is expressed by $\mathbf{E}_j$, such that

$$\hat{\mathbf{R}}_{\eta,j} = \frac{1}{L}\sum_n \boldsymbol{\eta}[n]\boldsymbol{\eta}^H[n] = \mathbf{R}_\eta + \mathbf{E}_j$$

Equation (9) indicates that the terms involving $\mathbf{n}_j$ or $\mathbf{E}_j$ account for all sample estimation error in $\hat{\mathbf{R}}_{x,j}$.

Elements of $\mathbf{n}_j$ and $\mathbf{E}_j$ are zero-mean random variables, while all other terms are deterministic. We use the standard deviations of $\mathbf{n}_j$ and $\mathbf{E}_j$ for a fair comparison of relative sizes in their expected contributions to $\hat{\mathbf{R}}_{x,j}$. For Gaussian $s[n]$ and $\boldsymbol{\eta}[n]$, stdv$\{\mathbf{E}_j\} = \sigma_z^2/\sqrt{L}$ where $\sigma_z^2 = \sigma_s^2 + \sigma_\eta^2$ [2], and stdv$\{\mathbf{n}_j\} = \mathcal{O}(\sqrt{L}\,\sigma_z)$. An order analysis of terms in $\hat{\mathbf{R}}_{x,j}$, taken in sequence as they appear in (9), yields $\hat{\mathbf{R}}_{x,j} = \mathcal{O}(\sigma_d^2) + \mathcal{O}((\sigma_d\,\sigma_z)/\sqrt{L}) + \mathcal{O}(\sigma_\eta^2) + \mathcal{O}(\sigma_s^2) + \mathcal{O}(\sigma_z^2/\sqrt{L})$.

Given the stated assumptions, $\frac{\sigma_d\,\sigma_z}{\sqrt{L}} \gg \sigma_s^2, \frac{\sigma_z^2}{\sqrt{L}}$, so we may neglect the final two terms. Indeed, any combination of signal power levels and L such that the first two terms dominate the last two will lead to scooping. The third term corresponds to $\sigma_\eta^2\mathbf{I}$, which is also neglected in the following analysis since its constant diagonal form does not affect eigenvectors. Thus

$$\hat{\mathbf{R}}_{x,j} \approx \sigma_d^2\left[\mathbf{v}_j\,\mathbf{v}_j^H + \frac{1}{L\sigma_d}\left(\mathbf{v}_j\,\mathbf{n}_j^H + \mathbf{n}_j\mathbf{v}_j^H\right)\right]. \qquad (11)$$

In the strong interferer case, the identified interferer subspace is dominated by these two leading terms, and the first two eigenvectors of the rank two matrix $\hat{\mathbf{R}}_{x,j}$ have the form $\hat{\mathbf{u}}_j = \mathbf{v}_j + \xi\mathbf{n}_j$. Using the approach of [3], the exact solution for the dominant eigenvector is found as follows. The eigen equation can be expressed as

$$\begin{aligned}\lambda \hat{\mathbf{u}}_j &= \hat{\mathbf{R}}_{x,j} \hat{\mathbf{u}}_j \\ \lambda(\mathbf{v}_j + \xi \mathbf{n}_j) &= \left[\sigma_d^2 \mathbf{v}_j \mathbf{v}_j^H + \frac{\sigma_d}{L}\left(\mathbf{v}_j \mathbf{n}_j^H + \mathbf{n}_j \mathbf{v}_j^H\right)\right](\mathbf{v}_j + \xi \mathbf{n}_j) \\ \lambda \mathbf{v}_j + \lambda \xi \mathbf{n}_j &= \left(\alpha + \rho^* + (\rho + \beta) L \sigma_d \, \xi\right) \mathbf{v}_j + \left(\frac{\alpha}{L\sigma_d} + \rho \xi\right) \mathbf{n}_j\end{aligned}$$

Where $\alpha = \sigma_d^2 \|\mathbf{v}_j\|^2$, $\beta = \frac{1}{L^2}\|\mathbf{n}_j\|^2$, $\rho = \frac{\sigma_d}{L}\mathbf{v}_j^H \mathbf{n}_j$, and $(\cdot)^*$ denotes complex conjugate. Matching coefficients of $\mathbf{v}_j$ and $\mathbf{n}_j$ respectively and solving for λ yields the two equations

$$\begin{aligned}\lambda &= \alpha + \rho^* + (\rho + \beta) L \sigma_d \, \xi, \text{ and} \\ \lambda &= \frac{\alpha}{L \sigma_d \, \xi} + \rho.\end{aligned}$$

Thus $L\sigma_d \xi = \frac{\alpha}{\lambda - \rho}$, which when substituted into the first equation leads to

$$\lambda^2 - (\alpha + \rho^* + \rho)\lambda - \alpha\beta + |\rho|^2 = 0.$$

This is solved with the quadratic formula to yield

$$\lambda = \frac{\alpha + 2\text{Re}\{\rho\} \pm \sqrt{(\alpha + 2\text{Re}\{\rho\})^2 - 4(|\rho|^2 - \alpha\beta)}}{2}.$$

Substituting this result into $\xi = \frac{\alpha}{L\sigma_d(\lambda - \rho)}$ and taking the positive radical term produces the needed parameter to determine the dominant eigenvector:

$$\xi = \frac{2/(L\sigma_d)}{1 + \sqrt{\left(1 + 2\frac{\text{Re}\{\rho\}}{\alpha}\right)^2 - 4\left(\frac{|\rho|^2}{\alpha^2} + \frac{\beta}{\alpha}\right)} - i4\frac{\text{Im}\{\rho\}}{\alpha}}$$

For strong interference $\sigma_\eta/\sigma_d \to 0$, implying that $\text{Re}\{\rho\}/\alpha$, $|\rho|/\alpha$, β/α, and $\text{Im}\{\rho\}/\alpha \to 0$. So

$$\lim_{\sigma_\eta/\sigma_d \to 0} \xi = 1/(L\sigma_d)$$

and the dominant eigenvector is

$$\hat{\mathbf{u}}_{j,1} \approx \hat{\mathbf{u}}_{j,1} \approx \mathbf{v}_j + \frac{1}{L\sigma_d}\mathbf{n}_j. \tag{12}$$

The projection matrix for interference canceling is $\hat{\mathbf{P}}_j = \mathbf{I} - \hat{\mathbf{u}}_{j,1}\hat{\mathbf{u}}_{j,1}^H/(\hat{\mathbf{u}}_{j,1}^H \hat{\mathbf{u}}_{j,1})$, which, given (12), will be approximately orthogonal to $\mathbf{v}_j + \mathbf{n}_j/(L\sigma_d)$.

Now consider the PSD estimator for this scenario using non-overlapping ($O = 0$) FFT windows which match the STI window length ($N = L$), a rectangular window ($\mathbf{G} = \mathbf{I}$ and $\gamma = 1/L$), and subspace projection beamforming. Using these parameters and substituting $\mathbf{w}_j = \mathbf{P}_j \mathbf{w}$ and (7) into (3) yields

$$\hat{\mathbf{S}}_y^T = \frac{1}{LM}\sum_{j=0}^{M-1}\left|\mathbf{w}^H \hat{\mathbf{P}}_j \, \text{DFT}_L\left\{\mathbf{v}_j \sigma_d e^{i\omega_d[n+jL]} + \boldsymbol{\eta}[n + jL]\right\}\right|^{\odot 2} \tag{13}$$

Let $\omega_k = 2\pi k/N$ be the kth FFT frequency bin, and pick k_d such that $\omega_{k_d} = \omega_d$, the interference frequency. Then evaluating the DFT in (13) at ω_{k_d} and using (10) yields

$$\begin{aligned}
\hat{S}_y(k_d) &= \frac{1}{LM}\sum_{j=0}^{M-1}\left|\mathbf{w}^H\hat{\mathbf{P}}_j\sum_{n=0}^{L-1}\left(\mathbf{v}_j\sigma_d e^{i\omega_d[n+jL]} + \boldsymbol{\eta}[n+jL]\right)e^{-i\omega_d n}\right|^2 \\
&= \frac{1}{LM}\sum_{j=0}^{M-1}\left|\mathbf{w}^H\hat{\mathbf{P}}_j\left(Le^{i\omega_d jL}\sigma_d\mathbf{v}_j + e^{i\omega_d jL}\mathbf{n}_j\right)\right|^2 \\
&= \frac{1}{LM}\sum_{j=0}^{M-1}\left|Le^{i\omega_d jL}\sigma_d\right|^2\left|\mathbf{w}^H\hat{\mathbf{P}}_j\left(\mathbf{v}_j + \frac{1}{L\sigma_d}\mathbf{n}_j\right)\right|^2 \\
&= \frac{L\sigma_d^2}{M}\sum_{j=0}^{M-1}\left|\mathbf{w}^H\hat{\mathbf{P}}_j\left(\mathbf{v}_j + \frac{1}{L\sigma_d}\mathbf{n}_j\right)\right|^2 \qquad (14)\\
&\approx 0
\end{aligned}$$

which is approximately zero due to the orthogonality of $\hat{\mathbf{P}}_j$ and $\mathbf{v}_j + \frac{1}{L\sigma_d}\mathbf{n}_j$. Therefore a null is placed at ω_d, in bin k_d of $\hat{\mathbf{S}}_y$.

B. Eliminating Spectral Scooping

Let $\mathcal{N}_j = \{n \,|\, jL \leq n \leq (j+1)L-1\}$ be the L–element set of all sample indices in the jth STI. Partition $\mathcal{N}_j$ into non-overlapping subsets $\mathcal{N}_j^a$ and $\mathcal{N}_j^b$ with L_a and L_b elements respectively such that $\mathcal{N}_j = \mathcal{N}_j^a \cup \mathcal{N}_j^b$, $\mathcal{N}_j^a \cap \mathcal{N}_j^b = \phi$ (the empty set), and $L = L_a + L_b$. Sample order is arbitrary. Define

$$\hat{\mathbf{R}}_{x,j}^a = \frac{1}{L_a}\sum_{n\in\mathcal{N}_j^a}\mathbf{x}[n]\mathbf{x}^H[n] \qquad (15)$$

which has dominant eigenvector $\hat{\mathbf{u}}_{j,1}^a \approx \mathbf{v}_j + 1/(L_a\sigma_d)\,\mathbf{n}_j^a$, where $\mathbf{n}_j^a$ is defined as in (10), but summing only over samples $n \in \mathcal{N}_j^a$. Let $\hat{\mathbf{P}}_j^a = \mathbf{I} - \hat{\mathbf{u}}_{j,1}^a(\hat{\mathbf{u}}_{j,1}^a)^H/((\hat{\mathbf{u}}_{j,1}^a)^H\hat{\mathbf{u}}_{j,1}^a)$, which by substituting from (12) can be expanded as

$$\begin{aligned}
\hat{\mathbf{P}}_j^a &\approx \mathbf{I} - \frac{(\mathbf{v}_j + \frac{1}{L\sigma_d}\mathbf{n}_j^a)(\mathbf{v}_j + \frac{1}{L\sigma_d}\mathbf{n}_j^a)^H}{\|\mathbf{v}_j + \frac{1}{L\sigma_d}\mathbf{n}_j^a\|^2} \\
&\approx \mathbf{I} - \frac{(\mathbf{v}_j + \frac{1}{L\sigma_d}\mathbf{n}_j^a)(\mathbf{v}_j + \frac{1}{L\sigma_d}\mathbf{n}_j^a)^H}{\|\mathbf{v}_j\|^2} \\
&\approx \mathbf{I} - \frac{\mathbf{v}_j\mathbf{v}_j^H}{\|\mathbf{v}_j\|^2} - \frac{\mathbf{n}_j^a\mathbf{v}_j^H}{L_a\sigma_d\|\mathbf{v}_j\|^2} - \frac{\mathbf{v}_j(\mathbf{n}_j^a)^H}{L_a\sigma_d\|\mathbf{v}_j\|^2} - \frac{\mathbf{n}_j^a(\mathbf{n}_j^a)^H}{L_a^2\sigma_d^2\|\mathbf{v}_j\|^2} \\
\hat{\mathbf{P}}_j^a &\approx \mathbf{I} - \tilde{\mathbf{v}}_j\tilde{\mathbf{v}}_j^H - \frac{\mathbf{v}_j(\mathbf{n}_j^a)^H + \mathbf{n}_j^a\tilde{\mathbf{v}}_j^H}{L_a\sigma_d\|\mathbf{v}_j\|}
\end{aligned}$$

where $\tilde{\mathbf{v}}_j = \mathbf{v}_j/\|\mathbf{v}_j\|$ and in the final approximation the last term was neglected since it is very small due to division by L_a^2.

Using $\hat{\mathbf{P}}_j^a$ in place of $\hat{\mathbf{P}}_j$ in (14), the projection product becomes

$$\begin{aligned}
\hat{\mathbf{P}}_j^a \left(\mathbf{v}_j + \frac{1}{L\sigma_d}\mathbf{n}_j\right) &\approx \\
&\left(\mathbf{I} - \tilde{\mathbf{v}}_j\tilde{\mathbf{v}}_j^H - \frac{\mathbf{v}_j(\mathbf{n}_j^a)^H + \mathbf{n}_j^a\tilde{\mathbf{v}}_j^H}{L_a\sigma_d\|\mathbf{v}_j\|}\right)(\mathbf{v}_j + \frac{1}{L\sigma_d}\mathbf{n}_j) \\
&= \mathbf{v}_j - \mathbf{v}_j - \frac{\mathbf{n}_j^a}{L_a\sigma_d} + \frac{\mathbf{n}_j^a + \mathbf{n}_j^b}{L\sigma_d} - \frac{\tilde{\mathbf{v}}_j^H\mathbf{n}_j}{L\sigma_d}\tilde{\mathbf{v}}_j - \frac{(\mathbf{n}_j)^H\tilde{\mathbf{v}}_j}{L_a\sigma_d}\tilde{\mathbf{v}}_j - \frac{(\tilde{\mathbf{v}}_j^H\mathbf{n}_j)\mathbf{n}_j^a + ((\mathbf{n}_j^a)^H\mathbf{n}_j)\mathbf{v}_j}{L_aL\sigma_d^2\|\mathbf{v}_j\|} \\
&\approx \frac{\mathbf{n}_j^b - \frac{L_b}{L_a}\mathbf{n}_j^a - \left(\tilde{\mathbf{v}}_j^H\mathbf{n}_j + \frac{L}{L_a}(\mathbf{n}_j^a)^H\tilde{\mathbf{v}}_j\right)\tilde{\mathbf{v}}_j}{L\sigma_d} \qquad (16)
\end{aligned}$$

where we have used $\mathbf{n}_j = \mathbf{n}_j^a + \mathbf{n}_j^b$, $L - L_a = L_b$, and the final term from the previous line was dropped in (16) due to division by large L_aL.

Substituting (16) into (14) and taking expectation yields

$$\begin{aligned}
E\{\hat{S}_y(k_d)\} &\approx \frac{1}{LM}\sum_{j=0}^{M-1} E\left\{\left|\mathbf{w}^H\mathbf{n}_j^b - \frac{L_b}{L_a}\mathbf{w}^H\mathbf{n}_j^a - \left(\tilde{\mathbf{v}}_j^H\mathbf{n}_j + \frac{L}{L_a}(\mathbf{n}_j^a)^H\tilde{\mathbf{v}}_j\right)\mathbf{w}^H\tilde{\mathbf{v}}_j\right|^2\right\} \\
&\approx \frac{1}{LM}\sum_{j=0}^{M-1} E\left\{\left|\mathbf{w}^H\mathbf{n}_j^b - \frac{L_b}{L_a}\mathbf{w}^H\mathbf{n}_j^a\right|^2\right\} \\
&\approx \frac{1}{LM}\sum_{j=0}^{M-1} \mathbf{w}^H E\left\{\mathbf{n}_j^b(\mathbf{n}_j^b)^H\right\}\mathbf{w} + \frac{L_b^2}{L_a^2}\mathbf{w}^H E\left\{\mathbf{n}_j^a(\mathbf{n}_j^a)^H\right\}\mathbf{w} \\
&\approx \frac{1}{LM}\sum_{j=0}^{M-1} \mathbf{w}^H\mathbf{w}L_b\sigma_\eta^2 + \mathbf{w}^H\mathbf{w}\frac{L_b^2}{L_a}\sigma_\eta^2 \\
&\approx \frac{L_b}{L_a}\sigma_\eta^2. \qquad (17)
\end{aligned}$$

To justify the approximation in the second line, we note that $\|\mathbf{w}\| = \|\tilde{\mathbf{v}}_k\| = 1$ and that $\tilde{\mathbf{v}}_j$ is the normalized array response vector for the interferer. Thus in the typical scenario where the interferer is observed in the sidelobe pattern of the quiescent beamformer, $\mathbf{w}^H\tilde{\mathbf{v}}_j$ is small, e.g. at least 20 dB below the mainlobe response. The final term under the magnitude square of the first line will be significantly attenuated relative to the first two terms. Line three exploits statistical independence of $\mathbf{n}_j^a$ and $\mathbf{n}_j^b$ due to noise being zero mean and temporally i.i.d. and sample sets $\mathcal{N}_j^a$ and $\mathcal{N}_j^b$ being disjoint. Line four has used the assumption that noise is i.i.d. across array elements, and the fact that $E\{|\mathbf{n}_j^a|^2\} = L_a\,\sigma_\eta^2$, and $E\{|\mathbf{n}_j^b|^2\} = L_b\,\sigma_\eta^2$. Line five relies on $\mathbf{w}^H\mathbf{w} = \|\mathbf{w}\|^2 = 1$ and some algebraic manipulation.

For $L_a = L_b = L/2$ the estimator in (17) is approximately unbiased, with $E\{\hat{S}_y(k_d)\} = \sigma_\eta^2$. This is the basis for our proposed algorithm to eliminate scooping. On the other hand, for $L_a = L$, $L_b = 0$, $E\{\hat{S}_y(k_d)\} = 0$, which is the full scooping case demonstrated by (14).

III. Results and Demo Code

A demonstration program to illustrate scooping behavior is available in [1]. This MATLAB (R) script shows how scooping occurs with both LCMV and subspace projection beamforming, and how the proposed algorithm corrects scooping. The results similar to those presented in the IEEE SP Letters paper can be generated with this code.

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