QPSK and OQPSK in Frequency Nonselective Fading

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QPSK and OQPSK
in Frequency Nonselective Fading

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Chapter 1

Introduction

In a frequency nonselective fading (or "flat fading") environment, the transmitted signal experiences an unknown attenuation and phase shift and the addition of thermal noise. In packetized communications, this unknown attenuation and phase shift may be estimated through the use of a known sequence of pilot symbols embedded in the packet. With any estimator, the estimate is not perfect and the resulting estimator error reduces the performance of the algorithm that uses the estimate. This report compares the impact of the estimator error for QPSK and Offset QPSK (OQPSK) and shows that the estimator errors impact QPSK and OQPSK differently. The degradation in OQPSK is more pronounced than the degradation in its non-offset counterpart.

1.1 Overview

We assume burst-mode communication as opposed to continuous or streaming communication. In each burst, a finite number of symbols are transmitted. In a frequency nonselective fading environment, the transmitted signal \( s(t) \) experiences an attenuation \( h \) and the addition of thermal noise \( w(t) \). Consequently, the received signal is

\[ r(t) = hs(t) + w(t). \]

Because we use complex-baseband equivalent signals, the channel attenuation \( h \) is a complex quantity, the magnitude of which defines the amplitude attenuation experienced during propagation and the phase of which defines the phase shift experienced during propagation. In this report, \( s(t) \) is either a QPSK or Offset QPSK signal comprised of \( L_0 \) pilot symbols and \( L_d \) data symbols whose arrangement is discussed in the next section.
CHAPTER 1. INTRODUCTION

The pilot symbols are used by the receiver to estimate the complex-valued quantity \( h \). The maximum likelihood estimator for \( h \) is derived for both QPSK and OQPSK. The developments start from basic principles. To maintain as general a development as possible, no preference is given to any particular receiver implementation, other than assuming the availability of complex-baseband equivalent continuous-time signals. Consequently, the derivations are carried out in the continuous time (or waveform) domain because waveforms are what are transmitted from an antenna, through the channel, and acquired by the antenna on the receiving end. The channel estimate is used by the detector to compensate for the amplitude and phase shifts caused by the channel. Because this report is restricted to QPSK and OQPSK, compensation by the amplitude is of minor importance: phase compensation is the primary use of the estimate for \( h \). One of the consequences of the additive noise is that the estimate for \( h \) is not perfect. Consequently, residual amplitude (minor) and phase shift (potentially major) errors are present in the decision-making process. The impact of the estimator error on both the estimator accuracy and on the bit error rate performance of (O)QPSK is examined in detail.

1.2 Packet Structure

This report assumes packetized communications where each packet consists of \( L_0 \) known pilot symbols and \( L_d \) data symbols. There are many possible arrangements involving the two sets of symbols. There are three arrangements involving a contiguous block of \( L_0 \) pilot symbols. These arrangements are illustrated in Figure 1.1. The preamble option, illustrated in Figure 1.1 (a) places the \( L_0 \) known pilot symbols at the beginning of the packet; the midamble option, illustrated in Figure 1.1 (b) places the \( L_0 \) known pilot symbols somewhere near the midpoint of the packet; and the postamble option, illustrated in Figure 1.1 (c) places the \( L_0 \) known pilot symbols at the end of the packet. For notational convenience, we assume that the time index origin is aligned with the beginning of the block of pilot symbols. The notation is completely general. For example, when the pilot symbols are arranged in a preamble as illustrated in Figure 1.1 (a), the symbols \( a(n) \) and \( b(n) \) are zero for \( n < 0 \). Similarly, when the pilot symbols are arranged in a postamble as illustrated in Figure 1.1 (c), the symbols \( a(n) \) and \( b(n) \) are zero for \( n \geq L_0 \).
1.3. THERMAL NOISE AND PROPER COMPLEX GAUSSIAN RANDOM VARIABLES

In the sections that follow, the complex-valued baseband representation is used. Consequently, complex-valued Gaussian random processes and variables are of interest. Reference is made to proper complex-valued Gaussian random variables and this needs to be defined. The definition used here is based on a zero-mean complex-valued Gaussian random variable. Extensions to complex-valued Gaussian random vectors and complex-valued Gaussian random processes (with arbitrary finite mean) are straight forward.

Let \( Z = X + jY \) be a zero-mean complex-valued Gaussian random variable. This means that the real part \( X \) and the imaginary part \( Y \) are jointly Gaussian zero-mean Gaussian random variables. The joint probability density function (pdf) of \( X \) and \( Y \) is

\[
f_{X,Y}(x, y) = \frac{1}{2\pi|R|^{1/2}} \exp\left\{ -\frac{1}{2} \begin{bmatrix} X & Y \end{bmatrix} R^{-1} \begin{bmatrix} X \\ Y \end{bmatrix} \right\} \tag{1.1}\]

Figure 1.1: Three options for the packet structure: (a) the preamble, (b) the midamble, (c) the postamble.

1.3 Thermal Noise and Proper Complex Gaussian Random Variables
where
\[
R = E\left\{ \begin{bmatrix} X \\ Y \end{bmatrix} \begin{bmatrix} X & Y \end{bmatrix} \right\} = E\left\{ \begin{bmatrix} XX & XY \\ YX & YY \end{bmatrix} \right\} = \begin{bmatrix} \sigma_{XX}^2 & \sigma_{XY}^2 \\ \sigma_{YX}^2 & \sigma_{YY}^2 \end{bmatrix}
\]
(1.2)
and \(|R|\) is the determinant of the matrix \(R\). A proper complex-valued Gaussian random variable is a complex-valued Gaussian random variable whose autocovariance matrix satisfies the conditions\(^1\):\[
\sigma_{XX}^2 = \sigma_{YY}^2, \quad \sigma_{XY}^2 = -\sigma_{YX}^2.
\]
(1.3)
The special case \(\sigma_{XY}^2 = \sigma_{YX}^2 = 0\) is the most relevant to the discussion that follows. Here, the real and imaginary parts are uncorrelated Gaussian random variables with equal variance. The autocorrelation matrix \(R\) corresponding to a proper complex-valued Gaussian random variable is
\[
R = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}
\]
(1.4)
where \(\sigma^2 = \sigma_{XX}^2 = \sigma_{YY}^2\). The quadratic form in the exponent reduces to
\[
\begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{\sigma^2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{x^2 + y^2}{\sigma^2}
\]
(1.5)
and the determinant of the autocorrelation matrix reduces to
\[
|R| = \left| \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix} \right| = \sigma^4.
\]
(1.6)
Inserting these into the pdf (1.1) gives
\[
f_{X,Y}(x,y) = \frac{1}{2\pi \sigma^2} \exp\left\{ -\frac{x^2 + y^2}{2\sigma^2} \right\}.
\]
(1.7)
The term \(x^2 + y^2\) can be interpreted as the magnitude squared of \(z = x + jy\) and \(\sigma^2\) can be interpreted as
\[
\sigma^2 = \frac{1}{2} E\{ZZ^*\} = \frac{1}{2} \left[ E\{X^2\} + E\{Y^2\} \right] = \frac{1}{2} \left[ E\{X^2\} + E\{Y^2\} \right] = \frac{1}{2} \left[ 2\sigma^2 \right].
\]
(1.8)
\(^1\)More formally, a complex random variable \(Z\) is defined by its complex-valued mean \(\mu_Z = E\{Z\}\), the covariance \(E\{(Z - \mu_Z)(Z - \mu_Z)^*\}\), and the pseudocovariance \(E\{(Z - \mu_Z)(Z - \mu_Z)\}\). The most strict definition of a proper complex-valued random variable is one whose pseudocovariance is zero. Expressing the pseudocovariance in terms of the covariances of the real and imaginary parts of \(Z\) produces the conditions listed here.
1.3. THERMAL NOISE AND PROPER COMPLEX GAUSSIAN RANDOM VARIABLES

These interpretations allow the pdf (1.7) to be re-expressed as

\[ f_Z(z) = \frac{1}{2\pi\sigma^2} \exp \left\{ -\frac{|z|^2}{2\sigma^2} \right\}. \]  
(1.9)

Some authors prefer to express the pdf of a proper, zero-mean complex-valued Gaussian random variable in terms of

\[ \sigma_2^2 = E \{ ZZ^* \}. \]  
(1.10)

This is identical to the definition (1.8) except for the factor 1/2. In this case the pdf is of the form

\[ f_Z(z) = \frac{1}{\pi\sigma_2^2} \exp \left\{ -\frac{|z|^2}{\sigma_2^2} \right\}. \]  
(1.11)

The complex-valued baseband equivalent of a white band-pass, zero-mean, Gaussian random process is a proper zero-mean complex-valued Gaussian random process. Let \( w_{bp}(t) \) be a white band-pass Gaussian random process centered at \( f_0 \). The power spectral density of \( w_{bp}(t) \) is

\[ S_{bp}(f) = \begin{cases} \frac{N_0}{2} & f_0 - B \leq |f| \leq f_0 + B \\ 0 & \text{otherwise} \end{cases} \]  
(1.12)

and is illustrated in Figure 1.2 (a). Now let \( w(t) \) be the complex-valued baseband equivalent of \( w_{bp}(t) \). The power spectral density of \( w(t) \) is given by [2]

\[ S_w(f) = \begin{cases} 2N_0 & |f| \leq B \\ 0 & \text{otherwise} \end{cases} \]  
(1.13)

and is illustrated in Figure 1.2 (b). Note we may write

\[ w(t) = w_R(t) + jw_I(t) \]

---

2Strictly speaking, a random process cannot be simultaneously white and band-pass. The convention adopted in the theory of bandpass communications is to label a random process that is constant over the frequencies of interest and zero everywhere else as a “white band-pass” random process [2].
where $w_R(t)$ and $w_I(t)$ are real-valued Gaussian random processes with power spectral densities

$$S_R(f) = S_I(f) = \begin{cases} N_0 & |f| \leq B \\ 0 & \text{otherwise} \end{cases} \quad (1.14)$$

as illustrated in Figure 1.2 (c). Note that

$$2BN_0 = \frac{1}{2} \mathbb{E}\{|w(t)|^2\} = \frac{1}{2} \mathbb{E}\{[w_R(t) + jw_I(t)][w_R(t) - jw_I(t)]\} = \frac{1}{2} \mathbb{E}\{w_R^2(t) + w_I^2(t) + j[w_I(t)w_R(t) - w_R(t)w_I(t)]\} = \frac{1}{2} \mathbb{E}\{w_R^2(t)\} + \frac{1}{2} \mathbb{E}\{w_I^2(t)\} + 0 \quad \text{(because } w(t) \text{ is proper)} \quad (1.15)$$

from which we must have

$$\mathbb{E}\{w_R^2(t)\} = \mathbb{E}\{w_I^2(t)\} = 2N_0B \quad (1.16)$$

as shown in Figure 1.2 (c).

### 1.4 Signal Energy and Noise Power

The signal energy for the complex-valued low-pass equivalent signal $s(t)$ is given by

$$E_{avg} = \frac{1}{2} \int_{L_pT_s} |s(t)|^2 dt \quad (1.17)$$

where $L_pT_s$ is the interval of support for a single symbol as explained in Section 2.1. Additional meaning to the definition (1.17) results from considering the case where $s(t)$ consists of one symbol. For QPSK, using (2.1) for the symbol index $k = 0$, we have

$$s(t) = [a(0) + jb(0)]p(t) \quad (1.18)$$

from which we obtain

$$|s(t)|^2 = [a(0)^2 + b(0)^2]p^2(t) = [A^2 + A^2]p^2(t) = 2A^2p^2(t). \quad (1.19)$$
1.4. SIGNAL ENERGY AND NOISE POWER

\[ f \]

\[ f_0 - f_0 \]

\[ N_0 \]

\[ S_{bp}(f) \]

\[ f \]

\[ 2B \]

\[ -f_0 - B \]

\[ -f_0 \]

\[ f_0 - B \]

\[ f_0 \]

\[ f_0 + B \]

\[ f \]

\[ S(f) \]

\[ 2N_0 \]

\[ -B \]

\[ 0 \]

\[ B \]

\[ S_R(f) \]

\[ N_0 \]

\[ -B \]

\[ 0 \]

\[ B \]

\[ S_I(f) \]

\[ N_0 \]

\[ -B \]

\[ 0 \]

\[ B \]

Figure 1.2: Power spectral densities of a white bandpass random process: (a) the power spectral density of the real-valued bandpass random process; (b) the power spectral density of the complex-valued low-pass equivalent process; (c) the power spectral densities of the real and imaginary components of the complex-valued low-pass equivalent process.
The average symbol energy is

\[ E_{\text{avg}} = \frac{1}{2} \int_{T_1}^{T_2} 2A^2 p^2(t)dt = A^2 \int_{T_1}^{T_2} p^2(t)dt. \]  

The integral term in (1.20) is the pulse shape energy. The average bit energy is

\[ E_b = \frac{E_{\text{avg}}}{2} = \frac{A^2}{2} \int_{T_1}^{T_2} p^2(t)dt. \]  

In this work, the definition (1.8) is used. Consequently, the form of the pdf of the complex-valued random variable \( Z \) is (1.9). The power spectral density complex-valued low-pass equivalent noise \( w(t) \) accompanying the complex-valued low-pass equivalent signal \( s(t) \) is \( 2N_0 \) [see Figure 1.2 (b)]. The power of this noise, in the bandwidth \( B \) is

\[ \text{noise power} = \frac{1}{2} E \{|w(t)|^2\} = \frac{1}{2} \int_{-\infty}^{\infty} S(f)df = \frac{1}{2} 4N_0B = 2N_0B. \]  

1.5 Log-Likelihood Function

As explained in Section 1.1, given the observation

\[ r(t) = hs(t) + w(t) \]  

an estimate of \( h \) is desired. The maximum likelihood approach is used throughout this work. The maximum likelihood estimator is based on the log-likelihood function, whose general form is developed here. The development follows what is now the standard technique:

1. The portion of the received signal (1.23) corresponding to the pilot sequence is sampled to produce a vector of samples.

2. The joint probability density function of the vector of noise samples is used to derive the joint probability density function of the vector of samples of \( r(t) \).

3. The log-likelihood function is derived.

4. The terms in the log-likelihood function that do not depend on \( h \) are discarded.
5. The sample rate is allowed to increase. The limiting form of the result is used as the log-likelihood function.

To create a discrete-time version of (1.23), we assume \( r(t) \) is first filtered by an ideal low-pass anti-aliasing filter whose transfer function is illustrated in Figure 1.3 where \( \Delta t \) is the time between samples. Assuming \( \Delta t \) is sufficiently small, signal component \( h_s(t) \) passes from the input to the output of this low-pass filter without modification. The resulting noise component is a complex-valued Gaussian random process whose power spectral density is a special case of Figure 1.2 (b) with \( B = 1/(2\Delta t) \). The autocorrelation function of the band-limited noise process is

\[
R(\tau) = \frac{1}{2} \{ w(t) w^*(t - \tau) \} = \frac{N_0}{\Delta t} \frac{\sin \left( \frac{\pi \tau}{\Delta t} \right)}{\pi \tau / \Delta t} \tag{1.24}
\]

Now suppose the pilot symbols embedded in \( s(t) \) are transmitted during the interval \( 0 \leq t \leq T_0 \). Let the vector formed from the \( \Delta t \)-spaced samples of \( r(t) \) for \( 0 \leq t \leq T_0 \) be

\[
r = [r(0), r(\Delta t), \ldots, r(n\Delta t), \ldots, r((N - 1)\Delta t)]^T
\]

where \( N = T_0 / \Delta t \). Because \( r(n\Delta t) = h_s(n\Delta t) + w(n\Delta t) \) we may write

\[
r = h_s + w \tag{1.25}
\]

where

\[
s = [s(0), s(\Delta t), \ldots, s(n\Delta t), \ldots, s((N - 1)\Delta t)]^T
\]
are known because the pilot symbols are assumed known, and
\[
\mathbf{w} = [w(0), w(\Delta t), \ldots, w(n\Delta t), \ldots, w((N - 1)\Delta t)]^T.
\]

The vector \(\mathbf{w}\) comprises proper complex-valued Gaussian random variables each with zero mean and autocorrelation function
\[
\frac{1}{2} \mathbb{E} \{ w(n\Delta t) w^*((n - k)\Delta t) \} = R(k\Delta t) = \begin{cases} \frac{N_0}{\Delta t} & k = 0 \\ 0 & k \neq 0. \end{cases}
\]
(1.26)

Consequently, the joint probability density function of the random vector \(\mathbf{w}\) is
\[
f(\mathbf{w}) = \frac{1}{\left(2\pi\frac{N_0}{\Delta t}\right)^N} \exp \left\{ -\frac{\Delta t}{2N_0} \mathbf{w}^\dagger \mathbf{w} \right\} = \frac{1}{\left(2\pi\frac{N_0}{\Delta t}\right)^N} \exp \left\{ -\frac{\Delta t}{2N_0} \sum_{n=0}^{N-1} |w(n\Delta t)|^2 \right\}. \tag{1.27}
\]

Leveraging this result, the joint probability density function of \(\mathbf{r}\) may be expressed as
\[
f(\mathbf{r}; h) = \frac{1}{\left(2\pi\frac{N_0}{\Delta t}\right)^N} \exp \left\{ -\frac{\Delta t}{2N_0} \sum_{n=0}^{N-1} |r(n\Delta t) - hs(n\Delta t)|^2 \right\}. \tag{1.28}
\]
This is the likelihood function. The log-likelihood function is
\[
\Lambda(h) = -N \log \left(2\pi\frac{N_0}{\Delta t}\right) - \frac{\Delta t}{2N_0} \sum_{n=0}^{N-1} |r(n\Delta t) - hs(n\Delta t)|^2. \tag{1.29}
\]
Because the first term on the right-hand side of (1.29) is not a function of \(h\), it may be discarded in the following development. Thus moving forward, we use as the log-likelihood function
\[
\Lambda(h) = -\frac{\Delta t}{2N_0} \sum_{n=0}^{N_1} |r(n\Delta t) - hs(n\Delta t)|^2. \tag{1.30}
\]
Now let \(\Delta t \to 0\). In the limit we have
\[
\Delta t \sum_{n=0}^{N-1} |r(n\Delta t) - hs(n\Delta t)|^2 \overset{\Delta t \to 0}{\to} \int_0^{T_0} |r(t) - hs(t)|^2 dt. \tag{1.31}
\]
This gives the final form for log-likelihood function used in this work:

$$
\Lambda(h) = -\frac{1}{2N_0} \int_0^{T_0} |r(t) - hs(t)|^2 dt.
$$

(1.32)

### 1.6 Computer Simulations

Computer simulations necessarily require discrete-time representations of the signals and noise. Equipped with the definitions (1.8), (1.20), and (1.21), the scaling of the noise and signal energy are the important issues here.

The computer simulations are based on $\Delta t$-spaced samples of the complex-valued low-pass equivalent continuous-time waveform $s(t)$. Assuming the Sampling Theorem has been satisfied (that is, $s(t)$ may be reconstructed from its samples $s(n\Delta t)$ for $n = 0, 1, \ldots$), the average bit energy (1.21) may be expressed as

$$
E_b = \frac{A^2}{2} \int_{T_1}^{T_2} p^2(t) dt \approx \frac{A^2}{2} \Delta t \sum_{n=N_1}^{n=N_2} p^2(n\Delta t)
$$

(1.33)

where $N_1 = T_1/\Delta t$ and $N_2 = T_2/\Delta t$. In the simulations, the pulse shape samples $p(n\Delta t)$ are scaled so that

$$
\sum_{n=N_1}^{n=N_2} p^2(n\Delta t) = 1.
$$

(1.34)

Consequently, the average bit energy is

$$
E_b = A^2 \Delta t.
$$

(1.35)

Moving to the noise, we assume the complex-valued low-pass equivalent noise process $w(t)$ has been filtered by the low-pass anti-aliasing filter shown in Figure 1.3. Consequently, $w(t)$ is a wide-sense stationary, complex-valued Gaussian random process with zero mean and power spectral density

$$
S_w(f) = \begin{cases} 
2N_0 & -\frac{1}{2\Delta t} \leq f \leq \frac{1}{2\Delta t} \\
0 & \text{otherwise}
\end{cases}
$$

(1.36)
[cf., Figure 1.2 (b) with $B = 1/(2\Delta t)$]. The autocorrelation is

$$R(\tau) = \frac{1}{2} \mathbb{E}\{w(t)w^*(t-\tau)\} = \frac{N_0}{\Delta t} \frac{\sin\left(\frac{\pi \tau}{\Delta t}\right)}{\pi \tau}.$$  

(1.37)

A vector of $N \Delta t$-spaced samples, denoted $w(n\Delta t)$ for $n = 0, 1, \ldots, N - 1$, comprises a vector of Gaussian random variables, each with zero mean. The autocorrelation of this sequence is

$$\frac{1}{2} \mathbb{E}\{w(n\Delta t)w^*((n-k)\Delta t)\} = R(k\Delta t) = \frac{N_0}{\Delta t} \frac{\sin\left(\frac{\pi k}{\Delta t}\right)}{\pi k} = \begin{cases} N_0/\Delta t & k = 0 \\ 0 & k \neq 0. \end{cases}$$  

(1.38)

Consequently, the random vector comprises $N$ independent complex-valued Gaussian random variables each with zero mean and variance $N_0/\Delta t$. In computer simulations, complex random variables are usually generated from a pair of real-valued random variables. That is, the $n$-th random variable in this random vector may be expressed as

$$w(n\Delta t) = w_R(n\Delta t) + jw_I(n\Delta t).$$  

(1.39)

Because $w(n\Delta t)$ is a proper complex-valued Gaussian random variable, the random variables $w_R(n\Delta t)$ and $w_I(n\Delta t)$ are uncorrelated Gaussian random variables. Furthermore, because $w(n\Delta t)$ is a sequence of uncorrelated complex-valued Gaussian random variables, $w_R(n\Delta t)$ is a sequence of uncorrelated Gaussian random variables and $w_I(n\Delta t)$ is a sequence of uncorrelated Gaussian random variables, with

$$w_R(n\Delta t) \sim N\left(0, \frac{N_0}{\Delta t}\right) \quad w_I(n\Delta t) \sim N\left(0, \frac{N_0}{\Delta t}\right).$$  

(1.40)

Let $\sigma^2 = N_0/\Delta t$ be the variance of the $w_R(n\Delta t)$ and the variance of $w_I(n\Delta t)$. The desire is to express $\sigma^2$ in terms of the “signal-to-noise ratio” $E_b/N_0$. This is accomplished by combining $N_0 = \Delta t\sigma^2$ with (1.35):

$$\frac{E_b}{N_0} = \frac{1}{2} \frac{A^2\Delta t}{\sigma^2\Delta t} = \frac{A^2}{2\sigma^2}.$$  

(1.41)
1.7. REPORT SUMMARY

Solving this for $\sigma^2$ produces the desired result:

$$\sigma^2 = \frac{A^2}{2 \left( \frac{E_b}{N_0} \right)}.$$ (1.42)

1.7  Report Summary

Chapter 2 is devoted to QPSK and includes the following:

1. The ML estimator for $h$ is derived and is given by (2.27). This estimator is unbiased and efficient.

2. An expression for the probability of error in the presence of estimation errors is derived and given by (2.121). This expression is a function of the received “signal-to-noise” ratio and the pilot sequence length.

Chapter 3 is devoted to OQPSK and includes the following:

1. The ML estimator for $h$ is derived and is given by (3.71). The estimator is unbiased on average, but biased for any particular data sequence.

2. The estimator error variance is inversely proportional to the the signal-to-noise ratio $E_b/N_0$ and the pilot sequence length $L_0$.

3. The estimator error variance is derived and given by (3.120). Equation (3.120) contains a term that does not decrease with increasing $E_b/N_0$ and defines a floor for the estimator error variance. That is, for $E_b/N_0$ sufficiently large, the estimator error variance is a constant. This constant represents the point at which thermal noise no longer dominates the estimator performance. Estimator performance is dominated by the I/Q interference effect quantified by $\mathcal{X}$.

4. The estimator error variance floor is inversely proportional to the square of the length of the pilot sequence. It is also a function of pulse shape autocorrelation function and the length of the pulse shape.

5. Comparing the estimator error variance (3.120) with the lower bound (C.29) shows that the estimator (3.71) is only asymptotically (in $L_0$) efficient.
6. Estimation errors introduce data-dependent I/Q interference into the decision variables for OQPSK. The estimation error and accompanying interference degrade the bit error rate performance.

7. The standard analysis techniques lead to an equation for the bit error probability given by (3.201). This expression requires the enumeration of all bit patterns that influence the decision variable. While this approach is manageable for full response pulse shapes (e.g., NRZ), it requires significant computation for partial response pulse shapes (e.g., SRRC).

8. Four examples illustrate the above points: Examples 1 and 2 are devoted to full-response pulse shapes (NRZ) whereas Examples 3 and 4 are devoted to partial response pulse shapes (SRRC). For the length-4 pilot sequence considered, the performance loss at $P_b = 10^{-6}$ is less than 1 dB for the NRZ pulse shape but almost 2 dB for the SRRC pulse shape. For the length-16 pilot sequence considered, the performance loss at $P_b = 10^{-6}$ is approximately 0.07 dB for the NRZ pulse shape and 0.1 dB for the SRRC pulse shape. This confirms the result predicted by the analysis that the performance loss decreases as $L_0$ increases. The improvement is due to the reduced estimation error.

Several auxiliary results are developed in Appendixes A, B, and C.
Chapter 2

QPSK

2.1 Signal Model

The transmitted signal is of the form

\[ s(t) = \sum_k [a(k) + jb(k)]p(t - kT_s) \] (2.1)

where \( a(k) + jb(k) \in \mathcal{A} \) is the \( k \)-th QPSK constellation point; \( p(t) \) is a real-valued unit-energy pulse shape; \( T_s \) is the symbol time (in s/symbol). The set of constellation points \( \mathcal{A} \) is

\[ \mathcal{A} = \{ A + jA, A - jA, -A + jA, -A - jA \} . \] (2.2)

As explained in Section 1.4 the average symbol energy \( E_{\text{avg}} \) is given by

\[ E_{\text{avg}} = \frac{1}{2} |a(k) + jb(k)|^2 = A^2. \] (2.3)

The pulse shape has support on the interval \( T_1 \leq t \leq T_2 \). For example, in the case of the NRZ pulse shape, \( T_1 = 0 \) and \( T_2 = T_s \). For the case of the SRRC pulse shape, \( T_2 = -T_1 = LT_s \) for a positive integer \( L \). For a discussion of appropriate values for \( L \) see Appendix A of [3]. In what follows the length of the pulse shape in symbols is

\[ L_p = \frac{T_2 - T_1}{T_s} . \] (2.4)
The pulse shape is also characterized by its autocorrelation function \( r_p(\tau) \) defined by

\[
r_p(\tau) = \int_{T_1}^{T_2} p(t)p(t-\tau)dt.
\] (2.5)

The Nyquist No-ISI condition constrains the values \( r_p(\tau) \) for \( \tau \) an integer multiple of the symbol time \( T_s \):

\[
r_p(kT_s) = \begin{cases} 
1 & k = 0 \\
0 & k \neq 0
\end{cases}.
\] (2.6)

The received signal is

\[
r(t) = h s(t) + w(t)
\] (2.7)

where \( h = h_R + jh_I \) is a complex-valued channel gain and \( w(t) = w_R(t) + jw_I(t) \) is a proper complex-valued zero-mean Gaussian random process with

\[
E[w_R(t)w_R(t-\tau)] = E[w_I(t)w_I(t-\tau)] = N_0\delta(\tau)
\]

\[
E[w_R(t)w_I(t-\tau)] = E[w_R(t)w_I(t-\tau)] = 0.
\] (2.8)

In the development that follows, perfect timing synchronization is assumed.
2.2 Channel Estimator

2.2.1 Derivation

The maximum-likelihood estimator observes \( r(t) \) for the interval spanning \( L_0 \) symbols times and produces an estimate for \( h \) — denoted \( \hat{h} \) — based on the maximum likelihood principle. Because the pulse shape has support on \( T_1 \leq t \leq T_2 \), the observation interval is \( T_1 \leq t \leq T_2 + (L_0 - 1)T_s \) and the log-likelihood function for \( h \) is

\[
\Lambda(h) = -\frac{1}{2N_0} \int_{T_1}^{T_2+(L_0-1)T_s} \left| r(t) - h \sum_{i=0}^{L_0-1} [a(i) + jb(i)]p(t - iT_s) \right|^2 dt. \tag{2.9}
\]

The maximum likelihood estimate for \( h \) is

\[
\hat{h} = \arg\max_h \left\{ \Lambda(h) \right\}. \tag{2.10}
\]

A necessary condition for \( \hat{h} \) is that it forces the first derivative of log-likelihood function to 0. The first derivative of log-likelihood function is

\[
\frac{\partial}{\partial h^*} \Lambda(h) = -\frac{1}{2N_0} \int_{T_1}^{T_2+(L_0-1)T_s} \left\{ r(t) - h \sum_{i=0}^{L_0-1} [a(i) + jb(i)]p(t - iT_s) \right\} \times \left\{ r^*(t) - h^* \sum_{l=0}^{L_0-1} [a(l) - jb(l)]p(t - lT_s) \right\} dt
\]

\[
= -\frac{1}{2N_0} \int_{T_1}^{T_2+(L_0-1)T_s} r(t) \sum_{i=0}^{L_0-1} [a(i) + jb(i)]p(t - iT_s) \sum_{l=0}^{L_0-1} [a(l) - jb(l)]p(t - lT_s) dt
\]

\[
= -\frac{1}{2N_0} \int_{T_1}^{T_2+(L_0-1)T_s} r(t) \sum_{i=0}^{L_0-1} a(i)p(t - iT_s) \sum_{l=0}^{L_0-1} [a(l) - jb(l)]p(l - lT_s) dt \tag{2.11}
\]
The first integral in (2.11) may be recast in a more usable form by partitioning the interval of integration as follows:

\[
\frac{1}{2\sigma^2} \int_{T_1}^{T_2 + (L_0 - 1)T_s} r(t) \sum_{l=0}^{L_0-1} [a(l) - jb(l)]p(t - lT_s)dt
\]

\[
= \frac{1}{2\sigma^2} \sum_{k=0}^{L_0 + L_p - 2} \int_{T_1 + kT_s}^{T_1 + (k+1)T_s} r(t) \sum_{l=0}^{L_0-1} [a(l) - jb(l)]p(t - lT_s)dt
\]

(2.12)

where \(L_p\) is the length of the pulse in symbols and is given by (2.4). Now restrict the symbol index \(l\) to include only the waveforms that are included in the interval of integration.\(^1\) The pulse shape \(p(t - lT_s)\) has support on the interval \(T_1 + lT_s \leq t \leq T_2 + lT_s\) and this interval coincides with the region of integration \(T_1 + kT_s \leq t \leq T_1 + (k + 1)T_s\) when

\[
T_2 + lT_s > T_1 + kT_s \\
\text{and} \\
T_1 + lT_s < T_1 + (k + 1)T_s
\]

Solving the first inequality gives the lower value for \(l\):

\[
T_2 + lT_s > T_1 + kT_s \\
lT_s > T_1 - T_2 + kT_s \\
l > \frac{T_s - T_1}{T_s} + k = -L_p + k.
\]

(2.13)

and solving the second inequality gives the upper value for \(l\):

\[
T_1 + lT_s < T_1 + (k + 1)T_s \\
lT_s < (k + 1)T_s \\
l < k + 1.
\]

(2.14)

Applying the results to the inner summation over \(l\) produces

\(^1\)In doing so, we temporarily extend the range of valid indexes to incorporate all symbol-pulse-shape products that could be included in the interval of integration. This is done to facilitate the relationship between “edge effects” and pilot symbol arrangement. See Comment 7 below.
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\[
\frac{1}{2N_0} \int_{T_1}^{T_2+(L_0-1)T_s} r(t) \sum_{l=0}^{L_0-1} \left[ a(l) - jb(l) \right] p(t - lT_s) dt \\
= \frac{1}{2N_0} \sum_{k=0}^{L_0+L_p-2} \sum_{l=-L_p+k+1}^{k} \left[ a(l) - jb(l) \right] \int_{T_1+(k+1)T_s}^{T_1+kT_s} r(t) p(t - lT_s) dt.
\]

(2.15)

The right-hand-side of (2.15) is of the form

\[
\sum_{k=0}^{L_0+L_p-2} \sum_{l=-L_p+k+1}^{k} F(k, l).
\]

(2.16)

With the aid of Figure 2.1, reversing the order of summation produces

\[
\sum_{k=0}^{L_0+L_p-2} \sum_{l=-L_p+k+1}^{k} F(k, l) = \sum_{l=-L_p+1}^{-1} \sum_{k=0}^{l+L_p-1} F(k, l) \\
+ \sum_{l=0}^{L_0-1} \sum_{k=l}^{l+L_p-1} F(k, l) + \sum_{l=L_0}^{L_0+L_p-2} \sum_{k=l}^{L_0+L_p-2} F(k, l).
\]

(2.17)

Applying this result produces

\[
\frac{1}{2N_0} \int_{T_1}^{T_2+(L_0-1)T_s} r(t) \sum_{l=0}^{L_0-1} \left[ a(l) - jb(l) \right] p(t - lT_s) dt \\
= \frac{1}{2N_0} \sum_{l=-L_p+1}^{-1} \sum_{k=0}^{l+L_p-1} \left[ a(l) - jb(l) \right] \int_{T_1+kT_s}^{T_1+(k+1)T_s} r(t) p(t - lT_s) dt \\
+ \frac{1}{2N_0} \sum_{l=0}^{L_0-1} \sum_{k=l}^{l+L_p-1} \left[ a(l) - jb(l) \right] \int_{T_1+kT_s}^{T_1+(k+1)T_s} r(t) p(t - lT_s) dt \\
+ \frac{1}{2N_0} \sum_{l=L_0}^{L_0+L_p-2} \sum_{k=l}^{L_0+L_p-2} \left[ a(l) - jb(l) \right] \int_{T_1+kT_s}^{T_1+(k+1)T_s} r(t) p(t - lT_s) dt
\]

\[
+ \frac{1}{2N_0} \sum_{l=L_0}^{L_0+L_p-2} \sum_{k=l}^{L_0+L_p-2} \left[ a(l) - jb(l) \right] \int_{T_1+kT_s}^{T_1+(k+1)T_s} r(t) p(t - lT_s) dt.
\]
\[\begin{align*}
= \frac{1}{2N_0} \sum_{l=-L_p+1}^{-1} [a(l) - jb(l)] \sum_{k=0}^{l+L_p-1} \frac{T_1 + (k+1)T_s}{T_1 + kT_s} \int r(t)p(t - lT_s)dt \\
+ \frac{1}{2N_0} \sum_{l=0}^{L_0-1} [a(l) - jb(l)] \sum_{k=l}^{l+L_p-1} \frac{T_1 + (k+1)T_s}{T_1 + kT_s} \int r(t)p(t - lT_s)dt \\
+ \frac{1}{2N_0} \sum_{l=L_0}^{L_0+L_p-2} [a(l) - jb(l)] \sum_{k=l}^{L_0+L_p-2} \frac{T_1 + (k+1)T_s}{T_1 + kT_s} \int r(t)p(t - lT_s)dt
\end{align*}\]

The upper limit of integration for the first and second integrals may be expressed as

\[T_1 + T_pT_s + lT_s = T_1 + \left( \frac{T_2 - T_1}{T_s} \right) T_s + lT_s\]
\[= T_1 + T_2 - T_1 + lT_s\]
\[= T_2 - lT_s\]  

and the upper limit of integration for the third integral may be expressed as

\[T_1 + T_pT_s + (L_0 - 1)T_s = T_1 + \left( \frac{T_2 - T_1}{T_s} \right) T_s + (L_0 - 1)T_s\]
\[= T_1 + T_2 - T_1 + (L_0 - 1)T_s\]
\[= T_2 + (L_0 - 1)T_s\]  

Substituting (2.19) and (2.20) into (2.18) produces
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\[
\frac{1}{2N_0} \int_{T_1}^{T_2+(L_0-1)T_s} r(t) \sum_{l=0}^{L_0-1} [a(l) - jb(l)] p(t - lT_s) dt
\]

\[
= \frac{1}{2N_0} \sum_{l=-L_p+1}^{-1} [a(l) - jb(l)] \int_{T_1}^{T_2+lT_s} r(t) p(t - lT_s) dt
\]

\[
+ \frac{1}{2N_0} \sum_{l=0}^{L_0-1} [a(l) - jb(l)] \int_{T_1+lT_s}^{T_2+lT_s} r(t) p(t - lT_s) dt
\]

\[
+ \frac{1}{2N_0} \sum_{l=L_0}^{L_0+L_p-2} [a(l) - jb(l)] \int_{T_1+lT_s}^{T_2+(L_0-1)T_s} r(t) p(t - lT_s) dt. \tag{2.21}
\]

The second term in (2.11) may be re-expressed in a more convenient form using the same procedure. Following exactly the same steps from (2.12) – (2.21) with \( \sum_{i=0}^{L_0-1} a(i) p(t - iT_s) \) in place of \( r(t) \) yields

\[
\frac{h}{2N_0} \int_{T_1}^{T_2+(L_0-1)T_s} \sum_{i=0}^{L_0-1} [a(i) + jb(i)] p(t - iT_s) \sum_{l=0}^{L_0-1} [a(l) - jb(l)] p(l - lT_s) dt
\]

\[
= \frac{h}{2N_0} \sum_{i=0}^{L_0-1} [a(i) + jb(i)] \sum_{l=-L_p+1}^{-1} [a(l) - jb(l)] \int_{T_1}^{T_2+lT_s} p(t - iT_s) p(t - lT_s) dt
\]

\[
+ \frac{h}{2N_0} \sum_{i=0}^{L_0-1} [a(i) + jb(i)] \sum_{l=-L_p+1}^{-1} [a(l) - jb(l)] \int_{T_1+lT_s}^{T_2+lT_s} p(t - iT_s) p(t - lT_s) dt
\]

\[
+ \frac{h}{2N_0} \sum_{i=0}^{L_0-1} [a(i) + jb(i)] \sum_{l=L_0}^{L_0+L_p-2} [a(l) - jb(l)] \int_{T_1+lT_s}^{T_2+(L_0-1)T_s} p(t - iT_s) p(t - lT_s) dt. \tag{2.22}
\]

Assembling these results produces the final form of the first derivative of the log-likelihood function. Substituting (2.21) and (2.22) into (2.11) gives

\[
\frac{\partial}{\partial h^*} \Lambda(h) = \frac{1}{2N_0} \sum_{l=-L_p+1}^{-1} [a(l) - jb(l)] \int_{T_1}^{T_2+lT_s} r(t) p(t - lT_s) dt \]

edge effect term
Figure 2.1: A graphical conceptualization of exchanging the order of summation leading to (2.17).
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\[ + \frac{1}{2N_0} \sum_{l=0}^{L_0-1} [a(l) - jb(l)] \int_{T_1+lT_s}^{T_2+lT_s} r(t)p(t-lT_s)dt \]
\[ + \frac{1}{2N_0} \sum_{l=L_0}^{L_0+L_p-2} [a(l) - jb(l)] \int_{T_1+lT_s}^{T_2+(L_0-1)T_s} r(t)p(t-lT_s)dt \]
\[ - \frac{h}{2N_0} \sum_{i=0}^{L_0-1} [a(i) + jb(i)] \sum_{l=-L_p+1}^{-1} [a(l) - jb(l)] \int_{T_1}^{T_2+lT_s} p(t-iT_s)p(t-lT_s)dt \]
\[ - \frac{h}{2N_0} \sum_{i=0}^{L_0-1} [a(i) + jb(i)] \sum_{l=0}^{L_0-1} [a(l) - jb(l)] \int_{T_1+lT_s}^{T_2+lT_s} r_p(t)(l-iT_s) dt \]
\[ - \frac{h}{2N_0} \sum_{i=0}^{L_0-1} [a(i) + jb(i)] \sum_{l=L_0}^{L_0+L_p-2} [a(l) - jb(l)] \int_{T_1+lT_s}^{T_2+(L_0-1)T_s} p(t-iT_s)p(t-lT_s)dt . \]  

(2.23)

Comments:

1. Each of the six terms on the right-hand side of (2.23) contain at least one summation. These summations were defined with the understanding that when the upper limit is less than the lower limit, the summation is zero.

2. The second integral in (2.23), associated with the variable \( x(lT_s) \), may be interpreted as the response to \( r(t) \) at \( t = lT_s \) of a filter matched to the pulse shape. The statistical properties of \( x(lT_s) \) are developed in Section 2.2.2. In summary, for pulse shapes that satisfy the Nyquist No-ISI condition,

\[ x(lT_s) = h[a(l) + jb(l)] + v(lT_s) \]

(2.24)

where \( v(lT_s) = v_R(lT_s) + jv_I(lT_s) \) is a complex-valued Gaussian random variable whose real and imaginary components have zero mean and variance \( N_0 \).

3. The first and third integrals on the right-hand side of (2.23), even though similar to the second term on right-hand side of (2.23), are not matched filter outputs because the region of
integration does not include an entire pulse shape. (This is due to a fixed value for one of the limits of integration.) Instead these are partial correlations with the pulse shape and constitute what are called “edge effect” terms [4]. The first of these edge effect terms involves the $L_p - 1$ symbols preceding the pilot symbols. The second of these edge effect terms involves the $L_0 + L_p - 2 - L_0 + 1 = L_p - 1$ symbols following the pilot symbols. Edge effect terms are illustrated in Figure 2.2 for the case $T_1 = -2T_s$, $T_2 = 2T_s$, and $L_0 = 4$. Observe that the integration interval contains complete copies of the pulse shapes carrying the $L_0 = 4$ pilot symbols $a(0), a(1), a(2), a(3)$. (The imaginary components $b(0), b(1), b(2), b(3)$ are omitted from Figure 2.2 for clarity.) But there are portions $L_p - 1 = 3$ pulse shapes [carrying the data symbols $a(4), a(5), a(6)$] also included in the interval of integration. These contributions of these terms are the edge terms.

4. The fifth integral in (2.23) is labeled $r_p([i - l]T_s)$ where $r_p(\tau)$ is the pulse shape autocorrelation function defined by (2.5). For pulse shapes that satisfy the Nyquist No-ISI condition, $r_p(\tau) = 0$ when $\tau$ is a non-zero integer multiple of the symbol time $T_s$. Consequently, all of the cross terms in the double summation term in (2.23) are zero.

5. The fourth and sixth integrals on the right-hand side of (2.23), even though similar to the fifth integral on the right-hand side of (2.23), are not pulse shape autocorrelations because the limits of integration do not include all of $p(t - lT_s)$. For this reason, these terms are also labeled “edge effect term”.

6. For the special case of full response pulse shapes, $T_1 = 0$, $T_2 = T_s$, and $L_p = 1$. Consequently, the “edge effect” terms of (2.23) disappear. In addition, $r_p(\tau) = 0$ for $\tau$ outside the interval $-T_s \leq \tau \leq T_s$ so that these pulse shapes satisfy the Nyquist No-ISI condition. The derivative of the log-likelihood function becomes

$$
\frac{\partial}{\partial h^*} \Lambda(h) = \frac{1}{2N_0} \sum_{l=0}^{L_0-1} [a(l) - jb(l)]x(lT_s) - \frac{h}{2N_0} \sum_{i=0}^{L_0-1} |a(i) + jb(i)|^2
$$

$$
= \frac{1}{2N_0} \sum_{l=0}^{L_0-1} [a(l) - jb(l)]x(lT_s) - h\frac{2L_0A^2}{2N_0}.
$$

(2.25)

7. For partial response pulse shapes, (2.23) applies to the case illustrated in Figure 1.1 (b) where the $L_0$ pilot symbols form a contiguous block inside the burst (say, for example in

\[\text{Comment 7.} \]
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the middle). For the arrangement illustrated in Figure 1.1 (a) the first and fourth terms on the right-hand side of (2.23) are zero because \(a(-L_p + 1) + jb(-L_p + 1) = \cdots = a(-1) + jb(-1) = 0\). For the arrangement illustrated in Figure 1.1 (c) the third and sixth terms on the right-hand side of (2.23) are zero because \(a(L_0) + jb(L_0) = \cdots = a(L_0 + L_p - 2) + jb(L_0 + L_p - 2) = 0\).

8. The edge effects represent information that could be used to compute the estimate for \(h\). The usual approach is to discard these terms in formulating the estimator. The justifications are as follows.

- Using the information contains in the edge terms is not practical because the \(L_p - 1\) data symbols before and/or after the pilot symbols are not known.
- For \(L_0\) sufficiently large, the energy lost by discarding the edge terms is negligibly small.

Omitting the edge terms from the right-hand side of (2.23) produces

\[
\frac{\partial}{\partial h^*} \Lambda(h) = \frac{1}{2N_0} \sum_{l=0}^{L_0-1} [a(l) - jb(l)]x(lT_s) - \frac{h}{2N_0} \sum_{i=0}^{L_0-1} [a(i) + jb(i)] \sum_{l=0}^{L_0-1} [a(l) - jb(l)]r_p([i-l]T_s). \tag{2.26}
\]

Note that (2.26) reduces to (2.25) for pulse shapes that satisfy the Nyquist No-ISI condition. This approach renders the exact solution for full response pulse shapes because there are no edge effects with full response pulse shapes (see Comment 6). For partial response pulse shapes, this approach is approximate. The approximation improves as \(L_0\) increases because the contribution of the neglected edge effect terms decreases relative to the contribution from the other terms.

Using the approximate expression for the derivative of the log-likelihood function as discussed in Comment 8 and assuming the pulse shape satisfies the Nyquist No-ISI condition, setting the derivative to zero and solving for \(h\) produces the desired estimator:

\[
\hat{h} = \frac{1}{2L_0A^2} \sum_{l=0}^{L_0-1} [a(l) - jb(l)]x(lT_s). \tag{2.27}
\]
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Figure 2.2: An illustration of the edge effects for \( T_1 = -2T_s, T_2 = 2T_s \) (hence, \( L_0 = 4 \)) and \( L_p = 4 \) pilot symbols.

\[ a(0) p(t) \]
\[ a(1) p(t - T_s) \]
\[ a(2) p(t - 2T_s) \]
\[ a(3) p(t - 3T_s) \]
\[ a(4) p(t - 4T_s) \]
\[ a(5) p(t - 5T_s) \]
\[ a(6) p(t - 6T_s) \]

Interval of integration: \( T_1 + (L_0 - 1)T_s \)

\( L_0 = 4 \) training symbols

\( L_p - 1 = 3 \) symbols involved with the "edge effects"
2.2.2 Statistical Properties of the Matched Filter Outputs

The estimator (2.27) is a function of the the sampled matched filter outputs \( x(lT_s) \) which are, in turn, random variables. From the previous section,

\[
x(lT_s) = \int_{T_1+T_s}^{T_2+lT_s} r(t)p(t-lT_s)dt.
\]

(2.28)

Using (2.1) and (2.7), \( x(lT_s) \) may be expressed as

\[
x(lT_s) = \int_{T_1+T_s}^{T_2+lT_s} \left[ h \sum_{k=0}^{L_0-1} a(k)p(t-kT_s) + w(t) \right] p(t-lT_s)dt
\]

\[
= h \sum_{k=0}^{L_0-1} \left[ a(k) + jb(k) \right] \int_{T_1+T_s}^{T_2+lT_s} p(t-kT_s)p(t-lT_s)dt + \int_{T_1+T_s}^{T_2+lT_s} w(t)p(t-lT_s)dt
\]

\[
= h \sum_{k=0}^{L_0-1} \left[ a(k) + jb(k) \right] r_p([k-l]T_s) + v(lT_s)
\]

(2.29)

where the noise term \( v(lT_s) \) may be expressed as

\[
v(lT_s) = \int_{T_1+T_s}^{T_2+lT_s} w(t)p(t-lT_s)dt
\]

\[
= \int_{T_1+T_s}^{T_2+lT_s} \left[ w_R(t) + jw_I(t) \right] p(t-lT_s)dt
\]

\[
= \int_{T_1+T_s}^{T_2+lT_s} w_R(t)p(t-lT_s)dt + j \int_{T_1+T_s}^{T_2+lT_s} w_I(t)p(t-lT_s)dt.
\]

(2.30)

Because \( w(t) \) is a proper complex-valued Gaussian random process, the random variable \( v(lT_s) = v_R(lT_s) + jv_I(lT_s) \) is a proper complex-valued Gaussian random variable whose real and imaginary parts are uncorrelated Gaussian random variables with means

\[
E[v_R(lT_s)] = \int_{T_1+T_s}^{T_2+lT_s} E[w_R(t)]p(t-lT_s)dt = 0
\]

(2.31)
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\[ E[v_I(lT_s)] = \int_{T_1 + lT_s}^{T_2 + lT_s} E[w_I(t)]p(t - lT_s)dt = 0 \] (2.32)

and autocorrelation functions

\[ E[v_R(iT_s)v_R(lT_s)] = E \left[ \int_{T_1 + iT_s}^{T_2 + iT_s} w_R(t)p(t - iT_s)dt \int_{T_1 + lT_s}^{T_2 + lT_s} w_R(\tau)p(\tau - lT_s)d\tau \right] \]

\[ = \int_{T_1 + iT_s}^{T_2 + iT_s} \int_{T_1 + lT_s}^{T_2 + lT_s} E[w_R(t)w_R(\tau)]p(t - iT_s)p(\tau - lT_s)d\tau dt \]

\[ = N_0 \int_{T_1 + iT_s}^{T_2 + iT_s} p(t - iT_s)p(t - lT_s)dt \]

\[ = N_0 r_p([l - i]T_s) \] (2.33)

and

\[ E[v_I(iT_s)v_I(lT_s)] = E \left[ \int_{T_1 + iT_s}^{T_2 + iT_s} w_I(t)p(t - iT_s)dt \int_{T_1 + lT_s}^{T_2 + lT_s} w_I(\tau)p(\tau - lT_s)d\tau \right] \]

\[ = \int_{T_1 + iT_s}^{T_2 + iT_s} \int_{T_1 + lT_s}^{T_2 + lT_s} E[w_I(t)w_I(\tau)]p(t - iT_s)p(\tau - lT_s)d\tau dt \]

\[ = N_0 \int_{T_1 + iT_s}^{T_2 + iT_s} p(t - iT_s)p(t - lT_s)dt \]

\[ = N_0 r_p([l - i]T_s) \] (2.34)

For pulse shapes that satisfy the Nyquist No-ISI condition, the \( l \)-th matched filter output is

\[ x(lT_s) = h[a(l) + jb(l)] + v(lT_s) \] (2.35)
where \( v(lT_s) = v_R(lT_s) + jv_I(lT_s) \) with

\[
v_R(lT_s) \sim N(0, N_0) \quad v_I(lT_s) \sim N(0, N_0).
\]

### 2.2.3 Estimator Performance

The performance of this (and any other) estimator is quantified using the first two moments of the estimator error

\[
e = \hat{h} - h.
\]

If the mean of the estimator error is zero, then the estimator is *unbiased*. The estimator error variance quantifies how much energy the estimator error injects into the overall system. The Cramér-Rao bound is a lower bound on the estimator error variance for any unbiased estimator. An estimator whose error variance achieves the Cramér-Rao bound is called an *efficient* estimator.

Using (2.35) the estimator (2.27) may be expressed as

\[
\hat{h} = \frac{1}{2L_0A^2} \sum_{l=0}^{L_0-1} [a(l) - jb(l)] \left[ h[a(l) + jb(l)] + v(lT_s) \right]
\]

\[
= \frac{h}{2L_0A^2} \sum_{l=0}^{L_0-1} \left[ a^2(l) + b^2(l) \right] + \frac{1}{2L_0A^2} \sum_{l=0}^{L_0-1} [a(l) - jb(l)]v(lT_s)
\]

\[
= \hat{h} + e.
\]

The estimator error \( e = e_R + je_I \) is the sum of zero-mean complex-valued Gaussian random variables. The real and imaginary parts of \( e \) are obtained as follows:

\[
e = \frac{1}{2L_0A^2} \sum_{l=0}^{L_0-1} [a(l) - jb(l)] \left[ v_R(lT_s) + jv_I(lT_s) \right]
\]

\[
= \frac{1}{2L_0A^2} \sum_{l=0}^{L_0-1} \left[ a(l)v_R(lT_s) + b(l)v_I(lT_s) \right] + j \frac{1}{2L_0A^2} \sum_{l=0}^{L_0-1} \left[ a(l)v_I(lT_s) - b(l)v_R(lT_s) \right].
\]

Because \( v(lT_s) \) is a *proper* complex-valued Gaussian random variable, \( v_R(lT_s) \) and \( v_I(lT_s) \) are uncorrelated regardless of index. Furthermore, the analysis of \( v_R(lT_s) \) and \( v_I(lT_s) \) immediately
preceeding (2.35) shows that for pulse shapes satisfying the Nyquist No-ISI condition, \( v_R(lT_s) \) and \( v_R(l'T_s) \) are uncorrelated Gaussian random variables for \( l' \neq l \) and \( v_I(lT_s) \) and \( v_I(l'T_s) \) are uncorrelated Gaussian random variables for \( l' \neq l \). Consequently, \( e_R \) is the sum of uncorrelated Gaussian random variables. Similarly, \( e_I \) is the sum of uncorrelated Gaussian random variables. The mean and variance of \( e_R \) are

\[
E[e_R] = E \left\{ \frac{1}{2L_0A^2} \sum_{l=0}^{L_0-1} \left[ a(l)v_R(lT_s) + b(l)v_I(lT_s) \right] \right\} = \frac{1}{2L_0A^2} \sum_{l=0}^{L_0-1} \left\{ a(l)E[v_R(lT_s)] + b(l)E[v_I(lT_s)] \right\} = 0 \tag{2.41}
\]

\[
E[e_R^2] = E \left\{ \frac{1}{2L_0A^2} \sum_{l=0}^{L_0-1} \left[ a(l)v_R(lT_s) + b(l)v_I(lT_s) \right]^2 \right\} = \frac{1}{2L_0A^2} \sum_{l=0}^{L_0-1} \left\{ a(l)^2E[v_R(lT_s)^2] + b(l)^2E[v_I(lT_s)^2] \right\} = \frac{N_0}{4L_0^2A^4} \sum_{l=0}^{L_0-1} \left[ a^2(l) + b^2(l) \right] = \frac{N_0}{4L_0^2A^4}2L_0A^2 = \frac{N_0}{2L_0A^2}. \tag{2.42}
\]

The mean and variance of \( e_I \) are

\[
E[e_I] = E \left\{ \frac{1}{2L_0A^2} \sum_{l=0}^{L_0-1} \left[ a(l)v_I(lT_s) - b(l)v_R(lT_s) \right] \right\} = \frac{1}{2L_0A^2} \sum_{l=0}^{L_0-1} \left\{ a(l)E[v_I(lT_s)] - b(l)E[v_R(lT_s)] \right\} = 0 \tag{2.43}
\]

\[
E[e_I^2] = E \left\{ \frac{1}{2L_0A^2} \sum_{l=0}^{L_0-1} \left[ a(l)v_I(lT_s) - b(l)v_R(lT_s) \right]^2 \right\} = \frac{1}{2L_0A^2} \sum_{l=0}^{L_0-1} \left\{ a(l)^2E[v_I(lT_s)^2] - b(l)^2E[v_R(lT_s)^2] \right\} = \frac{N_0}{4L_0^2A^4}2L_0A^2 = \frac{N_0}{2L_0A^2}.
\]
2.2. CHANNEL ESTIMATOR

\[
= \frac{1}{4L_0^2A^4} \sum_{l=0}^{L_0-1} \sum_{l'=0}^{L_0-1} \left\{ a(l)a(l')E[v_I(lT_s)v_I(l'T_s)] - a(l)b(l')E[v_I(lT_s)v_R(l'T_s)] + b(l)a(l')E[v_R(lT_s)v_I(l'T_s)] + b(l)b(l')E[v_R(lT_s)v_R(l'T_s)] \right\}
\]

\[
= \frac{1}{4L_0^2A^4} \sum_{l=0}^{L_0-1} \sum_{l'=0}^{L_0-1} \left\{ a(l)a(l')N_0\delta(l' - l) + b(l)b(l')N_0\delta(l' - l) \right\}
\]

\[
= \frac{N_0}{4L_0^2A^4} \sum_{l=0}^{L_0-1} \left[ a^2(l) + b^2(l) \right] = \frac{N_0}{4L_0^2A^4}2L_0A^2 = \frac{N_0}{4L_0^2A^4}. \tag{2.44}
\]

The cross correlation is

\[
E[e_Re_I] = E \left\{ \frac{1}{2L_0A^2} \sum_{l=0}^{L_0-1} \left[ a(l)v_R(lT_s) + b(l)v_I(lT_s) \right] \frac{1}{2L_0A^2} \sum_{l'=0}^{L_0-1} \left[ a(l')v_I(l'T_s) - b(l')v_R(l'T_s) \right] \right\}
\]

\[
= \frac{1}{4L_0^2A^4} \sum_{l=0}^{L_0-1} \sum_{l'=0}^{L_0-1} \left\{ a(l)a(l')E[v_R(lT_s)v_I(l'T_s)] - a(l)b(l')E[v_R(lT_s)v_R(l'T_s)] + b(l)a(l')E[v_I(lT_s)v_I(l'T_s)] - b(l)b(l')E[v_I(lT_s)v_R(l'T_s)] \right\}
\]

\[
= \frac{1}{4L_0^2A^4} \sum_{l=0}^{L_0-1} \sum_{l'=0}^{L_0-1} \left[ -a(l)b(l')N_0\delta(l' - l) + b(l)a(l')N_0\delta(l' - l) \right]
\]

\[
= \frac{N_0}{4L_0^2A^4} \sum_{l=0}^{L_0-1} \left[ -a(l)b(l) + b(l)a(l) \right] = 0. \tag{2.45}
\]

In summary, we have \(e_R\) and \(e_I\) are uncorrelated random variables with

\[
e_R \sim N(0, \sigma_e^2) \quad e_I \sim N(0, \sigma_e^2), \quad \sigma_e^2 = \frac{N_0}{4L_0^2A^4}. \tag{2.46}
\]

Consequently, the estimator (2.27) is unbiased. Comparing \(\sigma_e^2\) with the Cramér-Rao Bound (A.22) derived in Appendix A shows that the two are equal. Thus, the estimator (2.27) is efficient.

The performance of this estimator was simulated using a computer program. The pulse shape was the square-root raised cosine pulse shape with 50% excess bandwidth and \(L_p = 4\) (i.e., the pulse shape support is \(-2T_s \leq t \leq 2T_s\) making \(T_1 = -2T_s\) and \(T_2 = 2T_s\)). The true channel \(h = e^{j0.66\pi}\) was estimated using (2.27). The performance using both \(L_0 = 4\) and \(L_0 = 16\) was examined. The simulated estimator error variance is plotted in Figure 2.3 along with the Cramér-Rao bound (A.23) for each case. The simulation results show very nice agreement with analysis.
Figure 2.3: Simulation results for estimator error variance for $L_0 = 4$ and $L_0 = 16$. 
2.3 BER Performance of QPSK Using the Estimator (2.27)

Equipped with the estimate of $h$, the optimum QPSK detector computes the matched filter outputs $x(kT_s)$ for $k = L_0, L_0 + 1, \ldots, L_0 + L_d - 1$ and forms the decision variable

$$D(k) = \hat{h}^* x(kT_s).$$

(2.47)

Multiplication by $\hat{h}^*$ derotates the matched filter outputs by the phase of the channel estimate. When the estimate is perfect, this process performs perfect phase compensation. When the estimate contains an error, a residual phase shift remains. The residual phase shift increases the probability of error, as discussed below. The symbol decisions are

$$\hat{a}(k) - j\hat{b}(k) = \arg\min_{(a+jb) \in A} \{|D(k) - (a + jb)|^2\}.$$  

(2.48)

A conceptual block diagram of this system is illustrated in Figure 2.4.

Ultimately, we seek an expression for the probability of bit error. To get there, assume the gray-coded bit-to-symbol mapping shown in Figure 2.5. Here the bit decisions, based on $D(k) = D_R(k) + jD_I(k)$, may be formulated as follows:

$$\begin{align*}
\text{first bit} &= \begin{cases} 1 & D_R(k) \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{second bit} = \begin{cases} 1 & D_I(k) \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.49)
\end{align*}$$

The probability of bit error may be formulated as follows. Let $E_1$ be the event that a decision error occurs for the first bit and let $E_2$ be the event that a decision error occurs for the second bit. A straight-forward application of the total probability theorem gives

$$P(E_1) = P(E_1 \mid \text{first bit} = 1) P(\text{first bit} = 1) + P(E_1 \mid \text{first bit} = 0) P(\text{first bit} = 0)$$

$$= P(E_1 \mid a(k) = +A) P(a(k) = +A) + P(E_1 \mid a(k) = -A) P(a(k) = -A)$$

$$= P(E_1 \mid a(k) = +A, b(k) = +A) P(a(k) = +A, b(k) = +A)$$

$$+ P(E_1 \mid a(k) = +A, b(k) = -A) P(a(k) = -A, b(k) = -A)$$

$$+ P(E_1 \mid a(k) = -A, b(k) = +A) P(a(k) = -A, b(k) = +A)$$

$$+ P(E_1 \mid a(k) = -A, b(k) = -A) P(a(k) = -A, b(k) = -A).$$  

(2.50)
Figure 2.4: A block diagram for a QPSK detector using the ML estimator (2.27). This system assumes that the pilot symbols are the first $L_0$ symbols received. Other arrangements are accommodated using buffers and more sophisticated switch control.
2.3. BER PERFORMANCE OF QPSK USING THE ESTIMATOR (2.27)

Assuming equally likely and independent bits,

\[ P(E_1) = \frac{1}{4} P \left( E_1 \mid a(k) = +A, b(k) = +A \right) + \frac{1}{4} P \left( E_1 \mid a(k) = +A, b(k) = -A \right) \\
+ \frac{1}{4} P \left( E_1 \mid a(k) = -A, b(k) = +A \right) + \frac{1}{4} P \left( E_1 \mid a(k) = -A, b(k) = -A \right). \]  \tag{2.51}

Applying (2.49), \( P(E_1) \) may be expressed as

\[ P(E_1) = \frac{1}{4} P \left( D_R(k) < 0 \mid a(k) = +A, b(k) = +A \right) \\
+ \frac{1}{4} P \left( D_R(k) < 0 \mid a(k) = +A, b(k) = -A \right) \\
+ \frac{1}{4} P \left( D_R(k) \geq 0 \mid a(k) = -A, b(k) = +A \right) \\
+ \frac{1}{4} P \left( D_R(k) \geq 0 \mid a(k) = -A, b(k) = -A \right). \]  \tag{2.52}

Similarly, for the second bit, we have

\[ P(E_2) = P \left( E_2 \mid \text{second bit} = 1 \right) P \left( \text{second bit} = 1 \right) + P \left( E_2 \mid \text{second bit} = 0 \right) P \left( \text{second bit} = 0 \right) \\
= P \left( E_2 \mid b(k) = +A \right) P \left( b(k) = +A \right) + P \left( E_2 \mid b(k) = -A \right) P \left( b(k) = -A \right) \\
= P \left( E_2 \mid b(k) = +A, a(k) = +A \right) P \left( b(k) = +A, a(k) = +A \right) \\
+ P \left( E_2 \mid b(k) = +A, a(k) = -A \right) P \left( b(k) = +A, a(k) = -A \right) \\
+ P \left( E_2 \mid b(k) = -A, a(k) = +A \right) P \left( b(k) = -A, a(k) = +A \right) \\
+ P \left( E_2 \mid b(k) = -A, a(k) = -A \right) P \left( b(k) = -A, a(k) = -A \right) \]  \tag{2.53}

which, for equally likely and independent bits, reduces to

\[ P(E_2) = \frac{1}{4} P \left( E_2 \mid b(k) = +A, a(k) = +A \right) + \frac{1}{4} P \left( E_2 \mid b(k) = +A, a(k) = -A \right) \\
+ \frac{1}{4} P \left( E_2 \mid b(k) = -A, a(k) = +A \right) + \frac{1}{4} P \left( E_2 \mid b(k) = -A, a(k) = -A \right). \]  \tag{2.54}

Finally, applying (2.49), \( P(E_2) \) may be expressed as

\[ P(E_2) = \frac{1}{4} P \left( D_I(k) < 0 \mid b(k) = +A, a(k) = +A \right) \]
Putting this all together, we have

\[ P_b = \frac{1}{2} P(E_1) + \frac{1}{2} P(E_2). \tag{2.56} \]

Each of the terms in (2.52) and (2.55) may be computed as follows. Substituting (2.35) for \( x(kT_s) \), and (2.39) for \( \hat{h} \), the decision variable \( D(k) \) may be expressed as

\[ D(k) = \left[ h + e \right]^* \left[ h[a(k) + jb(k)] + v(kT_s) \right] \tag{2.57} \]

where \( e = e_R + je_I \) is a proper complex Gaussian random variable with

\[ e_R \sim N(0, \sigma_e^2) \quad e_I \sim N(0, \sigma_e^2), \quad \sigma_e^2 = \frac{N_0}{2L_0 A^2}. \tag{2.58} \]

and \( v(kT_s) = v_R(kT_s) + jv_I(kT_s) \) is a sequence of uncorrelated proper complex Gaussian random variables with

\[ v_R(kT_s) \sim N(0, N_0) \quad v_I(kT_s) \sim N(0, N_0). \tag{2.59} \]
2.3. BER PERFORMANCE OF QPSK USING THE ESTIMATOR (2.27) 41

The first term of (2.52). Here, \(a(k) = +A\) and \(b(k) = +A\) so that

\[
D(k) = \left[ h + e \right]^{*} \left[ h[A + jA] + v(kT_s) \right]
\]

(2.60)

where \(X\) and \(Y\) are independent proper complex-valued Gaussian random variables with

\[
X \sim N(h[A + jA], N_0) \quad Y \sim N(h, \sigma_e^2).
\]

(2.61)

The desired probability may be restated as

\[
P \left( D_R(k) < 0 \mid a(k) = +A, b(k) = +A \right) = P \left( \frac{1}{2} XY^* + \frac{1}{2} X^*Y < 0 \right).
\]

(2.62)

This is a special case of the form considered in Appendix B of [2].

In [2], the general quadratic form involving the complex-valued Gaussian random variables \(X\) and \(Y\) is investigated. The real-valued random variable

\[
D = A|X|^2 + B|Y|^2 + CXY^* + C^*X^*Y
\]

(2.63)

is defined for real valued constants \(A\) and \(B\) and a complex valued constant \(C\). (Here, \(A\) is a constant that is not related to our \(A\) used to denote the QPSK symbol values). The result is

\[
P(D < 0) = Q(a, b) - \frac{v_2/v_1}{1 + v_2/v_1} I_0(ab) \exp \left\{ -\frac{1}{2} (a^2 + b^2) \right\}
\]

(2.64)
where

\[ Q(a, b) = \int_b^\infty x \exp \left\{ -\frac{1}{2} (x^2 + a^2) \right\} I_0(ax) \, dx \quad \text{(Marcum Q-function)} \]

\[ I_0(x) = \text{0-th order modified Bessel function of the first kind} \]

\[ a = \sqrt{\frac{2v_1^2v_2(\alpha_1v_2 - \alpha_2)}{(v_1 + v_2)^2}} \]

\[ b = \sqrt{\frac{2v_1v_2^2(\alpha_1v_1 + \alpha_2)}{(v_1 + v_2)^2}} \]

\[ v_1 = \sqrt{\frac{w^2 + \frac{1}{4} (m_{XX}m_{YY} - |m_{XY}|^2) (|C|^2 - AB)}{|C|^2 - AB}} - w \]

\[ v_2 = \sqrt{\frac{w^2 + \frac{1}{4} (m_{XX}m_{YY} - |m_{XY}|^2) (|C|^2 - AB)}{|C|^2 - AB}} + w \]

\[ w = \frac{Am_{XX} + Bm_{YY} + Cm_{XY}^* + C^*m_{XY}}{4 (m_{XX}m_{YY} - |m_{XY}|^2) (|C|^2 - AB)} \]

\[ \alpha_1 = 2 (|C|^2 - AB) \left( |m_X|^2m_{YY} + |m_Y|^2m_{XX} - m_X^*m_Ym_{XY} - m_Xm_Y^*m_{XY} \right) \]

\[ \alpha_2 = Am_{XX}^2 + B|m_Y|^2 + Cm_Xm_Y^* + C^*m_X^*m_Y \]

with

\[ m_X = \mathbb{E}[X] \quad m_Y = \mathbb{E}[Y] \quad (2.66) \]

and

\[ m_{XX} = \frac{1}{2} \mathbb{E} \left[ |X|^2 \right] \quad m_{YY} = \frac{1}{2} \mathbb{E} \left[ |Y|^2 \right] \quad m_{XY} = \frac{1}{2} \mathbb{E} \left[ XY^* \right] \quad m_{YX} = \frac{1}{2} \mathbb{E} \left[ X^*Y \right]. \quad (2.67) \]

Applying this result to (2.62), we see that (2.62) is the special case of (2.63) for \( A = B = 0 \), \( C = \frac{1}{2} \). Here, first two moments of the random variables \( X \) and \( Y \) are

\[ m_X = h[A + jA], \quad m_Y = h, \quad (2.68) \]

and

\[ m_{XX} = N_0, \quad m_{YY} = \sigma_e^2 = \frac{N_0}{2L_0A^2}, \quad m_{XY} = m_{YX} = 0. \quad (2.69) \]
2.3. BER PERFORMANCE OF QPSK USING THE ESTIMATOR (2.27)

Consequently, the variables (2.65) are

\[
\begin{align*}
\alpha_2 &= |h|^2 A \\
\alpha_1 &= \frac{|h|^2 N_0}{2} \left( \frac{1}{L_0} + 1 \right) \\
w &= 0 \\
v_2 &= \frac{1}{N_0} \sqrt{2L_0 A^2} \\
v_1 &= \frac{1}{N_0} \sqrt{2L_0 A^2} \\
b &= \sqrt{R \left[ 1 + L_0 + \sqrt{2L_0} \right]} \\
a &= \sqrt{R \left[ 1 + L_0 - \sqrt{2L_0} \right]}
\end{align*}
\]

where

\[
R = \frac{|h|^2 A^2}{2N_0} = \frac{|h|^2 E_b}{N_0}
\]

is the received signal-to-noise ratio. Making the appropriate substitutions into (2.64) gives

\[
P\left( D_R(k) < 0 \mid a(k) = +A, b(k) = +A \right) =
Q \left( \sqrt{R \left[ 1 + L_0 - \sqrt{2L_0} \right]}, \sqrt{R \left[ 1 + L_0 + \sqrt{2L_0} \right]} \right)
- \frac{1}{2} I_0 \left( R \sqrt{1 + L_0^2} \right) \exp \left\{ -R(1 + L_0) \right\}.
\]

To simplify the notation below, we define

\[
\begin{align*}
R_{L-} &= R \left[ 1 + L_0 - \sqrt{2L_0} \right] \\
R_{L+} &= R \left[ 1 + L_0 + \sqrt{2L_0} \right].
\end{align*}
\]

Using the definitions (2.73) we have

\[
P\left( D_R(k) < 0 \mid a(k) = +A, b(k) = +A \right) =
Q \left( \sqrt{R_{L-}}, \sqrt{R_{L+}} \right) - \frac{1}{2} I_0 \left( R \sqrt{1 + L_0^2} \right) \exp \left\{ -R(1 + L_0) \right\}.
\]
The second term of (2.52). Here, \(a(k) = +A\) and \(b(k) = -A\) so that

\[
D(k) = \left[ h + e \right]^* \left[ h[A - jA] + v(kT_s) \right].
\] (2.75)

Proceeding as before, we have \(A = B = 0, C = \frac{1}{2}\) for (2.63) and

\[
m_X = h[A - jA], \quad m_Y = h,
\] (2.76)

\[
m_{XX} = N_0, \quad m_{YY} = \frac{N_0}{2L_0A^2}, \quad m_{XY} = m_{YX} = 0.
\] (2.77)

The terms required to compute (2.64) for this case are

\[
\alpha_2 = |h|^2 A
\]
\[
\alpha_1 = \frac{|h|^2 N_0}{2} \left( \frac{1}{L_0} + 1 \right)
\]
\[
w = 0
\]
\[
v_2 = \frac{1}{N_0} \sqrt{2L_0A^2}
\]
\[
v_1 = \frac{1}{N_0} \sqrt{2L_0A^2}
\]
\[
b = \sqrt{R \left[ 1 + L_0 + \sqrt{2L_0} \right]}
\]
\[
a = \sqrt{R \left[ 1 + L_0 - \sqrt{2L_0} \right]}
\] (2.78)

Because the constants (2.78) are identical to the constants (2.70), the second term of (2.52) is equal to the first term of (2.52):

\[
P \left( D_R(k) < 0 \Big| a(k) = +A, b(k) = -A \right)
= Q \left( \sqrt{R_{L-}}, \sqrt{R_{L+}} \right) - \frac{1}{2} L_0 \left( R \sqrt{1 + L_0^2} \right) \exp \left\{ -R(1 + L_0) \right\}. \] (2.79)

The third term of (2.52). Here, \(a(k) = -A\) and \(b(k) = +A\) to give

\[
D(k) = \left[ h + e \right]^* \left[ -h[A - jA] + v(kT_s) \right].
\] (2.80)
2.3. BER PERFORMANCE OF QPSK USING THE ESTIMATOR (2.27)

A bit error occurs when $D_R(k) \geq 0$. The conditional probability of error may be expressed as

$$P \left( D_R(k) \geq 0 \bigg| a(k) = -A, b(k) = +A \right) = 1 - P \left( D_R(k) < 0 \bigg| a(k) = -A, b(k) = +A \right).$$

(2.81)

The second term in the expression above may be computed as before. We have $A = B = 0$ and $C = \frac{1}{2}$ in (2.63) and

$$m_X = -h[A - jA], \quad m_Y = h,$$

$$m_{XX} = N_0, \quad m_{YY} = \frac{N_0}{2L_0A^2}, \quad m_{XY} = m_{YX} = 0.$$  

(2.82)

The terms required for (2.64) are

$$\alpha_2 = -|h|^2 A$$

$$\alpha_1 = \frac{|h|^2 N_0}{2} \left( \frac{1}{L_0} + 1 \right)$$

$$w = 0$$

$$v_2 = \frac{1}{N_0} \sqrt{2L_0A^2}$$

$$v_1 = \frac{1}{N_0} \sqrt{2L_0A^2}$$

$$a = \sqrt{R} \left[ 1 + L_0 + \sqrt{2L_0} \right] = \sqrt{R_{L+}}$$

$$b = \sqrt{R} \left[ 1 + L_0 - \sqrt{2L_0} \right] = \sqrt{R_{L-}}.$$ 

(2.84)

This gives

$$P \left( D_R(k) \geq 0 \bigg| a(k) = -A, b(k) = +A \right) = 1 - Q \left( \sqrt{R_{L+}}, \sqrt{R_{L-}} \right) + \frac{1}{2} I_0 \left( R \sqrt{1 + L_0^2} \right) \exp \left\{ -R(1 + L_0) \right\}.$$ 

(2.85)

Using the identity pilot the expression (2.85) simplifies to

$$P \left( D_R(k) \geq 0 \bigg| a(k) = -A, b(k) = +A \right) = Q \left( \sqrt{R_{L-}}, \sqrt{R_{L+}} \right) - \frac{1}{2} I_0 \left( R \sqrt{1 + L_0^2} \right) \exp \left\{ -R(1 + L_0) \right\}.$$ 

(2.86)
The fourth term of (2.52). Here, \( a(k) = -A \) and \( b(k) = -A \) to give

\[
D(k) = \left[ h + e \right]^* \begin{bmatrix} -h[A + jA] + v(kT_s) \end{bmatrix}.
\]  

(2.87)

A bit error occurs when \( D_R(k) \geq 0 \). The conditional probability of error may be expressed as

\[
P \left( D_R(k) \geq 0 \mid a(k) = -A, b(k) = -A \right) = 1 - P \left( D_R(k) < 0 \mid a(k) = -A, b(k) = -A \right).
\]  

(2.88)

The second term in the expression above may be computed as before. We have \( A = B = 0 \) and \( C = \frac{1}{2} \) in (2.63) and

\[
m_X = -h[A + jA], \quad m_Y = h,
\]

(2.89)

\[
m_{XX} = N_0, \quad m_{YY} = \frac{N_0}{2L_0 A^2}, \quad m_{XY} = m_{YX} = 0.
\]

(2.90)

The terms required for (2.64) are

\[
\alpha_2 = -|h|^2 A
\]

\[
\alpha_1 = \frac{|h|^2 N_0}{2} \left( \frac{1}{L_0} + 1 \right)
\]

\[
w = 0
\]

\[
v_2 = \frac{1}{N_0} \sqrt{2L_0 A^2}
\]

\[
v_1 = \frac{1}{N_0} \sqrt{2L_0 A^2}
\]

(2.91)

\[
a = \sqrt{R \left[ 1 + L_0 + \sqrt{2L_0} \right]} = \sqrt{R_{L+}}
\]

\[
b = \sqrt{R \left[ 1 + L_0 - \sqrt{2L_0} \right]} = \sqrt{R_{L-}}.
\]

Because the constants (2.91) are identical to the constants (2.91), the fourth term of (2.52) is equal to the second term of (2.52):

\[
P \left( D_R(k) \geq 0 \mid a(k) = -A, b(k) = -A \right) = Q \left( \sqrt{R_{L-}}, \sqrt{R_{L+}} \right) - \frac{1}{2} I_0 \left( R \sqrt{1 + L_0^2} \right) \exp \left\{ -R(1 + L_0) \right\}.
\]  

(2.92)
As an interim result, we have
\[
P(E_1) = Q \left( \sqrt{R_{L-}}, \sqrt{R_{L+}} \right) - \frac{1}{2} J_0 \left( R \sqrt{1 + L_0^2} \right) \exp \left\{ -R(1 + L_0) \right\}
\]
where \( R \) is given by (2.71) and \( R_{L-} \) and \( R_{L+} \) are given by (2.73).

The first term of (2.55). Here, \( a(k) = +A \) and \( b(k) = -A \) so that
\[
D(k) = \left[ h + e \right]^* \left[ h[A + jA] + v(kT_s) \right]
\]
where \( X \) and \( Y \) are independent proper complex-valued Gaussian random variables with
\[
X \sim N(h[A + jA], N_0) \quad Y \sim N(h, \sigma_e^2).
\]
The desired probability may be restated as
\[
P \left( D_1(k) < 0 \left| a(k) = +A, b(k) = +A \right. \right) = P \left( -\frac{1}{2} X Y^* + \frac{1}{2} X^* Y < 0 \right).
\]
This shows that the decision variable is a special case of (2.63) with \( A = B = 0 \) and \( C = j^{1/2} \).

Using
\[
m_X = h[A + jA], \quad m_Y = h, \quad m_{XX} = N_0, \quad m_{YY} = \frac{N_0}{2L_0 A^2}, \quad m_{XY} = m_{YX} = 0
\]

the constants $\alpha_2$, $\alpha_1$, $w$, $v_2$, $v_1$, $b$, and $a$ are

$$
\begin{align*}
\alpha_2 &= |h|^2 A \\
\alpha_1 &= \frac{|h|^2 N_0}{2} \left( \frac{1}{L_0} + 1 \right) \\
w &= 0 \\
v_2 &= \frac{1}{N_0} \sqrt{2L_0 A^2} \\
v_1 &= \frac{1}{N_0} \sqrt{2L_0 A^2} \\
b &= \sqrt{R \left[ 1 + L_0 + \sqrt{2L_0} \right]} = \sqrt{R_{L+}} \\
a &= \sqrt{R \left[ 1 + L_0 - \sqrt{2L_0} \right]} = \sqrt{R_{L-}}
\end{align*}
$$

and the first term of (2.55) is

$$
P \left( D_I(k) < 0 \bigg| a(k) = +A, b(k) = +A \right) =
Q \left( \sqrt{R_{L-}}, \sqrt{R_{L+}} \right) - \frac{1}{2} J_0 \left( R \sqrt{1 + L_0^2} \right) \exp \left\{ -R(1 + L_0) \right\}. \quad (2.100)
$$

**The second term of (2.55).** Here $a(k) = +A$ and $b(k) = -A$ so that

$$
D(k) = \left[ h + e \right]^* \left[ h[A - jA] + v(kT_s) \right] .
$$

(2.101)

Proceeding as before, we have $A = B = 0$, $C = -j/2$ for (2.63) and

$$
\begin{align*}
m_X &= h[A - jA], & m_Y &= h, \\
m_{XX} &= N_0, & m_{YY} &= \frac{N_0}{2L_0 A^2}, & m_{XY} = m_{YX} &= 0.
\end{align*}
$$

(2.102)
2.3. **BER PERFORMANCE OF QPSK USING THE ESTIMATOR** (2.27)

The terms required to compute (2.64) for this case are

\[
\begin{align*}
\alpha_2 &= |h|^2 A \\
\alpha_1 &= \frac{|h|^2 N_0}{2} \left( \frac{1}{L_0} + 1 \right) \\
w &= 0 \\
v_2 &= \frac{1}{N_0} \sqrt{2L_0 A^2} \\
v_1 &= \frac{1}{N_0} \sqrt{2L_0 A^2} \\
b &= \sqrt{R \left[ 1 + L_0 + \sqrt{2L_0} \right]} = \sqrt{R_{L+}} \\
a &= \sqrt{R \left[ 1 + L_0 - \sqrt{2L_0} \right]} = \sqrt{R_{L-}}
\end{align*}
\]

Because the constants (2.104) are identical to the constants (2.99), the second term of (2.55) is equal to the first term of (2.55):

\[
P \left( D_I(k) < 0 \bigg| a(k) = +A, b(k) = -A \right) = Q \left( \sqrt{R_{L-}}, \sqrt{R_{L+}} \right) - \frac{1}{2} I_0 \left( R \sqrt{1 + L_0^2} \right) \exp \left\{ -R(1 + L_0) \right\}. \tag{2.105}
\]

**The third term of (2.55).** Here, \(a(k) = -A\) and \(b(k) = +A\) to give

\[
D(k) = \left[ h + e \right]^* \left[ -h[A - jA] + v(kT_s) \right]. \tag{2.106}
\]

A bit error occurs when \(D_I(k) \geq 0\). The conditional bit error probability may be expressed as

\[
P \left( D_I(k) \geq 0 \bigg| a(k) = -A, b(k) = +A \right) = 1 - P \left( D_I(k) < 0 \bigg| a(k) = -A, b(k) = +A \right) \tag{2.107}
\]

The second term of this expression may be computed as before. We have \(A = B = 0, C = -j^{1/2}\) in (2.63) and

\[
m_X = -h[A - jA], \quad m_Y = h, \tag{2.108}
\]
\[ m_{XX} = N_0, \quad m_{YY} = \frac{N_0}{2L_0A^2}, \quad m_{XY} = m_{YX} = 0. \] (2.109)

The terms required to compute (2.64) for this case are

\[
\begin{align*}
\alpha_2 &= -|h|^2 A \\
\alpha_1 &= \frac{|h|^2 N_0}{2} \left( \frac{1}{L_0} + 1 \right) \\
w &= 0 \\
v_2 &= \frac{1}{N_0} \sqrt{2L_0A^2} \\
v_1 &= \frac{1}{N_0} \sqrt{2L_0A^2} \\
b &= \sqrt{R \left[ 1 + L_0 - \sqrt{2L_0} \right]} = \sqrt{R_{L-}} \\
a &= \sqrt{R \left[ 1 + L_0 + \sqrt{2L_0} \right]} = \sqrt{R_{L+}}
\end{align*}
\] (2.110)

This gives

\[
P \left( D_I(k) \geq 0 \mid a(k) = -A, b(k) = +A \right) = 1 - Q \left( \sqrt{R_{L+}}, \sqrt{R_{L-}} \right) + \frac{1}{2} I_0 \left( R \sqrt{1 + L_0^2} \right) \exp \left\{ -R(1 + L_0) \right\}. \] (2.111)

Using the identity

\[
Q(\alpha, \beta) + Q(\beta, \alpha) = 1 + I_0(\alpha \beta) \exp \left\{ -\frac{1}{2} (\alpha^2 + \beta^2) \right\}
\] (2.112)

the expression (2.85) simplifies to

\[
P \left( D_I(k) \geq 0 \mid a(k) = -A, b(k) = +A \right) = Q \left( \sqrt{R_{L-}}, \sqrt{R_{L+}} \right) - \frac{1}{2} I_0 \left( R \sqrt{1 + L_0^2} \right) \exp \left\{ -R(1 + L_0) \right\}. \] (2.113)

The fourth term of (2.55). Here, \( a(k) = -A \) and \( b(k) = -A \) to give

\[
D(k) = \left[ h + e \right]^* \left[ -h[A + jA] + v(kT_s) \right]. \] (2.114)
2.3. BER PERFORMANCE OF QPSK USING THE ESTIMATOR (2.27)

A bit error occurs when $D_I(k) \geq 0$. The conditional bit error probability may be expressed as

$$P \left( D_I(k) \geq 0 \Big| a(k) = -A, b(k) = -A \right)$$

$$= 1 - P \left( D_I(k) < 0 \Big| a(k) = -A, b(k) = -A \right)$$

(2.115)

The second term of this expression may be computed as before. We have $A = B = 0$, $C = -j^{1/2}$ in (2.63) and

$$m_X = -h[A + jA], \quad m_Y = h,$$

(2.116)

$$m_{XX} = N_0, \quad m_{YY} = \frac{N_0}{2L_0A^2}, \quad m_{XY} = m_{YX} = 0.$$  

(2.117)

The terms required to compute (2.64) for this case are

$$\alpha_2 = -|h|^2A$$

$$\alpha_1 = \frac{|h|^2N_0}{2} \left( \frac{1}{L_0} + 1 \right)$$

$$w = 0$$

$$v_2 = \frac{1}{N_0} \sqrt{2L_0A^2}$$

(2.118)

$$v_1 = \sqrt{2L_0A^2}$$

$$b = \sqrt{R \left[ 1 + L_0 - \sqrt{2L_0} \right]} = \sqrt{R_{L_-}}$$

$$a = \sqrt{R \left[ 1 + L_0 + \sqrt{2L_0} \right]} = \sqrt{R_{L_+}}$$

Because the constants (2.118) are identical to the constants (2.110), the fourth term of (2.55) is equal to the third term of (2.55):

$$P \left( D_I(k) \geq 0 \Big| a(k) = -A, b(k) = -A \right)$$

$$= Q \left( \sqrt{R_{L_-}}, \sqrt{R_{L_+}} \right) - \frac{1}{2} I_0 \left( R \sqrt{1 + L_0^2} \right) \exp \left\{ -R(1 + L_0) \right\}. \quad (2.119)$$

The interim result is

$$P(E_2) = Q \left( \sqrt{R_{L_-}}, \sqrt{R_{L_+}} \right) - \frac{1}{2} I_0 \left( R \sqrt{1 + L_0^2} \right) \exp \left\{ -R(1 + L_0) \right\}$$

(2.120)
where $R$ is given by (2.71) and $R_{L-}$ and $R_{L+}$ are given by (2.73).

**Summary** Finally putting this all together using (2.56) with (2.93) for $P(E_1)$ and (2.120) for $P(E_2)$ gives

$$P_b = Q\left(\sqrt{R_{L-}}, \sqrt{R_{L+}}\right) - \frac{1}{2} I_0\left(R\sqrt{1 + L_0^2}\right) \exp\left\{-R(1 + L_0)\right\}$$

(2.121)

where $R$ is given by (2.71) and $R_{L-}$ and $R_{L+}$ are given by (2.73). A plot of the bit error probability (2.121) for $L_0 = 4$ together with simulation results (described in the next section) is shown in Figure 2.6. The plot confirms the accuracy of the forgoing BER analysis.

Plots of (2.121) for increasing values of $L_0$ are shown in Figure 2.7. The plot shows that as $L_0$ increases, BER performance approaches the AWGN result. The AWGN result corresponds to using a perfect channel estimate. From this plot, one can determine the value of $R = |h|^2 E_b/N_0$ required to achieve $P_b = 10^{-6}$, denoted $R(10^{-6})$ below, and the loss, relative to the ideal estimator (which requires $R = 10.53$ dB to achieve $P_b = 10^{-6}$), denoted $\Delta R$ below. The performance loss as a function of $L_0$ is as follows:

<table>
<thead>
<tr>
<th>$L_0$</th>
<th>$R(10^{-6})$ dB</th>
<th>$\Delta R$ (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12.89</td>
<td>2.36</td>
</tr>
<tr>
<td>2</td>
<td>11.73</td>
<td>1.12</td>
</tr>
<tr>
<td>4</td>
<td>11.12</td>
<td>0.59</td>
</tr>
<tr>
<td>6</td>
<td>10.82</td>
<td>0.29</td>
</tr>
<tr>
<td>16</td>
<td>10.67</td>
<td>0.14</td>
</tr>
<tr>
<td>32</td>
<td>10.60</td>
<td>0.07</td>
</tr>
</tbody>
</table>
2.3. BER PERFORMANCE OF QPSK USING THE ESTIMATOR (2.27)

Figure 2.6: QPSK bit error rate simulations for $L_0 = 4$ and $L_d = 10000$ using $h = \exp(j0.66\pi)$. The plot also shows a plot of (2.121) [for $L_0 = 4$] as well as the AWGN curve for reference.

Figure 2.7: QPSK bit error probability for different values of $L_0$. The AWGN analysis is included for reference.
2.4 Computer Simulations

The bit error rate performance of QPSK using the estimator (2.27) was simulated using a computer program. Each packet comprises $L_0 = 4$ pilot symbols and $L_d = 10000$ data symbols. The true channel value was $h = e^{j0.66\pi}$. The simulated bit error rate performance is plotted in Figure 2.6 along with the bit error probability expression (2.121).

The following Matlab script was used to simulate the performance of the channel estimator and the bit error probability performance using the estimator.

```matlab
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Simulation Constants
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% QPSK symbols are +/- A +/- jA
% 01 | 11
% ------+------
% 00 | 10
% |  

j = 1i;
A = 1;
LUT = A*[1-j, -1-j, 1+j, 1-j];

h = exp(j*0.66*pi);

L0 = 4;
a = [1-j, -1-j, 1+j, 1-j];

% L0 = 1;
% a = 1+j;
% L0 = 8;
% a = [1-j, 1+j, -1-j, -1+j, 1+j, -1+j, 1-j, 1+j];

Ld = 10000;

SNRb = 0:11;
BER = zeros(size(SNRb));
its = 1000;

for idx1 = 1:length(SNRb)
    snrb = 10^(0.1*SNRb(idx1));
    nvar = A*A/(2*snrb);
    nstd = sqrt(nvar);
    bit_err_cnt = 0;
```
for idx2 = 1:its
    ibits = randi([0 1],1,Ld);
    qbits = randi([0 1],1,Ld);
    s = [a LUT(2*ibits+qbits+1)];
    w = nstd*(randn(size(s))+j*randn(size(s)));
    r = h*s + w;
    hhat = sqrt(2)/(4*L0*A*A)*r(1:L0)*a';
    D = hhat'*r(L0+1:L0+Ld);
    homer = real(D)>=0;
    bit_err_cnt = bit_err_cnt + sum(homer ~= ibits);
    homer = imag(D)>=0;
    bit_err_cnt = bit_err_cnt + sum(homer ~= qbits);
end
BER(idx1) = bit_err_cnt/(its*2*Ld);
fprintf('Eb/N0 = %d dB, bit errors = %d, BER = %g
',SNRb(idx1),bit_err_cnt,BER(idx1));
end
Chapter 3

Offset QPSK

3.1 Preliminaries

3.1.1 Signal Model

The transmitted signal is of the form

$$s(t) = \sum_k \left[ a(k)p(t - kT_s) + jb(k)p(t - kT_s - 0.5T_s) \right] \quad (3.1)$$

where $a(k) \in \{-A, +A\}$ and $b(k) \in \{-A, +A\}$ are the inphase and quadrature components of the $k$-th data symbol (or constellation point); $p(t)$ is a real-valued unit-energy pulse shape; and $T_s$ is the symbol time (in s/symbol). Note that the average symbol energy $E_s$ is given by

$$E_s = \frac{1}{2} |a(k) + jb(k)|^2 = A^2 \quad (3.2)$$

and the average bit energy $E_b$ is given by

$$E_b = \frac{E_s}{2} = \frac{A^2}{2}. \quad (3.3)$$

The pulse shape has support on the interval $T_1 \leq t \leq T_2$. For example, in the case of the NRZ pulse shape, $T_1 = 0$ and $T_2 = T_s$. For the case of the SRRC pulse shape, $T_2 = -T_1 = LT_s$ for a positive integer $L$. For a discussion of appropriate values for $L$ see Appendix A of [3]. The pulse
shape is also characterized by its autocorrelation function $r_p(\tau)$ defined by

$$r_p(\tau) = \int_{T_1}^{T_2} p(t)p(t-\tau)dt.$$  (3.4)

The Nyquist No-ISI condition constrains the values $r_p(\tau)$ for $\tau$ an integer multiple of the symbol time $T_s$:

$$r_p(kT_s) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}.$$  (3.5)

The signal contains $L_0$ pilot symbols and $L_d$ data symbols. Following the convention of Figure 1.1 the $L_0$ pilot symbols are

$$a(0) + jb(0), a(1) + jb(1), \ldots, a(L_0 - 1) + jb(L_0 - 1).$$

The received signal is

$$r(t) = hs(t) + w(t)$$  (3.6)

where $h$ is a complex-valued channel gain and $w(t) = w_R(t) + jw_I(t)$ is a proper complex-valued zero-mean Gaussian random process where

$$\begin{align*}
E[w_R(t)w_R(t-\tau)] &= E[w_I(t)w_I(t-\tau)] = N_0\delta(\tau) \\
E[w_R(t)w_I(t-\tau)] &= E[w_R(t)w_I(t-\tau)] = 0.
\end{align*}$$  (3.7)

In the development that follows, perfect timing synchronization is assumed.

### 3.1.2 Examples Used in This Chapter

We consider four examples in this chapter to illustrate the concepts that are developed and to demonstrate the performance of both the estimator and of OQPSK using the estimator. The examples use both short and long pilot sequences with full response and partial response pulse shapes.

**Example 1** In this example, the NRZ pulse shape is used with the pilot symbols organized in a preamble as illustrated in Figure 1.1 (a). This example focuses on the full response pulse shape
case characterized by \( L_p = 1 \) and a short pilot sequence using \( L_0 = 4 \). The pilot sequence is

\[
\begin{align*}
    a(0) &= +A & b(0) &= -A \\
    a(1) &= -A & b(1) &= -A \\
    a(2) &= +A & b(2) &= +A \\
    a(3) &= +A & b(3) &= -A \\
\end{align*}
\tag{3.8}
\]

The pulse shape autocorrelation function is given by

\[
r_p\left(\frac{kT_s}{2}\right) = \begin{cases} 
1 & k = 0 \\
0.5 & k = \pm 1 \\
0 & \text{otherwise}
\end{cases} \tag{3.9}
\]

**Example 2**  This example is identical to Example 1 except that a longer pilot sequence is used. The NRZ pulse shape \( (L_p = 1) \) is used with \( L_0 = 16 \) pilot symbols arranged in a preamble as illustrated in Figure 1.1 (a). The pilot sequence is

\[
\begin{align*}
    a(0) &= +A & b(0) &= -A \\
    a(1) &= +A & b(1) &= +A \\
    a(2) &= -A & b(2) &= +A \\
    a(3) &= +A & b(3) &= -A \\
    a(4) &= -A & b(4) &= +A \\
    a(5) &= +A & b(5) &= -A \\
    a(6) &= +A & b(6) &= -A \\
    a(7) &= +A & b(7) &= +A \\
    a(8) &= -A & b(8) &= -A \\
    a(9) &= +A & b(9) &= -A \\
    a(10) &= +A & b(10) &= -A \\
    a(11) &= -A & b(11) &= -A \\
    a(12) &= +A & b(12) &= -A \\
    a(13) &= -A & b(13) &= -A \\
    a(14) &= +A & b(14) &= +A \\
    a(15) &= +A & b(15) &= -A \\
\end{align*}
\tag{3.10}
\]
CHAPTER 3. OFFSET QPSK

Figure 3.1: The pulse shape autocorrelation function, sampled at 2 samples/symbol, for the square-root raised-cosine pulse shape with 50% excess bandwidth and $L_p = 4$.

The pulse shape autocorrelation function is given by

$$r_p \left( k \frac{T_s}{2} \right) = \begin{cases} 
1 & k = 0 \\
0.5 & k = \pm 1 \\
0 & \text{otherwise}
\end{cases}.$$  \hspace{1cm} (3.11)

**Example 3** In this example, a partial response pulse shape is used with pilot symbols organized in a preamble as illustrated in Figure 1.1 (a). The pulse shape is the square-root raised-cosine pulse shape with 50% excess bandwidth and $L_p = 4$; i.e., the pulse shape $p(t)$ has support on the interval $-2T_s \leq t \leq 2T_s$ which, in turn, means $r_p(t)$ has support on in the interval $-4T_s \leq t \leq 4T_s$ as shown in Figure 3.1. The $L_0 = 4$ pilot symbols are

$$
\begin{align*}
a(0) &= +A & b(0) &= -A \\
a(1) &= -A & b(1) &= -A \\
a(2) &= +A & b(2) &= +A \\
a(3) &= +A & b(3) &= -A
\end{align*}
\hspace{1cm} (3.12)
$$

**Example 4** In this example, we revisit the previous example but use a longer pilot sequence. The $L_0 = 16$ pilot symbols are organized in a preamble as illustrated in Figure 1.1 (a). The pulse shape is the square-root raised-cosine pulse shape with 50% excess bandwidth and $L_p = 4$; i.e., the pulse shape $p(t)$ has support on the interval $-2T_s \leq t \leq 2T_s$ which, in turn, means $r_p(t)$ has support on in the interval $-4T_s \leq t \leq 4T_s$ as shown in Figure 3.1. The $L_0 = 16$ pilot symbols are the same
3.1. PRELIMINARIES

ones used in Example 2:

\[
\begin{align*}
    a(0) &= +A & b(0) &= -A \\
    a(1) &= +A & b(1) &= +A \\
    a(2) &= -A & b(2) &= +A \\
    a(3) &= +A & b(3) &= -A \\
    a(4) &= -A & b(4) &= +A \\
    a(5) &= +A & b(5) &= -A \\
    a(6) &= +A & b(6) &= -A \\
    a(7) &= +A & b(7) &= +A \\
    a(8) &= -A & b(8) &= -A \\
    a(9) &= +A & b(9) &= -A \\
    a(10) &= +A & b(10) &= -A \\
    a(11) &= -A & b(11) &= -A \\
    a(12) &= +A & b(12) &= -A \\
    a(13) &= -A & b(13) &= -A \\
    a(14) &= +A & b(14) &= +A \\
    a(15) &= +A & b(15) &= -A
\end{align*}
\]

(3.13)
3.2 Channel Estimator

3.2.1 Derivation

The maximum-likelihood estimator observes $r(t)$ over an interval of duration $T_0 = T_2 - T_1 + (L_0 - 1)T_s + 0.5T_s$ and produces an estimate for $h$ — denoted $\hat{h}$ — based on the maximum likelihood principle. The log-likelihood function for $h$ is

$$
\Lambda(h) = -\frac{1}{2N_0} \int_{T_1}^{T_2+(L_0-1)T_s+0.5T_s} \left| r(t) - h \sum_i \left[ a(i)p(t - iT_s) + jb(i)p(t - iT_s - 0.5T_s) \right] \right|^2 dt.
$$

The maximum likelihood estimate for $h$ is

$$
\hat{h} = \arg\max_h \{ \Lambda(h) \}.
$$

A necessary condition for $\hat{h}$ is that it forces the first derivative of log-likelihood function to 0. The first derivative of log-likelihood function is

$$
\frac{\partial}{\partial h^*} \Lambda(h) = -\frac{1}{2N_0} \frac{\partial}{\partial h^*} \int_{T_1}^{T_2+(L_0-1)T_s+0.5T_s} \left\{ \left( r(t) - h \sum_i \left[ a(i)p(t - iT_s) + jb(i)p(t - iT_s - 0.5T_s) \right] \right) \right.
\times \left( r^*(t) - h^* \sum_l \left[ a(l)p(t - lT_s) - jb(l)p(t - lT_s - 0.5T_s) \right] \right) \}
\times \left. \sum_l \left[ a(l)p(t - lT_s) - jb(l)p(t - lT_s - 0.5T_s) \right] \right\} dt
$$

$$
= \frac{1}{2N_0} \int_{T_1}^{T_2+(L_0-1)T_s+0.5T_s} \left\{ \left( r(t) - h \sum_i \left[ a(i)p(t - iT_s) + jb(i)p(t - iT_s - 0.5T_s) \right] \right)
\times \sum_l \left[ a(l)p(t - lT_s) - jb(l)p(t - lT_s - 0.5T_s) \right] \right\} dt
$$

$$
= \frac{1}{2N_0} \int_{T_1}^{T_2+(L_0-1)T_s+0.5T_s} r(t) \sum_l a(l)p(t - lT_s) dt
$$

$$
- \frac{j}{2N_0} \int_{T_1}^{T_2+(L_0-1)T_s+0.5T_s} r(t) \sum_l b(l)p(t - lT_s - 0.5T_s) dt
$$
There are six integrals in this expression, each of which needs to be rewritten in a more useful form for the estimator.

The First Integral of (3.16) The first integral may be partitioned into $L_0 + L_P$ integrals as follows:

$$
\int_{T_1}^{T_2+(L_0-1)T_s+0.5T_s} r(t) \sum_{l} a(l)p(t - lT_s) \, dt =
\sum_{k=0}^{L_0 + L_P + 2} \int_{T_1 + kT_s}^{T_1 + (k+1)T_s} r(t) \sum_{l} a(l)p(t - lT_s) \, dt
\int_{T_2+(L_0-1)T_s+0.5T_s}^{T_2+(L_0-1)T_s+0.5T_s} r(t) \sum_{l} a(l)p(t - lT_s) \, dt.
$$

Now restrict the symbol index $l$ to include only the waveforms that are contained in the interval of integration.\(^1\) The pulse shape $p(t - lT_s)$ has support on the interval $T_1 + lT_s \leq t \leq T_2 + lT_s$. This support interval coincides with the integration interval of the first term on the right-hand-side of (3.17) when

$$
T_2 + lT_s > T_1 + kT_s \quad \text{and} \quad T_1 + lT_s < T_1 + (k+1)T_s.
$$

\(^1\)In doing so, we shall temporarily extend the range of valid indexes to incorporate all symbol-pulse-shape products that could be included in the interval of integration. This is done to facilitate the relationship between “edge effects” and pilot symbol arrangement. See the comments below.
Solving the first inequality gives the lower value for $l$:

\[
T_2 + lT_s > T_1 + kT_s \\
lT_s > T_1 - T_2 + kT_s \\
l > -\frac{T_2 - T_1}{T_s} + k = -L_p + k. \tag{3.18}
\]

and solving the second inequality gives the upper value for $l$:

\[
T_1 + lT_s < T_1 + (k + 1)T_s \\
lT_s < (k + 1)T_s \\
l < k + 1. \tag{3.19}
\]

The support interval for $p(t - lT_s)$ coincides with the integration interval of the second term on the right-hand-side of (3.17) when

\[
T_2 + lT_s > T_2 + (L_0 - 1)T_s \quad \text{and} \quad T_1 + lT_s < T_2 + (L_0 - 1)T_s + 0.5T_s.
\]

Solving the first inequality gives the lower value for $l$:

\[
T_2 + lT_s > T_2 + (L_0 - 1)T_s \\
lT_s > (L_0 - 1)T_s \\
l > L_0 - 1 \tag{3.20}
\]

and solving the second inequality gives the upper value for $l$:

\[
T_1 + lT_s < T_2 + (L_0 - 1)T_s + 0.5T_s \\
lT_s < T_2 - T_1 + L_0T_s - 0.5T_s \\
l < \frac{T_2 - T_1}{T_s} + L_0 - 0.5 = L_p + L_0 - 0.5. \tag{3.21}
\]

Applying the results to their respective inner summations over $l$ produces

\[
\int_{T_1}^{T_2 + (L_0 - 1)T_s + 0.5T_s} r(t) \sum_{l=0}^{L_0-1} a(l)p(t - lT_s)dt
\]
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The first term on the right-hand-side of (3.22) is of the form

\[ \sum_{k=0}^{L_0+L_p-2} \int_{T_1+kT_s}^{T_1+(k+1)T_s} r(t) \sum_{l=-L_p+k+1}^{k} a(l)p(t-lT_s)dt \]

\[ + \int_{T_2+(L_0-1)T_s+0.5T_s}^{T_2+(L_0+L_p-1)T_s+0.5T_s} r(t) \sum_{l=L_0}^{L_p+L_0-1} a(l)p(t-lT_s)dt \]

Applying this result produces

\[ \sum_{k=0}^{L_0+L_p-2} \sum_{l=-L_p+k+1}^{k} a(l) \int_{T_1+kT_s}^{T_1+(k+1)T_s} r(t)p(t-lT_s)dt \]

\[ + \sum_{l=L_0}^{L_p+L_0-1} \int_{T_2+(L_0-1)T_s}^{T_2+(L_0+L_p-1)T_s} r(t)p(t-lT_s)dt. \] (3.22)

The first term on the right-hand-side of (3.22) is of the form

\[ \sum_{k=0}^{L_0+L_p-2} \sum_{l=-L_p+k+1}^{k} a(l) \int_{T_1+kT_s}^{T_1+(k+1)T_s} r(t)p(t-lT_s)dt = \sum_{k=0}^{L_0+L_p-2} \sum_{l=-L_p+k+1}^{k} F(k, l). \] (3.23)

With the aid of Figure 2.1, reversing the order of summation produces

\[ \sum_{k=0}^{L_0+L_p-2} \sum_{l=-L_p+k+1}^{k} F(k, l) = \sum_{k=0}^{-1} \sum_{l=-L_p}^{l+L_p-1} F(k, l) \]

\[ + \sum_{l=0}^{L_0-1} \sum_{k=0}^{l+L_p-1} F(k, l) + \sum_{l=L_0}^{L_0+L_p-2} \sum_{k=0}^{L_0+L_p-2} F(k, l). \] (3.24)

Applying this result produces

\[ \int_{T_1}^{T_2+(L_0-1)T_s+0.5T_s} r(t) \sum_{l=0}^{L_0-1} a(l)p(t-lT_s)dt = \]

\[ \sum_{l=-L_p+1}^{-1} \sum_{k=0}^{l+L_p-1} a(l) \int_{T_1+kT_s}^{T_1+(k+1)T_s} r(t)p(t-lT_s)dt + \sum_{l=0}^{L_0-1} \sum_{k=0}^{l+L_p-1} a(l) \int_{T_1+kT_s}^{T_1+(k+1)T_s} r(t)p(t-lT_s)dt \]
Solving the first inequality gives the lower value for $l$ when

$$
L_0 + (k+1)T_s
$$

follows:

$$(3.16)$$

The second integral may be partitioned into $L_0$ integrals as follows:

$$
\int_{T_1 + T_s}^{T_2 + (L_0 - 1)T_s + 0.5T_s} r(t) \sum_{l} b(l)p(t - lT_s - 0.5T_s)dt
$$

$$
= \int_{T_1}^{T_1 + 0.5T_s} r(t) \sum_{l} b(l)p(t - lT_s - 0.5T_s)dt
$$

$$
+ \sum_{k=0}^{L_0 + L_p - 2} \int_{T_1 + (k+0.5)T_s}^{T_1 + (k+1.5)T_s} r(t) \sum_{l} b(l)p(t - lT_s - 0.5T_s)dt
$$

(3.26)

As before, we restrict the index $l$ to include only those terms whose corresponding waveforms are contained in the intervals of integration. The pulse shape $p(t - lT_s - 0.5T_s)$ has support on the interval $T_1 + lT_s + 0.5T_s \leq t \leq T_2 + lT_s + 0.5T_s$. This support interval coincides with the integration interval of the first term on the right-hand-side of (3.26) when

$$
T_2 + lT_s + 0.5T_s > T_1 \quad \text{and} \quad T_1 + lT_s + 0.5T_s < T_1 + 0.5T_s.
$$

Solving the first inequality gives the lower value for $l$:

$$
T_2 + lT_s + 0.5T_s > T_1
$$

$$
lT_s > T_1 - T_2 - 0.5T_s
$$
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\[ l > -\frac{T_2 - T_1}{T_s} - 0.5 = -L_p - 0.5 \]  

(3.27)

and solving the second inequality gives the upper value for \( l \):

\[ T_1 + lT_s + 0.5T_s < T_1 + 0.5T_s \]

\[ l < 0. \]  

(3.28)

The support interval for \( p(t - lT_s) \) coincides with the integration interval of the second term on the right-hand-side of (3.26) when

\[ T_2 + lT_s + 0.5T_s > T_1 + kT_s + 0.5T_s \quad \text{and} \quad T_1 + lT_s + 0.5T_s < T_1 + kT_2 + 1.5T_s. \]

Solving the first inequality gives the lower value for \( l \):

\[ T_2 + lT_s + 0.5T_s > T_1 + kT_s + 0.5T_s \]
\[ lT_s > T_1 - T_2 + kT_s \]
\[ l > -\frac{T_2 - T_1}{T_s} + k = -L_p + k \]  

(3.29)

and solving the second inequality gives the upper value for \( l \):

\[ T_1 + lT_s + 0.5T_s < T_1 + kT_2 + 1.5T_s \]
\[ lT_s < kT_2 + T_s \]
\[ l < k + 1. \]  

(3.30)

Applying the results to their respective inner summations over \( l \) produces

\[
\int_{T_1}^{T_2+(L_0-1)T_s+0.5T_s} r(t) \sum_{l=0}^{L_0-1} b(l)p(t - lT_s - 0.5T_s)dt \\
= \int_{T_1}^{T_1+0.5T_s} r(t) \sum_{l=-L_p}^{-1} b(l)p(t - lT_s - 0.5T_s)dt \\
+ \sum_{k=0}^{L_0+L_p-2} \int_{T_1+(k+1)T_s}^{T_1+(k+0.5)T_s} r(t) \sum_{l=-L_p+k+1}^{L_p+k} b(l)p(t - lT_s - 0.5T_s)dt
\]
The second term on the right-hand-side of (3.31) is of the form

\[ \sum_{k=0}^{L_0+L_p-2} \sum_{l=-L_p+k+1}^{k} b(l) \int_{T_1+(k+0.5)T_s}^{T_1+(k+1)T_s} r(t)p(t-lT_s - 0.5T_s)dt = \sum_{k=0}^{L_0+L_p-2} \sum_{l=-L_p+k+1}^{k} F(k,l). \] (3.32)

With the aid of Figure 2.1 reversing the order of summation produces

\[ \sum_{k=0}^{L_0+L_p-2} \sum_{l=-L_p+k+1}^{k} F(k,l) = \sum_{l=-L_p+1}^{-1} \sum_{k=0}^{l+L_p-1} F(k,l) \]

\[ + \sum_{l=0}^{L_0-1} \sum_{k=0}^{l+L_p-1} F(k,l) + \sum_{l=L_0}^{L_0+L_p-2} \sum_{k=l}^{L_0+L_p-2} F(k,l). \] (3.33)

Applying this result produces

\[ \int_{T_1}^{T_1+(L_0-1)T_s+0.5T_s} r(t) \sum_{l=0}^{-1} b(l) p(t-lT_s - 0.5T_s)dt = \]

\[ -1 \sum_{l=-L_p}^{T_1+0.5T_s} b(l) \int_{T_1}^{T_1+(k+1.5)T_s} r(t)p(t-lT_s - 0.5T_s)dt + \sum_{l=-L_p+1}^{-1} b(l) \sum_{k=0}^{l+L_p-1} \int_{T_1+(k+0.5)T_s}^{T_1+(k+1.5)T_s} r(t)p(t-lT_s - 0.5T_s)dt \]

\[ + \sum_{l=0}^{L_0-1} \sum_{k=l}^{l+L_p-1} \int_{T_1+(k+0.5)T_s}^{T_1+(k+1.5)T_s} r(t)p(t-lT_s - 0.5T_s)dt + \sum_{l=L_0}^{L_0+L_p-2} \sum_{k=l}^{L_0+L_p-2} \int_{T_1+(k+0.5)T_s}^{T_1+(k+1.5)T_s} r(t)p(t-lT_s - 0.5T_s)dt \]

\[ = \sum_{l=-L_p}^{-1} b(l) \int_{T_1}^{T_1+0.5T_s} r(t)p(t-lT_s - 0.5T_s)dt + \sum_{l=-L_p+1}^{-1} b(l) \int_{T_1+0.5T_s}^{T_1+0.5T_s} r(t)p(t-lT_s - 0.5T_s)dt \]
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\[ + \sum_{l=0}^{L_0-1} b(l) \int_{T_1+lT_s+0.5T_s}^{T_2+lT_s+0.5T_s} x(l+0.5T_s) r(t) p(t - lT_s - 0.5T_s) dt + \sum_{l=L_0}^{L_0+L_p-2} b(l) \int_{T_1+lT_s+0.5T_s}^{T_2+(L_0-1)T_s+0.5T_s} x(l+0.5T_s) r(t) p(t - lT_s - 0.5T_s) dt. \] (3.34)

The Third Integral of (3.16) The third integral of (3.16) is

\[ \int_{T_1}^{T_2+(L_0-1)T_s+0.5T_s} \sum_i a(i)p(t - iT_s) \sum_l a(l)p(t - lT_s) dt \]

and may be partitioned into \( L_0 + L_p \) integrals as follows:

\[ = \sum_{k=0}^{L_0+L_p-2} \int_{T_1+kT_s}^{T_1+(k+1)T_s} \sum_i a(i)p(t - iT_s) \sum_l a(l)p(t - lT_s) dt \]

\[ + \int_{T_2+(L_0-1)T_s}^{T_1+(k+1)T_s} \sum_i a(i)p(t - iT_s) \sum_l a(l)p(t - lT_s) dt. \] (3.35)

Now restrict the symbol indexes \( i \) and \( l \) to include only those waveforms contained in the interval of integration. Because the pulse shape \( p(t - iT_s) \) has support on the interval \( T_1 + iT_s \leq t \leq T_2 + iT_s \), the indexes that coincide with the integration interval of the first integral on the right-hand side of (3.35) when

\[ T_2 + iT_s > T_1 + kT_s \quad \text{and} \quad T_1 + iT_s < T_1 + (k+1)T_s. \]

Solving the first inequality gives the lower value for \( i \):

\[ T_2 + iT_s > T_1 + kT_s \]
\[ iT_s > -T_2 + T_1 + kT_s \]
\[ i > -\frac{T_2 - T_1}{T_s} + k = -L_p + k \] (3.36)
and solving the second inequality gives the upper value for $i$:

\[
T_1 + iT_s < T_1 + (k + 1)T_s \\
iT_s < (k + 1)T_s \\
i < k + 1.
\] (3.37)

The region of support for $p(t - iT_s)$ coincides with the integration interval of the second integral on the right-hand side of (3.35) when

\[
T_2 + iT_s > T_2 + (L_0 - 1)T_s \quad \text{and} \quad T_1 + iT_s < T_2 + (L_0 - 1)T_s + 0.5T_s
\]

so that the limits of $i$ are

\[
T_2 + iT_s > T_2 + (L_0 - 1)T_s \\
iT_s > (L_0 - 1)T_s \\
i > L_0 - 1
\] (3.38)

and

\[
T_1 + iT_s < T_2 + (L_0 - 1)T_s + 0.5T_s \\
iT_s < T_2 - T_1 + (L_0 - 1)T_s + 0.5T_s \\
i < \frac{T_2 - T_1}{T_s} + L_0 - 1 + 0.5 = L_p + L_0 - 0.5.
\] (3.39)

Similarly, the pulse shape $p(t - lT_s)$ has support on the interval $T_1 + lT_s \leq t \leq T_2 + lT_s$ and the indexes for which the region of support coincides with that of $p(t - iT_s)$ satisfy

\[
T_2 + lT_s > T_1 + iT_s \quad \text{and} \quad T_1 + lT_s < T_2 + iT_s
\]

so that the limits of $l$ are

\[
T_2 + lT_s > T_1 + iT_s \\
lT_s > -T_s + T_1 + iT_s \\
i > -L_p + i
\] (3.40)
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and

\[ T_1 + iT_s < T_2 + iT_s \]
\[ iT_s < T_2 - T_1 + iT_s \]
\[ i < L_p + i. \] (3.41)

Applying these results to their respective summations on \( i \) and \( l \) produces

\[ T_2 + (L_0 - 1)T_s + 0.5T_s \]
\[ \int_{T_1}^{L_0 + L_p - 2} \int_{T_1 + (k + 1)T_s}^{T_1 + (k + 1)T_s} \sum_{i=-L_p+k+1}^{L_p+i-1} \sum_{l=-L_p+i+1}^{L_p+i-1} a(i)p(t - iT_s)p(t - lT_s)dt \]
\[ = \sum_{k=0}^{L_0 + L_p - 2} \sum_{i=-L_p+k+1}^{L_p+i-1} \sum_{l=-L_p+i+1}^{L_p+i-1} a(i)a(l) \int_{T_1 + kT_s}^{T_1 + (k + 1)T_s} p(t - iT_s)p(t - lT_s)dt \]
\[ + \sum_{i=L_0}^{L_0 + L_p - 1} \sum_{l=-L_p+i+1}^{L_p+i-1} a(i)a(l) \int_{T_2 + (L_0 - 1)T_s}^{T_1 + (L_0 - 1)T_s + 0.5T_s} p(t - iT_s)p(t - lT_s)dt. \] (3.42)

The first term on the right-hand side of (3.42) is of the form

\[ \sum_{k=0}^{L_0 + L_p - 2} \sum_{i=-L_p+k+1}^{L_p+i-1} \sum_{l=-L_p+i+1}^{L_p+i-1} F(k, i, l). \] (3.43)

With the aid of Figure 2.1, moving the summation on \( k \) to the inner summation produces

\[ \sum_{k=0}^{L_0 + L_p - 2} \sum_{i=-L_p+k+1}^{L_p+i-1} \sum_{l=-L_p+i+1}^{L_p+i-1} F(k, i, l) = \sum_{k=0}^{L_0 + L_p - 1} \sum_{i=-L_p+1}^{L_p+i-1} \sum_{l=-L_p+i+1}^{L_p+i-1} F(k, i, l) \]
\[ + \sum_{i=0}^{L_0 - 1} \sum_{l=-L_p+i+1}^{L_p+i-1} \sum_{k=i}^{L_p+i-1} F(k, i, l) + \sum_{i=L_0}^{L_0 + L_p - 2} \sum_{l=-L_p+i+1}^{L_p+i-1} \sum_{k=i}^{L_p+i-1} F(k, i, l). \] (3.44)
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Applying this result to (3.42) produces

\[
T_{2+(L_0-1)T_s+0.5T_s} \int_{T_1} a(i) p(t - iT_s) \sum_l a(l) p(t - lT_s) dt
\]

\[
= \sum_{i=-L_p+1}^{L_p-1} \sum_{l=-L_p+i+1}^{L_p+i-1} a(i) a(l) \int_{T_1} p(t - iT_s) p(t - lT_s) dt
\]

\[
+ \sum_{i=0}^{L_0-1} \sum_{l=-L_p+i+1}^{L_p+i-1} a(i) a(l) \int_{T_1} p(t - iT_s) p(t - lT_s) dt
\]

\[
+ \sum_{i=L_0}^{L_0+L_p-2} \sum_{l=-L_p+i+1}^{L_p+i-1} a(i) a(l) \int_{T_1} p(t - iT_s) p(t - lT_s) dt
\]

\[
+ \sum_{i=L_0}^{L_0+L_p-2} \sum_{l=-L_p+i+1}^{L_p+i-1} a(i) a(l) \int_{T_1} p(t - iT_s) p(t - lT_s) dt
\]

Combining the inner-most summation-integral terms into a single integral produces the final result:

\[
T_{2+(L_0-1)T_s+0.5T_s} \int_{T_1} a(i) p(t - iT_s) \sum_l a(l) p(t - lT_s) dt
\]

\[
= \sum_{i=-L_p+1}^{L_p-1} \sum_{l=-L_p+i+1}^{L_p+i-1} a(i) a(l) \int_{T_1} p(t - iT_s) p(t - lT_s) dt
\]

\[
+ \sum_{i=0}^{L_0-1} \sum_{l=-L_p+i+1}^{L_p+i-1} a(i) a(l) \int_{T_1} p(t - iT_s) p(t - lT_s) dt
\]

\[
+ \sum_{i=L_0}^{L_0+L_p-2} \sum_{l=-L_p+i+1}^{L_p+i-1} a(i) a(l) \int_{T_1} p(t - iT_s) p(t - lT_s) dt
\]

\[
+ \sum_{i=L_0}^{L_0+L_p-2} \sum_{l=-L_p+i+1}^{L_p+i-1} a(i) a(l) \int_{T_1} p(t - iT_s) p(t - lT_s) dt. \quad (3.45)
\]
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The remaining three integrals are straight-forward variations on the procedures used for the first three integrals. The results are summarized as follows:

The Fourth Integral of (3.16)

\[
T_{2+(L_0-1)T_s+0.5T_s} \int_{T_1} \sum_i a(i)p(t - iT_s) \sum_l b(l)p(t - lT_s - 0.5T_s) dt
\]

\[
= \sum_{i=-L_p+1}^{i+L_p-1} \sum_{l=-L_p}^{l+L_p-1} a(i)b(l) \int_{T_1}^{T_2+iT_s} p(t - iT_s)p(t - lT_s - 0.5T_s) dt
\]

\[
+ \sum_{i=0}^{L_0-1} \sum_{l=i-L_p}^{i+L_p-1} a(i)b(l) \int_{T_1+iT_s}^{T_2+(L_0-1)T_s} p(t - iT_s)p(t - lT_s - 0.5T_s) dt
\]

\[
+ \sum_{i=L_0}^{L_0+L_p-2} \sum_{l=i-L_p}^{i+L_p-1} a(i)b(l) \int_{T_1+iT_s}^{T_2+(L_0-1)T_s+0.5T_s} p(t - iT_s)p(t - lT_s - 0.5T_s) dt.
\] (3.47)

The Fifth Integral of (3.16)

\[
T_{2+(L_0-1)T_s+0.5T_s} \int_{T_1} \sum_i b(i)p(t - iT_s - 0.5T_s) \sum_l a(l)p(t - lT_s) dt
\]

\[
= \sum_{i=-L_p}^{i+L_p} \sum_{l=-L_p+1}^{l+L_p-1} b(i)a(l) \int_{T_1}^{T_1+0.5T_s} p(t - iT_s - 0.5T_s)p(t - lT_s) dt
\]

\[
+ \sum_{i=-L_p+1}^{i+L_p} \sum_{l=-L_p}^{l+L_p-1} b(i)a(l) \int_{T_1+iT_s+0.5T_s}^{T_2+iT_s+0.5T_s} p(t - iT_s - 0.5T_s)p(t - lT_s) dt
\]

\[
+ \sum_{i=0}^{L_0-1} \sum_{l=i-L_p+1}^{i+L_p} b(i)a(l) \int_{T_1+iT_s+0.5T_s}^{T_2+iT_s+0.5T_s} p(t - iT_s - 0.5T_s)p(t - lT_s) dt.
\]
\[ L_0 + L_p - 2 \sum_{i=L_0}^{L_0-1} \sum_{l=-L_p+1}^{L_0-1} b(i) a(l) \int_{T_1 + iT_s + 0.5T_s}^{T_2 + (L_0 - 1)T_s + 0.5T_s} p(t - iT_s - 0.5T_s) p(t - lT_s) dt. \] 

(3.48)

The Sixth Integral of (3.16)

\[ \int_{T_1}^{T_2 + (L_0 - 1)T_s + 0.5T_s} \sum_{i=0}^{-1} \sum_{l=-L_p}^{L_0-1} b(i) b(l) \int_{T_1 + 0.5T_s}^{T_1 + 0.5T_s} p(t - iT_s - 0.5T_s) p(t - lT_s - 0.5T_s) dt 
+ \sum_{i=-L_p+1}^{i+L_p-1} \sum_{l=-L_p+1}^{L_0-1} b(i) b(l) \int_{T_1 + iT_s + 0.5T_s}^{T_2 + iT_s + 0.5T_s} p(t - iT_s - 0.5T_s) p(t - lT_s - 0.5T_s) dt 
+ \sum_{i=0}^{L_0-1} \sum_{l=-L_p+1}^{L_0-1} b(i) b(l) \int_{T_1 + iT_s + 0.5T_s}^{T_2 + iT_s + 0.5T_s + r_p[(l-i)T_s]} p(t - iT_s - 0.5T_s) p(t - lT_s - 0.5T_s) dt. \] 

(3.49)

Assembling the 24 terms defined in (3.25), (3.34), (3.46) – (3.49) to re-express (3.16) requires more space than is available. In its current form, the 24 terms apply to the arrangement illustrated in Figure 1.1 (b) where the \( L_0 \) contiguous pilot symbols form a midamble. For the case where the \( L_0 \) contiguous pilot symbols form a preamble, illustrated in Figure 1.1 (a), the \( a(i) \) and \( b(i) \) are zero for \( i < 0 \). Thus, all terms in (3.25), (3.34), (3.46) – (3.49) involving \( a(i) \) and \( b(i) \) for \( i < 0 \) may be discarded. For the case where the \( L_0 \) contiguous pilot symbols form a postamble, \( a(i) \) and \( b(i) \) are zero for \( i \geq L_0 \). In this case, all terms in (3.25), (3.34), (3.46) – (3.49) involving \( a(i) \) and \( b(i) \) for \( i \geq L_0 \) may be discarded. In any event, some general comments can be made.

Comments

1. The second integral in (3.25) is

\[ x(lT_s) = \int_{T_1 + lT_s}^{T_2 + lT_s} r(t) p(t - lT_s) dt \]

(3.50)
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and may be interpreted as the response to $r(t)$ at time $t = lT_s$ of a filter matched to the pulse shape. The other integrals in (3.25) of similar form cannot be considered matched filter outputs because the limits of integration do not include all of $p(t - lT_s)$. These terms correspond to pulses at the beginning or ending (i.e., the edges) of the block of pilot symbols. For this reason, the contributions of these terms are called “edge effects.” Similarly, the third integral in (3.34) is

$$x((l + 0.5)T_s) = \int_{T_1 + iT_s + 0.5T_s}^{T_2 + iT_s + 0.5T_s} r(t)p(t - lT_s - 0.5T_s)dt$$

(3.51)

and may be interpreted as matched filter outputs at time $t = lT_s + 0.5T_s$. Again, the other integrals in (3.34) with a similar form cannot be considered matched filter outputs because the limits of integration are not correct and are therefore contributors to the “edge effects.” Observe, that unlike its non-offset counterpart, QPSK requires matched filter outputs sampled at 2 samples/symbol.

2. There are four pulse shape autocorrelation functions in (3.46) – (3.49). The second term on the right-hand-side of (3.46) contains the integral

$$r_p([l - i]T_s) = \int_{T_1 + iT_s}^{T_2 + iT_s} p(t - iT_s)p(t - lT_s)dt.$$  

(3.52)

The third term on the right-hand-side of (3.47) contains the integral

$$r_p([l - i + 0.5]T_s) = \int_{T_1 + iT_s}^{T_2 + iT_s} p(t - iT_s)p(t - lT_s - 0.5T_s)dt.$$  

(3.53)

The third term on the right-hand-side of (3.48) contains the integral

$$r_p([l - i - 0.5]T_s) = \int_{T_1 + iT_s + 0.5T_s}^{T_2 + iT_s + 0.5T_s} p(t - iT_s - 0.5T_s)p(t - lT_s)dt.$$  

(3.54)
The third term on the right-hand-side of (3.49) contains the integral

$$r_p([l - i]T_s) = \int_{T_1 + iT_s + 0.5T_s}^{T_2 + iT_s + 0.5T_s} p(t - iT_s - 0.5T_s)p(t - lT_s - 0.5T_s)dt.$$  \hspace{1cm} (3.55)

Similar looking integrals in (3.46) – (3.49) are note pulse shape autocorrelations because the integration intervals are not correct. These integrals correspond to pulse shapes at beginning or end (or edge) of the block of pilot symbols and constitute another form of “edge effects.”

For pulse shapes that satisfy the Nyquist No-ISI condition, \( r_p(\tau) = 0 \) when \( \tau \) is a non-zero integer multiple of \( T_s \). Note however that there are many terms that include \( r_p(\tau) \) for \( \tau \) an integer \( \pm 0.5 \) times \( T_s \). This is a consequence of the requirement to sample the matched filter output at 2 samples/symbol.

3. For the special case of full response pulse shapes, \( T_1 = 0, T_2 = T_s \) so that \( L_p = 1 \). Further, \( r_p(\tau) = 0 \) for \( \tau \) outside the interval \(-T_s \leq \tau \leq T_s\). Applying these conditions to (3.25), (3.34) and (3.46) – (3.49) produces

$$\frac{\partial}{\partial h^*} \Lambda(h) = \frac{1}{2N_0} \left[ \sum_{l=0}^{L_0-1} a(l)x(lT_s) + a(L_0) \int_{L_0T_s}^{(L_0+0.5)T_s} r(t)p(t - L_0T_s)dt \right] - \frac{j}{2N_0} \left[ b(-1) \int_{0}^{0.5T_s} r(t)p(t + 0.5T_s)dt + \sum_{l=0}^{L_0-1} b(l)x(l + 0.5T_s) \right]$$

$$- \frac{h}{2N_0} \left[ \sum_{i=0}^{L_0-1} a^2(i)r_p(0) + a^2(L_0) \int_{L_0T_s}^{(L_0+0.5)T_s} p^2(t - L_0T_s)dt \right] + \frac{jh}{2N_0} \left[ \sum_{i=0}^{L_0-1} a(i) \left[ b(i) - 1 \right] r_p(-0.5T_s) + b(i) r_p(0.5T_s) + a(L_0)b(L_0 - 1) r_p(-0.5T_s) \right]$$

$$- \frac{jh}{2N_0} \left[ b(-1)a(0)r_p(-0.5T_s) + \sum_{i=0}^{L_0-1} b(i) \left[ a(i) r_p(-0.5T_s) + a(i + 1) r_p(0.5T_s) \right] \right] - \frac{h}{2N_0} \left[ b^2(-1) \int_{0}^{0.5T_s} p^2(t + 0.5T_s)dt + \sum_{i=0}^{L_0-1} b^2(i)r_p(0) \right]. \hspace{1cm} (3.56)$$

Rearranging the terms and simplifying using \( a^2(i) = A^2, b^2(i) = A^2, \) and \( r_p(0) = 1 \) gives
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\[
\frac{\partial}{\partial h^*} \Lambda(h) = \frac{1}{2N_0} \sum_{l=0}^{L_0-1} \left[ a(l)x(lT_s) - jb(l)x(|l + 0.5|T_s) \right] - 2hL_0A^2 \\
+ \frac{1}{2N_0} \left[ a(L_0) \int_r^{(L_0+0.5)T_s} r(t)p(t - L_0T_s)dt - ha^2(L_0) \int_{L_0T_s}^{(L_0+0.5)T_s} p^2(t - L_0T_s)dt \\
- jb(-1) \int_0^{0.5T_s} r(t)p(t + 0.5T_s)dt - hb^2(-1) \int_0^{0.5T_s} p^2(t + 0.5T_s)dt \right] \\
+ j\frac{\hbar}{2N_0} \sum_{i=0}^{L_0-1} a(i) \left[ b(i - 1)r_p(-0.5T_s) + b(i)r_p(0.5T_s) \right] + a(L_0)b(L_0 - 1)r_p(-0.5T_s) \\
- b(-1)a(0)r_p(-0.5T_s) - \sum_{i=0}^{L_0-1} b(i) \left[ a(i)r_p(-0.5T_s) + a(i + 1)r_p(0.5T_s) \right] \right] 
\]

(3.57)

The second through fifth lines of (3.57) are terms due to edge effects. There are two different contributors to edge effects for offset QPSK. The first contributor is due to the limited integration intervals for the pulses on the edges of the pilot symbol block. This contributor, denoted \( \epsilon \), is quantified by the terms in the second and third lines of (3.57):

\[
\epsilon = \frac{1}{2N_0} \left[ a(L_0) \int_r^{(L_0+0.5)T_s} r(t)p(t - L_0T_s)dt - ha^2(L_0) \int_{L_0T_s}^{(L_0+0.5)T_s} p^2(t - L_0T_s)dt \\
- jb(-1) \int_0^{0.5T_s} r(t)p(t + 0.5T_s)dt - hb^2(-1) \int_0^{0.5T_s} p^2(t + 0.5T_s)dt \right] . \quad (3.58)
\]

The second contributor to edge effects are due to the offset between the in-phase and quadrature components in the transmitted signal. This contributor, denoted \( X \), is quantified by the fourth and fifth lines of (3.57):

\[
X = \sum_{i=0}^{L_0-1} a(i) \left[ b(i - 1)r_p(-0.5T_s) + b(i)r_p(0.5T_s) \right] \\
- \sum_{i=0}^{L_0-1} b(i) \left[ a(i)r_p(-0.5T_s) + a(i + 1)r_p(0.5T_s) \right] . \quad (3.59)
\]

This expression is a special case of the more general form analyzed in Appendix B. Assum-
Observe that \( X \) is a function of the pilot bits \( a(0) \) and \( b(L_0 - 1) \). It is also a function of the data bits \( b(-1) \) and \( a(L_0) \). \( X \) involves products of the \( a \)'s in the pilot block and the \( b \)'s in the data block as well as the \( b \)'s in the pilot block and the \( a \)'s in the data block. Incorporating this result into (3.56) and rearranging produces

\[
\frac{\partial}{\partial h^*} \Lambda(h) = \frac{1}{2N_0} \left[ \sum_{l=0}^{L_0-1} \left[ a(l)x(lT_s) - jb(l)x([l + 0.5]T_s) \right] - 2hL_0A^2 \right] + \epsilon + X. \tag{3.61}
\]

The source of the edge effects is illustrated in Figure 3.2 for \( L_0 = 4 \). Observe that that interval of integration includes the complete pulse shapes carrying \( a(0), a(1), a(2), a(3) \) and \( b(0), b(1), b(2), b(3) \). Due to the offset, the trailing half of the pulse carrying \( b(-1) \) is included in the interval of integration as well as the leading half of the pulse carrying \( a(4) \). To include these terms in forming an estimate, \( a(L_0) \) and \( b(-1) \) must be known. Similarly, \( X \) depends on \( a(L_0) \) and \( b(-1) \). Usually, \( \epsilon \) and \( X \) are discarded from (3.61) on the way to producing a practical estimator. The justification for discarding the edge effect terms are the same as they were for QPSK, namely

- Using the information contains in the edge terms is not practical because the \( L_p - 1 \) data symbols before and/or after the pilot symbols are not known. (This applies to both \( \epsilon \) and \( X \).)
- For \( L_0 \) sufficiently large, the energy lost by discarding the edge terms is negligibly small. (This applies to \( \epsilon \).)

Discarding \( \epsilon \) and \( X \) from the right-hand side of (3.61) produces

\[
\frac{\partial}{\partial h^*} \Lambda(h) = \frac{1}{2N_0} \left[ \sum_{l=0}^{L_0-1} \left[ a(l)x(lT_s) - jb(l)x([l + 0.5]T_s) \right] - 2hL_0A^2 \right] \tag{3.62}
\]

from which the maximum-likelihood estimator is

\[
\hat{h} = \frac{1}{2L_0A^2} \sum_{l=0}^{L_0-1} \left[ a(l)x(lT_s) - jb(l)x(lT_s + 0.5T_s) \right]. \tag{3.63}
\]
4. For partial response pulse shapes, the first step in identifying a practical estimator is to eliminate the terms in (3.25), (3.34), and (3.46) – (3.49) that contain integrals whose integration intervals are not large enough to be matched filter outputs or pulse shape autocorrelation functions (i.e., terms that are similar to the edge effects term \( \epsilon \) identified in the previous comment). The result is

\[
\frac{\partial}{\partial h} \Lambda(h) = \frac{1}{2N_0} \left[ \sum_{l=0}^{L_0-1} \left[ a(l)x(lT_s) - jb(l)x([l + 0.5]T_s) \right] \right] \\
- \frac{h}{2N_0} \sum_{i=0}^{L_0-1} \sum_{l=i-L_p+1}^{i+L_p-1} \left[ a(i)a(l) + b(i)b(l) \right] r_p([l - i]T_s) \\
+ j \frac{h}{2N_0} \sum_{i=0}^{L_0-1} \sum_{l=i-L_p+1}^{i+L_p-1} a(i)b(l)r_p([l - i + 0.5]T_s) - \sum_{i=0}^{L_0-1} \sum_{l=i-L_p+1}^{i+L_p} b(i)a(l)r_p([l - i - 0.5]T_s) \right]
\]

(3.64)

For pulse shapes that satisfy the Nyquist No-ISI condition, the middle term reduces to

\[
\frac{h}{2N_0} \sum_{i=0}^{L_0-1} \sum_{l=i-L_p+1}^{i+L_p-1} \left[ a(i)a(l) + b(i)b(l) \right] r_p([l - i]T_s) = \frac{2hL_0A^2}{2N_0}.
\]

(3.65)

Incorporating these results into (3.64) produces

\[
\frac{\partial}{\partial h} \Lambda(h) = \frac{1}{2N_0} \left[ \sum_{l=0}^{L_0-1} \left[ a(l)x(lT_s) - jb(l)x([l + 0.5]T_s) \right] \right] - 2hL_0A^2 \\
+ j \frac{h}{2N_0} \sum_{i=0}^{L_0-1} \sum_{l=i-L_p+1}^{i+L_p-1} a(i)b(l)r_p([l - i + 0.5]T_s) - \sum_{i=0}^{L_0-1} \sum_{l=i-L_p+1}^{i+L_p} b(i)a(l)r_p([l - i - 0.5]T_s) \right].
\]

(3.66)

The expression

\[
\mathcal{X} = \sum_{i=0}^{L_0-1} \sum_{l=i-L_p+1}^{i+L_p-1} a(i)b(l)r_p([l - i + 0.5]T_s) - \sum_{i=0}^{L_0-1} \sum_{l=i-L_p+1}^{i+L_p} b(i)a(l)r_p([l - i - 0.5]T_s)
\]

(3.67)

is analyzed in detail in Appendix B. There, it is shown that \( \mathcal{X} \) reduces to
\[ X = \sum_{l=0}^{L_p-1} \sum_{l'=l-L_p}^{L_o-1} a(l)b(l') r_p([l' - l + 0.5]T_s) - \sum_{l'=0}^{L_p-1} \sum_{l=-L_p+1}^{L_o-1} b(l') a(l) r_p([l - l' - 0.5]T_s) \]
\[ + \sum_{l=L_0}^{L_o-1} \sum_{l'=L_p}^{L_p} a(l)b(l') r_p([l' - l + 0.5]T_s) - \sum_{l'=L_0}^{L_p} \sum_{l=0}^{L_o-1} b(l') a(l) r_p([l - l' - 0.5]T_s) \] (3.68)

where

\[ I_1 = \begin{cases} L_p - 1 & L_0 \geq L_p \\ L_0 - 1 & L_0 < L_p \end{cases} \quad I_2 = \begin{cases} L_p - 2 & L_0 \geq L_p \\ L_0 - 1 & L_0 < L_p \end{cases} \]
\[ I_3 = \begin{cases} L_0 - L_p + 1 & L_0 \geq L_p \\ 0 & L_0 < L_p \end{cases} \quad I_4 = \begin{cases} L_0 - L_p & L_0 \geq L_p \\ 0 & L_0 < L_p \end{cases} \]

This version of \( X \) shows that it is a function of \( a(i) \) and \( b(i) \) for \( i < 0 \) and \( i \geq L_0 \). As explained in the previous comment, this term involves up to \( L_p \) data symbols before and up to \( L_p \) data symbols after the block of pilot symbols. The data symbols interact with the pilot data to form products of the \( a \)'s in the pilot block and the \( b \)'s in the data block together with the \( b \)'s in the pilot block and the \( a \)'s in the data block. The I/Q cross term \( X \) is omitted in the development of a practical estimator because a practical estimator has no knowledge of the data symbols on the edges of the pilot symbol block. Omitting \( X \) from (3.66) produces

\[ \frac{\partial}{\partial h^*} \Lambda(h) = \frac{1}{2N_0} \left[ \sum_{l=0}^{L_o-1} \left[ a(l)x(lT_s) - jb(l)x([l + 0.5]T_s) \right] - 2hL_0A^2 \right] \] (3.69)

from which the maximum-likelihood estimator is

\[ \hat{h} = \frac{1}{2L_0A^2} \sum_{l=0}^{L_o-1} \left[ a(l)x(lT_s) - jb(l)x(lT_s + 0.5T_s) \right] . \] (3.70)

Note that this is identical to (3.63), the estimator for full response pulse shapes.

In summary, in the case of both partial response pulse shapes and full response pulse shapes, a relatively simple estimator for the complex-valued channel gain may be used. The estimator is
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Figure 3.2: An illustration of the edge effects for OQPSK using full response pulse shapes ($T_1 = 0$, $T_2 = T_s$, $L_p = 1$) for $L_0 = 4$ pilot symbols.
given by (3.63) [or (3.70)] and is repeated here:

\[ \hat{h} = \frac{1}{2L_0 A^2} \sum_{l=0}^{L_0-1} \left[ a(l)x(lT_s) - jb(l)x(lT_s + 0.5T_s) \right]. \]  

(3.71)

The relative simple estimator is obtained by discarding terms that depend on unknown symbol values. The unknown symbol values scale pulses, a fraction of whose time support coincides with the observation interval. These partial overlaps give rise to “edge effects.” Because the data symbols are unknown, it is difficult to envision how one might incorporate the edge effects into the estimator.

### 3.2.2 Statistical Properties of the Matched Filter Outputs

The matched filter outputs of interest are

\[ x(lT_s) = \int_{T_1+T_s}^{T_2+lT_s} r(t)p(t - lT_s)dt \]

\[ x(lT_s + 0.5T_s) = \int_{T_1+lT_s+0.5T_s}^{T_2+lT_s+0.5T_s} r(t)p(t - lT_s - 0.5T_s)dt. \]  

(3.72)

Using the substitution

\[ r(t) = h \left( \sum_k \left[ a(k)p(t - kT_s) + jb(k)p(t - kT_s - 0.5T_s) \right] \right) + w(t) \]  

(3.73)

the matched filter output \( x(lT_s) \) may be expressed as

\[ x(lT_s) = \int_{T_1+lT_s}^{T_2+lT_s} h \left( \sum_k \left[ a(k)p(t - kT_s) + jb(k)p(t - kT_s - 0.5T_s) \right] \right) + w(t) \right] p(t - lT_s)dt \]

\[ = h \sum_k a(k) \int_{T_1+lT_s}^{T_2+lT_s} p(t - kT_s)p(t - lT_s)dt \]

\[ r_p([k-l]T_s) \]
\[ + j h \sum_{k} b(k) \int_{T_{1} + lT_s}^{T_{2} + lT_s} p(t - kT_s - 0.5T_s) p(t - lT_s) dt \]
\[ r_{p}((k + l)T_s) \]
\[ + \int_{T_{1} + lT_s}^{T_{2} + lT_s} w(t) p(t - lT_s) dt. \] (3.74)

The summations over the index \( k \) should be restricted to those values of \( k \) for which the corresponding integral is nonzero. The first integral on the right-hand side of (3.74) is nonzero when

\[ T_{2} + kT_{s} > T_{1} + lT_{s} \quad \text{and} \quad T_{1} + kT_{s} < T_{2} + lT_{s}. \]

Solving the first inequality gives the lower bound on summation in the first term on the right-hand side of (3.74):

\[ T_{2} + kT_{s} > T_{1} + lT_{s} \]
\[ kT_{s} > -T_{2} + T_{1} + lT_{s} \]
\[ k > -\frac{T_{2} - T_{1}}{T_{s}} + l = -L_{p} + l, \] (3.75)

and solving the second inequality gives the upper bound on the summation in the first term on the right-hand side of (3.74):

\[ T_{1} + kT_{s} < T_{2} + lT_{s} \]
\[ kT_{s} < T_{2} - T_{1} + lT_{s} \]
\[ k < \frac{T_{2} - T_{1}}{T_{s}} + l = L_{p} + l. \] (3.76)

The second integral on the right-hand side of (3.74) is nonzero when

\[ T_{2} + kT_{s} + 0.5T_{s} > T_{1} + lT_{s} \quad \text{and} \quad T_{1} + kT_{s} + 0.5T_{s} < T_{2} + lT_{s}. \]

Solving the first inequality gives the lower bound on summation in the second term on the right-hand side of (3.74):

\[ T_{2} + kT_{s} + 0.5T_{s} > T_{1} + lT_{s} \]
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\[ k T_s + 0.5 T_s > -T_2 + T_1 + l T_s \]
\[ k + 0.5 > - \frac{T_2 - T_1}{T_s} + l = -L_p + l \]
\[ k > -L_p + l - 0.5, \]  
(3.77)

and solving the second inequality gives the upper bound on the summation in the second term on the right-hand side of (3.74):

\[ T_1 + k T_s + 0.5 T_s < T_2 + l T_s \]
\[ k T_s + 0.5 T_s < T_2 - T_1 + l T_s \]
\[ k + 0.5 < \frac{T_2 - T_1}{T_s} + l = L_p + l \]
\[ k < L_p + l - 0.5. \]  
(3.78)

The last term on the right-hand side of (3.74) is the noise term

\[ v(l T_s) = \int_{T_1+lT_s}^{T_2+lT_s} w(t)p(t-l T_s)dt \]  
(3.79)

which is a complex-valued Gaussian random variable. Writing \( v(l T_s) = v_R(l T_s) + jv_I(l T_s) \), the statistics of the real and imaginary parts may be expressed in terms of \( w_R(t) \) and \( w_I(t) \). The mean \( v_R(l T_s) \) is

\[ \mathbb{E} [v_R(l T_s)] = \mathbb{E} \left[ \int_{T_1+lT_s}^{T_2+lT_s} w_R(t)p(t-l T_s)dt \right] = \int_{T_1+lT_s}^{T_2+lT_s} \mathbb{E} [w_R(t)] p(t-l T_s)dt = 0 \]  
(3.80)

and the co-variance is

\[ \mathbb{E} [v_R(l T_s)v_R(l' T_s)] = \mathbb{E} \left[ \int_{T_1+lT_s}^{T_2+lT_s} w_R(t)p(t-l T_s)dt \int_{T_1+l'T_s}^{T_2+l'T_s} w_R(\tau)p(\tau-l' T_s)d\tau \right] \]
\[ = \int_{T_1+lT_s}^{T_2+lT_s} \int_{T_1+l'T_s}^{T_2+l'T_s} \mathbb{E} [w_R(t)w_R(\tau)] p(t-l T_s)p(\tau-l' T_s)d\tau dt \]
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\[ x(lT_s) = h \sum_{k=l-L_p}^{l+L_p-1} a(k)r_p([k - l]T_s) + jh \sum_{k=l-L_p}^{l+L_p-1} b(k)r_p([k - l + 0.5]T_s) + v(lT_s) \]

Similarly, the mean of \( v_I(lT_s) \) is

\[
E[v_I(lT_s)] = E \left[ \int_{T_1+IT_s}^{T_2+IT_s} w_1(t)p(t-lT_s)dt \right] = \int_{T_1+IT_s}^{T_2+IT_s} E[w_1(t)]p(t-lT_s)dt = 0 \tag{3.82}
\]

and the co-variance is

\[
E[v_I(lT_s)v_I(l'T_s)] = E \left[ \int_{T_1+IT_s}^{T_2+IT_s} \int_{T_1+l'T_s}^{T_2+l'T_s} w_1(t)p(t-lT_s)w_1(\tau)p(\tau-l'T_s)d\tau dt \right] = \int_{T_1+IT_s}^{T_2+IT_s} \int_{T_1+l'T_s}^{T_2+l'T_s} E[w_1(t)w_1(\tau)]p(t-lT_s)p(\tau-l'T_s)d\tau dt
\]

\[
= \int_{T_1+IT_s}^{T_2+IT_s} \int_{T_1+l'T_s}^{T_2+l'T_s} N_0 \delta(\tau - t)p(t-lT_s)p(\tau-l'T_s)d\tau dt = 0 \tag{3.83}
\]
\[ h_a(l) + jh \sum_{k=l-L_p}^{l+L_p-1} b(k) r_p([k - l + 0.5]T_s) + v(lT_s). \] (3.84)

The last step follows from the Nyquist No-ISI property.

Now for \( x([l + 0.5]T_s) \). Using the substitution (3.73), \( x([l + 0.5]T_s) \) may be expressed as

\[
x([l + 0.5]T_s) = h \sum_k a(k) \int_{T_1 + lT_s + 0.5T_s}^{T_2 + lT_s + 0.5T_s} p(t - kT_s) p(t - lT_s - 0.5T_s) dt \\
+ jh \sum_k b(k) \int_{T_1 + lT_s + 0.5T_s}^{T_2 + lT_s + 0.5T_s} p(t - kT_s - 0.5T_s) p(t - lT_s - 0.5T_s) dt \\
+ \int_{T_1 + lT_s + 0.5T_s}^{T_2 + lT_s + 0.5T_s} w(t) p(t - lT_s) dt.
\] (3.85)

The summations over the index \( k \) should be restricted to those values of \( k \) for which the corresponding integral is nonzero. The first integral on the right-hand side of (3.85) is nonzero when

\[
T_2 + kT_s > T_1 + lT_s + 0.5T_s \quad \text{and} \quad T_1 + kT_s < T_2 + lT_s + 0.5T_s.
\]

Solving the first inequality gives the lower bound on summation in the first term on the right-hand side of (3.85):

\[
T_2 + kT_s > T_1 + lT_s + 0.5T_s \\
kT_s > -T_2 + T_1 + lT_s + 0.5T_s \\
k > -\frac{T_2 - T_1}{T_s} + l + 0.5 = -L_p + l + 0.5, \tag{3.86}
\]

and solving the second inequality gives the upper bound on the summation in the first term on the right-hand side of (3.85):

\[
T_1 + kT_s < T_2 + lT_s + 0.5T_s \\
kT_s < T_2 - T_1 + lT_s + 0.5T_s
\]
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k < \frac{T_2 - T_1}{T_s} + l + 0.5 = L_p + l + 0.5.

(3.87)

The second integral on the right-hand side of (3.85) is nonzero when

\[ T_2 + kT_s + 0.5T_s > T_1 + lT_s + 0.5T_s \quad \text{and} \quad T_1 + kT_s + 0.5T_s < T_2 + lT_s + 0.5T_s. \]

Solving the first inequality gives the lower bound on summation in the second term on the right-hand side of (3.85):

\[ T_2 + kT_s + 0.5T_s > T_1 + lT_s + 0.5T_s \]

\[ kT_s > -T_2 + T_1 + lT_s \]

\[ k > -\frac{T_2 - T_1}{T_s} + l = -L_p + l, \]  

(3.88)

and solving the second inequality gives the upper bound on the summation in the second term on the right-hand side of (3.85):

\[ T_1 + kT_s + 0.5T_s < T_2 + lT_s + 0.5T_s \]

\[ kT_s < T_2 - T_1 + lT_s \]

\[ k < \frac{T_2 - T_1}{T_s} + l = L_p + l. \]  

(3.89)

The last term on the right-hand side of (3.85) is the noise term

\[ v([l + 0.5]T_s) = \int_{T_1 + lT_s + 0.5T_s}^{T_2 + lT_s + 0.5T_s} w(t)p(t - lT_s - 0.5T_s)dt \]  

(3.90)

which is a complex-valued Gaussian random variable. Writing \( v([l + 0.5]T_s) = v_R([l + 0.5]T_s) + jv_I([l + 0.5]T_s) \), the statistics of the real and imaginary parts may be expressed in terms of \( w_R(t) \) and \( w_I(t) \). The mean \( v_R([l + 0.5]T_s) \) is

\[ \mathbb{E}[v_R([l + 0.5]T_s)] = \mathbb{E} \left[ \int_{T_1 + lT_s + 0.5T_s}^{T_2 + lT_s + 0.5T_s} w_R(t)p(t - lT_s - 0.5T_s)dt \right] \]

\[ = \int_{T_1 + lT_s + 0.5T_s}^{T_2 + lT_s + 0.5T_s} \mathbb{E}[w_R(t)]p(t - lT_s - 0.5T_s)dt = 0 \]  

(3.91)
and the co-variance is

\[
E \left[ v_R([l + 0.5]T_s) v_R([l' + 0.5]T_s) \right]
= E \left[ \int_{T_1+IT_s+0.5T_s}^{T_2+l'T_s+0.5T_s} w_R(t) p(t - lT_s - 0.5T_s) dt \int_{T_1+l'T_s+0.5T_s}^{T_2+l'T_s+0.5T_s} w_R(\tau) p(\tau - l'T_s - 0.5T_s) d\tau \right]
= \int_{T_1+IT_s+0.5T_s}^{T_2+l'T_s+0.5T_s} \int_{T_1+l'T_s+0.5T_s}^{T_2+l'T_s+0.5T_s} E \left[ w_R(t) w_R(\tau) \right] p(t - lT_s - 0.5T_s) p(\tau - l'T_s - 0.5T_s) d\tau dt
= \int_{T_1+IT_s+0.5T_s}^{T_2+l'T_s+0.5T_s} \int_{T_1+l'T_s+0.5T_s}^{T_2+l'T_s+0.5T_s} N_0 \delta(\tau - t) p(t - lT_s - 0.5T_s) p(\tau - l'T_s - 0.5T_s) d\tau dt
= N_0 \int_{T_1+IT_s+0.5T_s}^{T_2+l'T_s+0.5T_s} p(t - lT_s - 0.5T_s) p(t - l'T_s - 0.5T_s) dt
= N_0 \tau_p([l' - l]T_s). \tag{3.92}
\]

Similarly, the mean of \( v_I([l + 0.5]T_s) \) is

\[
E \left[ v_I([l + 0.5]T_s) \right] = E \left[ \int_{T_1+IT_s+0.5T_s}^{T_2+l'T_s+0.5T_s} w_I(t) p(t - lT_s - 0.5T_s) dt \right]
= \int_{T_1+IT_s+0.5T_s}^{T_2+l'T_s+0.5T_s} E \left[ w_I(t) \right] p(t - lT_s - 0.5T_s) dt = 0 \tag{3.93}
\]

and the co-variance is

\[
E \left[ v_I([l + 0.5]T_s) v_I([l' + 0.5]T_s) \right]
= E \left[ \int_{T_1+IT_s+0.5T_s}^{T_2+l'T_s+0.5T_s} w_I(t) p(t - lT_s - 0.5T_s) dt \int_{T_1+IT_s+0.5T_s}^{T_2+l'T_s+0.5T_s} w_I(\tau) p(\tau - l'T_s - 0.5T_s) d\tau \right]
= \int_{T_1+IT_s+0.5T_s}^{T_2+l'T_s+0.5T_s} \int_{T_1+l'T_s+0.5T_s}^{T_2+l'T_s+0.5T_s} E \left[ w_I(t) w_I(\tau) \right] p(t - lT_s - 0.5T_s) p(\tau - l'T_s - 0.5T_s) d\tau dt
\]
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\[
\begin{align*}
T_2 + l'T_s + 0.5T_s & \quad T_2 + l'T_s + 0.5T_s \\
T_1 + l'T_s + 0.5T_s & \quad T_1 + l'T_s + 0.5T_s \\
T_2 + l'T_s + 0.5T_s & \quad T_2 + l'T_s + 0.5T_s \\
T_1 + l'T_s + 0.5T_s & \quad T_1 + l'T_s + 0.5T_s \\
\int_{T_1 + l'T_s + 0.5T_s} \int_{T_1 + l'T_s + 0.5T_s} N_0 \delta(\tau - t)p(t - l'T_s - 0.5T_s)p(\tau - l'T_s - 0.5T_s)d\tau dt \\
T_1 + l'T_s + 0.5T_s & \quad T_1 + l'T_s + 0.5T_s \\
= N_0 \int_{T_1 + l'T_s + 0.5T_s} p(t - l'T_s - 0.5T_s)p(t - l'T_s - 0.5T_s)dt \\
T_1 + l'T_s + 0.5T_s & \quad T_1 + l'T_s + 0.5T_s \\
= N_0 r_p([l' - l]T_s). & \quad (3.94)
\end{align*}
\]

For pulse shapes that satisfy the Nyquist No-ISI condition, the sequence \( v_R([l + 0.5]T_s) \) is uncorrelated and the sequence \( v_1([l + 0.5]T_s) \) is uncorrelated. The sequences \( v_R([l + 0.5]T_s) \) and \( v_1([l + 0.5]T_s) \) are uncorrelated from each other by virtue of the fact that \( w(t) \) is a proper complex-valued Gaussian random variable. Applying these results to (3.85) produces

\[
x([l + 0.5]T_s) = h \sum_{k=-L_p+1}^{l+L_p} a(k)r_p([k - l - 0.5]T_s) + jh \sum_{k=-L_p+1}^{l+L_p-1} b(k)r_p([k - l]T_s) + v([l + 0.5]T_s) \\
= h \sum_{k=-L_p+1}^{l+L_p} a(k)r_p([k - l - 0.5]T_s) + jhb(l) + v([l + 0.5]T_s). \quad (3.95)
\]

The last step follows from the Nyquist No-ISI condition.

In passing, note that the sequences \( v_R(lT_s) \) and \( v_R([l + 0.5]T_s) \) are correlated. The cross-correlation is

\[
E[v_R(lT_s)v_R([l' + 0.5]T_s)] \\
= E \left[ \int_{T_1 + l'T_s}^{T_2 + l'T_s + 0.5T_s} w_R(t)p(t - l'T_s)dt \int_{T_1 + l'T_s + 0.5T_s}^{T_2 + l'T_s + 0.5T_s} w_R(\tau)p(\tau - l'T_s - 0.5T_s)d\tau \right] \\
= \int_{T_1 + l'T_s}^{T_2 + l'T_s} \int_{T_1 + l'T_s + 0.5T_s}^{T_2 + l'T_s + 0.5T_s} E[w_R(t)w_R(\tau)]p(t - l'T_s)p(\tau - l'T_s - 0.5T_s)d\tau dt \\
= \int_{T_1 + l'T_s}^{T_2 + l'T_s} \int_{T_1 + l'T_s + 0.5T_s}^{T_2 + l'T_s + 0.5T_s} N_0 \delta(\tau - t)p(t - l'T_s)p(\tau - l'T_s - 0.5T_s)d\tau dt \\
= N_0 \int_{T_1 + l'T_s}^{T_2 + l'T_s} p(t - l'T_s)p(t - l'T_s - 0.5T_s)dt
\]
Similarly, the sequences \( v_I(lT_s) \) and \( v_I([l + 0.5]T_s) \) are correlated and the cross correlation function is identical to (3.96).

In summary we have

\[
x(lT_s) = h\alpha(l) + jh \sum_{k=l-L_p+1}^{l+L_p-1} b(k) r_p([k - l + 0.5]T_s) + v(lT_s)
\]

(3.97)

\[
x([l + 0.5]T_s) = h \sum_{k=l-L_p+1}^{l+L_p} a(k) r_p([k - l - 0.5]T_s) + jhb(l) + v([l + 0.5]T_s)
\]

(3.98)

where \( v(lT_s) = v_R(lT_s) + jv_I(lT_s) \) is a sequence of proper complex-valued Gaussian random variables with

\[
E[v_R(lT_s)] = E[v_I(lT_s)] = 0
\]

\[
E[v_R(lT_s)v_R([l' + 0.5]T_s)] = E[v_I(lT_s)v_I([l' + 0.5]T_s)] = N_0 r_p([l' - l]T_s);
\]

(3.99)

\[
v([l + 0.5T_s]) = v_R([l + 0.5T_s]) + jv_I([l + 0.5T_s]) \] is a sequence of proper complex-valued Gaussian random variables with

\[
E[v_R([l + 0.5T_s])] = E[v_I([l + 0.5T_s])] = 0
\]

\[
E[v_R([l + 0.5T_s])v_R([l' + 0.5]T_s)] = E[v_I([l + 0.5T_s])v_I([l' + 0.5]T_s)] = N_0 r_p([l' - l]T_s);
\]

(3.100)

and where

\[
E[v_R(lT_s)v_R([l + 0.5]T_s)] = E[v_I(lT_s)v_I([l + 0.5]T_s)] = N_0 r_p([l' - l + 0.5]T_s).
\]

(3.101)

### 3.2.3 Estimator Performance

The performance of this (and any other) estimator is quantified using the first two moments of the estimator error

\[
e = \hat{h} - h.
\]

(3.102)

If the mean of the estimator error is zero, then the estimator is unbiased. The estimator error variance quantifies how much energy the estimator error injects into the overall system. The
Cramér-Rao bound is a lower bound on the estimator error variance for any unbiased estimator. An estimator whose error variance achieves the Cramér-Rao bound is called an efficient estimator.

Using the expressions (3.84) and (3.95), the estimator may be expressed as

$$\hat{h} = \frac{1}{2L_0A^2} \sum_{l=0}^{L_0-1} \left[ a(l)x(lT_s) - jb(l)x([l + 0.5]T_s) \right]$$

$$= \frac{1}{2L_0A^2} \sum_{l=0}^{L_0-1} a(l) \left[ ha(l) + jh \sum_{k=l-L_p}^{l+L_p-1} b(k)r_p([k - l + 0.5]T_s) + v(lT_s) \right]$$

$$- j \frac{1}{2L_0A^2} \sum_{l=0}^{L_0-1} b(l) \left[ h \sum_{k=l-L_p+1}^{l+L_p} a(k)r_p([k - l - 0.5]T_s) + jhb(l) + v([l + 0.5]T_s) \right]$$

$$= \frac{h}{2L_0A^2} \sum_{l=0}^{L_0-1} \left[ a^2(l) + b^2(l) \right]$$

$$+ j \frac{h}{2L_0A^2} \left[ \sum_{l=0}^{L_0-1} a(l) \sum_{k=l-L_p}^{l+L_p-1} b(k)r_p([k - l + 0.5]T_s) - \sum_{l=0}^{L_0-1} b(l) \sum_{k=l-L_p+1}^{l+L_p} a(k)r_p([k - l - 0.5]T_s) \right]$$

$$+ \frac{1}{2L_0A^2} \sum_{l=0}^{L_0-1} \left[ a(l)v(lT_s) - jb(l)v([l + 0.5]T_s) \right]$$

$$= h + j \frac{h}{2L_0A^2} \mathcal{X} + \frac{1}{2L_0A^2} N_1 \tag{3.103}$$

The estimator error is

$$e = \hat{h} - h = j \frac{h}{2L_0A^2} \mathcal{X} + \frac{1}{2L_0A^2} N_1 \tag{3.105}$$

where $\mathcal{X}$ and $N_2$ are defined in Equation (3.103). The estimator error consists of a bias term and a Gaussian random variable.

The bias term is a scaled version of the I/Q cross term $\mathcal{X}$ analyzed in detail in Appendix B. In short, $\mathcal{X}$ is a function of the products of pilot symbols and the $L_p$ data symbols preceding the pilot symbols and the pilot symbols and the $L_p$ data symbols following the pilot symbols. This term introduces a data-dependent bias into the estimator. For independent and equally likely symbols, the bias term is zero is non-zero and introduces a bias the noticeably degrades the performance of both the estimator and the bit error rate performance.

The additive noise term $N_1$ is the sum of zero-mean complex-valued Gaussian random vari-
variables. The real and imaginary parts of $N_1$ are derived as follows:

$$N_1 = \sum_{l=0}^{L_0-1} \left[ a(l)v(lT_s) - jb(l)v([l + 0.5]T_s) \right]$$

$$= \sum_{l=0}^{L_0-1} \left[ a(l)\left\{ v_R(lT_s) + jv_I(lT_s) \right\} - jb(l) \left\{ v_R([l + 0.5]T_s) + jv_I([l + 0.5]T_s) \right\} \right]$$

$$= \sum_{l=0}^{L_0-1} \left[ a(l)v_R(lT_s) + b(l)v_I([l + 0.5]T_s) + j \left\{ a(l)v_I(lT_s) - b(l)v_R([l + 0.5]T_s) \right\} \right]$$

$$= \sum_{l=0}^{L_0-1} \left[ a(l)v_R(lT_s) + b(l)v_I([l + 0.5]T_s) \right] + j \sum_{l=0}^{L_0-1} \left[ a(l)v_I(lT_s) - b(l)v_R([l + 0.5]T_s) \right]$$

$$= N_{1,R} + N_{1,I}$$

(3.106)

Because $v(\cdot)$ is a proper complex-valued Gaussian random variable, $v_R(\cdot)$ and $v_I(\cdot)$ are uncorrelated regardless of the index. Furthermore, for pulse shapes that satisfy the Nyquist No-ISI condition, $T_s$-spaced samples of $v(t)$ are also uncorrelated. Consequently, $N_{1,R}$ is the sum of uncorrelated Gaussian random variables. Similarly, $N_{1,I}$ is the sum of uncorrelated Gaussian random variables. The mean and variance of $N_{1,R}$ are

$$\mathbb{E}[N_{1,R}] = \mathbb{E} \left\{ \sum_{l=0}^{L_0-1} \left[ a(l)v_R(lT_s) + b(l)v_I([l + 0.5]T_s) \right] \right\}$$

$$= \sum_{l=0}^{L_0-1} \left[ a(l)\mathbb{E}[v_R(lT_s)] + b(l)\mathbb{E}[v_I([l + 0.5]T_s)] \right] = 0 \quad (3.107)$$

$$\mathbb{E}[N_{1,R}^2] = \mathbb{E} \left[ \sum_{l=0}^{L_0-1} \left[ a(l)v_R(lT_s) + b(l)v_I([l + 0.5]T_s) \right] \sum_{l'=0}^{L_0-1} \left[ a(l')v_R(l'T_s) + b(l')v_I([l' + 0.5]T_s) \right] \right]$$

$$= \sum_{l=0}^{L_0-1} \sum_{l'=0}^{L_0-1} \left[ a(l)a(l')\mathbb{E}[v_R(lT_s)v_R(l'T_s)] + a(l)b(l')\mathbb{E}[v_R(lT_s)v_I([l' + 0.5]T_s)] \right. \right. \right. \right.$$

$$+ b(l)a(l')\mathbb{E}[v_I([l + 0.5]T_s)v_R(l'T_s)] + b(l)b(l')\mathbb{E}[v_I([l + 0.5]T_s)v_I([l' + 0.5]T_s)] \right]$$

$$= \sum_{l=0}^{L_0-1} \sum_{l'=0}^{L_0-1} \left[ a(l)a(l')N_0\delta(l' - l) + b(l)b(l')N_0\delta(l' - l) \right]$$
3.2. CHANNEL ESTIMATOR

\[ N_0 \sum_{l=0}^{L_0-1} \left[ a^2(l) + b^2(l) \right] \]
\[ = 2L_0A^2N_0. \quad (3.108) \]

The mean and variance of \( N_{2,l} \) are

\[
E [N_{1,l}] = E \left\{ \sum_{l=0}^{L_0-1} \left[ a(l)v_I(lT_s) - b(l)v_R([l + 0.5]T_s) \right] \right\} \\
= \sum_{l=0}^{L_0-1} \left[ a(l)E [v_I(lT_s)] - b(l)E [v_R([l + 0.5]T_s)] \right] = 0 \quad (3.109)
\]

\[
E [N_{1,l}^2] = E \left[ \sum_{l=0}^{L_0-1} \left[ a(l)v_I(lT_s) - b(l)v_R([l + 0.5]T_s) \right] \sum_{l'=0}^{L_0-1} \left[ a(l')v_I(l'T_s) - b(l')v_R([l' + 0.5]T_s) \right] \right] \\
= \sum_{l=0}^{L_0-1} \sum_{l'=0}^{L_0-1} \left[ a(l)a(l')E [v_I(lT_s)v_I(l'T_s)] - a(l)b(l')E [v_I(lT_s)v_R([l' + 0.5]T_s)] \\
- b(l)a(l')E [v_R([l + 0.5]T_s)v_I(l'T_s)] + b(l)b(l')E [v_R([l + 0.5]T_s)v_R([l' + 0.5]T_s)] \right] \\
= \sum_{l=0}^{L_0-1} \sum_{l'=0}^{L_0-1} \left[ a(l)a(l')N_0\delta(l' - l) + b(l)b(l')N_0\delta(l' - l) \right] \\
= N_0 \sum_{l=0}^{L_0-1} \left[ a^2(l) + b^2(l) \right] \\
= 2L_0A^2N_0. \quad (3.110)
\]

The cross correlation is

\[
E [N_{1,R}N_{1,I}] \\
= E \left[ \sum_{l=0}^{L_0-1} \left[ a(l)v_R(lT_s) + b(l)v_I([l + 0.5]T_s) \right] \sum_{l'=0}^{L_0-1} \left[ a(l')v_I(l'T_s) - b(l')v_R([l' + 0.5]T_s) \right] \right] \\
= \sum_{l=0}^{L_0-1} \sum_{l'=0}^{L_0-1} \left[ a(l)a(l')E [v_R(lT_s)v_I(l'T_s)] - a(l)b(l')E [v_R(lT_s)v_R([l' + 0.5]T_s)] \\
+ b(l)a(l')E [v_I([l + 0.5]T_s)v_I(l'T_s)] - b(l)b(l')E [v_I([l + 0.5]T_s)v_R([l' + 0.5]T_s)] \right] 
\]
Equation (3.111) is a special form of the interference term $\mathcal{X}$. This form is analyzed in Appendix B.3 where it is shown to be 0. Remarkably, $N_{2,R}$ and $N_{2,I}$ are uncorrelated.

The estimator error mean is

$$
E[e] = \mathcal{E} \left[j \frac{h}{2L_0 A^2} \mathcal{X} + \frac{1}{2L_0 A^2} N_1 \right]
= j \frac{h}{2L_0 A^2} E[\mathcal{X}] + \frac{1}{2L_0 A^2} E[N_1]
= j \frac{h}{2L_0 A^2} \times 0 + \frac{1}{2L_0 A^2} \times 0 = 0.
$$

This shows that the estimator is unbiased when averaging is performed over the unknown data symbols comprising $\mathcal{X}$.

Because $E[e] = 0$, the estimator error variance is $E[|e|^2]$. To derive an expression for the error variance, $|e|^2$ is expressed as follows:

$$
|e|^2 = \left\{ \frac{h_I}{2L_0} \left( \frac{\mathcal{X}}{A^2} \right) + \frac{N_{1,R}}{2L_0 A^2} \right\}^2 + \left\{ \frac{h_R}{2L_0} \left( \frac{\mathcal{X}}{A^2} \right) + \frac{N_{1,I}}{2L_0 A^2} \right\}^2
= \frac{h_I^2}{4L_0^2} \left( \frac{\mathcal{X}}{A^2} \right)^2 + \frac{h_I N_{1,R}}{2L_0^2 A^2} \left( \frac{\mathcal{X}}{A^2} \right) + \frac{N_{1,R}^2}{4L_0^2 A^4} + \frac{h_R^2}{4L_0^2} \left( \frac{\mathcal{X}}{A^2} \right)^2 + \frac{h_R N_{1,I}}{2L_0^2 A^2} \left( \frac{\mathcal{X}}{A^2} \right) + \frac{N_{1,I}^2}{4L_0^2 A^4}.
$$

Assuming $N_1$ and $\mathcal{X}$ are uncorrelated, the error variance may be expressed as

$$
E[|e|^2] = \frac{h_I^2}{4L_0^2} E \left[ \left( \frac{\mathcal{X}}{A^2} \right)^2 \right] - \frac{h_I}{2L_0^2 A^2} E[N_{1,R}] E \left[ \frac{\mathcal{X}}{A^2} \right] + \frac{1}{4L_0^2 A^4} E[N_{1,R}^2]
+ \frac{h_R^2}{4L_0^2} E \left[ \left( \frac{\mathcal{X}}{A^2} \right)^2 \right] + \frac{h_R}{2L_0^2 A^2} E[N_{1,I}] E \left[ \frac{\mathcal{X}}{A^2} \right] + \frac{1}{4L_0^2 A^4} E[N_{1,I}^2]
= \frac{h_R^2 + h_I^2}{4L_0^2} \times \sigma_X^2 \left( \frac{2L_0 A^2 N_0 + 2L_0 A^2 N_0}{4L_0^2 A^4} \right)
$$
where $\sigma^2_{\mathcal{X}}$ is the variance of $\mathcal{X}$ and is examined thoroughly in Section B.2. For the case where the pilot symbols are arranged in a preamble as illustrated in Figure 1.1 (a), the variance of $\mathcal{X}$ is given by (B.75) and (B.88). These two cases may be combined into the common form

$$
\sigma^2_{\mathcal{X}} = A^2 \left[ \sum_{i_1=0}^{I_1} \sum_{i_2=\max\{i_1,i'_1\}}^{I_1} a(i_1)a(i'_1) r_p([i_2-0.5T_s]) r_p([i_2-i'_1+0.5T_s]) \\
+ \sum_{i_3=0}^{I_3} \sum_{i_4=\max\{i_3,i'_3\}+1}^{I_3} b(i_3)b(i'_3) r_p([i_4-0.5T_s]) r_p([i_4-i'_3+0.5T_s]) \\
+ \sum_{i_5=I_5}^{L_0-1} \sum_{i_6=0}^{L_0} a(i_5)a(i'_5) r_p([i_6-i_5+0.5T_s]) r_p([i_6-i'_5+0.5T_s]) \\
+ \sum_{i_7=I_7}^{L_0} \sum_{i_8=0}^{L_0} b(i_7)b(i'_7) r_p([i_8-i_7-0.5T_s]) r_p([i_8-i'_7-0.5T_s]) \right]
$$

(3.116)

where

$$
I_1 = \begin{cases} 
L_p - 1 & L_0 \geq L_p \\
L_0 - 1 & L_0 < L_p 
\end{cases} \\
I_3 = \begin{cases} 
L_p - 2 & L_0 \geq L_p \\
L_0 - 1 & L_0 < L_p 
\end{cases} \\
I_5 = \begin{cases} 
L_0 - L_p + 1 & L_0 \geq L_p \\
0 & L_0 < L_p 
\end{cases} \\
I_7 = \begin{cases} 
L_0 - L_p & L_0 \geq L_p \\
0 & L_0 < L_p . 
\end{cases}
$$

(3.117) (3.118)

In the following it will prove convenient to parameterize system performance in terms of the received signal-to-noise ratio

$$
R = \frac{|h|^2 E_b}{N_0} = \frac{|h|^2 A^2}{2N_0}
$$

(3.119)

[see (3.3)]. Accordingly, we have

$$
\mathbb{E}[|e|^2] = \frac{|h|^2}{4L_0^2} \times \frac{\sigma^2_{\mathcal{X}}}{A^4} + \frac{|h|^2}{2L_0 R}
$$

(3.120)
1. The estimator error variance is inversely proportional to the received signal-to-noise ratio \( R = |h|^2 E_b/N_0 \) and the pilot sequence length \( L_0 \).

2. The first term on the right-hand side of (3.120) does not decrease with increasing \( R = |h|^2 E_b/N_0 \) and defines a floor for the estimator error variance. That is, for \( E_b/N_0 \) sufficiently large, the estimator error variance is a constant. This constant represents the point at which thermal noise no longer dominates the estimator performance. Estimator performance is dominated by the interference quantified by \( \mathcal{X} \).

3. The estimator error variance floor is inversely proportional to the square of the length of the pilot sequence. It is also a function of pulse shape autocorrelation function and the pilot symbols for \( L_p > 1 \). Consequently for \( L_p > 1 \), the pilot symbols may be selected to minimize the estimator error variance floor.

4. Comparing the estimator error variance (3.120) with the lower bound (C.29) shows that the estimator (3.71) is only asymptotically (in \( L_0 \)) efficient.

Many of these observations are best illustrated by example. The performance of the estimator for the scenarios of Examples 1 – 4 are summarized below.
3.2. CHANNEL ESTIMATOR

3.2.4 Examples

Example 1  The pulse shape and data pilot sequence are described in Section 3.1.2. Because \( L_p = 1 \) the \( \mathcal{X} \) reduces to

\[
\mathcal{X} = -Aa(4)r_p(0.5T_s) = -\frac{A}{2}a(4)
\]

(3.121)

where the last step follows from \( r_p(0.5) = \frac{1}{2} \). Consequently, \( \mathcal{X}^2 = A^4/4 \) so that

\[
\sigma^2_{\mathcal{X}} = E[\mathcal{X}^2] = \frac{A^4}{4}.
\]

(3.122)

It is easy to verify that (3.122) is a special case of (B.75). The estimator error variance is thus

\[
E[|\epsilon|^2] = \frac{|h|^2}{4 \times 16} \times \frac{1}{4} + \frac{|h|^2}{8R} = \frac{|h|^2}{256} + \frac{|h|^2}{8R}.
\]

(3.123)

The error variance (3.123) is plotted in Figure 3.3 for \( h = e^{j0.66\pi} \) along with the simulated estimator error and the Cramér-Rao bound (C.29) from Appendix C. The analysis (3.123) predicts a variance error floor at \( 1/256 = 3.90625 \times 10^{-3} \). This value is shown by the dash-dot line in Figure 3.3 and it is clear that the estimator error variance is limited by this value.
CHAPTER 3. OFFSET QPSK

Example 2  The pulse shape and data pilot sequence are described in Section 3.1.2. Because $L_p = 1$, $\mathcal{X}$ reduces to

$$\mathcal{X} = -A a(16) r_p(0.5 T_s) = -\frac{A}{2} a(16)$$

(3.124)

where the last step follows from $r_p(0.5) = 1/2$. Consequently, $\mathcal{X}^2 = A^4/4$ so that

$$\sigma_X^2 = \mathbb{E} [\mathcal{X}^2] = \frac{A^4}{4}.$$  

(3.125)

It is easy to verify that (3.125) is a special case of (B.75). The estimator error variance is thus

$$\mathbb{E} [|e|^2] = \frac{|h|^2}{4 \times 256} \times \frac{1}{4} + \frac{|h|^2}{32 R} = \frac{|h|^2}{4096} + \frac{|h|^2}{32 R}.$$  

(3.126)

The error variance (3.126) is plotted in Figure 3.4 for $h = e^{j0.66\pi}$ along with the simulated estimator error and the Cramér-Rao bound (C.29) from Appendix C. The analysis (3.123) predicts a variance error floor at $1/4096 = 3.44140625 \times 10^{-4}$. This value is shown by the dash-dot line in Figure 3.4 and it is clear that the estimator error variance is limited by this value.
Example 3  The pulse shape and data pilot sequence are described in Section 3.1.2. In this example we have \( L_p = 4 \) and \( L_0 = 4 \). Because the pilot block is arranged as shown in Figure 1.1 (a), the expression for \( \sigma_X^2 \) is a special case of (3.116) where \( a(i) = b(i) = 0 \) for \( i < 0 \). For the parameters used in this example, (3.116) evaluates to

\[
\sigma_X^2 = 0.5649A^4.
\]  

The estimator error variance is thus

\[
E[|\hat{h} - h|^2] = \frac{|h|^2}{4 \times 16} \times 0.5649 + \frac{|h|^2}{8R} = \frac{0.5649|h|^2}{64} + \frac{|h|^2}{8R}.
\]

The error variance (3.128) is plotted in Figure 3.5 for \( h = e^{j0.66\pi} \) along with the simulated estimator error and the Cramér-Rao bound (C.29) from Appendix C. The analysis (3.128) predicts an error variance floor at \( 0.5649/64 \approx 8.8 \times 10^{-3} \). This value is shown by the dot-dash line in Figure 3.5. It is clear that the estimator error variance is limited by this value.
Example 4  The pulse shape and data pilot sequence are described in Section 3.1.2. In this example, we have $L_p = 4$ and $L_0 = 16$. Because the pilot block is arranged as shown in Figure 1.1 (a), the expression for $\sigma^2_X$ is a special case of (3.116) where $a(i) = b(i) = 0$ for $i < 0$. For the parameters used in this example, (3.116) evaluates to

$$\sigma^2_X = 0.5636A^4. \tag{3.129}$$

The estimator error variance is thus

$$E[|e|^2] = \frac{|h|^2}{4 \times 256} \times 0.5636 + \frac{|h|^2}{32R} = \frac{0.5636|h|^2}{1024} + \frac{|h|^2}{32R}. \tag{3.130}$$

The error variance (3.130) is plotted in Figure 3.6 for $h = e^{j0.66\pi}$ along with the simulated estimator error and the Cramér-Rao bound (C.29) from Appendix C. The analysis (3.130) predicts an error variance floor at $0.5636/1024 \approx 5.5 \times 10^{-4}$. This value is shown by the dot-dash line in Figure 3.6. It is clear that the estimator error variance is limited by this value.
Comparisons  The results from Examples 1 – 4 are plotted on the same set of axes in Figure 3.7. The following observations are important.

1. The estimator performance as a function of pilot sequence length for a fixed pulse shape shows that the estimator error variance improves with increasing $L_0$ as predicted by (3.120). For the NRZ pulse shape, this is seen by comparing the curves with the square markers whereas for the SRRC pulse shape, this is seen by comparing the curves with the circle markers.

2. The estimator performance as a function of pulse shape for a fixed length is also interesting. For the length-4 pilot sequences, comparison of the NRZ and SRRC curves shows that the error variance floor is higher for the SRRC pulse shape than for the NRZ pulse shape. This is a direct consequence the pulse shape: the SRRC pulse shape is much longer than the NRZ pulse shape; the SRRC pulse shape has more to contribute to $\mathcal{X}$ than the NRZ pulse shape. The same phenomenon is true for the length-16 pilot sequences.
Minimum Error Variance Sequences  Because $\sigma_X^2$ is a function of the pilot symbols, the pilot symbols may be selected to minimize $\sigma_X^2$. For full response pulse shapes ($L_0 = 1$), the variance reduces to

$$\sigma_X^2 = E \left\{ [b(L_0 - 1)a(L_0)r_p(0.5T_s)]^2 \right\} = b^2(L_0 - 1)A^2r_p(0.5T_s) = A^4r_p^2(0.5T_s)$$

and is therefore the same for all pilot sequences. For partial response pulse shapes, the situation is different. Motivated by the examples, exhaustive searches were conducted for the pulse shape autocorrelation function used in Examples 3 and 4. For $L_0 = 4$, there are 64 pilot symbol sequences that achieve the minimum variance $\sigma_X^2 = 0.2048A^4$; for $L_0 = 16$, there are 280,480,624 pilot symbol sequences that achieve the minimum variance $\sigma_X^2 = 0.2048A^4$. An example of one of the length-4 minimum variance sequences is

$$a(0) = +A \quad b(0) = +A$$
$$a(1) = -A \quad b(1) = -A$$
$$a(2) = -A \quad b(2) = +A$$
$$a(3) = -A \quad b(3) = +A$$

(3.131)
An example of one of the length-16 minimum variance sequences is

\[
\begin{align*}
 a(0) &= -\mathcal{A} & b(0) &= -\mathcal{A} \\
 a(1) &= -\mathcal{A} & b(1) &= -\mathcal{A} \\
 a(2) &= -\mathcal{A} & b(2) &= -\mathcal{A} \\
 a(3) &= -\mathcal{A} & b(3) &= -\mathcal{A} \\
 a(4) &= +\mathcal{A} & b(4) &= +\mathcal{A} \\
 a(5) &= +\mathcal{A} & b(5) &= +\mathcal{A} \\
 a(6) &= +\mathcal{A} & b(6) &= +\mathcal{A} \\
 a(7) &= +\mathcal{A} & b(7) &= +\mathcal{A} \\
 a(8) &= +\mathcal{A} & b(8) &= +\mathcal{A} \\
 a(9) &= +\mathcal{A} & b(9) &= +\mathcal{A} \\
 a(10) &= -\mathcal{A} & b(10) &= +\mathcal{A} \\
 a(11) &= -\mathcal{A} & b(11) &= +\mathcal{A} \\
 a(12) &= +\mathcal{A} & b(12) &= +\mathcal{A} \\
 a(13) &= -\mathcal{A} & b(13) &= +\mathcal{A} \\
 a(14) &= +\mathcal{A} & b(14) &= +\mathcal{A} \\
 a(15) &= +\mathcal{A} & b(15) &= +\mathcal{A}
\end{align*}
\]

The estimator error variances for any one of these minimum variance sequences of length 4 or 16 is plotted in Figure 3.8. These curves show that the pilot sequence choice can significantly reduce the estimator error variance floor. For reference, the estimator error variance for the NRZ pulse shape is also included in the plot. The comparison shows that the minimum variance pilot sequences are slightly better than the variance when using the NRZ pulse shape. This is in contrast to the conclusions suggested by Figure 3.7.
3.3 BER Performance of OQPSK Using the Estimator (3.71)

Assuming the preamble arrangement of Figure 1.1 (a), the optimum OQPSK detector computes the matched filter outputs \( x(lT_s/2) \) for \( l = 0, 1, \ldots, 2(L_0 + L_d) - 1 \). The first \( 2L_0 \) matched filter outputs are used to compute the channel gain estimate \( \hat{h} \) based on (3.71). Equipped with the estimate of \( h \), the optimum OQPSK detector makes data symbol decisions based on the decision variables

\[
D(l) = \hat{h}^* x(lT_s/2).
\]  

Multiplication by \( \hat{h}^* \) derotates the matched filter outputs by the phase of the channel estimate. When the estimate is perfect, this process performs perfect phase compensation. On the other hand, when the estimate contains an error, residual phase shift remains. The residual phase shift increases the probability of bit error as described below. Writing \( D(l) = D_R(l) + jD_I(l) \), the decision rules may be expressed as

\[
\hat{a}(k) = \begin{cases} +A & D_R(2k) \geq 0 \\ -A & \text{otherwise} \end{cases} \quad \hat{b}(k) = \begin{cases} +A & D_I(2k + 1) \geq 0 \\ -A & \text{otherwise} \end{cases}
\]  

The decision variables are

\[
D(2k) = [h + e]^* [ha(k) + jhQ + v(kT_s)]
\]

\[
D(2k + 1) = [h + e]^* [jhb(k) + hI + v([k + 0.5]T_s)]
\]

where

\[
e = j\frac{hX}{2L_0A^2} + \frac{N_1}{2L_0A^2}, \quad [\text{see (3.105)}]
\]

\[
Q = \sum_{i=k-L_p}^{k+L_p} b(i)r_p([i - k + 0.5]T_s), \quad [\text{see (3.97)}]
\]

\[
I = \sum_{i=k-L_p+1}^{k+L_p} a(i)r_p(i - k - 0.5]T_s), \quad [\text{see (3.98)}]
\]

This shows that \( D(2k) \) is a function of \( X \) and \( Q \) whereas \( D(2k + 1) \) is a function of \( X \) and \( I \). The processing steps are outlined in the block diagram of Figure 3.9.

Let \( E_{2k} \) be the probability that \( \hat{a}(k) \) is incorrect and let \( E_{2k+1} \) be the probability that \( \hat{b}(k) \) is
Figure 3.9: A block diagram of the optimum OQPSK detector based on the estimator (3.71) and the symbol decision rules (3.134).
incorrect. We have

\[ P(E_{2k}) = P \left( D_R(2k) < 0 \mid a(k) = +A \right) P \left( a(k) = +A \right) + P \left( D_R(2k) \geq 0 \mid a(k) = -A \right) P \left( a(k) = -A \right) \]
\[ = \frac{1}{2} P \left( D_R(2k) < 0 \mid a(k) = +A \right) + \frac{1}{2} P \left( D_R(2k) \geq 0 \mid a(k) = -A \right) \quad (3.140) \]

where the last step assumes equally likely data bits. In recognition of the fact that \( D(2k) \) is a function of \( X \) and \( Q \), we change the notation and write

\[ P \left( E_{2k} \mid X, Q \right) = \frac{1}{2} P \left( D_R(2k) < 0 \mid a(k) = +A, X, Q \right) + \frac{1}{2} P \left( D_R(2k) \geq 0 \mid a(k) = -A, X, Q \right). \quad (3.141) \]

Similarly, we have

\[ P \left( E_{2k+1} \mid X, I \right) = \frac{1}{2} P \left( D_I(2k+1) < 0 \mid b(k) = +A, X, I \right) + \frac{1}{2} P \left( D_I(2k+1) \geq 0 \mid b(k) = -A, X, I \right). \quad (3.142) \]

Each of the terms involved in (3.141) and (3.142) are examined in the following.

**The first term of (3.141)** Here \( a(k) = +A \) so that the decision variable is

\[ D(2k) = [h + e]^* [hA + jhQ + v(kT_s)] \quad (3.143) \]

where \( e = e_R + je_I \) is a proper complex-valued Gaussian random variable,

\[ e_R \sim N \left( -\frac{h_I X}{2L_0 A^2}, \sigma_e^2 \right), \quad e_I \sim N \left( \frac{h_R X}{2L_0 A^2}, \sigma_e^2 \right), \quad \sigma_e^2 = \frac{N_0}{2L_0 A^2}, \quad (3.144) \]

and where \( v(kT_s) = v_R(kT_s) + jv_I(kT_s) \) is a proper complex-valued Gaussian random variable,

\[ v_R(kT_s) \sim N \left( 0, N_0 \right), \quad v_I(kT_s) \sim \left( 0, N_0 \right). \quad (3.145) \]
3.3. BER PERFORMANCE OF OQPSK USING THE ESTIMATOR (3.71)

The decision variable may be expressed as \( D(2k) = Y^*X \) where \( X \) and \( Y \) are independent complex-valued Gaussian random variables,

\[
X \sim N \left( h[A + jQ], N_0 \right), \quad Y \sim N \left( h \left[ 1 + j \frac{X}{2L_0A^2} \right], \frac{N_0}{L_0A^2} \right). \tag{3.146}
\]

Using the identity

\[
D_R(2k) = \frac{1}{2} D(2k) + \frac{1}{2} D^*(2k) = \frac{1}{2} XY^* + \frac{1}{2} X^*Y
\]

the desired probability is

\[
P \left( D_R(2k) < 0 \middle| a(k) = +A, X, Q \right) = P \left( \frac{1}{2} XY^* + \frac{1}{2} X^*Y < 0 \middle| X, Q \right) \tag{3.147}
\]

This is a special case of the form considered in Appendix B of [2]. In [2], the general quadratic form involving the complex-valued Gaussian random variables \( X \) and \( Y \) is investigated. The real-valued random variable

\[
D = A|X|^2 + B|Y|^2 + CXY^* + C^*X^*Y \tag{3.148}
\]

is defined for real valued constants \( A \) and \( B \) and a complex valued constant \( C \). (Here, \( A \) is a constant that is not related to our \( A \) used to denote the OQPSK symbol values). The result is

\[
P(D < 0) = Q(a, b) - \frac{v_2/v_1}{1 + v_2/v_1} I_0(ab) \exp \left\{ -\frac{1}{2} \left( a^2 + b^2 \right) \right\} \tag{3.149}
\]
where

$$Q(a, b) = \int_b^\infty x \exp \left\{ -\frac{1}{2} (x^2 + a^2) \right\} I_0(ax)dx \quad \text{(Marcum Q-function)}$$

$$I_0(x) = \text{0-th order modified Bessel function of the first kind}$$

$$a = \sqrt{\frac{2v_1^2v_2(\alpha_1v_2 - \alpha_2)}{(v_1 + v_2)^2}}$$

$$b = \sqrt{\frac{2v_1v_2^2(\alpha_1v_1 + \alpha_2)}{(v_1 + v_2)^2}}$$

$$v_1 = \sqrt{w^2 + \frac{1}{4 (m_{XX}m_{YY} - |m_{XY}|^2) (|C|^2 - AB)}} - w$$

$$v_2 = \sqrt{w^2 + \frac{1}{4 (m_{XX}m_{YY} - |m_{XY}|^2) (|C|^2 - AB)}} + w$$

$$w = \frac{Am_{XX} + Bm_{YY} + Cm_{XY} + C^*m_{XY}}{4(m_{XX}m_{YY} - |m_{XY}|^2) (|C|^2 - AB)}$$

$$\alpha_1 = 2 (|C|^2 - AB) \left( |m_X|^2 m_{YY} + |m_Y|^2 m_{XX} - m_X^*m_Y m_{XY} - m_X m_Y^* m_{XY}^* \right)$$

$$\alpha_2 = A|m_X|^2 + B|m_Y|^2 + Cm_X^*m_Y + C^*m_X^*m_Y$$

with

$$m_X = \mathbb{E}[X] \quad m_Y = \mathbb{E}[Y]$$

and

$$m_{XX} = \frac{1}{2} \mathbb{E}[|X|^2] \quad m_{YY} = \frac{1}{2} \mathbb{E}[|Y|^2] \quad m_{XY} = \frac{1}{2} \mathbb{E}[XY^*] \quad m_{YX} = \frac{1}{2} \mathbb{E}[X^*Y] .$$

Applying this result to (3.147), we see that (3.147) is the special case of (3.148) for $$A = B = 0, \quad C = 1/2$$. Here, first two moments of the random variables $$X$$ and $$Y$$ are

$$m_X = h[A + jQ], \quad m_Y = h \left[ 1 + j \frac{X}{2L_0A^2} \right] ,$$

and

$$m_{XX} = N_0, \quad m_{YY} = \frac{N_0}{2L_0A^2}, \quad m_{XY} = m_{YX} = 0.$$
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Consequently, the variables (3.150) are

\[
\alpha_2 = |h|^2 A \left[ 1 + \frac{1}{2L_0} \left( \frac{\mathcal{X}}{A^2} \right) \left( \frac{Q}{A} \right) \right]
\]

\[
\alpha_1 = \frac{|h|^2 N_0}{2} \left[ \left( 1 + \left( \frac{Q}{A} \right)^2 \right) \frac{1}{2L_0} + 1 + \frac{1}{4L_0^2} \left( \frac{\mathcal{X}}{A^2} \right)^2 \right]
\]

w = 0

\[
v_2 = \frac{A}{N_0} \sqrt{2L_0}
\]

\[
v_1 = \frac{A}{N_0} \sqrt{2L_0}
\]

\[
b_1 = \sqrt{\frac{R}{2}} \left[ \left( \left( \frac{Q}{A} \right) + \frac{1}{\sqrt{2L_0}} \left( \frac{\mathcal{X}}{A^2} \right) \right)^2 + \left( 1 + \sqrt{2L_0} \right)^2 \right]
\]

\[
a_1 = \sqrt{\frac{R}{2}} \left[ \left( \left( \frac{Q}{A} \right) - \frac{1}{\sqrt{2L_0}} \left( \frac{\mathcal{X}}{A^2} \right) \right)^2 + \left( 1 - \sqrt{2L_0} \right)^2 \right]
\]

where

\[
R = \frac{|h|^2 A^2}{2N_0} = |h|^2 \frac{E_b}{N_0}
\]

is the received signal-to-noise ratio. Making the appropriate substitutions into (3.147) gives

\[
P \left( D_R(k) < 0 \mid a(k) = +A, X, Q \right) = Q \left( a_1, b_1 \right) - \frac{1}{2} I_0 \left( a_1 b_1 \right) \exp \left\{ -\frac{1}{2} \left( a_1^2 + b_1^2 \right) \right\}.
\]

If so desired, one can make the substitutions

\[
a_1 b_1 = \frac{R}{2} \sqrt{\left( \frac{Q}{A} \right)^4 + \frac{1}{4L_0^2} \left( \frac{\mathcal{X}}{A^2} \right)^4 + 2 \left( \frac{Q}{A} \right)^2 \left( \frac{\mathcal{X}}{A^2} \right)^2 - 8 \left( \frac{Q}{A} \right) \left( \frac{\mathcal{X}}{A^2} \right) + (1 + 2L_0)^2}
\]

\[
a_1^2 + b_1^2 = R \left[ \left( \frac{Q}{A} \right)^2 + \frac{1}{2L_0} \left( \frac{\mathcal{X}}{A^2} \right)^2 + 1 + 2L_0 \right]
\]

The second term of (3.141) Here \(a(k) = -A\) so that the decision variable is

\[
D(2k) = [h + e]^* [h(-A + jQ) + v(kT_s)]
\]
and is of the form $D(2k) = Y^*X$ where $X$ and $Y$ are independent complex-valued Gaussian random variables,
\[
X \sim N \left( h[-A + jQ], N_0 \right), \quad Y \sim N \left( h \left[ 1 + j \frac{\mathcal{X}}{2L_0 A^2} \right], \frac{N_0}{L_0 A^2} \right) \quad (3.161)
\]
whose moments are
\[
m_X = h[A + jQ], \quad m_Y = h \left[ 1 + j \frac{\mathcal{X}}{2L_0 A^2} \right], \quad (3.162)
\]
and
\[
m_{XX} = N_0, \quad m_{YY} = \frac{N_0}{2L_0 A^2}, \quad m_{XY} = m_{YX} = 0. \quad (3.163)
\]
Using the identity
\[
D_R(2k) = \frac{1}{2} D(2k) + \frac{1}{2} D^*(2k) = \frac{1}{2} XY^* + \frac{1}{2} X^*Y
\]
the desired probability is
\[
P \left( D_R(2k) \geq 0 \mid a(k) = -A, \mathcal{X}, Q \right) = 1 - P \left( \frac{1}{2} XY^* + \frac{1}{2} X^*Y < 0 \mid \mathcal{X}, Q \right) \quad (3.164)
\]
The probability on the right-hand side of (3.164) is given by (3.149) where the variables (3.150) evaluate to
\[
\alpha_2 = |h|^2 A \left[ -1 + \frac{1}{2L_0} \left( \frac{\mathcal{X}}{A^2} \right) \left( \frac{Q}{A} \right) \right]
\]
\[
\alpha_1 = \frac{|h|^2 N_0}{2} \left[ \left( 1 + \left( \frac{Q}{A} \right)^2 \right) \frac{1}{2L_0} + 1 + \frac{1}{4L_0^2} \left( \frac{\mathcal{X}}{A^2} \right)^2 \right]
\]
\[w = 0\]
\[v_2 = \frac{A}{N_0} \sqrt{2L_0}\]
\[v_1 = \frac{A}{N_0} \sqrt{2L_0}\]
\[b_2 = \sqrt{\frac{R}{2}} \left[ \left( \frac{Q}{A} + \frac{1}{\sqrt{2L_0}} \left( \frac{\mathcal{X}}{A^2} \right) \right)^2 + \left( 1 - \sqrt{2L_0} \right)^2 \right]
\]
\[a_2 = \sqrt{\frac{R}{2}} \left[ \left( \frac{Q}{A} - \frac{1}{\sqrt{2L_0}} \left( \frac{\mathcal{X}}{A^2} \right) \right)^2 + \left( 1 + \sqrt{2L_0} \right)^2 \right].\]
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Thus, we have

\[ P \left( \frac{1}{2} XY^* + \frac{1}{2} X^* Y < 0 \middle| \mathcal{X}, Q \right) = Q(a_2, b_2) - \frac{1}{2} I_0(a_2 b_2) \exp \left\{ -\frac{1}{2} \left( a_2^2 + b_2^2 \right) \right\}. \]  

(3.166)

Now, using the identity

\[ Q(\alpha, \beta) + Q(\beta, \alpha) = 1 + I_0(\alpha \beta) \exp \left\{ -\frac{1}{2} (\alpha^2 + \beta^2) \right\} \]  

(3.167)

the probability (3.164) may be expressed as

\[ P \left( D_R(2k) \geq 0 \middle| a(k) = -A, \mathcal{X}, Q \right) = Q(b_2, a_2) - \frac{1}{2} I_0(a_2 b_2) \exp \left\{ -\frac{1}{2} \left( a_2^2 + b_2^2 \right) \right\}. \]  

(3.168)

The first term of (3.142) Here \( b(k) = +A \) so that the decision variable is

\[ D(2k + 1) = [h + e]^*[h(jA + \bar{I}) + v([k + 0.5]T_s)] \]  

(3.169)

and is of the form \( D(2k + 1) = Y^* X \) where \( X \) and \( Y \) are independent complex-valued Gaussian random variables,

\[ X \sim N \left( h[jA + \bar{I}], N_0 \right), \quad Y \sim N \left( h \left[ 1 + j \frac{\mathcal{X}}{2L_0 A^2} \right], \frac{N_0}{L_0 A^2} \right) \]  

(3.170)

whose moments are

\[ m_X = h[jA + \bar{I}], \quad m_Y = h \left[ 1 + j \frac{\mathcal{X}}{2L_0 A^2} \right], \]  

(3.171)

and

\[ m_{XX} = N_0, \quad m_{YY} = \frac{N_0}{2L_0 A^2}, \quad m_{XY} = m_{YX} = 0. \]  

(3.172)

Using the identity

\[ D_I(2k + 1) = -\frac{1}{2} D(2k + 1) + \frac{1}{2} D^*(2k + 1) = -\frac{1}{2} XY^* + \frac{1}{2} X^* Y \]  

the desired probability is

\[ P \left( D_I(2k + 1) < 0 \middle| b(k) = +A, \mathcal{X}, \bar{I} \right) = P \left( -\frac{1}{2} XY^* + \frac{1}{2} X^* Y < 0 \middle| \mathcal{X}, \bar{I} \right). \]  

(3.173)
The probability on the right-hand side of (3.173) is given by (3.149) where the variables (3.150) evaluate to

$$\alpha_2 = |h|^2 A \left[ 1 - \frac{1}{2L_0} \left( \frac{\mathcal{X}}{A^2} \right) \left( \frac{\mathcal{T}}{A} \right) \right]$$

$$\alpha_1 = \frac{|h|^2 N_0}{2} \left[ \left( 1 + \left( \frac{\mathcal{T}}{A} \right)^2 \right) \frac{1}{2L_0} + 1 + \frac{1}{4L_0^2} \left( \frac{\mathcal{X}}{A^2} \right)^2 \right]$$

$$w = 0$$

$$v_2 = \frac{A}{N_0} \sqrt{2L_0}$$

$$v_1 = \frac{A}{N_0} \sqrt{2L_0}$$

$$b_3 = \sqrt{\frac{R}{2}} \left[ \left( \frac{\mathcal{T}}{A} \right) - \frac{1}{\sqrt{2L_0}} \left( \frac{\mathcal{X}}{A^2} \right)^2 + \left( 1 + \sqrt{2L_0} \right)^2 \right]$$

$$a_3 = \sqrt{\frac{R}{2}} \left[ \left( \frac{\mathcal{T}}{A} \right) + \frac{1}{\sqrt{2L_0}} \left( \frac{\mathcal{X}}{A^2} \right)^2 + \left( 1 - \sqrt{2L_0} \right)^2 \right].$$

Thus, we have

$$P \left( D_I(2k + 1) < 0 \middle| b(k) = +A, \mathcal{X}, \mathcal{T} \right) = Q(a_3, b_3) - \frac{1}{2} I_0(a_3b_3) \exp \left\{ -\frac{1}{2} (a_3^2 + b_3^2) \right\}. \quad (3.175)$$

The second term of (3.142) Here $b(k) = -A$ so that the decision variable is

$$D(2k + 1) = [h + e]^* [h(-jA + \mathcal{T}) + v([k + 0.5]T_s)] \quad (3.176)$$

and is of the form $D(2k + 1) = Y^* X$ where $X$ and $Y$ are independent complex-valued Gaussian random variables,

$$X \sim \mathcal{N} \left( h[-jA + \mathcal{T}], N_0 \right), \quad Y \sim \mathcal{N} \left( h \left[ 1 + j \frac{\mathcal{X}}{2L_0A^2} \right], \frac{N_0}{L_0A^2} \right) \quad (3.177)$$

whose moments are

$$m_X = h[-jA + \mathcal{T}], \quad m_Y = h \left[ 1 + j \frac{\mathcal{X}}{2L_0A^2} \right], \quad (3.178)$$
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and

\[ m_{XX} = N_0, \quad m_{YY} = \frac{N_0}{2L_0A^2}, \quad m_{XY} = m_{YX} = 0. \]  (3.179)

Using the identity

\[ D_I(2k+1) = -\frac{1}{2}D(2k+1) + \frac{1}{2}D^*(2k+1) = -\frac{1}{2}XY^* + \frac{1}{2}X^*Y \]

the desired probability is

\[
P \left( D_I(2k+1) \geq 0 \left| b(k) = -A, \mathcal{X}, \mathcal{I} \right. \right) = 1 - P \left( D_I(2k+1) < 0 \left| b(k) = -A, \mathcal{X}, \mathcal{I} \right. \right)
\]

\[= 1 - P \left( -\frac{1}{2}XY^* + \frac{1}{2}X^*Y < 0 \left| \mathcal{X}, \mathcal{I} \right. \right). \]  (3.180)

The probability on the right-hand side of (3.180) is given by (3.149) where the the variables (3.150) evaluate to

\[
\alpha_2 = -|h|^2 A \left[ 1 + \frac{1}{2L_0} \left( \frac{\mathcal{X}}{A^2} \right) \left( \frac{\mathcal{I}}{A} \right) \right]
\]

\[
\alpha_1 = \frac{|b|^2 N_0}{2} \left[ \left( 1 + \left( \frac{\mathcal{I}}{A} \right)^2 \right) \frac{1}{2L_0} + 1 + \frac{1}{4L_0^2} \left( \frac{\mathcal{X}}{A^2} \right)^2 \right]
\]

\[ w = 0 \]

\[ v_2 = \frac{A}{N_0} \sqrt{2L_0} \]

\[ v_1 = \frac{A}{N_0} \sqrt{2L_0} \]

\[
b_4 = \sqrt{\frac{R}{2}} \left[ \left( \frac{\mathcal{I}}{A} \right) - \frac{1}{\sqrt{2L_0}} \left( \frac{\mathcal{X}}{A^2} \right)^2 + \left( 1 - \sqrt{2L_0} \right)^2 \right]
\]

\[
a_4 = \sqrt{\frac{R}{2}} \left[ \left( \frac{\mathcal{I}}{A} \right) + \frac{1}{\sqrt{2L_0}} \left( \frac{\mathcal{X}}{A^2} \right)^2 + \left( 1 + \sqrt{2L_0} \right)^2 \right].
\]

Thus, we have

\[
P \left( -\frac{1}{2}XY^* + \frac{1}{2}X^*Y < 0 \left| \mathcal{X}, \mathcal{I} \right. \right) = Q(a_4, b_4) - \frac{1}{2} f_0(a_4b_4) \exp \left\{ -\frac{1}{2} \left( a_4^2 + b_4^2 \right) \right\}. \]  (3.182)
Now, using the identity
\[ Q(\alpha, \beta) + Q(\beta, \alpha) = 1 + I_0(\alpha \beta) \exp \left\{ -\frac{1}{2} (\alpha^2 + \beta^2) \right\} \] (3.183)
the probability (3.180) may be expressed as
\[ P \left( D_I(2k+1) \geq 0 \bigg| b(k) = -A, X, I \right) = Q(b_4, a_4) - \frac{1}{2} I_0(a_4 b_4) \exp \left\{ -\frac{1}{2} (a_4^2 + b_4^2) \right\}. \] (3.184)

**The Average Bit Error Probability**

Assembling the previous results, we have
\[ P(\text{bit error}|X, Q, I) = \frac{1}{4} P(E_{2k}|a(k) = +A, X, Q) + \frac{1}{4} P(E_{2k}|a(k) = -A, X, Q) \]
\[ + \frac{1}{4} P(E_{2k+1}|b(k) = +A, X, I) + \frac{1}{4} P(E_{2k+1}|a(k) = -A, X, I). \] (3.185)

It may seem a touch unsettling to recognize that the conditional bit error probabilities (3.157), (3.168), (3.175), and (3.184) are different. This is to be expected when one realizes that the bit error probability for \( a(k) \) depends on the data bit sequence \( b(l) \). This is a result of the incomplete cancellation of the quadrature component from the real part \( D(2k) \) due to the imperfect channel estimate. Similarly, the bit error probability for \( b(k) \) depends on the data bit sequence \( a(l) \) due to incomplete cancellation of the inphase component from the imaginary part of \( D(2k+1) \) caused by the imperfect channel estimate. All four terms are equal when averaged over all possible values of the interfering bits.

Note that \( X \) is a function of the pulse shape autocorrelation function \( r_p(\tau) \) and the pilot bits
\[ a_p = a(L_0 - L_p + 1), \ldots, a(L_0 - 1) \quad \text{for } L_0 \geq L_p \] (3.186)
\[ b_p = b(L_0 - L_p), \ldots, a(L_0 - 1) \]
or
\[ a_p = a(0), \ldots, a(L_0 - 1) \quad \text{for } L_0 < L_p \] (3.187)
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and the data bits

\[ a_X = a(L_0), a(L_0 + 1), \ldots, a(L_0 + L_p - 1) \]

\[ b_X = b(L_0), b(L_0 + 1), \ldots, b(L_0 + L_p - 2). \] (3.188)

This means for a given pilot sequence, there are \( 2^{L_p} \) possible sequences of \( a \)'s and \( 2^{L_p-1} \) possible sequences of \( b \)'s. Consequently, there are \( 2^{2L_p-1} \) possible values of \( X \):

\[ X \in \{ X_0, X_1, \ldots, X_{2^{2L_p-1} - 1} \}. \] (3.189)

The term \( Q \) is a function of the pulse shape autocorrelation function \( r_p(\tau) \) and the data bits

\[ b_Q = b(k - L_p), b(k - L_p + 1), \ldots, b(k + L_p - 1). \] (3.190)

Consequently, there are \( 2^{2L_p} \) possible values of \( Q \) denoted

\[ Q \in \{ Q_0, Q_1, \ldots, Q_{2^{2L_p} - 1} \}. \] (3.191)

The term \( I \) is a function of the pulse shape autocorrelation function \( r_p(\tau) \) and the data bits

\[ a_I = a(k-L_p+1), a(k-L_p+2), \ldots, a(k+L_p). \] (3.192)

Consequently, there are \( 2^{2L_p} \) possible values of \( I \) denoted

\[ I \in \{ I_0, I_1, \ldots, I_{2^{2L_p} - 1} \}. \] (3.193)

Note that for \( k > L_0 + 2L_p - 2 \) the sequences \( a_X \) and \( a_I \) are disjoint. The same is true for sequences \( b_X \) and \( b_Q \).

The average conditional bit error probabilities are now derived. Starting with the first term in (3.185), we have

\[
P( E_{2k} | a(k) = +A ) = \sum_{\ell_1=0}^{2^{2L_p-1} - 1} \sum_{\ell_2=0}^{2^{2L_p-1}} P( E_{2k} | a(k) = +A, X = X_{\ell_1}, Q = Q_{\ell_2} ) P( X = X_{\ell_1}, Q = Q_{\ell_2} )
\] (3.194)
which, for \( k > L_0 + 2L_p - 2 \) and independent data bits, reduces to

\[
P(E_{2k} | a(k) = +A) = \frac{1}{2^{2L_p - 2L_p}} \sum_{\ell_1=0}^{2^{2L_p - 1} - 1} \sum_{\ell_2=0}^{2^{2L_p - 1} - 1} P(E_{2k} | a(k) = +A, \mathcal{X} = \mathcal{X}_{\ell_1}, \mathcal{Q} = \mathcal{Q}_{\ell_2}).
\]  

(3.195)

Similarly, the second through fourth terms in (3.185) are

\[
P(E_{2k} | a(k) = -A) = \frac{1}{2^{2L_p - 2L_p}} \sum_{\ell_1=0}^{2^{2L_p - 1} - 1} \sum_{\ell_2=0}^{2^{2L_p - 1} - 1} P(E_{2k} | a(k) = -A, \mathcal{X} = \mathcal{X}_{\ell_1}, \mathcal{Q} = \mathcal{Q}_{\ell_2}),
\]  

(3.196)

\[
P(E_{2k+1} | b(k) = +A) = \frac{1}{2^{2L_p - 2L_p}} \sum_{\ell_1=0}^{2^{2L_p - 1} - 1} \sum_{\ell_2=0}^{2^{2L_p - 1} - 1} P(E_{2k+1} | b(k) = +A, \mathcal{X} = \mathcal{X}_{\ell_1}, \mathcal{I} = \mathcal{I}_{\ell_2}),
\]  

(3.197)

\[
P(E_{2k+1} | b(k) = -A) = \frac{1}{2^{2L_p - 2L_p}} \sum_{\ell_1=0}^{2^{2L_p - 1} - 1} \sum_{\ell_2=0}^{2^{2L_p - 1} - 1} P(E_{2k+1} | b(k) = -A, \mathcal{X} = \mathcal{X}_{\ell_1}, \mathcal{I} = \mathcal{I}_{\ell_2}).
\]  

(3.198)

As it turns out, all four of these terms are equal. Consequently, we may write

\[
P_b = \frac{1}{2^{2L_p - 2L_p}} \sum_{\ell_1=0}^{2^{2L_p - 1} - 1} \sum_{\ell_2=0}^{2^{2L_p - 1} - 1} P(E_{2k} | a(k) = +A, \mathcal{X} = \mathcal{X}_{\ell_1}, \mathcal{Q} = \mathcal{Q}_{\ell_2})
\]  

(3.199)

where the summand is given by the right-hand side of (3.157). In (3.157), the \( a \) and \( b \) are functions of \( \mathcal{X} \) and \( \mathcal{Q} \). To make this dependence explicit, the notation is changed slightly as follows

\[
a \rightarrow a(\mathcal{X}, \mathcal{Q}), \quad b \rightarrow b(\mathcal{X}, \mathcal{Q}).
\]  

(3.200)

We may now write
\[ P_b = \frac{1}{2^{2L_p-1}2^{2L_p}} \sum_{\ell_1=0}^{2^{2L_p-1}-1} \sum_{\ell_2=0}^{2^{2L_p-1}-1} \left[ Q \left( a(\mathcal{X}_{\ell_1}, Q_{\ell_2}), b(\mathcal{X}_{\ell_1}, Q_{\ell_2}) \right) \right. \\
- \frac{1}{2} I_0 \left( a(\mathcal{X}_{\ell_1}, Q_{\ell_2}) b(\mathcal{X}_{\ell_1}, Q_{\ell_2}) \right) \exp \left\{ \frac{1}{2} \left( a^2(\mathcal{X}_{\ell_1}, Q_{\ell_2}) + b^2(\mathcal{X}_{\ell_1}, Q_{\ell_2}) \right) \right\} \]  

(3.201)

The final comment to be made is that (3.201) is a function of the pilot bit sequence. This dependence will be explored through the following examples.
**Example 1**  The term $\mathcal{X}$ is a special case of (B.17) which reduces to

$$
\mathcal{X} = b(L_0 - 1)a(L_0) r_p(0.5 T_s) = b(3)a(4) r_p(0.5 T_s) = -\frac{A}{2} a(4).
$$

This shows that $\mathcal{X}$ is a function of the pilot bits

$$
a_p = \text{null}
$$

$$
b_p = b(3) = -A
$$

[cf. Equation (3.186)] and the data bits

$$
a_{\mathcal{X}} = a(4)
$$

$$
b_{\mathcal{X}} = \text{null}
$$

[cf. Equation (3.188)]. For the given pilot sequence, the 2 possible values for $\mathcal{X}$ are

$$
\mathcal{X}_0 = 0.5A^2
$$

$$
\mathcal{X}_1 = -0.5A^2.
$$

The quadrature interference term is

$$
Q = \sum_{l=k-1}^{l=k} b(l) r_p([l - k + 0.5] T_s) = b(k - 1) r_p(-0.5 T_s) + b(k) r_p(0.5 T_s)
$$

$$
= \frac{1}{2} b(k - 1) + \frac{1}{2} b(k).
$$

Here we have

$$
b_Q = [b(k - 1), b(k)].
$$

The four values of $Q$ are

$$
Q_0 = -A
$$

$$
Q_1 = 0
$$

$$
Q_2 = 0
$$

$$
Q_3 = +A.
$$

The average bit error probability (3.201) for this example is plotted in Figure 3.10 along with
Figure 3.10: The average bit error probability (3.201) versus received signal-to-noise ratio for Example 1. Also included are simulation results and the performance for a perfect channel estimate. We observe that the imperfect channel estimate causes a loss of approximately 1 dB at a bit error probability of $10^{-6}$.
Example 2  As before, the term $X$ is a special case of (B.17) which reduces to

$$X = b(L_0 - 1)a(L_0)r_p(0.5T_s) = b(15)a(16)r_p(0.5T_s) = -\frac{A}{2}a(16). \quad (3.209)$$

This shows that $X$ is a function of the pilot bits

$$a_p = \text{null}$$

$$b_p = b(15) = -A \quad (3.210)$$

[cf. Equation (3.186)] and the data bits

$$a_X = a(16)$$

$$b_X = \text{null} \quad (3.211)$$

[cf. Equation (3.188)]. For the given pilot sequence, the 2 possible values for $X(-A, a(16))$:

$$X_0 = 0.5A^2$$

$$X_1 = -0.5A^2. \quad (3.212)$$

The quadrature interference term is

$$Q = \sum_{l=k-1}^{k} b(l)r_p([l - k + 0.5]T_s) = b(k - 1)r_p(-0.5T_s) + b(k)r_p(0.5T_s)$$

$$= \frac{1}{2}b(k - 1) + \frac{1}{2}b(k). \quad (3.213)$$

Here, we have

$$b_Q = [b(k - 1), b(k)]. \quad (3.214)$$

The four values of $Q(b(k - 1), b(k))$ are

$$Q_0 = -A$$

$$Q_1 = 0$$

$$Q_2 = 0 \quad (3.215)$$

$$Q_3 = +A.$$
Figure 3.11: The average bit error probability (3.201) versus received signal-to-noise ratio for Example 2. Also included are simulation results and the performance for a perfect channel estimate. Simulation results. Also included is the curve representing the performance for a perfect channel estimate. We observe that the imperfect channel estimate causes a loss of approximately 0.07 dB at a bit error probability of $10^{-6}$. 
Example 3  The term $\mathcal{X}$ is a special case of (B.17) with $L_p = 4$:

$$
\mathcal{X} = \sum_{l=0}^{3} \sum_{l'=4}^{l+3} a(l)b(l')r_p([l' - l + 0.5]T_s) - \sum_{l'=0}^{3} \sum_{l=4}^{l'+4} b(l')a(l)r_p([l - l' - 0.5]T_s) \tag{3.216}
$$

and is a function of the pilot bits

$$
a_p = [a(0), a(1), a(2), a(3)] = [+A, -A, +A, +A] \tag{3.217}
b_p = [b(0), b(1), b(2), b(3)] = [-A, -A, +A, -A]
$$

[cf. Equation (3.187)] and the $2L_p - 1 = 7$ data bits

$$
a_X = [a(4), a(5), a(6), a(7)] 
b_X = [b(4), b(5), b(6)]. \tag{3.218}
$$

[cf. Equation (3.188)]. For the given pilot sequence, there are $2^7 = 128$ possible values for $[a_X, b_X]$. Consequently, there are 128 possible values for $\mathcal{X}$. The 128 values of $\mathcal{X}/A^2$ are plotted in Figure 3.12. Here, the abscissa is the decimal equivalent\(^2\) of 7-bit binary pattern

$$a(4), b(4), a(5), b(5), a(6), b(6), a(7)$$

that define $a_X$ and $b_X$.

The quadrature interference term is

$$Q = \sum_{l=k-4}^{k+3} b(l)r_p([l - k + 0.5]T_s) \tag{3.219}
$$

and is a function of the $2L_p = 8$ data bits

$$b_Q = [b(k - 4), b(k - 3), \ldots, b(k), \ldots, b(k + 2), b(k + 3)]. \tag{3.220}
$$

Because there are $2^8 = 256$ possibilities for $b_Q$, there are 256 possible values of $Q$. The 256 values of $Q/A$ are plotted in Figure 3.12 where, again, the abscissa is the binary-to-decimal equivalent of $b_Q$ where $b(k - 4)$ is the MSB and $b(k + 3)$ is the LSB.

The average bit error probability (3.201) for this example is plotted in Figure 3.13 along with

\(^2\)The mapping $a(i) = +A \rightarrow 1$, $a(i) = -A \rightarrow 0$ for $i = 4, \ldots, 7$ was used. An identical mapping for $b(i)$ for $i = 4, \ldots, 6$ was used. In the binary-to-decimal mapping, $a(4)$ is the MSB and $a(7)$ is the LSB.
Figure 3.12: (Top) the 128 values of $X/A^2$ for Example 3; (Bottom) the 256 values of $Q/A$ for Example 3.

simulation results. Also included is the curve representing the performance for a perfect channel estimate. We observe that the imperfect channel estimate causes a loss of approximately 2 dB at a bit error probability of $10^{-6}$. 
Figure 3.13: The average bit error probability (3.201) versus received signal-to-noise ratio for Example 3. Also included are simulation results and the performance for a perfect channel estimate.
Example 4 Here, the term $\mathcal{X}$ is a special case of (B.17) with $L_p = 4$:

$$\mathcal{X} = \sum_{l=11}^{15} \sum_{l'=16}^{l+3} a(l)b(l')r_p([l' - l + 0.5]T_s) - \sum_{l'=10}^{15} \sum_{l=16}^{l'+4} b(l')a(l)[l - l' - 0.5]T_s). \quad (3.221)$$

This shows that $\mathcal{X}$ is a function of the pilot bits

$$a_p = [a(11), a(12), a(13), a(14), a(15)] \quad (3.222)$$

$$b_p = [b(10), b(11), b(12), b(13), b(14), b(15)]$$

[cf. Equation (3.187)], and the $2L_p - 1 = 7$ data bits

$$a_X = [a(16), a(17), a(18), a(19)] \quad (3.223)$$

$$b_X = [b(16), b(17), b(18)]$$

[cf. Equation (3.188)]. For the given pilot sequence, there are $2^7 = 128$ possible values for $[a_X, b_X]$. Consequently, there are 128 possible values for $\mathcal{X}$. The 128 values of $\mathcal{X}/A^2$ are plotted in Figure 3.14. Here, the abscissa is the decimal equivalent$^3$ of 7-bit binary pattern

$$a(4), b(4), a(5), b(5), a(6), b(6), a(7)$$

that define $a_X$ and $b_X$.

The quadrature interference term is

$$Q = \sum_{l=k-4}^{k+3} b(l)r_p([l - k + 0.5]T_s) \quad (3.224)$$

and is a function of the $2L_p = 8$ data bits

$$b_Q = [b(k - 4), b(k - 3), \ldots, b(k), \ldots, b(k + 2), b(k + 3)] \quad (3.225)$$

Because there are $2^8 = 256$ possibilities for $b_Q$, there are 256 possible values of $Q$. The 256 values of $Q/A$ are plotted in Figure 3.14 where, again, the abscissa is the binary-to-decimal equivalent of $b_Q$ where $b(k - 4)$ is the MSB and $b(k + 3)$ is the LSB.

The average bit error probability (3.201) for this example is plotted in Figure 3.15 along with

$^3$The mapping $a(i) = +A \rightarrow 1$, $a(i) = -A \rightarrow 0$ for $i = 4, \ldots, 7$ was used. An identical mapping for $b(i)$ for $i = 4, \ldots, 6$ was used. In the binary-to-decimal mapping, $a(4)$ is the MSB and $a(7)$ is the LSB.
Figure 3.14: (Top) the 128 values of $X/A^2$ for Example 4; (Bottom) the 256 values of $Q/A$ for Example 4.

Simulation results. Also included is the curve representing the performance for a perfect channel estimate. We observe that the imperfect channel estimate causes a loss of approximately 0.1 dB at a bit error probability of $10^{-6}$. 
3.3. BER PERFORMANCE OF OQPSK USING THE ESTIMATOR (3.71)

Figure 3.15: The average bit error probability (3.201) versus received signal-to-noise ratio for Example 4. Also included are simulation results and the performance for a perfect channel estimate.
Figure 3.16: Bit error probability comparisons for $L_0 = 4$. See Examples 1 and 3. The lines are evaluations of the analytical expressions and the markers are the simulation results.

Comparisons Figure 3.16 plots the bit error probabilities for the $L_0 = 4$ sequences used in Examples 1 and 3. The bit error probability for the SRRC pulse shape (Example 3) is worse than the bit error probability for the NRZ pulse shape (Example 1) because the SRRC pulse shape autocorrelation function has more ISI at odd multiples of the $1/2T_s$ than the NRZ pulse shape (cf., Figure 3.7). Also included in Figure 3.16 is the bit error probability using the minimum error variance pilot sequence (3.131). This is one of the 64 minimum-variance sequences. A comparison of Figures 3.8 and 3.16 reveals a curious observation. The results of Figure 3.8 document that the estimator error variance for the NRZ pulse and the minimum-variance sequence for the SRRC pulse shape are essentially the same. This might suggest that the corresponding bit error probabilities should be the same. This is clearly not the case, as demonstrated in Figure 3.16. The issue is that whereas the estimator error variance is a function of $X$ [see (3.120)], the bit error probability is a function of both $X$ and $Q$ [see (3.201)].

Figure 3.17 plots the bit error probabilities for the $L_0 = 16$ sequences used in Examples 2 and 4. The bit error probability for the SRRC pulse shape (Example 4) is worse than the bit error probability for the NRZ pulse shape (Example 2) because the SRRC pulse shape autocorrelation function has more ISI at odd multiples of the $1/2T_s$ than the NRZ pulse shape (cf., Figure 3.7). The difference in bit error probability performance is small, just under 0.2 dB. Also included is
3.3. BER PERFORMANCE OF OQPSK USING THE ESTIMATOR (3.71)

Figure 3.17: Bit error probability comparisons for $L_0 = 16$. See Examples 2 and 4. The lines are evaluations of the analytical expressions and the markers are the simulation results.

the BER performance using the minimum-variance pilot sequence (3.132). As expected, the BER performance is better using this sequence than Example 4, but worse than Example 2.
3.4 Computer Simulations

The following Matlab script was used to simulate the estimator performance for the NRZ pulse shape (Examples 1 and 2):

```matlab
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% 
% Simulation Constants %%%%%%%%%%%%%%%%%% 
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% 
SNR = [1:12];
h = exp(j*0.66*pi);
L0 = 4;
pilot_bits = [1 0 0 0 1 1 1 0];
% L0 = 16; 
%pilot_bits = [1 0 1 1 0 1 1 0 0 1 1 0 1 0 1 0 0 0 1 0 0 0 1 1 0 0];
Lp = 1;
Ld = lp;
p = 1/sqrt(2)*ones(1,2);
start_xx_index = 2;
end_xx_index = start_xx_index + 2*(L0+Ld)-1;

% |
% 01 | 11 
% |
% -------
% |
% 00 | 10 
% |

LUT = [-1-j, -1+j, 1-j, 1+j];
a_bit = pilot_bits(1:2:end);
b_bit = pilot_bits(2:2:end);
a_index = 2*a_bit+b_bit;
a = LUT(a_index+1);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% 
% Simulation Loop %%%%%%%%%%%%%%%%%%%%%% 
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

temp = zeros(1,2*(L0+Ld)-1);
simherr = zeros(size(SNR));
simhvar = zeros(size(SNR));

its = 50000;
for ii = 1:length(SNR)
    snr = 10.^(SNR(ii)/10);
nvar = h'*h/(2*snr);
nstd = sqrt(nvar);
h_err = 0;
h_var = 0;
    for i = 1:its
```

% create random data bits
ad_bit = randi(2,1,Ld)-1; %ad_bit = randint(1,Ld,2);
bd_bit = randi(2,1,Ld)-1; %bd_bit = randint(1,Ld,2);

% create QPSK packet at 2 samples/symbol
ad = LUT(2*ad_bit + bd_bit + 1);
temp(1:2:end) = real([a ad]);
II = conv(p,temp);
temp(1:2:end) = imag([a ad]);
QQ = conv(p,temp);
s = [II 0] + j*[0 QQ];

% generate noise and add to QPSK packet
w = nstd*(randn(size(s)) + j*randn(size(s)));
r = h*s + w;

% matched filter
xx = conv(r,p);
xx = xx(start_xx_index:end_xx_index);

% estimate h
hhat = xx(1:2:2*L0)*real(a).’ - j*xx(2:2:2*L0)*imag(a).’;
hhat = hhat/(2*L0);
this_err = h - hhat;
h_err = h_err + this_err;
h_var = h_var + this_err * this_err’;

end
simherr(ii) = h_err/its;
simhvar(ii) = h_var/its;
disp(’-----------------------------------------------’);
disp([’SNR = ’,num2str(SNR(ii)),’ dB’]);
disp([’estimate error mean = ’,num2str(h_err/its)]);
disp([’estimator error variance = ’,num2str(h_var/its)]);
end
The following Matlab script was used to simulate the estimator performance for the SRRC pulse shape (Examples 3 and 4):

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%% Simulation Constants %%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

SNR = [1:12];
h = exp(j*0.66*pi);
L0 = 4;
pilot_bits = [1 0 0 0 1 1 1 0];
%L0 = 16;
%pilot_bits = [1 0 1 1 0 1 1 1 0 1 1 0 1 1 1 0 1 1 0 1 0 0 0 1 0 0 1 1 1 0];
Lsrrc = 2;
Lp = 2*Lsrrc;
Ld = Lp;
p = rcosine(1,2,'fir/sqrt',0.5,Lsrrc);
start_xx_index = 2*2*Lsrrc+1;
end_xx_index = 2*L0 + 2*Ld + 8*Lsrrc - 2*2*Lsrrc;

% |
% 01 | 11
% |
% ------+------
% |
% 00 | 10
% |

LUT = [-1-j, -1+j, 1-j, 1+j];
a_bit = pilot_bits(1:2:end);
b_bit = pilot_bits(2:2:end);
a_index = 2*a_bit+b_bit;
a = LUT(a_index+1);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%% Simulation Loop %%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

temp = zeros(1,2*(L0+Ld-1)+1);
simherr = zeros(size(SNR));
simhvar = zeros(size(SNR));

its = 50000;
for ii = 1:length(SNR)
    snr = 10.^(SNR(ii)/10);
nvar = h'*h/(2*snr);
nstd = sqrt(nvar);
h_err = 0;
h_var = 0;
for i = 1:its
    %
```
% create random data bits
ad_bit = randi([0 1],1,Ld)-1;
b_d_bit = randi([0 1],1,Ld)-1;
% create OQPSK packet at 2 samples/symbol
ad = LUT(2*ad_bit + bd_bit + 1);
temp(1:2:end) = real([a ad]);
II = conv(p,temp);
temp(1:2:end) = imag([a ad]);
QQ = conv(p,temp);
s = [II 0] + j*[0 QQ];
% generate noise and add to OQPSK packet
w = nstd*randn(size(s)) + j*randn(size(s));
r = h*s + w;
% matched filter
xx = conv(r,p);
xx = xx(start_xx_index:end_xx_index);
% estimate h
hhat = xx(1:2:2*L0)*real(a).’ - j*xx(2:2:2*L0)*imag(a).’;
hhat = hhat/(2*L0);
this_err = h - hhat;
h_err = h_err + this_err;
h_var = h_var + this_err.*this_err;
end
simherr(ii) = h_err/its;
simhvar(ii) = h_var/its;
disp('-------------------------------------------------');
disp(['SNR = ',num2str(SNR(ii)),' dB']);
disp(['estimate error mean = ',num2str(h_err/its)]);
disp(['estimator error variance = ',num2str(h_var/its)]);
The following Matlab script was used to simulate the BER performance for the NRZ pulse shape (Examples 1 and 2):

```matlab
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%% Simulation Constants %%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
SNR = [1:12];
h = exp(j*0.66*pi);
L0 = 4;
pilot_bits = [1 0 0 0 1 1 0 0];
% L0 = 16;
% pilot_bits = [1 0 1 1 0 1 1 0 1 0 1 1 0 1 0 0 1 0 0 1 0 0 1 1 1 0];
Ld = 1000;
Lp = 1;
p = 1/sqrt(2)*ones(1,2);
start_xx_index = 2;
end_xx_index = start_xx_index + 2*(L0+Ld)-1;

% |
% 01 | 11
% |
% --+-----
% 00 | 10
% |

LUT = [-1-j, -1+j, 1-j, 1+j];
a_bit = pilot_bits(1:2:end);
b_bit = pilot_bits(2:2:end);
a_index = 2*a_bit+b_bit;
a = LUT(a_index+1);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%% Simulation Loop %%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
temp = zeros(1,2*(L0+Ld-1)+1);
simBER = zeros(size(SNR));
its = 20000;
for ii = 1:length(SNR)
    snr = 10.^((SNR(ii)/10));
nvar = h'*h/(2*snr);
nstd = sqrt(nvar);
bit_err_cnt = 0;
    for i = 1:its
        % create random data bits
        ad_bit = randi([0 1],1,Ld);
```
bd_bit = randi([0 1],1,Ld);

% create OQPSK packet at 2 samples/symbol
% start of %
ad = LUT(2*ad_bit + bd_bit + 1);
temp(1:2:end) = real([a ad]);
II = conv(p,temp);
temp(1:2:end) = imag([a ad]);
QQ = conv(p,temp);
s = [II 0] + j*[0 QQ];
%
% generate noise and add to OQPSK packet
% start of %
w = nstd*(randn(size(s)) + j*randn(size(s)));
r = h*s + w;
%
% matched filter
% start of %
xx = conv(r,p);
xx = xx(start_xx_index:end_xx_index);
%
% estimate h
% start of %
hhat = xx(1:2:2*L0)*real(a)' - j*xx(2:2:2*L0)*imag(a)';
hhat = hhat/(2*L0);
%
% apply estimate to detection
% start of %
xdata = hhat'*xx;
abit_hat = real(xdata(2*L0+1:2:end)) > 0;
bit_err_cnt = bit_err_cnt + sum(abit_hat ~= ad_bit);
bbit_hat = imag(xdata(2*L0+2:2:end)) > 0;
bit_err_cnt = bit_err_cnt + sum(bbit_hat ~= bd_bit);
end

simBER(ii) = bit_err_cnt/(2*Ld*its);
disp('-------------------------------------------------');
disp(['SNR = ',num2str(SNR(ii)),' dB']);
disp([',num2str(bit_err_cnt),, bit errors counted: BER = ',num2str(bit_err_cnt/(2*Ld*its))]);
end
The following Matlab script was used to simulate the BER performance for the SRRC pulse shape (Examples 3 and 4):

```matlab
SNR = [1:12];
h = exp(j*0.66*pi);
L0 = 4;
pilot_bits = [1 0 0 0 1 1 1 0];
% L0 = 16;
% pilot_bits = [1 0 1 1 0 1 1 0 1 1 1 0 1 1 0 1 1 0 1 0 0 0 1 1 1 0];
Ld = 1000;
Lsrrc = 2;
Lp = 2*Lsrrc;
p = rcosine(1,2,'fir/sqrt',0.5,Lsrrc);
start_xx_index = 2*2*Lsrrc+1;
end_xx_index = 2*L0 + 2*Ld + 8*Lsrrc - 2*2*Lsrrc;

LUT = [-1-j, -1+j, 1-j, 1+j];
a_bit = pilot_bits(1:2:end);
b_bit = pilot_bits(2:2:end);
a_index = 2*a_bit+b_bit;
a = LUT(a_index+1);

temp = zeros(1,2*(L0+Ld-1)+1);
simBER = zeros(size(SNR));
its = 20000;
for ii = 1:length(SNR)
    snr = 10.'(SNR(ii)/10);
nvar = h'*h/(2*snr);
nstd = sqrt(nvar);
bit_err_cnt = 0;
for i = 1:its
    % create random data bits
    %
```
ad_bit = randi([0 1],1,Ld);
bд_bit = randi([0 1],1,Ld);
%
% create OQPSK packet at 2 samples/symbol
ad = LUT(2*ad_bit + bd_bit + 1);
temp(1:2:end) = real([a ad]);
II = conv(p,temp);
temp(1:2:end) = imag([a ad]);
QQ = conv(p,temp);
s = [II 0] + j*[0 QQ];
%
% generate noise and add to OQPSK packet
w = nstd*(randn(size(s)) + j*randn(size(s)));
r = h*s + w;
%
% matched filter
xx = conv(r,p);
xx = xx(start_xx_index:end_xx_index);
%
% estimate h
hhat = xx(1:2:2*L0)*real(a).* - j*xx(2:2:2*L0)*imag(a).*;
hhat = hhat/(2*L0);
%
% apply estimate to detection
xdata = hhat*xx;
abit_hat = real(xdata(2*L0+1:2:end)) > 0;
bbit_hat = imag(xdata(2*L0+2:2:end)) > 0;
bit_err_cnt = bit_err_cnt + sum(abit_hat ~= ad_bit);
bbit_err_cnt = bit_err_cnt + sum(bbit_hat ~= bd_bit);
end
simBER(ii) = bit_err_cnt/(2*Ld*its);
disp('-------------------------------------------------');
disp(['SNR = ',num2str(SNR(ii)),' dB']);
disp(['bit errors counted: BER = ',num2str(bit_err_cnt/(2*Ld*its))]);
end
Bibliography


Appendix A

The Cramér-Rao Bound Using QPSK

The Cramér-Rao bound is only defined for real-valued parameters. Writing $h = h_R + jh_I$, the log-likelihood function (2.9) may be expressed in terms of the real-valued parameters $h_R$ and $h_I$ as

$$
\Lambda(h_R, h_I) = -\frac{1}{2N_0} \int_{T_1}^{T_2 + T_0} \left| r(t) - (h_R + jh_I) \sum_{k=0}^{L_0-1} a(k)p(t - kT_s) \right|^2 dt. \quad (A.1)
$$

Let $I(t, h_R, h_I)$ be the integrand. Expanding the integrand produces

$$
I(t, h_R, h_I) = |r(t)|^2 - (h_R - jh_I)r(t) \sum_{l=0}^{L_0-1} a^*(l)p(t - lT_s)
$$

$$
- (h_R + jh_I)r^*(t) \sum_{i=0}^{L_0-1} a(i)p(t - iT_s)
$$

$$
+ (h_R^2 + h_I^2) \sum_{i=0}^{L_0-1} a(i)p(t - iT_s) \sum_{l=0}^{L_0-1} a^*(l)p(t - lT_s). \quad (A.2)
$$

The two first derivatives of the integrand are

$$
\frac{\partial}{\partial h_R} I(t, h_R, h_I) = -r(t) \sum_{l=0}^{L_0-1} a^*(l)p(t - lT_s) - r^*(t) \sum_{i=0}^{L_0-1} a(i)p(t - iT_s)
$$

$$
+ 2h_R \sum_{i=0}^{L_0-1} a(i)p(t - iT_s) \sum_{l=0}^{L_0-1} a^*(l)p(t - lT_s) \quad (A.3)
$$

$$
\frac{\partial}{\partial h_I} I(t, h_R, h_I) = jr(t) \sum_{l=0}^{L_0-1} a^*(l)p(t - lT_s) - jr^*(t) \sum_{i=0}^{L_0-1} a(i)p(t - iT_s)
$$
and the four second partial derivatives of the integrand are

\[
\frac{\partial^2 I(t, h_R, h_I)}{\partial h_R \partial h_R} = 2 \sum_{i=0}^{L_0-1} a(i) p(t - iT_s) \sum_{l=0}^{L_0-1} a^*(l) p(t - lT_s) \tag{A.5}
\]

\[
\frac{\partial^2 I(t, h_R, h_I)}{\partial h_I \partial h_R} = 0 \tag{A.6}
\]

\[
\frac{\partial^2 I(t, h_R, h_I)}{\partial h_R \partial h_I} = 0 \tag{A.7}
\]

\[
\frac{\partial^2 I(t, h_R, h_I)}{\partial h_I \partial h_I} = 2 \sum_{i=0}^{L_0-1} a(i) p(t - iT_s) \sum_{l=0}^{L_0-1} a^*(l) p(t - lT_s). \tag{A.8}
\]

Using these results, the desired partial derivatives of the log-likelihood function are computed:

\[
\frac{\partial^2 \Lambda(h_R, h_I)}{\partial h_R \partial h_R} = - \frac{1}{2N_0} \int_{T_1}^{T_2 + T_0} \frac{\partial^2 I(t, h_R, h_I)}{\partial h_R \partial h_R} dt
\]

\[
= - \frac{1}{N_0} \int_{T_1}^{T_2 + T_0} \sum_{i=0}^{L_0-1} a(i) \sum_{l=0}^{L_0-1} a^*(l) p(t - iT_s) p(t - lT_s) dt
\]

\[
= - \frac{1}{N_0} \sum_{i=0}^{L_0-1} a(i) \sum_{l=0}^{L_0-1} a^*(l) \int_{T_1 + lT_s}^{T_2 + iT_s} p(t - iT_s) p(t - lT_s) dt
\]

\[
= - \frac{1}{N_0} \sum_{i=0}^{L_0-1} a(i) \sum_{l=0}^{L_0-1} a^*(l) r_p([i - l]T_s)
\]

\[
= - \frac{1}{N_0} \sum_{i=0}^{L_0-1} |a(i)|^2 = - \frac{2L_0 A^2}{N_0} \tag{A.9}
\]

where the last step assumes the pulse shape satisfies the Nyquist No-ISI condition. Similarly,

\[
\frac{\partial^2 \Lambda(h_R, h_I)}{\partial h_I \partial h_R} = - \frac{1}{2N_0} \int_{T_1}^{T_2 + T_0} \frac{\partial^2 I(t, h_R, h_I)}{\partial h_I \partial h_R} dt = 0 \tag{A.10}
\]
\[
\frac{\partial^2 \Lambda(h_R, h_I)}{\partial h_R \partial h_I} = -\frac{1}{2N_0} \int_{T_1}^{T_2 + T_0} \frac{\partial^2 I(t, h_R, h_I)}{\partial h_R \partial h_I} \, dt = 0.
\]  

(A.11)

Finally,
\[
\frac{\partial^2 \Lambda(h_R, h_I)}{\partial h_I \partial h_I} = -\frac{1}{2N_0} \int_{T_1}^{T_2 + T_0} \frac{\partial^2 I(t, h_R, h_I)}{\partial h_I \partial h_I} \, dt
\]
\[
= -\frac{1}{N_0} \sum_{i=0}^{L_0-1} a(i) \sum_{l=0}^{L_0-1} a^*(l) \int_{T_1}^{T_2 + lT_s} p(t - iT_s)p(t - lT_s) \, dt
\]
\[
= -\frac{1}{N_0} \sum_{i=0}^{L_0-1} a(i) \sum_{l=0}^{L_0-1} a^*(l) r_p([i - l]T_s)
\]
\[
= -\frac{1}{N_0} \sum_{i=0}^{L_0-1} |a(i)|^2 = -\frac{2L_0A^2}{N_0}
\]  

(A.12)

where the last step assumes the pulse shape satisfies the Nyquist No-ISI condition.

The Fisher information matrix \( \mathbf{J} \) is
\[
\mathbf{J} = \begin{bmatrix}
J_{h_R,h_R} & J_{h_R,h_I} \\
J_{h_I,h_R} & J_{h_I,h_I}
\end{bmatrix}
\]  

(A.13)

where
\[
J_{h_R,h_R} = -E \left[ \frac{\partial^2 \Lambda(h_R, h_I)}{\partial h_R \partial h_R} \right] = \frac{2L_0A^2}{N_0}
\]  

(A.14)
\[
J_{h_R,h_I} = -E \left[ \frac{\partial^2 \Lambda(h_R, h_I)}{\partial h_R \partial h_I} \right] = 0
\]  

(A.15)
\[
J_{h_I,h_R} = -E \left[ \frac{\partial^2 \Lambda(h_R, h_I)}{\partial h_I \partial h_R} \right] = 0
\]  

(A.16)
\[
J_{h_I,h_I} = -E \left[ \frac{\partial^2 \Lambda(h_R, h_I)}{\partial h_I \partial h_I} \right] = \frac{2L_0A^2}{N_0}
\]  

(A.17)
The estimator error covariance matrix is bounded by the inverse of the Fisher information matrix:

\[
\begin{bmatrix}
\mathbb{E} \left[ (\hat{h}_R - h_R)(\hat{h}_R - h_R) \right] & \mathbb{E} \left[ (\hat{h}_R - h_R)(\hat{h}_I - h_I) \right] \\
\mathbb{E} \left[ (\hat{h}_I - h_I)(\hat{h}_R - h_R) \right] & \mathbb{E} \left[ (\hat{h}_I - h_I)(\hat{h}_I - h_I) \right]
\end{bmatrix} \geq \mathbf{J}^{-1} = \begin{bmatrix} \frac{N_0}{2L_0A^2} & 0 \\ 0 & \frac{N_0}{2L_0A^2} \end{bmatrix}.
\]  

(A.18)

From this, we see that

\[
\mathbb{E} \left[ (\hat{h}_R - h_R)^2 \right] \geq \frac{N_0}{2L_0 A^2} \tag{A.19}
\]

\[
\mathbb{E} \left[ (\hat{h}_I - h_I)^2 \right] \geq \frac{N_0}{2L_0 A^2}. \tag{A.20}
\]

To obtain the final result, the magnitude squared of the estimator error must be expressed in terms of the real and imaginary parts of the estimate and the true value:

\[
|h - \hat{h}|^2 = |\hat{h}_R + j\hat{h}_I - (h_R + jh_I)|^2 = |(\hat{h}_R - h_R) + j(\hat{h}_I - h_I)|^2
\]

\[
= (\hat{h}_R - h_R)^2 + (\hat{h}_I - h_I)^2. \tag{A.21}
\]

Applying this result produces

\[
\mathbb{E} \left[ |h - \hat{h}|^2 \right] = \mathbb{E} \left[ (\hat{h}_R - h_R)^2 + (\hat{h}_I - h_I)^2 \right] \geq \frac{N_0}{2L_0 A^2} + \frac{N_0}{2L_0 A^2} = \frac{N_0}{L_0 A^2}. \tag{A.22}
\]

Using the substitutions \( E_b = \frac{1}{2A^2} \), the Cramér-Rao bound is

\[
\mathbb{E} \left[ |h - \hat{h}|^2 \right] \geq \frac{1}{2L_0 \frac{E_b}{N_0}}. \tag{A.23}
\]
Appendix B

The Interference Term in OQPSK

B.1 Alternate Forms

The interference term is

$$\mathcal{X} = \sum_{l=0}^{L_0-1} \sum_{k=-L_p}^{l+L_p-1} a(l)b(k)r_p([k - l + 0.5]T_s) - \sum_{l'=-0}^{L_0-1} \sum_{k'=l'-L_p+1}^{l'+L_p} b(l')a(k')r_p([k' - l' - 0.5]T_s)$$ (B.1)

where $a(i) \in \{-A, +A\}$ and $b(i) \in \{-A, +A\}$ represent the pilot or data symbols, $L_0$ is the number of pilot symbols, and

$$r_p(\tau) = \int_{T_1}^{T_2} p(t)p(t-\tau)d\tau$$ (B.2)

is the autocorrelation of the pulse shape $p(t)$ that has support on the interval $T_1 \leq t \leq T_2$. Because $r_p(\tau)$ is an autocorrelation, it is symmetric: $r_p(-\tau) = r_p(\tau)$. Most of the terms in (B.1) sum to zero. To obtain an alternate expression for $\mathcal{X}$, the summations are re-written using new sets of indexes that help identify the terms that sum to zero.

Define a new index for the first term (B.1). Let $i = k - l$. Then $k = l - L_p \rightarrow i = l - L_p - l = -L_p$ and $k = l + L_p - 1 \rightarrow i = l + L_p - 1 - l = L_p - 1$. Similarly, define a new index for the second term. Let $i' = k' - l'$. Then $k' = l' - L_p + 1 \rightarrow i' = l' - L_p + 1 - l' = -L_p + 1$ and $k' = l' + L_p \rightarrow i' = l' + L_p - l' = L_p$. Rewriting the term using the new indices gives

$$\mathcal{X} = \sum_{l=0}^{L_0-1} \sum_{i=-L_p}^{L_p} a(l)b(i+l)r_p([i + 0.5]T_s) - \sum_{l'=0}^{L_0-1} \sum_{i'=-L_p+1}^{L_p} b(l')a(i' + l')r_p([i' - 0.5]T_s).$$ (B.3)
The two double summations have many terms in common. These terms cancel. To identify the terms that cancel, it is necessary to re-order the second double summation. The approach is to rewrite the inner summation as a sum along a “line” defined by \( i' + l' = l \) and sum over \( l \). The details depend on the relationship between \( L_0 \) and \( L_p \). There are three cases that need to be considered.

### B.1.1 Case 1: \( L_0 \geq 2L_p \)

The index range covered by the double-summation of the second term in (B.3) is

\[
0 \leq l' \leq L_0 - 1 \quad \text{and} \quad -L_p + 1 \leq i' \leq L_p.
\]

Traversing this range using \( i' + l' = l \) forms diagonal lines such as those illustrated in Figure B.1. The summation region covered by this substitution is shown by the shaded region in Figure B.1. Note that as \( l \) ranges from 0 to \( L_0 - 1 \), the boundary conditions change and define three different sub-cases whose boundaries are marked by the heavy lines in Figure B.1. The second term of (B.3) may be written as

\[
\begin{align*}
\sum_{l_0-1}^{L_p} \sum_{l_0-L_p}^{L_p} b(l')a(i' + l')r_p([i' - 0.5]T_s) &= \sum_{l=0}^{L_p-1} \sum_{l'=0}^{L_p-1} b(l')a(l)r_p([-l' + l - 0.5]T_s) \\
+ & \sum_{l=L_p}^{L_0-1} \sum_{l'=l-L_p}^{L_p} b(l')a(l)r_p([-l' + l - 0.5]T_s) + \sum_{l'=L_0-L_p}^{L_0-1} \sum_{l'=l-L_p}^{L_p} b(l')a(l)r_p([-l' + l - 0.5]T_s) \\
+ & \sum_{l'=0}^{L_p-2} \sum_{l'=L_p+1}^{L_P} b(l')a(i' + l')r_p([i' - 0.5]T_s) + \sum_{l'=L_0-L_p}^{L_0-1} \sum_{l'=l'-L_p}^{L_p} b(l')a(i' + l')r_p([i' - 0.5]T_s).
\end{align*}
\]

Motivated by the sub-cases shown in Figure B.1 the outer summation of first term in (B.3) may be partitioned into three summations as follows:

\[
\begin{align*}
\sum_{i=-L_p}^{L_p-1} \sum_{l=0}^{L_0-1} a(l)b(i + l)r_p([i + 0.5]T_s) &= \sum_{l=0}^{L_p-1} \sum_{i=-L_p}^{L_p-1} a(l)b(i + l)r_p([i + 0.5]T_s) \\
+ & \sum_{l=L_p}^{L_0-L_p} \sum_{i=-L_p}^{L_p-1} a(l)b(i + l)r_p([i + 0.5]T_s) + \sum_{i=-L_p}^{L_0-L_p} \sum_{l=L_p}^{L_p-1} a(l)b(i + l)r_p([i + 0.5]T_s).
\end{align*}
\]

Substituting (B.4) for the second term in (B.3) and (B.5) for the first term in (B.3), the difference
Figure B.1: A graphical representation of reorganizing the double-summation of the second term in (B.3) for the case $L_0 \geq 2L_p$. 
(B.3) may be written, after some minor rearrangements, as
\[
X = \sum_{l=0}^{L_p-1} \sum_{i=-L_p}^{L_p-1} a(l)b(i + l)r_p([i + 0.5]T_s) - \sum_{l=0}^{L_p-1} \sum_{l'=0}^{L_p-1} b(l')a(l)r_p([-l' + l - 0.5]T_s) \\
\Delta_{1,1} + \sum_{l=L_0-L_p}^{L_0-1} \sum_{i=-L_p}^{L_p-1} a(l)b(i + l)r_p([i + 0.5]T_s) - \sum_{l=L_0-L_p}^{L_0-1} \sum_{l'=l-L_p}^{L_0-1} b(l')a(l)r_p([-l' + l - 0.5]T_s) \\
\Delta_{1,2} + \sum_{l=L_0-L_p+1}^{L_0-1} \sum_{i=-L_p}^{L_p-1} a(l)b(i + l)r_p([i + 0.5]T_s) - \sum_{l=L_0-L_p+1}^{L_0-1} \sum_{l'=l-L_p}^{L_0-1} b(l')a(l)r_p([-l' + l - 0.5]T_s) \\
\Delta_{1,3} - \sum_{l'=0}^{L_0-2} \sum_{l'=-L_p+1}^{-l'-1} b(l')a(i' + l')r_p([i' - 0.5]T_s) - \sum_{l'=L_0-L_p}^{L_0-1} \sum_{l'=-L_p}^{-l'+L_0} b(l')a(i' + l')r_p([i' - 0.5]T_s).
\]  
(B.6)

Alternate expressions for \(\Delta_{1,1}, \Delta_{1,2},\) and \(\Delta_{1,3}\) are derived as follows.

\(\Delta_{1,1}:\) Using the substitution \(l' = l + i\) with the inner summation of the first term of \(\Delta_{1,1}, \Delta_{1,1}\) may be written as

\[
\Delta_{1,1} = \sum_{l=0}^{L_p-1} a(l) \left[ \sum_{l'=l-L_p}^{l+L_p-1} b(l')r_p([l' - l + 0.5]T_s) - \sum_{l'=0}^{l+L_p-1} b(l')r_p([-l' + l - 0.5]T_s) \right] \\
= \sum_{l=0}^{L_p} a(l) \left[ \sum_{l'=l-L_p}^{l+L_p-1} b(l')r_p([l' - l + 0.5]T_s) \\
+ \sum_{l'=0}^{l+L_p-1} b(l')r_p([-l' + l - 0.5]T_s) - \sum_{l'=l-L_p}^{l+L_p-1} b(l')r_p([-l' + l - 0.5]T_s) \right]. \quad (B.7)
\]

If the pulse shape autocorrelation function is symmetric, then

\[
r_p([-l' + l - 1/2]T_s) = r_p([-l' - l + 0.5]T_s) = r_p([l' - l + 0.5]T_s). \quad (B.8)
\]
In this case, the last two terms inside the square brackets of (B.7) sum to zero so that

\[ \Delta_{1,1} = \sum_{l=0}^{L_p-1} a(l) \sum_{l'=l-L_p}^{-1} b(l') r_p([l' - l + 0.5]T_s). \] (B.9)

\[ \Delta_{1,1}, \Delta_{1,2} : \text{Using the substitution } l' = l + i \text{ with the inner summation of the first term of } \Delta_{1,2}, \Delta_{1,2} \text{ may be written as} \]

\[ \Delta_{1,2} = \sum_{l=L_0-L_p}^{L_0-L_p} a(l) \left[ \sum_{l'=l-L_p}^{l+L_p-1} b(l') r_p([l' - l + 0.5]T_s) - \sum_{l'=l-L_p}^{l+L_p-1} b(l') r_p([-l' + l - 0.5]T_s) \right]. \] (B.10)

If the pulse shape is symmetric, then by (B.8) the difference inside the square brackets of (B.10) is zero.

\[ \Delta_{1,2} = 0. \] (B.11)

\[ \Delta_{1,3} : \text{Using the substitution } l' = l + i \text{ with the inner summation of the first term of } \Delta_{1,3}, \Delta_{1,3} \text{ may be written as} \]

\[ \Delta_{1,3} = \sum_{l=L_0-L_p+1}^{L_0-1} a(l) \left[ \sum_{l'=l-L_p}^{l+L_p-1} b(l') r_p([l' - l + 0.5]T_s) - \sum_{l'=l-L_p}^{L_0-1} b(l') r_p([-l' + l - 0.5]T_s) \right] + \left[ \sum_{l'=L_0}^{l+L_p-1} b(l') r_p([l' - l + 0.5]T_s) - \sum_{l'=l-L_p}^{L_0-1} b(l') r_p([-l' + l - 0.5]T_s) \right]. \] (B.12)

If the pulse shape is symmetric, then by (B.8) the first and third terms inside the square brackets of (B.12) sum to zero so that

\[ \Delta_{1,3} = \sum_{l=L_0-L_p+1}^{L_0-1} a(l) \sum_{l'=L_0}^{l+L_p-1} b(l') r_p([l' - l + 0.5]T_s). \] (B.13)

Substituting (B.9), (B.11), and (B.13) into (B.6) gives:
\[ X = \sum_{l=0}^{L_p-1} \sum_{l'=l-L_p}^{L_p-2} a(l)b(l')r_p([l'+l+0.5]T_s) - \sum_{l'=0}^{L_p-2} \sum_{i'=L_p+1}^{L_p-1} b(l')a(i'+l')r_p([i'-l+0.5]T_s) \]
\[ + \sum_{l=L_0-L_p+1}^{L_0-1} \sum_{l'=L_0}^{L_p-1} a(l)b(l')r_p([l'-l+0.5]T_s) - \sum_{l'=L_0-L_p}^{L_0-1} \sum_{l'=L_0}^{L_p} b(l')a(i'+l')r_p([i'-l-0.5]T_s). \]  

(B.14)

There are two ways to re-express (B.14). The first way uses a common indexing convention for the symbols and makes explicit the symbols involved in the interference term. This is useful in computing the variance of \( X \) in Section B.2. (The variance of \( X \) is required to compute the variance of the estimation error in Section 3.2.3.) The second way uses a common indexing convention for the pulse shape autocorrelation function \( r_p(\cdot) \) thus making explicit the values of \( r_p(\cdot) \) involved in the edge effects.

It is easier to see the symbols involved in the interference term by using a common indexing convention for the symbols. Using the substitution \( i = i'+l' \), the second term on the right-hand side of (B.14) may be expressed as

\[ \sum_{l'=0}^{L_p-2} \sum_{i'=L_p+1}^{L_p-1} b(l')a(i'+l')r_p([i'-0.5]T_s) = \sum_{l=0}^{L_p-2} \sum_{l'=l-L_p+l+1}^{L_p-1} b(l')a(l)r_p([l-l'-0.5]T_s) \]  

(B.15)

and, using the same substitution, the fourth term on the right-hand side of (B.14) may be expressed as

\[ \sum_{l=L_0-L_p}^{L_0-1} \sum_{l'=L_0}^{L_p} b(l')a(l'+l')r_p([l'-0.5]T_s) = \sum_{l'=L_0-L_p}^{L_0-1} \sum_{l=L_0}^{l'+L_p} b(l')a(l)r_p([l-l'-0.5]T_s). \]  

(B.16)

Substituting produces the final result:

\[ X = \sum_{l=0}^{L_p-1} \sum_{l'=l-L_p}^{L_p-2} a(l)b(l')r_p([l'+l+0.5]T_s) - \sum_{l'=0}^{L_p-2} \sum_{l=L_p+1}^{L_p-1} b(l')a(l)r_p([l-l'-0.5]T_s) \]

interference due to data symbols preceding the pilot symbols

\[ + \sum_{l=L_0-L_p+1}^{L_0-1} \sum_{l'=L_0}^{L_p-1} a(l)b(l')r_p([l'-l+0.5]T_s) - \sum_{l'=L_0-L_p}^{L_0-1} \sum_{l'=L_0}^{L_p} b(l')a(i'+l')r_p([i'-l-0.5]T_s). \]

interference due to data symbols following the pilot symbols

(B.17)
The edge effects due to the data symbols preceding the pilot symbols involve the symbols \(a(-L_p + 1), \ldots, a(L_p - 1)\) and \(b(-L_p), \ldots, b(L_p - 2)\): in other words, the \(L_p - 1\) \(a\)’s and the \(L_p\) \(b\)’s preceding the pilot symbols. The edge effects due to the data symbols following the pilot symbols involve the symbols \(a(0 - L_p + 1), \ldots, a(L_0 + L_p - 1)\) and \(b(L_0 - L_p), \ldots, b(L_0 + L_p - 2)\): in other words, the \(L_p\) \(a\)’s and the \(L_p - 1\) \(b\)’s following the pilot symbols. The vector-matrix version of (B.17) is given by (B.18) below. This form is useful for computing \(\mathcal{X}'\) in Matlab.

It is easier to see the pulse shape autocorrelation terms involved in \(\mathcal{X}\) by modifying the first and third terms on the right-hand side of (B.14) to create a single index for \(r_p(\cdot)\). The first term on the right-hand side of (B.14) may be reexpressed using the substitution \(i = l' - l + 1\) to produce

\[
\sum_{l=0}^{L_p - 1} \sum_{l'=l-L_p}^{-1} a(l)b(l')r_p([l' - l + 0.5]T_s) = \sum_{l=0}^{L_p - 1} \sum_{l'=l-1}^{-1} a(l)b(l + i - 1)r_p([i - 0.5]T_s). \tag{B.19}
\]

The third term on the right-hand side of (B.14) may be reexpressed using the substitution \(i = l' - l + 1\) to produce

\[
\sum_{l=L_0-L_p+1}^{L_0-1} \sum_{l'=L_0}^{l+L_p-1} a(l)b(l')r_p([l' - l + 0.5]T_s) = \sum_{l=L_0-L_p+1}^{L_0-1} \sum_{i=L_0-l+1}^{L_p} a(l)b(l + i - 1)r_p([i - 0.5]T_s). \tag{B.20}
\]

Substituting gives

\[
\mathcal{X}' = \sum_{l=0}^{L_p - 1} \sum_{i=-L_p+1}^{-1} a(l)b(l + i - 1)r_p([i - 0.5]T_s) - \sum_{l'=0}^{L_p - 2} \sum_{l'=L_p+1}^{-l'-1} b(l')a(i' + l')r_p([i' - 0.5]T_s)
+ \sum_{l=L_0-L_p+1}^{L_0-1} \sum_{i=L_0-l+1}^{L_p} a(l)b(l + i - 1)r_p([i - 0.5]T_s)
- \sum_{l'=L_0-L_p}^{L_0-1} \sum_{i'=L_0-1}^{L_p} b(l')a(i' + l')r_p([i' - 0.5]T_s) \tag{B.21}
\]

Additional insight is gained by exchanging the order of summation in each of the four terms of (B.21). With the aid of Figure B.2 the first term of (B.21) may be rewritten as

\[
\sum_{l=0}^{L_p - 1} \sum_{i=-L_p+1}^{-l} a(l)b(l + i - 1)r_p([i - 0.5]T_s) = \sum_{i=-L_p+1}^{0} \sum_{l=0}^{-i} a(l)b(l + i - 1)r_p([i - 0.5]T_s). \tag{B.22}
\]
\[ (I - dT + 0T)p \quad (I - dT + 0T)q \]
\[ (\mathcal{I} - dT + 0T)p \quad (\mathcal{I} - dT + 0T)q \]
\[ (I - dT - 0T)p \quad (I - dT - 0T)q \]
\[ (\mathcal{I} - dT - 0T)p \quad (\mathcal{I} - dT - 0T)q \]
\[ (I + dT - 0T)p \quad (I + dT - 0T)q \]
\[ (\mathcal{I} + dT - 0T)p \quad (\mathcal{I} + dT - 0T)q \]
\[ (I + dT - dT)p \quad (I + dT - dT)q \]
\[ (\mathcal{I} + dT - dT)p \quad (\mathcal{I} + dT - dT)q \]
\[ (I - dT - dT)p \quad (I - dT - dT)q \]
\[ (\mathcal{I} - dT - dT)p \quad (\mathcal{I} - dT - dT)q \]
\[ (I + dT - dT)p \quad (I + dT - dT)q \]
\[ (\mathcal{I} + dT - dT)p \quad (\mathcal{I} + dT - dT)q \]
\[ (I - dT - dT)p \quad (I - dT - dT)q \]
\[ (\mathcal{I} - dT - dT)p \quad (\mathcal{I} - dT - dT)q \]
\[ (I + dT - dT)p \quad (I + dT - dT)q \]
\[ (\mathcal{I} + dT - dT)p \quad (\mathcal{I} + dT - dT)q \]

\[ \begin{bmatrix}
(I - dT + 0T)p \\
(\mathcal{I} - dT + 0T)q \\
\vdots \\
(0p)
\end{bmatrix} \begin{bmatrix}
(s\mathcal{I}[\mathcal{I} - dT])^d, \\
0 \\
\vdots \\
0
\end{bmatrix} \begin{bmatrix}
(I - 0T)q \\
0 \\
0 \\
0
\end{bmatrix} \]

\[ \begin{bmatrix}
(\mathcal{I} - dT + 0T)q \\
(I - dT + 0T)p \\
\vdots \\
(0q)
\end{bmatrix} \begin{bmatrix}
(s\mathcal{I}[\mathcal{I} - dT])^d, \\
0 \\
\vdots \\
0
\end{bmatrix} \begin{bmatrix}
(I - 0T)p \\
(I + dT - 0T)p \\
\vdots \\
(0q)
\end{bmatrix} +

\[ \begin{bmatrix}
(I - dT)p \\
\mathcal{I} - dT - 0T)q \\
\vdots \\
(0)q
\end{bmatrix} \begin{bmatrix}
(s\mathcal{I}[\mathcal{I} + dT - dT])^d, \\
0 \\
\vdots \\
0
\end{bmatrix} \begin{bmatrix}
(I - 0T)p \\
(I + dT - 0T)p \\
\vdots \\
(0q)
\end{bmatrix} -

\[ \begin{bmatrix}
(\mathcal{I} - dT - dT)p \\
\mathcal{I} - dT - dT)q \\
\vdots \\
(0)q
\end{bmatrix} \begin{bmatrix}
(s\mathcal{I}[\mathcal{I} + dT - dT])^d, \\
0 \\
\vdots \\
0
\end{bmatrix} \begin{bmatrix}
(I - 0T)p \\
(I + dT - 0T)p \\
\vdots \\
(0q)
\end{bmatrix} = \mathcal{X}
With the aid of Figure B.3 the second term of (B.21) may be rewritten as
\[
\sum_{l=0}^{L_p-2} \sum_{i=-L_p+1}^{-l-1} b(l)a(i + l)r_p([i - 0.5]T_s) = \sum_{i=-L_p+1}^{-1} \sum_{l=0}^{-i-1} b(l)a(i + l)r_p([i - 0.5]T_s). \tag{B.23}
\]

The component of $\mathcal{X}$ due to data symbols preceding the pilot symbols is
\[
\sum_{i=-L_p+1}^{0} \sum_{l=0}^{-i} a(l)b(l + i - 1)r_p([i - 0.5]T_s) - \sum_{i=-L_p+1}^{-1} \sum_{l=0}^{-i-1} b(l')a(i' + l')r_p([i' - 0.5]T_s)
\]
\[
= a(0)b(1)r_p(-0.5T_s)
\]
\[
+ \sum_{k=-L_p+1}^{-1} \left[ \sum_{l=0}^{-i} a(l)b(l + i - 1) - \sum_{l=0}^{-i-1} b(l)a(i + l) \right] r_p([i - 0.5]T_s). \tag{B.24}
\]

Moving on the terms involving the data symbols that follow the pilot symbols, exchanging the order of summation in the third term in (B.21) gives (see Figure B.4)
\[
\sum_{l=L_0-L_p+1}^{L_0-1} \sum_{i=L_0-l+1}^{L_p} a(l)b(i + l - 1)r_p([i - 0.5]T_s) = \sum_{i=2}^{L_p} \sum_{l=L_0-i+1}^{L_0-1} a(l)b(i + l - 1)r_p([i - 0.5]T_s). \tag{B.25}
\]

With the aid of Figure B.5 the fourth term of (B.21) may be rewritten as
\[
\sum_{i=1}^{L_0-1} \sum_{i'=L_0-L_p}^{L_p} b(l')a(i + l')r_p([i - 0.5]T_s) = \sum_{i=1}^{L_p} \sum_{i'=L_0-i}^{L_0-1} b(l')a(i + l')r_p([i - 0.5]T_s). \tag{B.26}
\]

The component of $\mathcal{X}$ due to data symbols following the pilot symbols is
\[
\sum_{i=2}^{L_p} \sum_{l=L_0-i+1}^{L_0-1} a(l)b(l + i - 1)r_p([i - 0.5]T_s) - \sum_{i=1}^{L_0-1} \sum_{i'=L_0-i}^{L_0-1} b(l')a(i + l')r_p([i - 0.5]T_s)
\]
\[
= \sum_{i=2}^{L_p} \left[ \sum_{l=L_0-i+1}^{L_0-1} a(l)b(l + i - 1) - \sum_{i'=L_0-i}^{L_0-1} b(l')a(i + l') \right] r_p([i - 0.5]T_s)
\]
\[
- b(L_0 - 1)a(L_0)r_p(0.5T_s). \tag{B.27}
\]

A simple counting exercise shows that there are $2i - 1$ terms involving $r_p([i - 0.5]T_s)$ for $i =
Figure B.2: A visual representation of changing the order of summation in the first term of (B.21).
Figure B.3: A visual representation of changing the order of summation in the second term of (B.21).
For $L_p$, in summary, assembling (B.24) and (B.27) gives

$$X = a(0)b(1)r_p(-0.5T_s)$$

$$+ \sum_{k=-L_p+1}^{-1} \left[ \sum_{l=0}^{-i} a(l)b(l+i-1) - \sum_{l=0}^{-i-1} b(l)a(i+l) \right] r_p([i - 0.5]T_s)$$

interference due to data symbols preceding the pilot symbols

$$+ \sum_{i=2}^{L_p} \left[ \sum_{l=L_0-i+1}^{L_0-1} a(l)b(l+i-1) - \sum_{l'=L_0-i}^{L_0-1} b(l')a(i+l') \right] r_p([i - 0.5]T_s)$$

$$- b(L_0 - 1)a(0)r_p(0.5T_s).$$

(B.28)

To get a feel for what these terms look like, we evaluate (B.17) for small values of $L_p$ (this case is the case that applies most often for small $L_p$):

$L_p = 1 : X = a(0)b(-1)r_p(-0.5T_s) - b(L_0 - 1)a(0)r_p(0.5T_s)$

(B.29)

$L_p = 2 : X = a(0)b(-2)r_p(-1.5T_s) + a(0)b(-1)r_p(-0.5T_s)$$

+ a(1)b(-1)r_p(-1.5T_s) - b(0)a(-1)r_p(-1.5T_s)$$

+ a(L_0 - 1)b(L_0)r_p(1.5T_s) - b(L_0 - 2)a(L_0)r_p(1.5T_s)$$

- b(L_0 - 1)a(L_0)r_p(0.5T_s) - b(L_0 - 1)a(L_0 + 1)r_p(1.5T_s)$

(B.30)

$L_p = 3 : X = a(0)b(-3)r_p(-2.5T_s) + a(0)b(-2)r_p(-1.5T_s)$$

+ a(0)b(-1)r_p(-0.5T_s) - b(0)a(-2)r_p(-2.5T_s)$$

- b(0)a(-1)r_p(-1.5T_s) + a(1)b(-2)r_p(-2.5T_s)$$

+ a(1)b(-1)r_p(-1.5T_s) - b(1)a(-1)r_p(-2.5T_s)$$

+ a(2)b(-1)r_p(-2.5T_s)$$

- a(L_0)b(L_0 - 3)r_p(2.5T_s) + a(L_0 - 2)b(L_0)r_p(2.5T_s)$$

- a(L_0)b(L_0 - 2)r_p(1.5T_s) - a(L_0 + 1)b(L_0 - 2)r_p(2.5T_s)$$

+ a(L_0 - 1)b(L_0)r_p(1.5T_s) + a(L_0 - 1)b(L_0 + 1)r_p(2.5T_s)$$

- b(L_0 - 1)a(L_0)r_p(0.5T_s) - b(L_0 - 1)a(L_0 + 1)r_p(1.5T_s)$$

- b(L_0 - 1)a(L_0 + 2)r_p(2.5T_s).$$

(B.31)

The terms on the right-hand side are ordered such that terms due to the edge effects caused by the
Figure B.4: A visual representation of changing the order of summation in the third term of (B.21)
Figure B.5: A visual representation of changing the order of summation in the fourth term of (B.21).
data symbols preceding the pilot symbols are first; whereas the terms due to the edge effects caused by the data symbols following the pilot symbols are second. (Note that the first set of terms is zero when the pilot symbols are organized in a preamble. The second set of terms is zero when the pilot symbols are organized in a postamble. Both sets of terms are present when the pilot symbols form a midamble. See Figure 1.1.) Several observations are important:

1. $\mathcal{X}$ is formed by products of data symbols (outside the block of pilot symbols) and pilot symbols.

2. The data-symbol-pilot-symbol products are those that occur within the span of the pulse shape autocorrelation function $r_p(\cdot)$. For this reason, $\mathcal{X}$ only contains products of symbols whose indexes differ by $L_p$ or less.

3. $\mathcal{X}$ is a function of the pulse shape autocorrelation function evaluated at odd multiples of $T_s/2$.

4. It might appear that $\mathcal{X}$ is a function of the number of pilot symbols $L_0$, but this is not true. The appearance of $L_0$ in (B.17) is an artifact of indexing convention. For the case where there are data symbols that follow the pilot symbols (e.g., the preamble and midamble arrangements of Figure 1.1), the interference term $\mathcal{X}$ depends on the last $L_p$ data symbols preceding the pilot symbol block and the first $L_p - 1$ data symbols following the pilot symbol block.

**B.1.2 Case 2: $L_p \leq L_0 < 2L_p$**

Traversing the double-summation range

$$0 \leq l' \leq L_0 - 1 \quad \text{and} \quad -L_p + 1 \leq i' \leq L_p$$

of the second term in (B.3) using $i' + l' = l$ forms diagonal lines such as those illustrated in Figure B.6. The summation region covered by this substitution is shown by the shaded region in Figure B.6. Note that as $l$ ranges from 0 to $L_0 - 1$, the boundary conditions change and define three different sub-cases whose boundaries are marked by the heavy lines in Figure B.6. The
second term of (B.3) may be written as

\[
\sum_{l'=0}^{L_p-1} \sum_{l'=-L_p+1}^{L_0} b(l') a(i' + l') r_p([i' - 0.5]T_s) = \sum_{l=0}^{L_0-L_p} \sum_{l'=-L_p+1}^{L_p-1} b(l') a(l) r_p([-l' + l - 0.5]T_s)
\]

Motivated by the sub-cases shown in Figure B.6 the outer summation of the first term in (B.3) may be partitioned into three summations as follows:

\[
\sum_{l=0}^{L_0-1} \sum_{i=-L_p}^{L_p-1} a(l) b(i + l) r_p([i + 0.5]T_s) = \sum_{l=0}^{L_0-L_p} \sum_{i=-L_p}^{L_p-1} a(l) b(i + l) r_p([i + 0.5]T_s)
\]

\[
+ \sum_{l=L_0-L_p+1}^{L_p-1} \sum_{i=-L_p}^{L_p-1} a(l) b(i + l) r_p([i + 0.5]T_s) + \sum_{l=L_p}^{L_0-1} \sum_{i=-L_p}^{L_p-1} a(l) b(i + l) r_p([i + 0.5]T_s). \quad (B.33)
\]

Substituting (B.32) for the second term in (B.3) and (B.33) for the first term in (B.3), the difference (B.3) may be written, after some minor rearrangements, as

\[
\mathcal{X} = \sum_{l=0}^{L_0-L_p} \sum_{i=-L_p}^{L_p-1} a(l) b(i + l) r_p([i + 0.5]T_s) - \sum_{l=0}^{L_0-L_p} \sum_{l'=-L_p+1}^{L_p-1} b(l') a(l) r_p([-l' + l - 0.5]T_s)
\]

\[
+ \sum_{l=L_0-L_p+1}^{L_p-1} \sum_{i=-L_p}^{L_p-1} a(l) b(i + l) r_p([i + 0.5]T_s) - \sum_{l=L_0-L_p+1}^{L_p-1} \sum_{l'=-L_p+1}^{L_p-1} b(l') a(l) r_p([-l' + l - 0.5]T_s)
\]

\[
+ \sum_{l=L_p}^{L_0-1} \sum_{i=-L_p}^{L_p-1} a(l) b(i + l) r_p([i + 0.5]T_s) - \sum_{l=L_p}^{L_0-1} \sum_{l'=-L_p}^{L_0-1} b(l') a(l) r_p([-l' + l - 0.5]T_s)
\]

\[
\Delta_{2.1}, \Delta_{2.2}, \Delta_{2.3}
\]
Figure B.6: A graphical representation of reorganizing the double-summation of the second term in (B.3) for the case $L_p \leq L_0 < 2L_p$. 

\[
\begin{align*}
  \sum_{i=0}^{L_0 - 1} \sum_{i'=0}^{L_0 - i - 1} b(i') \alpha(i' + i + L_p) r_p(i - i' + L_p - 1) - l' + i + L_p - 0.5 \big[T_s \big] \\
  \sum_{i=0}^{L_0 - 1} \sum_{i'=0}^{L_0 - i - 1} b(i') \alpha(i' + i + L_p) r_p(i - i' + L_p - 1) - l' + i + L_p - 0.5 \big[T_s \big]
\end{align*}
\]
Alternate expressions for \( \Delta_{2,1} \), \( \Delta_{2,2} \), and \( \Delta_{2,3} \) are derived as follows.

\[ \Delta_{2,1} : \text{Using the substitution } l' = l + i \text{ with the inner summation of the first term of } \Delta_{2,1}, \Delta_{2,1} \text{ may be written as} \]

\[
\Delta_{2,1} = \sum_{l=0}^{L_0-L_p} a(l) \left[ \sum_{l'=l-L_p}^{l+L_p-1} b(l') r_p([l' - l + 0.5]T_s) - \sum_{l'=0}^{l+L_p-1} b(l') r_p([-l' + l - 0.5]T_s) \right]
\]

\[
= \sum_{l=0}^{L_0-L_p} a(l) \left[ \sum_{l'=l-L_p}^{l+L_p-1} b(l') r_p([l' - l + 0.5]T_s) + \sum_{l'=0}^{l+L_p-1} b(l') r_p([l' - l + 0.5]T_s) \right.
\]

\[ \left. - \sum_{l'=0}^{l+L_p-1} b(l') r_p([-l' + l - 0.5]T_s) \right]. \tag{B.35} \]

If the pulse shape is symmetric, then by (B.8) the second and third terms inside the square brackets of (B.35) sum to zero so that

\[ \Delta_{2,1} = \sum_{l=0}^{L_0-L_p} a(l) \sum_{l'=l-L_p}^{-1} b(l') r_p([l' - l + 0.5]T_s). \tag{B.36} \]

\[ \Delta_{2,2} : \text{Using the substitution } l' = l + i \text{ with the inner summation of the first term of } \Delta_{2,2}, \Delta_{2,2} \text{ may be written as} \]

\[
\Delta_{2,2} = \sum_{l=L_0-L_p+1}^{L_p-1} a(l) \left[ \sum_{l'=l-L_p}^{l+L_p-1} b(l') r_p([l' - l + 0.5]T_s) - \sum_{l'=0}^{L_0-1} b(l') r_p([-l' + l - 0.5]T_s) \right]
\]

\[
= \sum_{l=L_0-L_p+1}^{L_p-1} a(l) \left[ \sum_{l'=l-L_p}^{-1} b(l') r_p([l' - l + 0.5]T_s) + \sum_{l'=0}^{L_0-1} b(l') r_p([l' - l + 0.5]T_s) \right.
\]

\[ \left. + \sum_{l'=L_0}^{l+L_p-1} b(l') r_p([l' - l + 0.5]T_s) - \sum_{l'=0}^{L_0-1} b(l') r_p([-l' + l - 0.5]T_s) \right]. \tag{B.37} \]

If the pulse shape is symmetric, then by (B.8) the second and fourth terms inside the square brackets
of (B.37) sum to zero so that

$$\Delta_{2,2} = \sum_{l=L_0-L_p+1}^{L_p-1} a(l) \sum_{l'=l-L_p}^{-1} b(l')r_p([l' - l + 0.5]T_s) + \sum_{l=L_0-L_p+1}^{L_p-1} a(l) \sum_{l'=-L_p}^{l+L_p-1} b(l')r_p([l' - l + 0.5]T_s).$$  \hspace{1cm} (B.38)$$

$$\Delta_{2,3}$$: Using the substitution \( l' = l + i \) with the inner summation of the first term of \( \Delta_{2,3} \), \( \Delta_{2,3} \) may be written as

$$\Delta_{2,3} = \sum_{l=L_p}^{L_0-1} a(l) \left[ \sum_{l'=l-L_p}^{l+L_p-1} b(l')r_p([l' - l + 0.5]T_s) - \sum_{l'=l-L_p}^{L_0-1} b(l')r_p([-l' + l - 0.5]T_s) \right]$$

$$= \sum_{l=L_p}^{L_0-1} a(l) \left[ \sum_{l'=l-L_p}^{L_0-1} b(l')r_p([l' - l + 0.5]T_s) + \sum_{l'=L_0}^{l+L_p-1} b(l')r_p([l' - l + 0.5]T_s) \right]$$

$$- \sum_{l'=l-L_p}^{L_0-1} b(l')r_p([-l' + l - 0.5]T_s).$$  \hspace{1cm} (B.39)$$

If the pulse shape is symmetric, then by (B.8) the first and third terms inside the square brackets of (B.39) sum to zero so that

$$\Delta_{2,3} = \sum_{l=L_p}^{L_0-1} a(l) \sum_{l'=l-L_p}^{-1} b(l')r_p([l' - l + 0.5]).$$  \hspace{1cm} (B.40)$$

Substituting (B.36), (B.38), and (B.40) into (B.34) gives

$$X = \sum_{l=0}^{L_0-L_p} a(l) \sum_{l'=l-L_p}^{-1} b(l')r_p([l' - l + 0.5]T_s) + \sum_{l=L_0-L_p+1}^{L_p-1} a(l) \sum_{l'=l-L_p}^{-1} b(l')r_p([l' - l + 0.5]T_s)$$

$$+ \sum_{l=L_0-L_p+1}^{L_p-1} a(l) \sum_{l'=L_0}^{l+L_p-1} b(l')r_p([l' - l + 0.5]T_s) + \sum_{l=L_0-1}^{L_p-1} a(l) \sum_{l'=L_0}^{l+L_p-1} b(l')r_p([l' - l + 0.5]T_s)$$

$$- \sum_{l'=0}^{L_0-L_p+1} b(l')a(i' + l')r_p([i' - 0.5]T_s) - \sum_{l'=L_0-L_p}^{L_p} b(l')a(i' + l')r_p([i' - 0.5]T_s)$$

$$- \sum_{l'=L_0-L_p}^{L_p} b(l')a(i' + l')r_p([i' - 0.5]T_s) - \sum_{l'=0}^{L_0-L_p} b(l')a(i' + l')r_p([i' - 0.5]T_s)$$

$$= \sum_{l=0}^{L_p-1} a(l) \sum_{l'=l-L_p}^{-1} b(l')r_p([l' - l + 0.5]T_s) - \sum_{l'=0}^{L_0-L_p} b(l')a(i' + l')r_p([i' - 0.5]T_s)$$

$$- \sum_{l'=L_0-L_p}^{L_p} b(l')a(i' + l')r_p([i' - 0.5]T_s) - \sum_{l'=0}^{L_0-L_p} b(l')a(i' + l')r_p([i' - 0.5]T_s)$$
\[ + \sum_{l=L_0-L_p+1}^{L_0-1} a(l) \sum_{l'=-L_p}^{L_p-1} b(l') r_p([l' - l + 0.5]T_s) - \sum_{l=0}^{L_0-1} \sum_{l'=0}^{L_p} b(l') a(l') r_p([l' - l - 0.5]T_s). \]

This result is identical to (B.14). Case 1 and Case 2 have the same alternate representations. The two alternate forms for (B.41) are given by (B.17) and (B.28).

**B.1.3 Case 3:** \( L_0 < L_p \)

Traversing the double-summation range

\[ 0 \leq l' \leq L_0 - 1 \quad \text{and} \quad -L_p + 1 \leq i' \leq L_p \]

of the second term in (B.3) using \( i' + l' = l \) forms diagonal lines such as those illustrated in Figure B.7. The summation region covered by this substitution is shown by the shaded region in Figure B.7. The second term of (B.3) may be written as

\[ \sum_{l'=0}^{L_0-1} \sum_{l'=-L_p+1}^{L_p} b(l') a(i' + l') r_p([i' - 0.5]T_s) = \sum_{l=0}^{L_0-1} \sum_{l'=0}^{L_p} b(l') a(l') r_p([-l' + l - 0.5]T_s) \]

\[ + \sum_{l'=0}^{L_0-1} \sum_{l'=-L_p+1}^{L_p} b(l') a(i' + l') r_p([i' - 0.5]T_s) + \sum_{l=0}^{L_0-1} \sum_{l'=0}^{L_p} b(l') a(l') r_p([-l' + l - 0.5]T_s). \]

Using (B.42), the difference (B.3) may be written as

\[ X = \sum_{l=0}^{L_0-1} \sum_{i=-L_p}^{L_p-1} a(l) b(i + l) r_p([i + 0.5]T_s) - \sum_{l=0}^{L_0-1} \sum_{l'=0}^{L_p} b(l') a(l') r_p([-l' + l - 0.5]T_s) \]

\[ - \sum_{l'=0}^{L_0-1} \sum_{i'=-L_p+1}^{L_p} b(l') a(i' + l') r_p([i' - 0.5]T_s) - \sum_{l=0}^{L_0-1} \sum_{l'=0}^{L_p} b(l') a(l') r_p([-l' + l - 0.5]T_s). \]

Using the substitution \( l' = l + i \) with the inner summation of the first term of \( \Delta_3 \), \( \Delta_3 \) may be
Figure B.7: A graphical representation of reorganizing the double-summation of the second term in (B.3) for the case $L_0 < L_p$. 
written as

\[ \Delta_3 = \sum_{l=0}^{L_0-1} a(l) \left[ \sum_{l'=l-L_p}^{l+L_p-1} b(l') r_p([l' - l + 0.5]T_s) - \sum_{l'=0}^{L_0-1} b(l') r_p([-l' + l - 0.5]T_s) \right] \]

\[ = \sum_{l=0}^{L_0-1} a(l) \left[ \sum_{l'=l-L_p}^{l+L_p-1} b(l') r_p([l' - l + 0.5]T_s) + \sum_{l'=0}^{L_0-1} b(l') r_p([l' - l + 0.5]T_s) \right. \\
+ \left. \sum_{l'=L_0}^{l+L_p-1} b(l') r_p([l' - l + 0.5]T_s) - \sum_{l'=0}^{L_0-1} b(l') r_p([-l' + l - 0.5]T_s) \right] \]  \hspace{1cm} (B.44)

If the pulse shape is symmetric, then by (B.8) the second and fourth terms inside the square brackets of (B.44) sum to zero so that

\[ \Delta_3 = \sum_{l=0}^{L_0-1} a(l) \sum_{l'=l-L_p}^{l+L_p-1} b(l') r_p([l' - l + 0.5]T_s) + \sum_{l=0}^{L_0-1} a(l) \sum_{l'=L_0}^{l+L_p-1} b(l') r_p([l' - l + 0.5]T_s). \]  \hspace{1cm} (B.45)

Substituting (B.45) into (B.43) gives:

\[ X = \sum_{l=0}^{L_0-1} \sum_{l'=l-L_p}^{l+L_p-1} a(l) b(l') r_p([l' - l + 0.5]T_s) - \sum_{l=0}^{L_0-1} \sum_{l'=l-L_p}^{l+L_p-1} b(l') a(\cdot + l') r_p([\cdot - 0.5]T_s) \]

\[ + \sum_{l=0}^{L_0-1} \sum_{l'=L_0}^{l+L_p-1} a(l) b(l') r_p([l' - l + 0.5]T_s) - \sum_{l=0}^{L_0-1} \sum_{l'=l-L_p}^{l+L_p-1} b(l') a(\cdot + l') r_p([\cdot - 0.5]T_s). \]  \hspace{1cm} (B.46)

As with Case 1, there are two ways to re-express (B.46). The first way uses a common indexing convention for the symbols and makes explicit the symbols involved in the interference term. This is useful in computing the variance of \( X \) in Section B.2. (The variance of \( X \) is required to compute the variance of the estimation error in Section 3.2.3.) The second way uses a common indexing convention for the pulse shape autocorrelation function \( r_p(\cdot) \) thus making explicit the values of \( r_p(\cdot) \) involved in the edge effects.

The form involving single variables as symbol indexes requires modifications to the second and fourth terms on the right-hand side of (B.46). Using the substitution \( l = i' + l' \), the inner
B.1. ALTERNATE FORMS

The summation of the second term of (B.46) may be written as

\[
\sum_{i'=-L_p+1}^{-l'-1} a(i' + l') r_p([i' - 0.5] T_s) = \sum_{l=-L_p+l'+1}^{1} a(l) r_p([l - l' - 0.5] T_s) \tag{B.47}
\]

and, using the same substitution, the inner summation of the fourth term of (B.46) may be written as

\[
\sum_{i'=-l'+L_0}^{L_p} a(i' + l') r_p([i' - 0.5] T_s) = \sum_{l=L_0}^{l'+L_p} a(l) r_p([l - l' - 0.5] T_s). \tag{B.48}
\]

Substituting gives the desired result:

\[
\mathcal{X} = \sum_{l=0}^{L_0-1} \sum_{l'=l-L_p}^{-1} a(l) b(l') r_p([l' - l + 0.5] T_s) - \sum_{l=0}^{L_0-1} \sum_{l'=0}^{-L_p+l'+1} b(l') a(l) r_p([l - l' - 0.5] T_s)
\]

\[
+ \sum_{l=0}^{L_0-1} \sum_{l'=L_0}^{l-L_p-1} a(l) b(l') r_p([l' - l + 0.5] T_s) - \sum_{l=0}^{L_0-1} \sum_{l'=L_0}^{l+L_p} b(l') a(l) r_p([l - l' - 0.5] T_s). \tag{B.49}
\]

The interference due to the data symbols preceding the pilot symbols involve the symbols \(a(-L_p + 1), \ldots, a(L_0 - 1)\) and \(b(-L_p), \ldots, b(L_0 - 1)\): in other words, the \(L_p - 1\) \(a\)'s and the \(L_p\) \(b\)'s preceding the pilot symbols. The edge effects due to the data symbols following the pilot symbols involve the symbols \(a(0), \ldots, a(L_0 + L_p - 1)\) and \(b(0), \ldots, b(L_0 + L_p - 2)\): in other words, the \(L_p\) \(a\)'s and the \(L_p - 1\) \(b\)'s following the pilot symbols. The vector/matrix version of (B.49) is given by (B.50) below. This form is useful for computing \(\mathcal{X}\) in Matlab®.

The form involving single a single variable indexing the pulse shape autocorrelation function requires modifications to the first and third terms on the right-hand side of (B.46). Using the substitution \(i = l' - l + 1\) the inner summation in the first sum on the right-hand side of (B.46) may be expressed as

\[
\sum_{l'=l-L_p}^{l'-1} b(l') r_p([l' - l + 0.5] T_s) = \sum_{i=-L_p+1}^{-l} b(l + i - 1) r_p([i - 0.5] T_s). \tag{B.51}
\]

and, using the same substitution, the inner summation in the third sum on the right-hand side of
APPENDIX B. THE INTERFERENCE TERM IN OQPSK

\[
\begin{align*}
&\begin{bmatrix}
(I - dT + oT)\nu \\
(I - dT + oT)\nu \\
(I - dT + oT)\nu \\
(0)\nu
\end{bmatrix}
\begin{bmatrix}
(\tau L[\tau^0 - d\tau])^d_{\nu} & (\tau L[\tau^1 - d\tau])^d_{\nu} & \cdots & (\tau L[\tau^0 - o\tau])^d_{\nu} \\
0 & (\tau L[\tau^0 - d\tau])^d_{\nu} & \cdots & (\tau L[\tau^1 - o\tau])^d_{\nu} \\
\vdots & \vdots & & \vdots \\
0 & \cdots & & (\tau L[\tau^0 - o\tau])^d_{\nu}
\end{bmatrix}
- \begin{bmatrix}
(I - o\tau)\eta & \cdots & (0)\eta \\
(0)\eta & \cdots & \cdots & (0)\eta
\end{bmatrix}
+ \begin{bmatrix}
(I - o\tau)\eta & \cdots & (0)\eta \\
(0)\eta & \cdots & \cdots & (0)\eta
\end{bmatrix}
= \chi
\end{align*}
\]
(B.46) may be expressed as

\[ \sum_{l'=L_0}^{l+L_p-1} b(l') r_p([l' - l + 0.5]T_s) = \sum_{i=L_0-l+1}^{L_p} b(l + i - 1) r_p([i - 0.5]T_s). \] (B.52)

Substituting gives

\[ \mathcal{X} = \sum_{l=0}^{L_0-1} \sum_{i=-L_p+1}^{-l} a(l) b(l + i - 1) r_p([i - 0.5]T_s) - \sum_{l'=0}^{L_0-1} \sum_{i'=L_p+1}^{-l'-1} b(l') a(i' + l') r_p([i' - 0.5]T_s) 
\]

\[ + \sum_{l=0}^{L_0-1} \sum_{i=L_0-l+1}^{L_p} a(l) b(l + i - 1) r_p([i - 0.5]T_s) - \sum_{l'=0}^{L_0-1} \sum_{i'=L_p+1}^{-l'-1} b(l') a(i' + l') r_p([i' - 0.5]T_s). \] (B.53)

Additional insight is gained by exchanging the order of summation in each of the four terms of (B.53). With the aid of Figure B.8, the first term of (B.53) may be rewritten as

\[ \sum_{l=0}^{L_0-1} \sum_{i=-L_p+1}^{-l} a(l) b(l + i - 1) r_p([i - 0.5]T_s) = -\sum_{i=0}^{L_0} \sum_{l=0}^{l+L_p-1} a(l) b(l + i - 1) r_p([i - 0.5]T_s) 
\]

\[ + \sum_{i=0}^{L_0} \sum_{l=0}^{L_p} a(l) b(l + i - 1) r_p([i - 0.5]T_s). \] (B.54)

With the aid of Figure B.9, the second term of (B.53) may be rewritten as

\[ \sum_{l=0}^{L_0-1} \sum_{i=-L_p+1}^{-l} b(l) a(l + i) r_p([i - 0.5]T_s) = -\sum_{l=0}^{L_0} \sum_{i=-L_p+1}^{-l} b(l) a(l + i) r_p([i - 0.5]T_s) 
\]

\[ + \sum_{l=0}^{L_0} \sum_{i=-L_0+1}^{-l-1} b(l) a(l + i) r_p([i - 0.5]T_s). \] (B.55)

The component of \( \mathcal{X} \) due to the data symbols preceding the pilot symbols is the difference between (B.54) and (B.55):

\[ -\sum_{i=-L_p+1}^{L_0} \sum_{l=0}^{L_0-1} a(l) b(l + i - 1) r_p([i - 0.5]T_s) + \sum_{i=-L_0+1}^{0} \sum_{l=0}^{-i} a(l) b(l + i - 1) r_p([i - 0.5]T_s) \]
\[ \sum_{l=0}^{L_0-1} \sum_{i=-L_0+1}^{L_0-1} a(l) b(i) + l - 1 - 1) \tau_l (i - 0.5 \tau_s) \]

Figure B.8: A visual representation of changing the order of summation in the first term of (B.53).
Figure B.9: A visual representation of changing the order of summation in the second term of (B.53).
\[- \sum_{i=\mathcal{L}}^{\mathcal{L}-1} \sum_{l=0}^{\mathcal{L}-1} b(l)a(l+i)r_p([i - 0.5]T_s) - \sum_{i=\mathcal{L}}^{\mathcal{L}-1} \sum_{l=0}^{\mathcal{L}-1} b(l)a(l + i)r_p([i - 0.5]T_s) \]

\[
= \sum_{i=\mathcal{L}}^{\mathcal{L}-1} \sum_{l=0}^{\mathcal{L}-1} a(l)b(i + l - 1)r_p([i - 0.5]T_s) - \sum_{i=\mathcal{L}}^{\mathcal{L}-1} \sum_{l=0}^{\mathcal{L}-1} b(l)a(l + i)r_p([i - 0.5]T_s) + \sum_{i=\mathcal{L}}^{\mathcal{L}-1} \sum_{l=0}^{\mathcal{L}-1} a(l)b(i + l - 1) - \sum_{i=\mathcal{L}}^{\mathcal{L}-1} \sum_{l=0}^{\mathcal{L}-1} b(l)a(l + i) \]

\[
= \sum_{i=\mathcal{L}}^{\mathcal{L}-1} \left( \sum_{l=0}^{\mathcal{L}-1} a(l)b(i + l - 1) - \sum_{l=0}^{\mathcal{L}-1} b(l)a(l + i) \right) r_p([i - 0.5]T_s) + a(0)b(-1)r_p(-0.5T_s) \]

\[
= \sum_{i=\mathcal{L}}^{\mathcal{L}-1} \left( \sum_{l=0}^{\mathcal{L}-1} a(l)b(i + l - 1) - \sum_{l=0}^{\mathcal{L}-1} b(l)a(l + i) \right) r_p([i - 0.5]T_s) + a(0)b(-1)r_p(-0.5T_s) \]

\[
+ \sum_{i=\mathcal{L}}^{\mathcal{L}-1} a(-i)b(-1) + \sum_{l=0}^{\mathcal{L}-1} \left( a(l)b(i + l - 1) - b(l)a(l + i) \right) r_p([i - 0.5]T_s). \tag{B.56} \]

Moving on the the terms involving the data symbols following the pilot symbols, using the diagram of Figure B.10 shows that exchanging the order of summation in the third term of (B.53) gives

\[
\sum_{l=0}^{\mathcal{L}} \sum_{i=\mathcal{L}+l+1}^{\mathcal{L}+l} a(l)b(i + l - 1)r_p([i - 0.5]T_s) = \sum_{i=2}^{\mathcal{L}} \sum_{l=\mathcal{L}+i+1}^{\mathcal{L}+i} a(l)b(i + l - 1)r_p([i - 0.5]T_s) \]

\[
+ \sum_{i=\mathcal{L}+1}^{\mathcal{L}} \sum_{l=0}^{\mathcal{L}-1} a(l)b(i + l - 1)r_p([i - 0.5]T_s). \tag{B.57} \]

With the aid of Figure B.11, exchanging the order of summation in the fourth term of (B.53) gives
Figure B.10: A visual representation of changing the order of summation in the third term of (B.53).
Figure B.11: A visual representation of changing the order of summation in the fourth term of (B.53).
\[ \sum_{i=0}^{L_0-1} \sum_{l=0}^{L_p} b(l) a(i + l) r_p(p[i - 0.5] T_s) = \sum_{i=1}^{L_0-1} \sum_{l=L_0-i}^{L_0-1} b(l) a(i + l) r_p(p[i - 0.5] T_s) \]

\[ + \sum_{i=L_0+1}^{L_0-1} \sum_{l=0}^{L_0-1} b(l) a(i + l) r_p(p[i - 0.5] T_s). \] (B.58)

The component in \( \mathcal{X} \) due to the data symbols following the pilot symbols is the difference between (B.57) and (B.58):

\[ \sum_{i=2}^{L_0} \sum_{l=L_0-i+1}^{L_0-1} a(l) b(i + l - 1) r_p(p[i - 0.5] T_s) + \sum_{i=L_0+1}^{L_0} \sum_{l=0}^{L_0-1} a(l) b(i + l - 1) r_p(p[i - 0.5] T_s) \]

\[ - \sum_{i=1}^{L_0} \sum_{l=L_0-i}^{L_0-1} b(l) a(i + l) r_p(p[i - 0.5] T_s) - \sum_{i=L_0+1}^{L_0} \sum_{l=0}^{L_0-1} b(l) a(i + l) r_p(p[i - 0.5] T_s) \]

\[ = \sum_{i=2}^{L_0} \sum_{l=L_0-i+1}^{L_0-1} a(l) b(i + l - 1) r_p(p[i - 0.5] T_s) - \sum_{i=1}^{L_0} \sum_{l=L_0-i}^{L_0-1} b(l) a(i + l) r_p(p[i - 0.5] T_s) \]

\[ + \sum_{i=L_0+1}^{L_0} \sum_{l=0}^{L_0-1} a(l) b(i + l - 1) r_p(p[i - 0.5] T_s) - \sum_{i=L_0+1}^{L_0} \sum_{l=0}^{L_0-1} b(l) a(i + l) r_p(p[i - 0.5] T_s). \]

\[ = -b(L_0 - 1) a(L_0) r_p(0.5 T_s) \]

\[ + \sum_{i=2}^{L_0} \left( \sum_{l=L_0-i+1}^{L_0-1} a(l) b(i + l - 1) - \sum_{l=L_0-i}^{L_0-1} b(l) a(i + l) \right) r_p(p[i - 0.5] T_s) \]

\[ + \sum_{i=L_0+1}^{L_0} \left( \sum_{l=0}^{L_0-1} a(l) b(i + l - 1) - \sum_{l=0}^{L_0-1} b(l) a(i + l) \right) r_p(p[i - 0.5] T_s). \]

\[ = -b(L_0 - 1) a(L_0) r_p(0.5 T_s) \]

\[ + \sum_{i=2}^{L_0} \left( \sum_{l=L_0-i+1}^{L_0-1} a(l) b(i + l - 1) - b(l) a(i + l) \right) r_p(p[i - 0.5] T_s) \]

\[ + \sum_{i=L_0+1}^{L_0} \left( \sum_{l=0}^{L_0-1} a(l) b(i + l - 1) - b(l) a(i + l) \right) r_p(p[i - 0.5] T_s). \] (B.59)
In summary, assembling (B.56) and (B.59) gives

\[
X = \sum_{i=-L_p+1}^{-L_0} \left( \sum_{l=0}^{L_0-1} \left[ a(l) b(i+l-1) - b(l) a(l+i) \right] \right) r_p([i - 0.5] T_s) + a(0) b(-1) r_p(-0.5 T_s) \\
+ \sum_{i=-L_0+1}^{-1} \left( a(-i) b(-1) + \sum_{l=0}^{-i-1} \left[ a(l) b(i+l-1) - b(l) a(l+i) \right] \right) r_p([i - 0.5] T_s)
\]

interference due to data symbols preceding the pilot symbols

\[
+ \sum_{i=2}^{L_0} \left( \sum_{l=L_0-i+1}^{L_0-1} \left[ a(l) b(i+l-1) - b(l) a(i+l) \right] - b(L_0-i) a(L_0) \right) r_p([i - 0.5] T_s)
- b(L_0-1) a(L_0) r_p(0.5 T_s) + \sum_{i=L_0+1}^{L_p} \left( \sum_{l=0}^{L_0-1} \left[ a(l) b(i+l-1) - b(l) a(i+l) \right] \right) r_p([i - 0.5] T_s)
\]

interference due to data symbols following the pilot symbols

(B.60)

### B.2 Variance of the Interference Term Under The Assumption of IID Data Symbols

The variance of $X$, denoted

\[
\sigma_X^2 = \mathbb{E} \left[ X^2 \right], \tag{B.61}
\]

shows up in the analysis of the estimator error variance. An expression for $\sigma_X^2$ is possible when the data symbols are assumed independent and equally likely. More formally, we assume for $l, k < 0$ or $l, k \geq L_0$

\[
\mathbb{E}[a(k)a(l)] = A^2 \delta(l - k) \\
\mathbb{E}[b(k)b(l)] = A^2 \delta(l - k) \tag{B.62}
\]

\[
\mathbb{E}[a(k)b(l)] = 0.
\]

The mathematical expression for $\sigma_X^2$ depends on the relationship between $L_p$ and $L_0$. In the following, we derive expressions for $\sigma_X^2$ for the case where the pilot symbols are arranged in a preamble as shown in Figure 1.1 (b). Mathematical expressions for the other two arrangements in Figure 1.1 are special cases.
B.2. VARIANCE OF THE INTERFERENCE TERM

B.2.1 Cases 1 and 2: \( L_0 \geq L_p \)

Here, \( \mathcal{X} \) is given by (B.17). Clearly, under the given assumptions,

\[
E[\mathcal{X}] = 0. \tag{B.63}
\]

Consequently, the variance of \( \mathcal{X} \) is the mean square value. Squaring \( \mathcal{X} \) and applying expectation gives

\[
E[\mathcal{X}^2] = \sum_{i_1=0}^{L_p-1} \sum_{i_2=1-L_p}^{L_p-1} \sum_{i_3=0}^{L_p-2} \sum_{i_4=0}^{L_p-1} E[a(i_1)b(i_2)a(i'_3)b(i'_2)] r_p([i_2-i_1+0.5]T_s) r_p([i'_2-i'_1+0.5]T_s)
\]

\[
- 2 \sum_{i_2=1-L_p}^{L_p-1} \sum_{i_3=0}^{L_p-2} \sum_{i_4=0}^{L_p-1} E[a(i_1)b(i_2)b(i_3)a(i_4)] r_p([i_2-i_1+0.5]T_s) r_p([i_4-i_3-0.5]T_s)
\]

\[
+ \sum_{i_3=0}^{L_p-2} \sum_{i_4=0}^{L_p-1} E[b(i_3)a(i_4)b(i'_3)a(i'_4)] r_p([i_4-i_3-0.5]T_s) r_p([i'_4-i'_3-0.5]T_s)
\]

\[
+ 2 \sum_{i_1=0}^{L_p-1} \sum_{i_2=1-L_p}^{L_p-1} \sum_{i_3=0}^{L_p-2} \sum_{i_5=L_0-L_p}^{L_p-1} E[a(i_1)b(i_2)a(i_5)b(i_6)] r_p([i_2-i_1+0.5]T_s) r_p([i_6-i_5+0.5]T_s)
\]

\[
- 2 \sum_{i_2=1-L_p}^{L_p-1} \sum_{i_5=L_0-L_p}^{L_p-1} \sum_{i_6=L_0}^{L_p-1} E[a(i_1)b(i_2)b(i_7)a(i_8)] r_p([i_2-i_1+0.5]T_s) r_p([i_8-i_7-0.5]T_s)
\]

\[
- 2 \sum_{i_3=0}^{L_p-2} \sum_{i_5=L_0-L_p}^{L_p-1} \sum_{i_6=L_0}^{L_p-1} E[b(i_3)a(i_4)a(i_5)b(i_6)] r_p([i_4-i_3-0.5]T_s) r_p([i_6-i_5+0.5]T_s)
\]

\[
+ 2 \sum_{i_3=0}^{L_p-2} \sum_{i_5=L_0-L_p}^{L_p-1} \sum_{i_6=L_0}^{L_p-1} E[b(i_3)a(i_4)b(i_7)a(i_8)] r_p([i_4-i_3-0.5]T_s) r_p([i_8-i_7-0.5]T_s)
\]

\[
+ \sum_{i_5=L_0-L_p}^{L_p-1} \sum_{i_6=L_0}^{L_p-1} E[a(i_5)b(i_6)a(i'_5)b(i'_6)] r_p([i_6-i_5+0.5]T_s) r_p([i'_6-i'_5+0.5]T_s)
\]

\[
- 2 \sum_{i_5=L_0-L_p}^{L_p-1} \sum_{i_6=L_0}^{L_p-1} E[a(i_5)b(i_6)b(i_7)a(i_8)] r_p([i_6-i_5+0.5]T_s) r_p([i_8-i_7-0.5]T_s)
\]

\[
+ \sum_{i_7=L_0-L_p}^{L_p-1} E[b(i_7)a(i_8)b(i'_7)a(i'_8)] r_p([i_8-i_7-0.5]T_s) r_p([i'_8-i'_7-0.5]T_s). \tag{B.64}
\]
The ten terms of (B.64) are simplified as follows.

**The First Term of (B.64)** Because

\[ 0 \leq i_1 < L_0, \quad i_2 < 0, \quad 0 \leq i'_1 < L_0, \quad i'_2 < 0, \]

the only random quantities involved in the expectation are \( b(i_2) \) and \( b(i'_2) \) [\( i_1 \) and \( i'_1 \) always index pilot symbols]. Consequently,

\[
E[a(i_1)b(i_2)a(i'_1)b(i'_2)] = a(i_1)a(i'_1)E[b(i_2)b(i'_2)] = A^2a(i_1)a(i'_1)\delta(i'_2 - i_2).
\]

Incorporating this into the first term of (B.64) gives

\[
\text{first term of (B.64) = } \sum_{i_1=0}^{L_p-1} \sum_{i_2=i_1-L_p}^{L_p-1} \sum_{i'_1=0}^{L_p-1} \sum_{i'_2=i'_1-L_p}^{L_p-1} A^2a(i_1)a(i'_1)\delta(i'_2 - i_2)r_p([i_2 - i_1 + 0.5]T_s)r_p([i'_2 - i'_1 + 0.5]T_s) = A^2 \sum_{i_1=0}^{L_p-1} \sum_{i'_1=0}^{L_p-1} \sum_{i_2=\max\{i_1,i'_1\}-L_p}^{L_p-1} a(i_1)a(i'_1)r_p([i_2 - i_1 + 0.5]T_s)r_p([i_2 - i'_1 + 0.5]T_s). \tag{B.65}
\]

**The Second Term of (B.64)** Because

\[ 0 \leq i_1 < L_0, \quad i_2 < 0, \quad 0 \leq i_3 < L_0, \quad i_4 < 0, \]

the only random quantities are \( b(i_2) \) and \( a(i_4) \) [\( i_1 \) and \( i_3 \) always index pilot symbols]. Consequently,

\[
E[a(i_1)b(i_2)b(i_3)a(i_4)] = a(i_1)b(i_3)E[b(i_2)a(i_4)] = 0.
\]

This shows that

\[
\text{second term of (B.64) = 0.} \tag{B.66}
\]

**The Third Term of (B.64)** Because

\[ 0 \leq i_3 < L_0, \quad i_4 < 0, \quad 0 \leq i'_3 < L_0, \quad i'_4 < 0, \]
B.2. VARIANCE OF THE INTERFERENCE TERM

The only random quantities involved in the expectation are $a(i_4)$ and $a(i'_4)$. Consequently,

$$
E[b(i_3)a(i_4)b(i'_3)a(i'_4)] = b(i_3)b(i'_3)E[a(i_4)a(i'_4)] = A^2 b(i_3)b(i'_3)\delta(i'_4 - i_4).
$$

Incorporating this into the third term of (B.64) gives

$$
\text{third term of (B.64)} = A^2 \sum_{i_3=0}^{L_p-2} \sum_{i_4=-L_p+i_3+1}^{-1} \sum_{i'_3=0}^{L_p-2} \sum_{i'_4=-L_p+i'_3+1}^{-1} b(i_3)b(i'_3)\delta(i'_4 - i_4)r_p([i_4 - i_3 - 0.5]T_s)r_p([i'_4 - i'_3 - 0.5]T_s).
$$

The Fourth Term of (B.64) Because

$$
0 \leq i_1 < L_0, \quad i_2 < 0, \quad 0 < i_5 < L_0, \quad i_6 \geq L_0,
$$

the only random quantities involved in the expectation are $b(i_2)$ and $b(i_6)$. Consequently,

$$
E[a(i_1)b(i_2)a(i_5)b(i_6)] = a(i_1)a(i_5)E[b(i_2)b(i_6)] = A^2 a(i_1)a(i_5)\delta(i_6 - i_2) = 0
$$

because $i_2$ and $i_6$ never coincide. This shows that

$$
fourth \text{ term of (B.64)} = 0.
$$

The Fifth Term of (B.64) Because

$$
0 \leq i_1 < L_0, \quad i_2 < 0, \quad 0 \leq i_7 < L_0, \quad i_8 \geq L_0,
$$

the only random quantities involved in the expectation are $b(i_2)$ and $a(i_8)$. Consequently,

$$
E[a(i_1)b(i_2)b(i_7)a(i_8)] = a(i_1)b(i_2)E[b(i_7)a(i_8)] = 0.
$$

This shows that

$$
fifth \text{ term of (B.64)} = 0.
$$
The Sixth Term of (B.64)  Because

\[ 0 \leq i_3 < L_0, \quad i_4 < 0, \quad 0 < i_5 < L_0, \quad i_6 \geq L_0, \]

the only random quantities involved in the expectation are \( a(i_4) \) and \( b(i_6) \). Consequently,

\[
E \left[ b(i_3) a(i_4) a(i_5) b(i_6) \right] = b(i_3) a(i_5) E [a(i_4) b(i_6)] = 0.
\]

This shows that

sixth term of (B.64) = 0. \hfill (B.70)

The Seventh Term of (B.64)  Because

\[ 0 \leq i_3 < L_0, \quad i_4 < 0, \quad 0 < i_7 < L_0, \quad i_8 \geq L_0, \]

the only random quantities involved in the expectation are \( a(i_4) \) and \( a(i_8) \). Consequently,

\[
E \left[ b(i_3) a(i_4) b(i_7) a(i_8) \right] = b(i_3) b(i_7) E [a(i_4) a(i_8)] = A^2 b(i_3) b(i_7) \delta(i_8 - i_4) = 0
\]

because \( i_4 \) and \( i_8 \) never coincide. This shows that

seventh term of (B.64) = 0. \hfill (B.71)

The Eighth Term of (B.64)  Because

\[ 0 \leq i_5 < L_0, \quad i_6 < 0, \quad 0 < i_6' < L_0, \quad i_6' \geq L_0, \]

the only random quantities involved in the expectation are \( b(i_6) \) and \( b(i_6') \). Consequently,

\[
E \left[ a(i_5) b(i_6) a(i_6') b(i_6') \right] = a(i_5) a(i_6') E [b(i_6) b(i_6')] = A^2 a(i_5) a(i_6') \delta(i_6' - i_6).
\]

Incorporating this into the eighth term of (B.64) gives

\[
\text{eighth term of (B.64)} = A^2 \sum_{i_5=L_0-L_p+1}^{L_0-1} \sum_{i_6=L_0}^{i_5+L_p-1} \sum_{i_6'=L_0-L_p+1}^{L_0-1} \sum_{i_6'=L_0}^{i_6'+L_p-1} a(i_5) a(i_6') \delta(i_6' - i_6) r_p \left( [i_6 - i_5 + 0.5]T_s \right) r_p \left( [i_6' - i_6' + 0.5]T_s \right)
\]
B.2. VARIANCE OF THE INTERFERENCE TERM

\[ A^2 \sum_{i_5 = L_0 - L_p + 1}^{L_0 - 1} \sum_{i_5' = L_0 - L_p + 1}^{L_0 - 1} \sum_{i_6 = L_0}^{\min \{i_5, i_5'\} + L_p - 1} a(i_5)a(i_5')r_p([i_6 - i_5 + 0.5]T_s)r_p([i_6' - i_5' + 0.5]T_s). \]  

(B.72)

The Ninth Term of (B.64)  Because

\[ 0 \leq i_5 < L_0, \quad i_6 < 0, \quad 0 \geq i_7 < L_0, \quad i_8 \geq L_0, \]

the only random quantities involved in the expectation are \(b(i_6)\) and \(a(i_8)\). Consequently,

\[ E \left[ a(i_5)b(i_6)a(i_7) \right] = a(i_5)b(i_7)E \left[ b(i_6)a(i_8) \right] = 0. \]

This shows that

\[ \text{ninth term of (B.64)} = 0. \]  

(B.73)

The Tenth Term of (B.64)  Because

\[ 0 \leq i_7 < L_0, \quad i_8 < 0, \quad 0 \geq i_7' < L_0, \quad i_8' \geq L_0, \]

the only random quantities involved in the expectation are \(a(i_8)\) and \(a(i_8')\). Consequently,

\[ E \left[ b(i_7)a(i_8)b(i_7')a(i_8') \right] = b(i_7)b(i_7')E \left[ a(i_8)a(i_8') \right] = A^2b(i_7)b(i_7')\delta(i_8' - i_8). \]

Incorporating this into the tenth term of (B.64) gives

\[ \text{tenth term of (B.64)} = \]

\[ A^2 \sum_{i_7 = L_0 - L_p}^{L_0 - 1} \sum_{i_8 = L_0}^{i_7 + L_p} \sum_{i_8' = L_0}^{i_7' + L_p} \sum_{i_5 = L_0}^{\min \{i_7, i_7'\} + L_p} b(i_7)b(i_7')\delta(i_8' - i_8)r_p([i_8 - i_7 - 0.5]T_s)r_p([i_8' - i_7' - 0.5]T_s) \]

= \[ A^2 \sum_{i_7 = L_0 - L_p}^{L_0 - 1} \sum_{i_8 = L_0}^{L_0 - 1} \sum_{i_8' = L_0}^{\min \{i_7, i_7'\} + L_p} b(i_7)b(i_7')r_p([i_8 - i_7 - 0.5]T_s)r_p([i_8' - i_7' - 0.5]T_s). \]  

(B.74)

Putting It All Together  The final result is the sum of (B.65) – (B.74):
\[ \mathbb{E}[\mathcal{X}^2] = A^2 \left[ \sum_{i_2 = 0}^{L_p-1} \sum_{i_2^0 = \text{max}(i_1,i_2')} - 1 \sum_{i_2 = 0}^{L_p-1} \sum_{i_2^0 = \text{max}(i_1,i_2')} - 1 \sum_{i_1 = 0}^{L_p-1} \sum_{i_1^0 = 0}^{L_p-1} - 1\right. \]

\[+ \sum_{i_3 = 0}^{L_p-1} \sum_{i_3^0 = 0}^{L_p-1} \sum_{i_4 = -L_p + \text{max}(i_3,i_3')} + 1 \sum_{i_4 = 0}^{L_p-1} \sum_{i_4^0 = \text{max}(i_3,i_3')} + 1 \sum_{i_6 = 0}^{L_p-1} \sum_{i_6^0 = \text{max}(i_5,i_6')} + 1 \sum_{i_6 = 0}^{L_p-1} \sum_{i_6^0 = \text{max}(i_5,i_6')} + 1 \sum_{i_7 = 0}^{L_p-1} \sum_{i_7^0 = \text{max}(i_6,i_7')} + 1 \sum_{i_7 = 0}^{L_p-1} \sum_{i_7^0 = \text{max}(i_6,i_7')} + 1 \sum_{i_8 = 0}^{L_p-1} \sum_{i_8^0 = \text{max}(i_7,i_8')} + 1 \sum_{i_8 = 0}^{L_p-1} \sum_{i_8^0 = \text{max}(i_7,i_8')} + 1 \sum_{i_9 = 0}^{L_p-1} \sum_{i_9^0 = \text{max}(i_8,i_9')} + 1 \sum_{i_9 = 0}^{L_p-1} \sum_{i_9^0 = \text{max}(i_8,i_9')} + 1 \sum_{i_10 = 0}^{L_p-1} \sum_{i_10^0 = \text{max}(i_9,i_10')} + 1 \sum_{i_10 = 0}^{L_p-1} \sum_{i_10^0 = \text{max}(i_9,i_10')} + 1 \sum_{i_11 = 0}^{L_p-1} \sum_{i_11^0 = \text{max}(i_10,i_11')} + 1 \sum_{i_11 = 0}^{L_p-1} \sum_{i_11^0 = \text{max}(i_10,i_11')} + 1 \sum_{i_12 = 0}^{L_p-1} \sum_{i_12^0 = \text{max}(i_11,i_12')} + 1 \sum_{i_12 = 0}^{L_p-1} \sum_{i_12^0 = \text{max}(i_11,i_12')} + 1 \sum_{i_13 = 0}^{L_p-1} \sum_{i_13^0 = \text{max}(i_12,i_13')} + 1 \sum_{i_13 = 0}^{L_p-1} \sum_{i_13^0 = \text{max}(i_12,i_13')} + 1 \sum_{i_14 = 0}^{L_p-1} \sum_{i_14^0 = \text{max}(i_13,i_14')} + 1 \sum_{i_14 = 0}^{L_p-1} \sum_{i_14^0 = \text{max}(i_13,i_14')} + 1 \left. \right] \]
B.2. VARIANCE OF THE INTERFERENCE TERM

\[ + 2 \sum_{i_3=0}^{L_0-1} \sum_{i_4=-L_p+i_3+1}^{i_3} \sum_{i_8=L_0} \sum_{i_8=0}^{L_0-1} [i_8 - i_3 - 0.5] T_s r_p([i_8 - i_7 - 0.5] T_s) \]

The ten terms of (B.77) are simplified as follows.

The First Term of (B.77)  Because

\[ 0 \leq i_1 < L_0, \quad i_2 < 0, \quad 0 \leq i'_1 < L_0, \quad i'_2 < 0, \]

the only random quantities involved in the expectation are \( b(i_2) \) and \( b(i'_2) \) \([i_1 \text{ and } i'_1 \text{ always index pilot symbols}]. \) Consequently,

\[ E[a(i_1)b(i_2)a(i'_1)b(i'_2)] = a(i_1)a(i'_1)E[b(i_2)b(i'_2)] = A^2a(i_1)a(i'_1)\delta(i'_2 - i_2). \]

Incorporating this into the first term of (B.77) gives

First term of (B.77) =

\[ \sum_{i_1=0}^{L_0-1} \sum_{i_2=-L_p}^{i_1} \sum_{i'_1=0}^{i_1-L_p} \sum_{i'_2=-L_p}^{i'_1-L_p} A^2a(i_1)a(i'_1)\delta(i'_2 - i_2) r_p([i_2 - i_1 + 0.5] T_s) r_p([i'_2 - i'_1 + 0.5] T_s) \]

\[ = A^2 \sum_{i_1=0}^{L_0-1} \sum_{i'_1=0}^{i_1-L_p} \sum_{i_2=\max\{i_1,i'_1\}-L_p}^{L_0-1} a(i_1)a(i'_1)r_p([i_2 - i_1 + 0.5] T_s) r_p([i_2 - i'_1 + 0.5] T_s). \quad (B.78) \]

The Second Term of (B.77)  Because

\[ 0 \leq i_1 < L_0, \quad i_2 < 0, \quad 0 \leq i_3 < L_0, \quad i_4 < 0, \]
the only random quantities are \( b(i_2) \) and \( a(i_4) \) \([i_1 \text{ and } i_3 \text{ always index pilot symbols}]\). Consequently,

\[
E[a(i_1)b(i_2)b(i_3)a(i_4)] = a(i_1)b(i_3)E[b(i_2)a(i_4)] = 0.
\]

This shows that

second term of (B.77) = 0. \hspace{1cm} (B.79)

The Third Term of (B.77)  
Because

\[
0 \leq i_3 < L_0, \quad i_4 < 0, \quad 0 \leq i'_3 < L_0, \quad i'_4 < 0,
\]

the only random quantities involved in the expectation are \( a(i_4) \) and \( a(i'_4) \). Consequently,

\[
E[b(i_3)a(i_4)b(i'_3)a(i'_4)] = b(i_3)b(i'_3)E[a(i_4)a(i'_4)] = A^2b(i_3)b(i'_3)\delta(i'_4 - i_4).
\]

Incorporating this into the third term of (B.77) gives

third term of (B.77) =

\[
A^2 \sum_{i_3=0}^{L_0-1} \sum_{i'_3=0}^{L_0-1} \sum_{i_4=-L_p+i_3+1}^{L_0-1} \sum_{i'_4=-L_p+i'_3+1} \; b(i_3)b(i'_3)\delta(i'_4 - i_4)r_p([i_4 - i_3 - 0.5]T_s) r_p([i'_4 - i'_3 - 0.5]T_s).
\]

The Fourth Term of (B.77)  
Because

\[
0 \leq i_1 < L_0, \quad i_2 < 0, \quad 0 < i_5 < L_0, \quad i_6 \geq L_0,
\]

the only random quantities involved in the expectation are \( b(i_2) \) and \( b(i_6) \). Consequently,

\[
E[a(i_1)b(i_2)a(i_5)b(i_6)] = a(i_1)a(i_5)E[b(i_2)b(i_6)] = A^2a(i_1)a(i_5)\delta(i_6 - i_2) = 0
\]

because \( i_2 \) and \( i_6 \) never coincide. This shows that

fourth term of (B.77) = 0. \hspace{1cm} (B.81)
The Fifth Term of (B.77) Because

\[ 0 \leq i_1 < L_0, \quad i_2 < 0, \quad 0 \leq i_7 < L_0, \quad i_8 \geq L_0, \]

the only random quantities involved in the expectation are \( b(i_2) \) and \( a(i_8) \). Consequently,

\[
E[a(i_1)b(i_2)b(i_7)a(i_8)] = a(i_1)b(i_7)E[b(i_2)a(i_8)] = 0.
\]

This shows that

fifth term of (B.77) = 0. \quad (B.82)

The Sixth Term of (B.77) Because

\[ 0 \leq i_3 < L_0, \quad i_4 < 0, \quad 0 < i_5 < L_0, \quad i_6 \geq L_0, \]

the only random quantities involved in the expectation are \( a(i_4) \) and \( b(i_6) \). Consequently,

\[
E[b(i_3)a(i_4)a(i_5)b(i_6)] = b(i_3)a(i_5)E[a(i_4)b(i_6)] = 0.
\]

This shows that

sixth term of (B.77) = 0. \quad (B.83)

The Seventh Term of (B.77) Because

\[ 0 \leq i_3 < L_0, \quad i_4 < 0, \quad 0 < i_7 < L_0, \quad i_8 \geq L_0, \]

the only random quantities involved in the expectation are \( a(i_4) \) and \( a(i_8) \). Consequently,

\[
E[b(i_3)a(i_4)b(i_7)a(i_8)] = b(i_3)b(i_7)E[a(i_4)a(i_8)] = A^2 b(i_3)b(i_7) \delta(i_8 - i_4) = 0
\]

because \( i_4 \) and \( i_8 \) never coincide. This shows that

seventh term of (B.77) = 0. \quad (B.84)

The Eighth Term of (B.77) Because

\[ 0 \leq i_5 < L_0, \quad i_6 < 0, \quad 0 < i_5' < L_0, \quad i_6' \geq L_0, \]
the only random quantities involved in the expectation are $b(i_6)$ and $b(i_6')$. Consequently,

$$E[a(i_5)b(i_6)a(i_6')b(i_6')] = a(i_5)a(i_6')E[b(i_6)b(i_6')] = A^2 a(i_5)a(i_6')\delta(i_6' - i_6).$$

Incorporating this into the eighth term of (B.77) gives

$$\text{eighth term of (B.77)} = A^2 \sum_{i_5 = 0}^{L_0 - 1} \sum_{i_6' = 0}^{L_0 - 1} \sum_{i_6 = L_0}^{L_0 - 1} \sum_{i_6'' = L_0}^{L_0 - 1} a(i_5)a(i_6')\delta(i_6' - i_6)r_p([i_6 - i_5 + 0.5]T_s)r_p([i_6' - i_5' + 0.5]T_s)$$

$$= A^2 \sum_{i_5 = 0}^{L_0 - 1} \sum_{i_6' = 0}^{L_0 - 1} \sum_{i_6 = L_0}^{L_0 - 1} a(i_5)a(i_6')r_p([i_6 - i_5 + 0.5]T_s)r_p([i_6' - i_5' + 0.5]T_s). \quad (B.85)$$

The Ninth Term of (B.77) Because

$$0 \leq i_5 < L_0, \quad i_6 < 0, \quad 0 \geq i_7 < L_0, \quad i_8 \geq L_0,$$

the only random quantities involved in the expectation are $b(i_6)$ and $a(i_8)$. Consequently,

$$E[a(i_5)b(i_6)b(i_7)a(i_8)] = a(i_5)b(i_7)E[b(i_6)a(i_8)] = 0.$$  

This shows that

$$\text{ninth term of (B.77)} = 0. \quad (B.86)$$

The Tenth Term of (B.77) Because

$$0 \leq i_7 < L_0, \quad i_8 < 0, \quad 0 \geq i_7' < L_0, \quad i_8' \geq L_0,$$

the only random quantities involved in the expectation are $a(i_8)$ and $a(i_8')$. Consequently,

$$E[b(i_7) a(i_8) b(i_7') a(i_8')] = b(i_7)b(i_7')E[a(i_8)a(i_8')] = A^2 b(i_7)b(i_7')\delta(i_8' - i_8).$$

Incorporating this into the tenth term of (B.77) gives

$$\text{tenth term of (B.77)} = \ldots$$
B.3. A RELATED FORM

\[
A^2 \sum_{i_7=0}^{L_0-1} \sum_{i_8=L_0}^{L_0-1} \sum_{i_7'=0}^{L_0} \sum_{i_8'=0}^{L_0} b(i_7)b(i_7') \delta(i_8 - i_8') r_p([i_8 - i_7 - 0.5]T_s) r_p([i_8' - i_7' - 0.5]T_s)
\]

\[
= A^2 \sum_{i_7=0}^{L_0-1} \sum_{i_8=L_0}^{L_0-1} \sum_{i_8'=L_0}^{L_0} b(i_7)b(i_7') r_p([i_8 - i_7 - 0.5]T_s) r_p([i_8' - i_7' - 0.5]T_s). \tag{B.87}
\]

B.2.3 Putting It All Together

The final result is the sum of (B.78) – (B.87):

\[
E[A^2] = A^2 \left[ \sum_{i_1=0}^{L_0-1} \sum_{i_1'=0}^{L_0-1} \sum_{i_2=2}^{L_0} a(i_1)a(i_1') r_p([i_2 - i_1 + 0.5]T_s) r_p([i_2 - i_1' + 0.5]T_s)
\]

\[
+ \sum_{i_3=0}^{L_0-1} \sum_{i_3'=0}^{L_0-1} \sum_{i_4=2}^{L_0} b(i_3)b(i_3') r_p([i_4 - i_3 - 0.5]T_s) r_p([i_4 - i_3' - 0.5]T_s)
\]

\[
+ \sum_{i_5=0}^{L_0-1} \sum_{i_5'=0}^{L_0-1} \sum_{i_6=2}^{L_0} a(i_5)a(i_5') r_p([i_6 - i_5 + 0.5]T_s) r_p([i_6 - i_5' + 0.5]T_s)
\]

\[
+ \sum_{i_7=0}^{L_0-1} \sum_{i_7'=0}^{L_0} \sum_{i_8=L_0}^{L_0} b(i_7)b(i_7') r_p([i_8 - i_7 - 0.5]T_s) r_p([i_8' - i_7' - 0.5]T_s) \right]. \tag{B.88}
\]

B.3 A Related Form

A variant of (B.1) shows up in the analysis of the additive noise term in the estimator error as described in Section 3.2.3. The variant is

\[
\mathcal{X}' = \sum_{l=0}^{L_0-1} \sum_{k'=0}^{L_0-l} a(l)b(k) r_p([k - l + 0.5]T_s) - \sum_{l'=0}^{L_0-1} \sum_{k'=0}^{L_0-l} b(l') a(k') r_p([k' - l' + 0.5]T_s) \tag{B.89}
\]

The only difference between (B.89) and (B.1) is the limits on the inner summation. As it turns out, this difference is key, because it forces \(\mathcal{X}' = 0\) for symmetric \(r_p(t)\). To see that this is so, reverse the order of the double summation of the second term on the right-hand side of (B.89):

\[
\sum_{l'=0}^{L_0-1} \sum_{k'=0}^{L_0-1} b(l') a(k') r_p([k' - l' + 0.5]T_s) = \sum_{k'=0}^{L_0-1} \sum_{l'=0}^{L_0-1} b(l') a(k') r_p([k' - l' + 0.5]T_s)
\]
For \( r_p(-[l' - k' + 0.5]T_s) = r_p([l' - k' + 0.5]T_s) \) this becomes

\[
\sum_{l'=0}^{L_0-1} \sum_{k'=0}^{L_0-1} b(l')a(k')r_p([l' - k' + 0.5]T_s) = \sum_{k'=0}^{L_0-1} \sum_{l'=0}^{L_0-1} b(l')a(k')r_p([l' - k' + 0.5]T_s). \tag{B.91}
\]

Because the indexes \( l' \) and \( k' \) are dummy variables, replace them with \( l' = k \), and \( k' = l \). The result is

\[
\sum_{k'=0}^{L_0-1} \sum_{l'=0}^{L_0-1} b(l')a(k')r_p([l' - k' + 0.5]T_s) = \sum_{l=0}^{L_0-1} \sum_{k=0}^{L_0-1} b(k)a(l)r_p([k - l + 0.5]T_s) \tag{B.92}
\]

which is equal to the first term in (B.89). Thus

\[
\chi' = \sum_{l=0}^{L_0-1} \sum_{k=0}^{L_0-1} a(l)b(k)r_p([k - l + 0.5]T_s) - \sum_{l'=0}^{L_0-1} \sum_{k'=0}^{L_0-1} b(l')a(k')r_p([k' - l' - 0.5]T_s) = 0. \tag{B.93}
\]
Appendix C

The Cramér-Rao Bound Using OQPSK

The Cramér-Rao bound is only defined for real-valued parameters. Writing $h = h_R + jh_I$, the log-likelihood function (2.9) may be expressed in terms of the real-valued parameters $h_R$ and $h_I$ as

$$\Lambda(h_R, h_I) = -\frac{1}{2N_0} \int_{T_1}^{T_2+(L_0-0.5)T_s} \left| r(t) - (h_R + jh_I) \sum_l \left[ a(l)p(t - lT_s) + jb(l)p(t - lT_s - 0.5T_s) \right] \right|^2 \, dt. \quad (C.1)$$

Let $I(t, h_R, h_I)$ be the integrand. The desired derivatives are

$$\frac{\partial^2}{\partial h_R \partial h_R} \Lambda(h_R, h_I) = -\frac{1}{2N_0} \frac{\partial^2}{\partial h_R \partial h_R} \int_{T_1}^{T_2+(L_0-0.5)T_s} I(t, h_R, h_I) \, dt$$

$$= -\frac{1}{2N_0} \int_{T_1}^{T_2+(L_0-0.5)T_s} \frac{\partial^2}{\partial h_R \partial h_R} I(t, h_R, h_I) \, dt$$

$$\frac{\partial^2}{\partial h_R \partial h_I} \Lambda(h_R, h_I) = -\frac{1}{2N_0} \frac{\partial^2}{\partial h_R \partial h_I} \int_{T_1}^{T_2+(L_0-0.5)T_s} I(t, h_R, h_I) \, dt$$

$$= -\frac{1}{2N_0} \int_{T_1}^{T_2+(L_0-0.5)T_s} \frac{\partial^2}{\partial h_R \partial h_I} I(t, h_R, h_I) \, dt$$
\[
\frac{\partial^2}{\partial h_I \partial h_R} \Lambda(h_R, h_I) = - \frac{1}{2N_0} \frac{\partial^2}{\partial h_I \partial h_R} \int_{T_1}^{T_2+(L_0-0.5)T_s} I(t, h_R, h_I) dt \\
= - \frac{1}{2N_0} \int_{T_1}^{T_2+(L_0-0.5)T_s} \frac{\partial^2}{\partial h_I \partial h_R} I(t, h_R, h_I) dt
\]

\[
\frac{\partial^2}{\partial h_I \partial h_I} \Lambda(h_R, h_I) = - \frac{1}{2N_0} \frac{\partial^2}{\partial h_I \partial h_I} \int_{T_1}^{T_2+(L_0-0.5)T_s} I(t, h_R, h_I) dt \\
= - \frac{1}{2N_0} \int_{T_1}^{T_2+(L_0-0.5)T_s} \frac{\partial^2}{\partial h_I \partial h_I} I(t, h_R, h_I) dt.
\]

This shows that the partial derivatives of the integrand are required. Expanding the integrand produces

\[
I(t, h_R, h_I) = |r(t)|^2 - (h_R - jh_I)r(t) \sum_{l} \left[ a(l)p(t - lT_s) - jb(l)p(t - lT_s - 0.5T_s) \right] \\
- (h_R + jh_I)r^*(t) \sum_{l} \left[ a(l)p(t - lT_s) + jb(l)p(t - lT_s - 0.5T_s) \right] \\
+ (h_R^2 + h_I^2) \sum_{l} \left[ a(l)p(t - lT_s) + jb(l)p(t - lT_s - 0.5T_s) \right] \\
\times \sum_{l'} \left[ a(l')p(t - l'T_s) - jb(l')p(t - l'T_s - 0.5T_s) \right]. \quad \text{(C.2)}
\]

The two first derivatives of the integrand are

\[
\frac{\partial}{\partial h_R} I(t, h_R, h_I) = -r(t) \sum_{l'} \left[ a(l')p(t - l'T_s) - jb(l')p(t - l'T_s - 0.5T_s) \right] \\
- r^*(t) \sum_{l} \left[ a(l)p(t - lT_s) + jb(l)p(t - lT_s - 0.5T_s) \right] \\
+ 2h_R \sum_{l} \left[ a(l)p(t - lT_s) + jb(l)p(t - lT_s - 0.5T_s) \right] \\
\times \sum_{l'} \left[ a(l')p(t - l'T_s) - jb(l')p(t - l'T_s - 0.5T_s) \right]. \quad \text{(C.3)}
\]
\[
\frac{\partial}{\partial h_I} I(t, h, h_I) = j r(t) \sum_{l'} \left[ a(l')p(t - l'T_s) - j b(l')p(t - l'T_s - 0.5T_s) \right]
\]
\[
- j r^*(t) \sum_{l} \left[ a(l)p(t - l'T_s) + j b(l)p(t - l'T_s - 0.5T_s) \right]
\]
\[
+ 2h_I \sum_{l} \left[ a(l)p(t - l'T_s) + j b(l)p(t - l'T_s - 0.5T_s) \right]
\]
\[
\times \sum_{l'} \left[ a(l')p(t - l'T_s) - j b(l')p(t - l'T_s - 0.5T_s) \right] \quad \text{(C.4)}
\]

and the four second partial derivatives of the integrand are

\[
\frac{\partial^2 I(t, h, h_I)}{\partial h_R \partial h_R} = 2 \sum_{l} \left[ a(l)p(t - l'T_s) + j b(l)p(t - l'T_s - 0.5T_s) \right]
\]
\[
\times \sum_{l'} \left[ a(l')p(t - l'T_s) - j b(l')p(t - l'T_s - 0.5T_s) \right] \quad \text{(C.5)}
\]

\[
\frac{\partial^2 I(t, h, h_I)}{\partial h_I \partial h_R} = 0 \quad \text{(C.6)}
\]

\[
\frac{\partial^2 I(t, h, h_I)}{\partial h_R \partial h_I} = 0 \quad \text{(C.7)}
\]

\[
\frac{\partial^2 I(t, h, h_I)}{\partial h_I \partial h_I} = 2 \sum_{l} \left[ a(l)p(t - l'T_s) + j b(l)p(t - l'T_s - 0.5T_s) \right]
\]
\[
\times \sum_{l'} \left[ a(l')p(t - l'T_s) - j b(l')p(t - l'T_s - 0.5T_s) \right]. \quad \text{(C.8)}
\]

Using these results, the desired partial derivatives of the log-likelihood function are computed. Expanding (C.5) gives

\[
\frac{\partial^2 I(t, h, h_I)}{\partial h_R \partial h_R} = 2 \sum_{l} \sum_{l'} a(l)a(l')p(t - l'T_s)p(t - l'T_s)
\]
\[
- j^2 \sum_{l} \sum_{l'} a(l)b(l')p(t - l'T_s)p(t - l'T_s - 0.5T_s)
\]
\[
+ j^2 \sum_{l} \sum_{l'} b(l)a(l')p(t - l'T_s - 0.5T_s)p(t - l'T_s)
\]
\[
+ 2 \sum_{l} \sum_{l'} b(l)b(l')p(t - l'T_s - 0.5T_s)p(t - l'T_s - 0.5T_s). \quad \text{(C.9)}
\]

The desired second derivative of the log-likelihood function is
\[
\frac{\partial^2 \Lambda(h_R, h_I)}{\partial h_R \partial h_R} = -\frac{1}{2N_0} \int_{T_1}^{T_2 + (L_0 - 0.5)T_s} \frac{\partial^2 I(t, h_R, h_I)}{\partial h_R \partial h_R} dt
\]

\[
= -\frac{1}{N_0} \int_{T_1}^{T_2 + (L_0 - 0.5)T_s} \sum_l \sum_{l'} a(l)a(l')p(t - lT_s)p(t - l'T_s)dt
\]

\[
+ \frac{j}{N_0} \int_{T_1}^{T_2 + (L_0 - 0.5)T_s} \sum_l \sum_{l'} b(l)b(l')p(t - lT_s)p(t - l'T_s - 0.5T_s)dt
\]

\[
- \frac{j}{N_0} \int_{T_1}^{T_2 + (L_0 - 0.5)T_s} \sum_l \sum_{l'} a(l)a(l')p(t - lT_s - 0.5T_s)p(t - l'T_s)dt
\]

\[
- \frac{1}{N_0} \int_{T_1}^{T_2 + (L_0 - 0.5)T_s} \sum_l \sum_{l'} b(l)b(l')p(t - lT_s - 0.5T_s)p(t - l'T_s - 0.5T_s)dt. \quad (C.10)
\]

The four integrals on the right-hand side of (C.10) are identical to the last four integrals on the right-hand side of (3.16). Expressions for these four integrals derived in Section 3.2 and are given by (3.46) – (3.49). The results are summarized as follows:

\[
\int_{T_1}^{T_2 + (L_0 - 0.5)T_s} \sum_l \sum_{l'} a(l)a(l')p(t - lT_s)p(t - l'T_s)dt
\]

\[
= \sum_{l=0}^{L_0-1} \sum_{l'=l-L_p+1}^{l+L_p-1} a(l)a(l')r_p([l' - l]T_s) + \text{edge effects} \quad (C.11)
\]

\[
\int_{T_1}^{T_2 + (L_0 - 0.5)T_s} \sum_l \sum_{l'} b(l)b(l')p(t - lT_s)p(t - l'T_s - 0.5T_s)dt
\]

\[
= \sum_{l=0}^{L_0-1} \sum_{l'=l-L_p}^{l+L_p-1} a(l)b(l')r_p([l' - l - 0.5]T_s) + \text{edge effects} \quad (C.12)
\]
\[
T_2 + (L_0 - 0.5)T_s \int_{T_1} \sum_{l} \sum_{l'} b(l)a(l')p(t - lT_s - 0.5T_s)p(t - l'T_s)dt \\
= \sum_{l=0}^{L_0-1} \sum_{l' = l - L_p + 1} b(l)a(l')r_p([l' - l - 0.5]T_s) + \text{edge effects} \quad \text{(C.13)}
\]

\[
T_2 + (L_0 - 0.5)T_s \int_{T_1} \sum_{l} \sum_{l'} b(l)b(l')p(t - lT_s - 0.5T_s)p(t - l'T_s - 0.5T_s)dt \\
= \sum_{l=0}^{L_0-1} \sum_{l' = l - L_p + 1} b(l)b(l')r_p([l' - l]T_s) + \text{edge effects}. \quad \text{(C.14)}
\]

Assuming the pulse shape satisfies the Nyquist No-ISI condition, we have

\[
\frac{\partial^2 \Lambda(h_R, h_I)}{\partial h_R \partial h_R} = -\frac{j}{N_0} \left[ L_0 A^2 + \text{edge effects} \right] \\
+ \frac{j}{N_0} \left[ \sum_{l=0}^{L_0-1} \sum_{l' = l - L_p} b(l)b(l')r_p([l' - l - 0.5]T_s) + \text{edge effects} \right] \\
- \frac{j}{N_0} \left[ \sum_{l=0}^{L_0-1} \sum_{l' = l - L_p + 1} b(l)a(l')r_p([l' - l + 0.5]T_s) + \text{edge effects} \right] \\
- \frac{1}{N_0} \left[ L_0 A^2 + \text{edge effects} \right]. \quad \text{(C.15)}
\]

Simplifying gives

\[
\frac{\partial^2 \Lambda(h_R, h_I)}{\partial h_R \partial h_R} = -\frac{2L_0 A^2}{N_0} + \frac{j}{N_0} \mathcal{X} + \text{edge effects} \quad \text{(C.16)}
\]

where

\[
\mathcal{X} = \sum_{l=0}^{L_0-1} \sum_{l' = l - L_p} a(l)b(l')r_p([l' - l + 0.5]T_s) - \sum_{l=0}^{L_0-1} \sum_{l' = l - L_p + 1} b(l)a(l')r_p([l' - l - 0.5]T_s) \quad \text{(C.17)}
\]

is explored in detail in Appendix B. Note that the constants $1/N_0$ and $j/N_0$ have been absorbed into the “edge effects” terms as necessary.
The other three second partial derivatives of the log-likelihood function are
\[
\frac{\partial^2 \Lambda(h_R, h_I)}{\partial h_I \partial h_R} = 0 \tag{C.18}
\]
\[
\frac{\partial^2 \Lambda(h_R, h_I)}{\partial h_R \partial h_I} = 0 \tag{C.19}
\]
\[
\frac{\partial^2 \Lambda(h_R, h_I)}{\partial h_I \partial h_I} = -\frac{2L_0A^2}{N_0} + \frac{j}{N_0} \mathcal{X} + \text{edge effects} \tag{C.20}
\]
where \( \mathcal{X} \) is given by (C.17). The result (C.20) follows exactly the same steps used to obtain (C.16).

The entries of the Fisher information matrix
\[
J = \begin{bmatrix}
J_{h_R,h_R} & J_{h_R,h_I} \\
J_{h_I,h_R} & J_{h_I,h_I}
\end{bmatrix}
\tag{C.21}
\]
are given by
\[
J_{h_R,h_R} = -E \left[ \frac{\partial^2 \Lambda(h_R, h_I)}{\partial h_R \partial h_R} \right] = \frac{2L_0A^2}{N_0} - \frac{j}{N_0} E[\mathcal{X}] - E[\text{edge effects}] \tag{C.22}
\]
\[
J_{h_R,h_I} = -E \left[ \frac{\partial^2 \Lambda(h_R, h_I)}{\partial h_R \partial h_I} \right] = 0 \tag{C.23}
\]
\[
J_{h_I,h_R} = -E \left[ \frac{\partial^2 \Lambda(h_R, h_I)}{\partial h_I \partial h_R} \right] = 0 \tag{C.24}
\]
\[
J_{h_I,h_I} = -E \left[ \frac{\partial^2 \Lambda(h_R, h_I)}{\partial h_I \partial h_I} \right] = \frac{2L_0A^2}{N_0} - \frac{j}{N_0} E[\mathcal{X}] - E[\text{edge effects}] \tag{C.25}
\]
Assuming the (non-pilot) data symbols involved in the edge effects terms are equally likely, close examination of \( E \) and of the edge effect terms in (3.25), (3.34), and (3.46) – (3.49) shows that all of these terms are zero mean. Consequently, the Fisher information matrix is
\[
J = \begin{bmatrix}
\frac{2L_0A^2}{N_0} & 0 \\
0 & \frac{2L_0A^2}{N_0}
\end{bmatrix}
\tag{C.26}
\]
To obtain the final result, the magnitude squared of the estimator error must be expressed in terms of the real and imaginary parts of the estimate and the true value:
\[
\left| \hat{h} - h \right|^2 = \left| \hat{h}_R + j\hat{h}_I - (h_R + jh_I) \right|^2 = \left| (\hat{h}_R - h_R) + j(\hat{h}_I - h_I) \right|^2
\]
\[ = \left( \hat{h}_R - h_R \right)^2 + \left( \hat{h}_I - h_I \right)^2. \]  
(C.27)

Applying this result produces
\[
E \left[ \left| \hat{h} - h \right|^2 \right] = E \left[ \left( \hat{h}_R - h_R \right)^2 + \left( \hat{h}_I - h_I \right)^2 \right] \geq \frac{N_0}{2L_0A^2} + \frac{N_0}{2L_0A^2} = \frac{N_0}{L_0A^2}. \]
(C.28)

Using the substitutions \( E_b = \frac{1}{2}A^2 \) the Cramér-Rao bound is
\[
E \left[ \left| \hat{h} - h \right|^2 \right] \geq \frac{1}{2L_0N_0} E_b. \]
(C.29)