Nonlinear Observability Analysis of Bearing-only Cooperative Localization using Graph Theory

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0.1 Introduction

In this report we investigate the nonlinear observability properties of bearing-only cooperative localization. We establish a link between observability and a graph representing measurements and communication between the robots.

0.2 Observability of an edge between two robots

To determine the observability matrix of G_2^0, we first find the Lie derivatives of η_{ij}. The zeroth order Lie derivatives is:

$$L_g^0 h(X) = \eta_{ij}. \tag{1}$$

Differentiating $L_g^0 h(X)$, we obtain

$$\begin{aligned} \nabla L_g^0 h(X) &= \begin{bmatrix} H_{1ij}^\top & -1 & -H_{1ij}^\top & 0 \end{bmatrix}, \text{ where} \\ H_{1ij}^\top &= [-a \ b]^\top = \begin{bmatrix} -\frac{y_i - y_j}{R_{ij}^2} & \frac{x_i - x_j}{R_{ij}^2} \end{bmatrix}. \end{aligned} \tag{2}$$

This leads to the first Lie derivative

$$L_g^1(h) = \frac{\partial}{\partial X} L_g^0 h(X) \dot{X} = H_{1ij}^\top (f_i - f_j) - \omega_1. \tag{3}$$

Differentiating $L_{f(h)}^1$ we obtain

$$\nabla L_g^1(h) = \begin{bmatrix} H_{2ij}^\top & H_{1ij}^\top F_i & -H_{2ij}^\top & -H_{1ij}^\top F_j \end{bmatrix} \tag{4}$$

where $H_{2ij}^\top = (f_i - f_j)^\top J_{1ij}$, $f_i = V_i[\cos\psi_i \ \ \sin\psi_i]^\top$, $F_i = \frac{\partial f_i}{\partial \psi_i}$, and

$$J_{1ij} = \frac{\partial H_{ij}}{\partial X_i} = \begin{bmatrix} 2a_{ij}b_{ij} & (a_{ij}^2 - b_{ij}^2) \\ (a_{ij}^2 - b_{ij}^2) & -2a_{ij}b_{ij} \end{bmatrix}. \tag{5}$$

Following a similar derivation, we obtain the second Lie derivative as

$$L_g^2(h) = (f_i - f_j)^\top J_{1ij}(f_i - f_j) + H_{ij}^\top (F_i\omega_i - F_j\omega_j). \tag{6}$$

Differentiating $L_g^2(h)$ we obtain

$$\nabla L_g^2(h) = \begin{bmatrix} H_{3ij}^\top & H_{4ij}^\top & -H_{3ij}^\top & H_{5ij}^\top \end{bmatrix} \tag{7}$$

where $H_{3ij}^\top = (f_i - f_j)^\top J_{2ij} + (F_i\omega_i - F_j\omega_j)^\top J_{1ij}$, $J_{2ij} = 2(a^2 + b^2)[J_{2ij}^a J_{2ij}^b](f_i - f_j)$, $J_{2ij}^a = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, $J_{2ij}^b = \begin{bmatrix} -b & a \\ a & b \end{bmatrix}$, $H_{4ij}^\top = (f_i - f_j)^\top J_{1ij}^\top F_i + H_{2ij}^\top F_i - \omega_i H_{1ij}^\top f_i$, and $H_{5ij}^\top = -(f_i - f_j)^\top J_{1ij}^\top F_j - H_{2ij}^\top F_j + \omega_j H_{1ij}^\top f_j$.

Higher order Lie derivatives are linear combinations of $(f_i - f_j)$ and $(F_i\omega_i - F_j\omega_j)$ and therefore do not contribute to the rank of the observability matrix. The observability matrix for G_2^0 can now be written as

$$O^{ij} = \frac{\partial}{\partial X}\Omega = \begin{bmatrix} H_{1ij}^\top & -1 & -H_{1ij}^\top & 0 \\ H_{2ij}^\top & H_{1ij}^\top F_i & -H_{2ij}^\top & -H_{1ij}^\top F_j \\ H_{3ij}^\top & H_{4ij}^\top & -H_{3ij}^\top & H_{5ij}^\top \end{bmatrix}. \tag{8}$$

Conditions for the maximum rank of (8) are stated in the following lemma.

Lemma 1. *The observability matrix given in (8) has rank three if and only if $V_j > 0$, $V_i \neq V_j$ or $\psi_i \neq \psi_j$, and $\omega_i \neq H_{1ij}^\top(f_i - f_j)$.*

Proof. We first prove the sufficiency of the listed conditions. The rank of observability matrix in (8) is three iff all the three rows are linearly independent. To show the linear independence we perform gaussian elimination on the rows of (8), and show that the reduced-row echelon form (RREF) of the observability matrix O_ij is,

$$\bar{O}_{ij} \triangleq \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & y_i - y_j \\ 0 & 1 & 0 & 0 & -1 & x_j - x_i \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}. \tag{9}$$

We can expand the observability matrix and write,

$$O = \begin{bmatrix} -a & b & -1 & a & -b & 0 \\ H_{21} & H_{22} & H_{23} & -H_{21} & -H_{22} & H_{26} \\ H_{31} & H_{32} & H_{33} & -H_{31} & -H_{32} & H_{36} \end{bmatrix} \tag{10}$$

If all the conditions of lemma 1 are satisfied we can perform gaussian elimination on O to find the basis which spans the observability space for two robots.

We first perform these operation on R_2 and R_3. (1) $R_2 = R_2 - \frac{R_1}{-a}H_{21}$ (2) $R_3 = R_3 - \frac{R_1}{-a}H_{31}$, and obtain

$$O = \begin{bmatrix} -a & b & -1 & a & -b & 0 \\ 0 & H'_{22} & H'_{23} & 0 & -H'_{22} & H_{26} \\ 0 & H'_{32} & H'_{33} & 0 & -H'_{32} & H_{36} \end{bmatrix} \tag{11}$$

$R_3 = R_3 - \frac{R_2}{H'_{22}}H'_{32}$

$$O = \begin{bmatrix} -a & b & -1 & a & -b & 0 \\ 0 & H'_{22} & H'_{23} & 0 & -H'_{22} & H_{26} \\ 0 & 0 & H''_{33} & 0 & 0 & H'_{36} \end{bmatrix} \tag{12}$$

Plugging the values it can shown that $H''_{33} = -H'_{36}$.

$R_3 = \frac{R_3}{H''_{33}}$

$$O = \begin{bmatrix} -a & b & -1 & a & -b & 0 \\ 0 & H'_{22} & H'_{23} & 0 & -H'_{22} & H_{26} \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \tag{13}$$

$R_2 = R_2 - R_3 H'_{23}$ $R_1 = R_1 + R_3$

$$O = \begin{bmatrix} -a & b & 0 & a & -b & -1 \\ 0 & H'_{22} & 0 & 0 & -H'_{22} & H'_{26} \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \tag{14}$$

$R_2 = \frac{R_2}{H'_{22}}$ and using $\frac{H'_{26}}{H'_{22}} = x_j - x_i$,

$$O = \begin{bmatrix} -a & b & 0 & a & -b & -1 \\ 0 & 1 & 0 & 0 & -1 & x_j - x_i \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \tag{15}$$

$R_1 = R_1 - R_2 b$ and $R_1 = \frac{R_1}{-a}$

$$O = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & \frac{-1-b(x_j - x_i)}{-a} \\ 0 & 1 & 0 & 0 & -1 & x_j - x_i \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \tag{16}$$

Plugging in values of a and b we can show that $\frac{-1-b(x_j-x_i)}{-a} = y_i - y_j$ and then we can write

$$O = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & y_i - y_j \\ 0 & 1 & 0 & 0 & -1 & x_j - x_i \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \tag{17}$$

It can be easily verified that all the rows in (9) are linearly independent and therefore, $rank(O_{ij}) = 3$. To prove the necessity of the lemma.

(a) Assume that $V_i > 0$ and $V_j = 0$. Plugging these into (8) we obtain, $O^{ij} = \begin{bmatrix} H_{1ij}^\top & -1 & -H_{1ij}^\top & 0 \\ f_i^\top J_{1ij} & H_{1ij}^\top F_i & -f_i^\top J_{1ij} & 0 \\ f_i^\top J_{2ij} & 2 f_i^\top J_{1ij}^\top F_i & -f_i^\top J_{2ij} & 0 \end{bmatrix}$ It can be seen that second and third rows are not linearly independent and therefore, $rank(O^{ij}) = 2 < 3$.

(b) Note that if $V_i = V_j$ and $\psi_i = \psi_j$ (i.e., the robots are moving in formation) then $\dot{\eta}_{ij} = 0$, leading to insufficient measurements for observability. Similarly, when $\omega_i = H_{1ij}^\top (f_i - f_j)$, $\dot{\eta}_{ij} = 0$ leading to a rank 1 observability matrix. □

Here we derive the observability conditions for an edge η_{il} between a robot and landmark.

The zeroth order Lie derivative is $L^0_{f(h)} = \eta_{il}$ Differentiating $L^0_{f(h)}$ we obtain, $\frac{\partial}{\partial X} L^0_{f(h)} = \begin{bmatrix} H_{1il}^\top & -1 \end{bmatrix}$, where H_{1il} is as defined in Equation (2), leading to a first order Lie derivative of $L^1_{f(h)} = H_{1il}^\top f_i - \omega_i$. Differentiating $L^1_{f(h)}$ we obtain $\nabla L^1_{f(h)} = \begin{bmatrix} H_{2il}^\top & H_{1il}^\top F_i \end{bmatrix}$ where, $H_{2il}^\top = f_i^\top J_{1il}$ and J_{1il} is similar to J_{1ij} defined in (5). Second and higher order Lie derivatives will be multiples of f_i

and F_i. We can write the observability matrix entries for the observing robot as

$$O_{il} = \begin{bmatrix} H_{1il}^\top & -1 \\ H_{2il}^\top & H_{1il}^\top F_i \end{bmatrix}. \tag{18}$$

Conditions for the maximum rank of (18) are stated in following lemma.

Lemma 2. *The number of linearly independent rows contributed by edge ε_{il} between a robot and a landmark is two if and only if $V_i > 0$ and $\omega_i \neq H_{1ip}^\top f_i$.*

Proof. If all the conditions in above lemma are satisfied we can perform gaussian elimination (similarly to lemma 1) on the rows of (18), and can write its RREF as,

$$\bar{O}_{il} = \begin{bmatrix} 1 & 0 & y_i - y_l \\ 0 & 1 & x_l - x_i \end{bmatrix}. \tag{19}$$

which has rank two. If $V_i = 0$ or $\omega_i = H_{1il}^\top f_i$, then $\dot{\eta}_{ip} = 0$, leading to an observability matrix rank of 1. $\square$

Definition 1. *A RPMG G_n^l is called a proper RPMG if all the edges between robot nodes satisfies the conditions of lemma 1 and all the edges between robots and landmarks satisfies lemma 2.*

From (9) we can write the basis for the row space of an observability matrix of an edge between two robots in the proper graph G_n^l as,

$$\begin{bmatrix} 0_{3\times 3(i-1)} & O_i^{ij} & 0_{3\times(3(j-1)-3i)} & O_j^{ij} & 0_{3\times(n-3j)} \end{bmatrix}, \tag{20}$$

where $\bar{O}_i^{ij} = \mathbf{I}_3$, and $\bar{O}_j^{ij} = \begin{bmatrix} -1 & 0 & y_i - y_j \\ 0 & -1 & x_j - x_i \\ 0 & 0 & -1 \end{bmatrix}$.

Similarly, from (19) we can write the basis for the row space of an observability matrix of an edge between a robot and a landmark in the proper G_n^l as,

$$[0_{2\times 3(i-1)})\ \bar{O}_{il}\ 0_{2\times 3(n-i)})]. \tag{21}$$

0.3 Observability of three nodes

Lemma 3. *If a three node proper RPMG G_3^0 is connected then the rank of observability matrix is six.*

Proof. There are four possible configurations of a connected graph G_3^0, shown as sub-figures (a) through (d) in Fig. 1. We can write the observability matrix

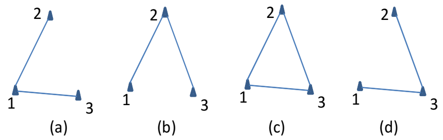

Figure 1: The observability conditions between these four possible configurations of a connected, 3-node RPMG are identical.

for these configurations using (20) as

$$O_a = \begin{bmatrix} O_1^{12} & O_2^{12} & \mathbf{0} \\ O_1^{13} & \mathbf{0} & O_3^{13} \end{bmatrix} \quad O_b = \begin{bmatrix} O_1^{12} & O_j^{12} & \mathbf{0} \\ \mathbf{0} & O_2^{23} & O_3^{23} \end{bmatrix}$$

$$O_c = \begin{bmatrix} O_1^{12} & O_2^{12} & \mathbf{0} \\ \mathbf{0} & O_2^{23} & O_3^{23} \\ O_1^{13} & \mathbf{0} & O_3^{13} \end{bmatrix} \quad O_d = \begin{bmatrix} O_1^{13} & \mathbf{0} & O_3^{13} \\ \mathbf{0} & O_2^{23} & O_3^{23} \end{bmatrix}.$$

First we perform gaussian elimination on O_a. Plugging values of matrices O_1^{12}, O_2^{12}, O_1^{13}, and O_3^{13} we get,

$$O_a = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & y_1 - y_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & x_2 - x_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & y_1 - y_3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & x_3 - x_1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{22}$$

By performing following elementary operations $R4 = R4 - R1$, $R5 = R5 - R2$, and $R6 = R6 - R3$ we get

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & y_1 - y_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & x_2 - x_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -(y_1 - y_2) & -1 & 0 & y_1 - y_3 \\ 0 & 0 & 0 & 0 & 1 & -(x_2 - x_1) & 0 & -1 & x_3 - x_1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \tag{23}$$

$R1 = R1 + R4$, $R2 = R2 + R5$, and $R3 = R3 + R1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & y_1 - y_3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & x_3 - x_1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -(y_1 - y_2) & -1 & 0 & y_1 - y_3 \\ 0 & 0 & 0 & 0 & 1 & -(x_2 - x_1) & 0 & -1 & x_3 - x_1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \tag{24}$$

$R4 = R4 + R6(y_1 - y_2)$ and $R5 = R5 + R6(x_2 - x_1)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & y_1 - y_3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & x_3 - x_1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & y_1 - y_3 - (y_1 - y_2) \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & x_3 - x_1 - (x_2 - x_1) \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \tag{25}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & y_1 - y_3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & x_3 - x_1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & y_2 - y_3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & x_3 - x_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \tag{26}$$

Therefore,

$$\bar{O}_a = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} & O_3^{13} \\ \mathbf{0} & \mathbf{I}_3 & O_3^{23} \end{bmatrix} \tag{27}$$

Similarly, we can perform gaussian elimination and elementary ro operations on O_b and O_c and show that $\bar{O}_a = \bar{O}_b = \bar{O}_c = O_d$ and there rank is six. □

0.4 Observability with landmarks

Consider RPMG (G_2^1) with two robot nodes and one landmark. We can write observability matrix of (G_2^1) where landmark k is connected to node i,

$$\mathcal{O} \quad = \quad \begin{bmatrix} O_i^{ij} & O_j^{ij} \\ O_{ik} & 0_{2\times3} \end{bmatrix}. \tag{28}$$

Plugging values of matrices O_i^{ij}, and O_{ik} we get,

$$O = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & y_i - y_j \\ 0 & 1 & 0 & 0 & -1 & x_j - x_i \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & y_i - y_l & 0 & 0 & 0 \\ 0 & 1 & x_l - x_i & 0 & 0 & 0 \end{bmatrix} \tag{29}$$

First $R4 = R4 - R1$ and $R5 = R5 - R2$

$$= \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & y_i - y_j \\ 0 & 1 & 0 & 0 & -1 & x_j - x_i \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & y_i - y_l & 1 & 0 & -(y_i - y_j) \\ 0 & 0 & x_l - x_i & 0 & 1 & -(x_j - x_i) \end{bmatrix} \tag{30}$$

$R4 = R4 - R3(y_i - y_l)$ and $R5 = R5 - R3(x_l - x_i)$

$$= \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & y_i - y_j \\ 0 & 1 & 0 & 0 & -1 & x_j - x_i \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -(y_i - y_j) + y_i - y_l \\ 0 & 0 & 0 & 0 & 1 & -(x_j - x_i) + x_l - x_i \end{bmatrix} \tag{31}$$

$$= \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & y_i - y_j \\ 0 & 1 & 0 & 0 & -1 & x_j - x_i \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & y_j - y_l \\ 0 & 0 & 0 & 0 & 1 & x_l - x_j \end{bmatrix} \tag{32}$$

$$= \begin{bmatrix} O_i^{ij} & O_j^{ij} \\ 0_{2\times 3} & O_j^{jl} \end{bmatrix}. \tag{33}$$

This is the observability matrix for landmark l connected to node i is equal to landmark p connected to node j.