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Observability analysis of cooperative localization using graph theory and lie derivatives

Rajnikant Sharma
raj.drdo@gmail.com

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Report

On

Observability Analysis of Cooperative Localization using Graph Theory And Lie Derivatives

Department of Electrical and Computer Engineering

Brigham Young University

Provo, UT

Prepared by

Rajnikant Sharma and Randal W. Beard

Chapter 1

Observability Analysis of Cooperative Localization using Graph Theory and Lie Derivatives

1.1 Bearing-only Cooperative Localization

Consider n robots moving in a horizontal plane performing cooperative localization. We can write the equations of motion for the i^{th} robot as,

$$\dot{X}_i = g_i(X_i, u_i) \triangleq \begin{pmatrix} V_i \cos\psi_i \\ V_i \sin\psi_i \\ \omega_i \end{pmatrix}, \tag{1.1}$$

where $X_i = [x_i \; y_i \; \psi_i]^\top \in \mathbb{R}^3$ is the robot state, including robot location $(x_i, \; y_i)$ and robot heading ψ_i, and $u_i = [V_i, w_i]^\top$ is the control input vector. We assume that onboard introspective sensors (e.g., encoders) provide linear speed V_i and angular speed ω_i of the robot. Without loss of generality, we assume that robots cannot move backward $(V_i \geq 0, \; i = 1 \cdots n)$. Each vehicle has an exteroceptive sensor to measure relative bearing to other vehicles and known landmarks. Relative bearing from the i^{th} robot to the j^{th} robot or landmark can be written as,

$$\eta_{ij} = \tan^{-1}\left(\frac{y_j - y_i}{x_j - x_i}\right) - \psi_i. \tag{1.2}$$

For cooperative localization, each robot exchanges their local sensor measurements (velocity, turn rate, and bearing to landmarks and other robots) with their neighbors. Let N_i^M be the set of neighbors for which robot i can obtain bearing measurements, and let N_i^C be the set of neighbors with which robots i can communicate. In this paper, we assume that $N_i^M = N_i^C$ and we will therefore denote the set of neighbors as N_i. To represent the connection topology of the robots we use a relative position measurement graph (RPMG)[1] which is defined as follows.

Definition 1 *An RPMG for n robots performing cooperative localization with l different known landmarks is a directed graph $G_n^l \triangleq \{\mathcal{V}_{n,l}, \mathcal{E}_{n,l}\}$, where $\mathcal{V}_{n,l} = \{1, \cdots, n, n+1, \cdots, n+l\}$ is the node set consisting of n robot*

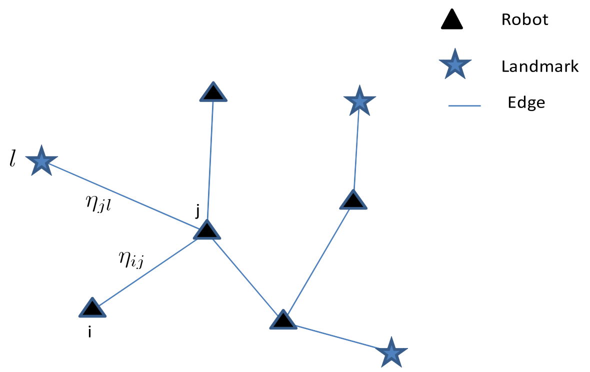

Figure 1.1: Relative position measurement graph (RPMG). The nodes of an RPMG represent vehicle states and the edges represent bearing measurements between nodes.

nodes and l landmark nodes and $\mathcal{E}_{n,l}(t) \subset \{\mathcal{V}_{n,0} \times \mathcal{V}_{n,l}\} = \{\eta_{ij}\},\ i \in \{1, \cdots, n\},\ j \in \{1, \cdots, n, n+1, \cdots, n+l\}$ is the edge set representing the availability of a relative bearing measurement. We use m to denote the number of edges in the RPMG. An example RPMG (G_5^3 with $m = 7$) is shown in Fig. 1.1.

Remark 1 *We define RPMG as a directed graph because an edge between two nodes is not bidirectional. There are two reasons for RPMG with bearing measurement directional. Firstly, If j^{th} vehicle is in field-of-view i^{th} vehicle it is not necessary that opposite is true. Secondly, even if both the vehicles are in each other's field-of-view because of unsymmetrical nature of the bearing measurement, which depends on the heading of the vehicle, $\eta_{ij} \neq \eta_{ji}$ except when both the vehicles are heading towards each other. Later in the paper we will show that an edge can be treated as an bidirectional edge because under certain conditions both η_{ij} and η_{ji} contribute the same number of linearly independent rows to the observability matrix. Such an RPMG will be called as a proper RPMG. However, the edge between a landmark and a vehicle is directional because of the obvious reason.*

Without loss of generality we assume that a robot maintains safe distance from other robots $R_{ij} > 0,\ \forall i, j = 1, \cdots n$ and landmarks $R_{ik} > 0,\ \forall i = 1, \cdots n;\ k = 1, \cdots, l$.

1.1.1 Lie Derivatives and Nonlinear Observability

To determine the observability of the entire system represented by the RPMG we use the nonlinear observability rank criteria developed by Hermann and Krener [2] which is summarized in the next paragraph.

Consider a system model with the following form

$$\Sigma: \begin{array}{c} \dot{X} = g(X, u) = [g_1^\top(X_1, u_1), \cdots, g_n^\top(X_n, u_n)]^\top \\ y = h(X, Z) = [h_1^\top(X, Z) \cdots h_m^\top(X, Z)]^\top \end{array} \tag{1.3}$$

where $X = [X_1^\top X_2^\top \cdots X_n^\top]^\top \in \mathbb{R}^{3n}$ is the state of the system, $Z = [Z_1^\top Z_2^\top \cdots Z_l^\top]^\top \in \mathbb{R}^{2l}$ is the position vector of all landmarks, $Z_i = [x_i\ y_i]^\top$ is the position vector of i^{th} landmark, $h_i : \mathbb{R}^{3n} \times \mathbb{R}^{2l} \mapsto \mathbb{R}$ is the measurement model of the i^{th} sensor, $u \in \Lambda \subseteq \mathbb{R}^{2n}$ is the control input vector, and $g : \mathbb{R}^{3n} \times \Lambda \mapsto \mathbb{R}^{3n}$. We consider the special case of system (1.3) where the process function g can be separated into a summation of independent functions, each one excited by a different component of the control input vector, i.e.,

$$\dot{X} = g(X, u) = f_{v_1} V_1 + f_{\omega_1} \omega_1 + f_{v_2} V_2 + f_{\omega_2} \omega_2 + \cdots + f_{v_n} V_n + f_{\omega_n} \omega_n \tag{1.4}$$

The zeroth-order Lie derivative of any (scalar) function is the function itself, i.e., $L^0 h_k(X, Z) = h_k(X, Z)$. The first-order Lie derivative of function $h_k(X, Z)$ with respect to f_{v_i} is defined as

$$L^1_{f_{v_i}} h = \nabla L^0 h \cdot f_{v_i} \tag{1.5}$$

∇ represents the gradient operator, and $\cdot$ denotes the vector inner product. Considering that $L^1 f_{v_i} h_k(X, Z)$ is a scalar function itself, the second-order Lie derivative of $h_k(X, Z)$ with respect to f_{v_i} is

$$L^2_{f_{v_i} f_{v_i}} h = \nabla L^1_{f_{v_i}} h \cdot f_{v_i} \tag{1.6}$$

Higher order Lie derivatives are computed similarly. Additionally, it is possible to define mixed Lie derivatives, i.e., with respect to different functions of the process model. For example, the second-order Lie derivative of h_k with respect to f_{v_i} and f_{v_j}, given its first derivative with respect to f_{v_i} ,is

$$L^2_{f_{v_i} f_{v_j}} h = \nabla L^1_{f_{v_i}} h \cdot f_{v_j} \tag{1.7}$$

Based on the preceding expressions for the Lie derivative e observability matrix is defined as the matrix with rows

$$O = \left\{ \nabla L^p_{f_{v_i}, \cdots, f_{v_j}, f_{\omega_i}, \cdots, f_{\omega_j}} h_k(X, Z) | i, j = 1, \cdots, n; k = 1, \cdots, m; p \in \mathbb{N} \right\} \tag{1.8}$$

The important role of this matrix in the observability analysis of a nonlinear system is captured by [[2], Ths. 3.1 and 3.11], repeated next.

Theorem 1 *A system is locally weakly observable iff the system satisfies the observability rank condition (observability matrix (cf.* (1.8)*)has full rank).*

We assume that the robot sensors have limited sensor range ρ_{sensor} and limited field of view. Therefore, agents can only measure the bearing of those robots and landmarks that are located within the footprint of the sensor. Therefore, the graph G_n^m will likely have a time varying topology.

1.2 Graph-based Observability Analysis

In this section we obtain the conditions for the observability of the graph G_n^l. First we derive the observability matrix for a single edge in the graph G_n^l. Next we obtain the conditions for the observability of the graph G_n^0 without landmarks followed by observability analysis of the graph G_n^l with landmarks.

1.2.1 Observability due to an Edge

In a graph G_n^l there are two types of edges: an edge between two robots, and an edge between a robot and a landmark. We derive the conditions for the maximum rank of the observability matrix and obtain the basis of the observability space for these edges. These bases will serve as building block for the observability conditions for the graph G_n^l.

Edge between two robots

To determine the observability matrix of the graph G_2^0, we first find the Lie derivatives of $h(X, Z) = \eta_{ij}$. We rearrange the nonlinear kinematic equations in the following convenient form for computing Lie derivatives:

$$\dot{X} = \begin{bmatrix} \dot{X}_i \\ \dot{X}_j \end{bmatrix} = \underbrace{\begin{bmatrix} c\psi_i \\ s\psi_i \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{f_{v_i}} V_i + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{f_{\omega_i}} \omega_i + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ c\psi_j \\ s\psi_j \\ 0 \end{bmatrix}}_{f_{v_j}} V_j + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{f_{\omega_j}} \omega_j. \tag{1.9}$$

We hereafter compute the necessary Lie derivatives of η_{ij} and their gradients.

1) Zeroth-order Lie derivative ($L^0 h$)

$$L^0 h = \eta_{ij}$$

with gradient

$$\nabla L^0 h = \begin{bmatrix} -\frac{\Delta y_{ij}}{R_{ij}^2} & \frac{\Delta x_{ij}}{R_{ij}^2} & -1 & \frac{\Delta y_{ij}}{R_{ij}^2} & -\frac{\Delta x_{ij}}{R_{ij}^2} & 0 \end{bmatrix}$$

where, $\Delta y_{ij} = x_i - x_j$,$\Delta x_{ij} = x_i - x_j$, and $R_{ij}^2 = \Delta^2 x_{ij} + \Delta^2 y_{ij}$.

2) First-order Lie derivatives($L^1_{f_{v_i}} h$, $L^1_{f_{v_j}} h$, $L^1_{f_{\omega_i}} h$)

$$L^1_{f_{v_i}} h = \nabla L^0 h \cdot f_{v_i} = \frac{\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i}{R_{ij}^2}$$

$$L^1_{f_{v_j}} h = \nabla L^0 h \cdot f_{v_j} = -\frac{(\Delta x_{ij} s\psi_j - \Delta y_{ij} c\psi_j)}{R_{ij}^2}$$

$$L^1_{f_{\omega_i}} h = \nabla L^0 h \cdot f_{\omega_i} = -1.$$

In order to simplify higher-order Lie derivatives, we multiply $L^1_{f_{v_i}}$ and $L^1_{f_{v_j}}$ with R^2_{ij}. The scaled first-order Lie-derivatives can be written as

$$L^1_{f_{v_i}} h = \Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i$$
$$L^1_{f_{v_j}} h = -(\Delta x_{ij} s\psi_j - \Delta y_{ij} c\psi_j)$$

with non zero gradients

$$\nabla L^1_{f_{v_i}} h = \begin{bmatrix} s\psi_i & -c\psi_i & \Delta x_{ij} c\psi_i + \Delta y_{ij} s\psi_i & -s\psi_i & c\psi & 0 \end{bmatrix}$$
$$\nabla L^1_{f_{v_j}} h = \begin{bmatrix} -s\psi_j & c\psi_j & 0 & s\psi_j & -c\psi_j & -(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j) \end{bmatrix}.$$

3) Second-order Lie derivatives($L^2_{f_{v_i} f_{v_i}} h$, $L^2_{f_{v_j} f_{v_j}} h$, $L^2_{f_{v_i} f_{v_j}} h$, $L^2_{f_{v_i} f_{\omega_i}} h$, $L^2_{f_{v_j} f_{\omega_j}} h$)

$$L^2_{f_{v_i} f_{v_i}} h = \nabla L^1_{f_{v_i}} h \cdot f_{v_i} = s\psi_i c\psi_i - s\psi_i c\psi_i = 0$$
$$L^2_{f_{v_j} f_{v_j}} h = \nabla L^1_{f_{v_j}} h \cdot f_{v_j} = s\psi_j c\psi_j - s\psi_j c\psi_j = 0$$
$$L^2_{f_{v_i} f_{v_j}} h = \nabla L^1_{f_{v_i}} h \cdot f_{v_j} = -s\psi_i c\psi_j + s\psi_j c\psi_i$$
$$L^2_{f_{v_i} f_{\omega_i}} h = \nabla L^1_{f_{v_i}} h \cdot f_{\omega_i} = \Delta x_{ij} c\psi_i + \Delta y_{ij} s\psi_i$$
$$L^2_{f_{v_j} f_{\omega_j}} h = \nabla L^1_{f_{v_j}} h \cdot f_{\omega_j} = -(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j)$$

with non zero gradients

$$\nabla L^2_{f_{v_i} f_{v_j}} h = \begin{bmatrix} 0 & 0 & -(c\psi_i c\psi_j + s\psi_i s\psi_j) & 0 & 0 & c\psi_i c\psi_j + s\psi_i s\psi_j \end{bmatrix}$$
$$\nabla L^2_{f_{v_i} f_{\omega_i}} h = \begin{bmatrix} c\psi_i & s\psi_i & \Delta y_{ij} c\psi_i - \Delta x_{ij} s\psi_i & c\psi_i & s\psi_i & 0 \end{bmatrix}$$
$$\nabla L^2_{f_{v_j} f_{\omega_j}} h = \begin{bmatrix} -c\psi_j & -s\psi_j & 0 & s\psi_j & -c\psi_j & -(\Delta y_{ij} c\psi_j - \Delta x_{ij} s\psi_j) \end{bmatrix}.$$

4) Third-order Lie derivatives ($L^3_{f_{v_i} f_{v_j} f_{\omega_i}} h$, $L^3_{f_{v_i} f_{v_j} f_{\omega_j}} h$, $L^3_{f_{v_i} f_{\omega_i} f_{v_i}} h$, $L^3_{f_{v_j} f_{\omega_j} f_{v_j}} h$ $L^3_{f_{v_i} f_{\omega_i} f_{\omega_i}} h$, $L^3_{f_{v_j} f_{\omega_j} f_{\omega_j}} h$)

$$L^3_{f_{v_i} f_{v_j} f_{\omega_i}} h = \nabla L^2_{f_{v_i} f_{v_j}} h \cdot f_{\omega_i} = -(c\psi_i c\psi_j + s\psi_i s\psi_j)$$
$$L^3_{f_{v_i} f_{v_j} f_{\omega_j}} h = \nabla L^2_{f_{v_i} f_{v_j}} h \cdot f_{\omega_j} = (c\psi_i c\psi_j + s\psi_i s\psi_j)$$
$$L^3_{f_{v_i} f_{\omega_i} f_{v_i}} h = \nabla L^2_{f_{v_i} f_{\omega_i}} h \cdot f_{v_i} = 1$$
$$L^3_{f_{v_j} f_{\omega_j} f_{v_j}} h = \nabla L^2_{f_{v_i} f_{\omega_j}} h \cdot f_{v_j} = 1$$
$$L^3_{f_{v_i} f_{\omega_i} f_{\omega_i}} h = \nabla L^2_{f_{v_i} f_{\omega_i}} h \cdot f_{\omega_i} = -(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i) = -L^1_{f_{v_i}}$$
$$L^3_{f_{v_j} f_{\omega_j} f_{\omega_j}} h = \nabla L^2_{f_{v_i} f_{\omega_j}} h \cdot f_{\omega_j} = \Delta x_{ij} s\psi_j - \Delta y_{ij} c\psi_j = -L^1_{f_{v_j}}$$

with non zero gradients

$$\nabla L^3_{f_{v_i} f_{v_j} f_{\omega_i}} h = (s\psi_i c\psi_j - c\psi_i s\psi_j) \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} = -\frac{s\psi_i c\psi_j - c\psi_i s\psi_j}{c\psi_i c\psi_j + s\psi_i s\psi_j} \nabla L^2_{f_{v_i} f_{v_j}} h$$
$$\nabla L^3_{f_{v_i} f_{v_j} f_{\omega_j}} h = -(s\psi_i c\psi_j - c\psi_i s\psi_j) \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} = \frac{s\psi_i c\psi_j - c\psi_i s\psi_j}{c\psi_i c\psi_j + s\psi_i s\psi_j} \nabla L^2_{f_{v_i} f_{v_j}} h$$
$$\nabla L^3_{f_{v_i} f_{\omega_i} f_{\omega_i}} h = -(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i) = -\nabla L^1_{f_{v_i}} h$$
$$\nabla L^3_{f_{v_j} f_{\omega_j} f_{\omega_j}} h = \Delta x_{ij} s\psi_j - \Delta y_{ij} c\psi_j = -\nabla L^1_{f_{v_j}} h.$$

Therefore, third order and higher-order Lie derivatives do not contribute any independent row to the observability matrix.

Following lemma states conditions for the maximum rank of the observability matrix of an edge between two robots G_2^0.

Lemma 1 (Sufficient Condition) *Maximum rank of the observability matrix of an edge between two robots is three if following conditions are satisfied*

1. $V_i > 0$.
2. $V_j > 0$.
3. $\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i \neq 0$ *and* $\Delta x_{ij} s\psi_j - \Delta y_{ij} c\psi_j \neq 0$.

Proof: If $V_i > 0$, $V_j > 0$, $\omega_i \neq 0$, and $\omega_j \neq 0$, then the observability matrix can be written using gradients of Lie derivatives related to f_{v_i}, f_{v_j}, f_{ω_i}, and f_{ω_j} as

$$O_{ij} = \left[R_{ij}^2 (\nabla L^0 h)^\top \ (\nabla L^1_{f_{v_i}} h)^\top \ (\nabla L^1_{f_{v_j}} h)^\top \ (\nabla L^2_{f_{v_i} f_{v_j}} h)^\top \ (\nabla L^2_{f_{v_i} f_{\omega_i}} h)^\top \ (\nabla L^2_{f_{v_j} f_{\omega_j}} h)^\top \right]^\top$$
$$= \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R_{ij}^2 & \Delta y_{ij} & -\Delta x_{ij} & 0 \\ s\psi_i & -c\psi_i & (\Delta x_{ij} c\psi_i + \Delta y_{ij} s\psi_i) & -s\psi_i & c\psi_i & 0 \\ -s\psi_j & c\psi_j & 0 & s\psi_j & -c\psi_j & -(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j) \\ 0 & 0 & -(c\psi_i c\psi_j + s\psi_i s\psi_j) & 0 & 0 & (c\psi_i c\psi_j + s\psi_i s\psi_j) \\ c\psi_i & s\psi_i & (\Delta y_{ij} c\psi_i - \Delta x_{ij} s\psi_i) & -c\psi_i & -s\psi_i & 0 \\ -c\psi_j & -s\psi_j & 0 & c\psi_j & s\psi_j & -(\Delta y_{ij} c\psi_j - \Delta x_{ij} s\psi_j) \end{bmatrix}. \quad (1.10)$$

We perform Gaussian elimination to show that there are three independent rows in the above observability matrix. It should be noted that the columns related to x_j and y_j (4^{th} and 5^{th}) are same but opposite to columns related to x_i and y_i (1^{st} and 2^{nd}) respectively. Therefore, to save space, we remove 4^{th} and 5^{th} column from the observability matrix. The resulting observability matrix has four columns and the order of column is x_i, y_i, ω_i, and ω_j.

$$T_1 = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R_{ij}^2 & 0 \\ s\psi_i & -c\psi_i & (\Delta x_{ij} c\psi_i + \Delta y_{ij} s\psi_i) & 0 \\ -s\psi_j & c\psi_j & 0 & -(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j) \\ 0 & 0 & -(c\psi_i c\psi_j + s\psi_i s\psi_j) & (c\psi_i c\psi_j + s\psi_i s\psi_j) \\ c\psi_i & s\psi_i & (\Delta y_{ij} c\psi_i - \Delta x_{ij} s\psi_i) & 0 \\ -c\psi_j & -s\psi_j & 0 & -(\Delta y_{ij} c\psi_j - \Delta x_{ij} s\psi_j) \end{bmatrix}$$

First, we choose $-\Delta y_{ij}$ as pivot to perform the Gaussian elimination. Because $R_{ij} > 0$, without loss of generality, we assume $|\Delta y_{ij}| > 0$ and perform following elementary operations on the rows of the observability matrix. (1) $R_2 = R_2 + \frac{R_1}{\Delta y_{ij}} s\psi_i$ (2) $R_3 = R_3 - \frac{R_1}{\Delta y_{ij}} s\psi_j$ (3) $R_5 = R_5 + \frac{R_1}{\Delta y_{ij}} c\psi_i$ (4) $R_6 = R_6 - \frac{R_1}{\Delta y_{ij}} c\psi_j$ to obtain

$$T_2 = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R_{ij}^2 & 0 \\ 0 & \frac{1}{\Delta y_{ij}}(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i) & -\frac{\Delta x_{ij}}{\Delta y_{ij}}(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i) & 0 \\ 0 & -\frac{1}{\Delta y_{ij}}(\Delta x_{ij} s\psi_j - \Delta y_{ij} c\psi_j) & \frac{R_{ij}^2}{\Delta y_{ij}} s\psi_j & -(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j) \\ 0 & 0 & -(c\psi_i c\psi_j + s\psi_i s\psi_j) & (c\psi_i c\psi_j + s\psi_i s\psi_j) \\ 0 & \frac{1}{\Delta y_{ij}}(\Delta x_{ij} c\psi_i + \Delta y_{ij} s\psi_i) & -\frac{\Delta x_{ij}}{\Delta y_{ij}}(\Delta x_{ij} c\psi_i + \Delta y_{ij} c\psi_i) & 0 \\ 0 & -\frac{1}{\Delta y_{ij}}(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j) & \frac{R_{ij}^2}{\Delta y_{ij}} c\psi_j & -(\Delta y_{ij} c\psi_j - \Delta x_{ij} s\psi_j) \end{bmatrix}.$$

For second pivot we choose $\frac{1}{\Delta y_{ij}}(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i)$. If condition (5) of the lemma is true then this pivot is non zero. We perform following elementary operations on the rows of T_2.

(1) $R_3 = R_3 + \frac{R_2(\Delta x_{ij} s\psi_j - \Delta y_{ij} c\psi_j)}{(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i)}$ (2) $R_5 = R_5 + \frac{R_2(\Delta x_{ij} c\psi_i + \Delta y_{ij} s\psi_i)}{(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i)}$ (3) $R_6 = R_6 - \frac{R_2(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j)}{(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i)}$ to obtain

$$T_3 = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R_{ij}^2 & 0 \\ 0 & \frac{1}{\Delta y_{ij}}(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i) & -\frac{\Delta x_{ij}}{\Delta y_{ij}}(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i) & 0 \\ 0 & 0 & (\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j) & -(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j) \\ 0 & 0 & -(c\psi_i c\psi_j + s\psi_i s\psi_j) & (c\psi_i c\psi_j + s\psi_i s\psi_j) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (\Delta y_{ij} c\psi_j - \Delta x_{ij} s\psi_j) & -(\Delta y_{ij} c\psi_j - \Delta x_{ij} s\psi_j) \end{bmatrix}.$$

We know that if $\Delta x_{ij} s\psi_j - \Delta y_{ij} c\psi_j \neq 0$ then $\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j \neq 0$ and we can perform following elementary row operations taking $(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j)$ as third pivot.

Perform elementary row operation $R_4 = R_4 + \frac{R_3}{(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j)}$ and $R_6 = R_6 - \frac{R_3}{(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j)}$ to obtain

$$T_4 = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R_{ij}^2 & 0 \\ 0 & \frac{1}{\Delta y_{ij}}(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i) & -\frac{\Delta x_{ij}}{\Delta y_{ij}}(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i) & 0 \\ 0 & 0 & (\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j) & -(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly, the maximum rank of the observability rank is three. Next we remove the rows with zeros to obtain.

$$T_5 = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R_{ij}^2 & 0 \\ 0 & \frac{1}{\Delta y_{ij}}(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i) & -\frac{\Delta x_{ij}}{\Delta y_{ij}}(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i) & 0 \\ 0 & 0 & (\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j) & -(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j) \end{bmatrix}.$$

Next elementary row operation is $R_3 = \frac{R_3}{(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j)}$ with resulting observability matrix

$$T_6 = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R_{ij}^2 & 0 \\ 0 & \frac{1}{\Delta y_{ij}}(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i) & -\frac{\Delta x_{ij}}{\Delta y_{ij}}(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i) & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Perform elementary row operation $R_2 = R_2 + R_3 \frac{\Delta x_{ij}}{\Delta y_{ij}}(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i)$ to obtain

$$T_7 = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & 0 & -R_{ij}^2 \\ 0 & \frac{1}{\Delta y_{ij}}(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i) & 0 & -\frac{\Delta x_{ij}}{\Delta y_{ij}}(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i) \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Perform elementary row operation $R_2 = \frac{R_2 \Delta y_{ij}}{(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i)}$ to obtain

$$T_8 = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & 0 & -R_{ij}^2 \\ 0 & 1 & 0 & -\Delta x_{ij} \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Perform elementary row operation $R_1 = R_1 - R_2 \Delta x_{ij}$ to obtain

$$T_9 = \begin{bmatrix} -\Delta y_{ij} & 0 & 0 & -(\Delta y_{ij})^2 \\ 0 & 1 & 0 & -\Delta x_{ij} \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Perform elementary row operation $R_1 = -\frac{R_1}{\Delta y_{ij}}$ to obtain

$$T_{10} = \begin{bmatrix} 1 & 0 & 0 & \Delta y_{ij} \\ 0 & 1 & 0 & -\Delta x_{ij} \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Putting back the two columns related to x_j and y_j the row reduced echelon form(RREF) of the observability matrix can be written as

$$\bar{O}_{ij} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & \Delta y_{ij} \\ 0 & 1 & 0 & 0 & -1 & -\Delta x_{ij} \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}. \tag{1.11}$$

It can be verified that above matrix has three independent and these rows corresponds to $L^0 h$,$L^1_{f_{v_i}}$, and $L^1_{f_{v_j}}$. Therefore, the rank of the observability matrix of an edge is three if $V_i > 0$, $V_j > 0$, $\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i \neq 0$, and $\Delta x_{ij} s\psi_j - \Delta y_{ij} c\psi_j \neq 0$. ■

Lemma 2 *[Necessary condition] Rank of the observability matrix is less then three if*

1. $V_j = 0$
2. $\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i = 0$, $\Delta x_{ij} s\psi_j - \Delta y_{ij} c\psi_j = 0$, $w_i = 0$, *and* $w_j = 0$.

Proof: We hereafter show that when any of these two conditions is true, the rank of the observability matrix is always less then three.

1. If $V_j = 0$ then we do not consider Lie derivatives with respect to f_{v_j} and write the observability matrix as,

$$O_{ij} = \left[R^2_{ij} (\nabla L^0 h)^\top \ (\nabla L^1_{f_{v_i}} h)^\top \ (\nabla L^2_{f_{v_i} f_{\omega_i}} h)^\top \right]^\top$$
$$= \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R^2_{ij} & \Delta y_{ij} & -\Delta x_{ij} & 0 \\ s\psi_i & -c\psi_i & (\Delta x_{ij} c\psi_i + \Delta y_{ij} s\psi_i) & -s\psi_i & c\psi_i & 0 \\ c\psi_i & s\psi_i & (\Delta y_{ij} c\psi_i - \Delta x_{ij} s\psi_i) & -c\psi_i & -s\psi_i & 0 \end{bmatrix}.$$

To show that the rank of the above matrix is less than three we perform Gaussian elimination. It should be noted that the columns related to x_j and y_j (4^{th} and 5^{th}) are same but opposite to columns related to x_i and y_i (1^{st} and 2^{nd}) respectively. Therefore, to save space, we remove 4^{th} and 5^{th} column from the observability matrix. The resulting observability matrix has four columns and the order of column is x_i, y_i, ω_i, and ω_j.

$$T_1 = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R^2_{ij} & 0 \\ s\psi_i & -c\psi_i & (\Delta x_{ij} c\psi_i + \Delta y_{ij} s\psi_i) & 0 \\ c\psi_i & s\psi_i & (\Delta y_{ij} c\psi_i - \Delta x_{ij} s\psi_i) & 0 \end{bmatrix}.$$

First, we choose $-\Delta y_{ij}$ as pivot to perform the Gaussian elimination. Because $R_{ij} > 0$, without loss of generality, we assume $|\Delta y_{ij}| > 0$ and perform following elementary operations on the rows of the observability matrix. (1) $R_2 = R_2 + \frac{R_1}{\Delta y_{ij}} s\psi_i$ (2) $R_3 = R_3 + \frac{R_1}{\Delta y_{ij}} c\psi_i$.

$$T_2 = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R^2_{ij} & 0 \\ 0 & \frac{1}{\Delta y_{ij}} (\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i) & -\frac{\Delta x_{ij}}{\Delta y_{ij}} (\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i) & 0 \\ 0 & \frac{1}{\Delta y_{ij}} (\Delta x_{ij} c\psi_i + \Delta y_{ij} s\psi_i) & -\frac{\Delta x_{ij}}{\Delta y_{ij}} (\Delta x_{ij} c\psi_i + \Delta y_{ij} c\psi_i) & 0 \end{bmatrix}.$$

Assume that $(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i) \neq 0$ and perform $R_3 = R_3 - \frac{R_2 (\Delta x_{ij} c\psi_i + \Delta y_{ij} s\psi_i)}{(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i)}$.

$$T_3 = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R^2_{ij} & 0 \\ 0 & \frac{1}{\Delta y_{ij}} (\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i) & -\frac{\Delta x_{ij}}{\Delta y_{ij}} (\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly rank of the observability is less then three. Perform elementary row operation $R_2 = \frac{R_2 \Delta x_{ij}}{(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i)}$ to obtain

$$T_4 = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R^2_{ij} & 0 \\ 0 & 1 & -\Delta x_{ij} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Perform elementary row operation $R_1 = R_1 - R_2\Delta x_{ij}$ to obtain

$$T_5 = \begin{bmatrix} -\Delta y_{ij} & 0 & -(\Delta y_{ij})^2 & 0 \\ 0 & 1 & -\Delta x_{ij} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Perform elementary row operation $R_1 = -\frac{R_1}{\Delta y_{ij}}$ to obtain

$$T_6 = \begin{bmatrix} 1 & 0 & \Delta y_{ij} & 0 \\ 0 & 1 & -\Delta x_{ij} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Putting back the columns related to x_j and y_j in T_6 we can write the RREF of the observability matrix as

$$\bar{O}_{ij} = \begin{bmatrix} 1 & 0 & \Delta y_{ij} & -1 & 0 & 0 \\ 0 & 1 & -\Delta x_{ij} & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It can easily verified that two rows of the above matrix are linearly independent, therefore, the rank of the observability matrix with $V_j = 0$ is two.

2. If $V_i > 0$, $V_j > 0$, $\omega_i = 0$, and $\omega_j = 0$ then we can write the observability matrix as

$$\begin{aligned} O_{ij} &= \Big[R_{ij}^2(\nabla L^0 h)^\top \ (\nabla L^1_{f_{v_i}} h)^\top \ (L^1_{f_{v_j}} h)^\top \ (\nabla L^2_{f_{v_i} f_{v_j}} h)^\top\Big]^\top \\ &= \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R_{ij}^2 & \Delta y_{ij} & -\Delta x_{ij} & 0 \\ s\psi_i & -c\psi_i & (\Delta x_{ij} c\psi_i + \Delta y_{ij} s\psi_i) & -s\psi_i & c\psi_i & 0 \\ -s\psi_j & c\psi_j & 0 & s\psi_j & -c\psi_j & -(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j) \\ 0 & 0 & -(c\psi_i c\psi_j + s\psi_i s\psi_j) & 0 & 0 & (c\psi_i c\psi_j + s\psi_i s\psi_j) \end{bmatrix}. \end{aligned}$$

To show that the rank of the above matrix is less than three we perform Gaussian elimination. It should be noted that the columns related to x_j and y_j (4^{th} and 5^{th}) are same but opposite to columns related to x_i and y_i (1^{st} and 2^{nd}) respectively. Therefore, to save space, we remove 4^{th} and 5^{th} column from the observability matrix. The resulting observability matrix has four columns and the order of column is x_i, y_i, ω_i, and ω_j.

$$T_1 = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R_{ij}^2 & 0 \\ s\psi_i & -c\psi_i & (\Delta x_{ij} c\psi_i + \Delta y_{ij} s\psi_i) & 0 \\ -s\psi_j & c\psi_j & 0 & -(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j) \\ 0 & 0 & -(c\psi_i c\psi_j + s\psi_i s\psi_j) & (c\psi_i c\psi_j + s\psi_i s\psi_j) \end{bmatrix}.$$

First, we choose $-\Delta y_{ij}$ as pivot to perform the Gaussian elimination. Because $R_{ij} > 0$, without loss of generality, we assume $|\Delta y_{ij}| > 0$ and perform following elementary operations on the rows of the observability matrix. (1) $R_2 = R_2 + \frac{R_1}{\Delta y_{ij}} s\psi_i$ (2) $R_3 = R_3 - \frac{R_1}{\Delta y_{ij}} s\psi_j$ (3) $R_4 = R_4 + \frac{R_1}{\Delta y_{ij}} \Delta x_{ij}$.

$$T_2 = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R_{ij}^2 & 0 \\ 0 & \frac{1}{\Delta y_{ij}}(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i) & -\frac{\Delta x_{ij}}{\Delta y_{ij}}(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i) & 0 \\ 0 & -\frac{1}{\Delta y_{ij}}(\Delta x_{ij} s\psi_j - \Delta y_{ij} c\psi_j) & \frac{R_{ij}^2}{\Delta y_{ij}} s\psi_j & -(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j) \\ 0 & 0 & -(c\psi_i c\psi_j + s\psi_i s\psi_j) & (c\psi_i c\psi_j + s\psi_i s\psi_j) \end{bmatrix}.$$

If$(\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i) = 0$ and $(\Delta x_{ij} s\psi_j - \Delta y_{ij} c\psi_j) = 0$ then

$$T_2 = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R_{ij}^2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{R_{ij}^2}{\Delta y_{ij}} s\psi_j & -(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j) \\ 0 & 0 & -(c\psi_i c\psi_j + s\psi_i s\psi_j) & (c\psi_i c\psi_j + s\psi_i s\psi_j) \end{bmatrix}.$$

Perform elementary row operation $R_4 = -\frac{R_4}{(c\psi_i c\psi_j + s\psi_i s\psi_j)}$ to obtain

$$T_3 = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R_{ij}^2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{R_{ij}^2}{\Delta y_{ij}} s\psi_j & -(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j) \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Perform elementary row operation $R_3 = R_3 - R_2 \frac{R_{ij}^2}{\Delta y_{ij}} s\psi_j$ to obtain

$$T_4 = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R_{ij}^2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j) + \frac{R_{ij}^2}{\Delta y_{ij}} s\psi_j \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

where

$$\begin{aligned} -(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j) + \frac{R_{ij}^2}{\Delta y_{ij}} s\psi_j &= \frac{-\Delta x_{ij}\Delta y_{ij} c\psi_j - (\Delta y_{ij})^2 s\psi_j + R_{ij}^2 s\psi_j}{\Delta y_{ij}} \\ &= \frac{-\Delta x_{ij}\Delta y_{ij} c\psi_j + (\Delta x_{ij})^2 s\psi_j}{\Delta y_{ij}} \\ &= \frac{\Delta x_{ij}}{\Delta y_{ij}} (\Delta x_{ij} s\psi_j - \Delta y_{ij} c\psi_j) = 0, \end{aligned}$$

therefore,

$$T_4 = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R_{ij}^2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Putting back the two columns related to x_j and y_j the observability matrix can be written as

$$\bar{O}_{ij} = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R_{ij}^2 & -\Delta y_{ij} & \Delta x_{ij} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}.$$

It can be verified that two non zero rows in the above observability matrix are linearly independent, therefore, rank of the observability matrix is two if $\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i = 0$, $\Delta x_{ij} s\psi_j - \Delta y_{ij} c\psi_j = 0$, $w_i = 0$, and $w_j = 0$.

■

Remark 2 *In Lemma 2 if the vehicle which is not measuring the bearing is not moving ($V_j = 0$), then it will restrict the ability to estimate the relative heading. Next, if $\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i = 0$, $\Delta x_{ij} s\psi_j - \Delta y_{ij} c\psi_j = 0$, $w_i = 0$, and $w_j = 0$ then both the robots move along the line joining the two robots for all time. In other words bearing measurement is always zero, therefore, does not provide new information for state estimation.*

Corollary 1 *Following conditions*

1. $V_i \neq 0$.
2. $w_i \neq 0$.
3. $w_j \neq 0$.

are not necessary for $rank(O_{ij}) = 3$.

Proof:

1. In the proof of Lemma 1 it is shown that first three rows of the observability matrix comprising of $L^0 h$, $L^1_{f_{v_i}}$, and $L^1_{f_{v_j}}$ are linearly independent if $v_i > 0$, $v_j > 0$, and $\Delta x_{ij} s\psi_i - \Delta y_{ij} c\psi_i \neq 0$ and $\Delta x_{ij} s\psi_j - \Delta y_{ij} c\psi_j \neq 0$. Therefore, rank of the observability matrix can be three with $\omega_i = 0$ and $\omega_j = 0$.

2. If $v_i = 0$ observability matrix will not have rows related to Lie derivative with respect to input f_{v_i}. Therefore, we can write the observability matrix as

$$O_{ij} = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R_{ij}^2 & \Delta y_{ij} & -\Delta x_{ij} & 0 \\ -s\psi_j & c\psi_j & 0 & s\psi_j & -c\psi_j & -(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j) \\ -c\psi_j & -s\psi_j & 0 & c\psi_j & s\psi_j & -(\Delta y_{ij} c\psi_j - \Delta x_{ij} s\psi_j) \end{bmatrix}.$$

It should be noted that the columns related to x_j and y_j (4^{th} and 5^{th}) are same but opposite to columns related to x_i and y_i (1^{st} and 2^{nd}) respectively. Therefore, to save space, we remove 4^{th} and 5^{th} column from the observability matrix. The resulting observability matrix has four columns and the order of column is x_i, y_i, ω_i, and ω_j.

$$T_1 = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R_{ij}^2 & 0 \\ -s\psi_j & c\psi_j & 0 & -(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j) \\ -c\psi_j & -s\psi_j & 0 & -(\Delta y_{ij} c\psi_j - \Delta x_{ij} s\psi_j) \end{bmatrix}.$$

We perform gaussian elimination find the number of linearly independent rows in the observability matrix.
Perform elementary operation $R_2 = R_2 - \frac{R_1 s\psi_j}{\Delta y_{ij}}$ and $R_3 = R_3 - \frac{R_1 c\psi_j}{\Delta y_{ij}}$ to obtain

$$T_2 = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R_{ij}^2 & 0 \\ 0 & -\frac{1}{\Delta y_{ij}}(\Delta x_{ij} s\psi_j - \Delta y_{ij} c\psi_j) & \frac{R_{ij}^2}{\Delta y_{ij}} s\psi_j & -(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j) \\ 0 & -\frac{1}{\Delta y_{ij}}(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j) & \frac{R_{ij}^2}{\Delta y_{ij}} c\psi_j & -(\Delta y_{ij} c\psi_j - \Delta x_{ij} s\psi_j) \end{bmatrix}.$$

Perform elementary operation $R_3 = R_3 - \frac{R_2 c\psi_j}{s\psi_j}$ to obtain

$$T_3 = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R_{ij}^2 & 0 \\ 0 & -\frac{1}{\Delta y_{ij}}(\Delta x_{ij} s\psi_j - \Delta y_{ij} c\psi_j) & \frac{R_{ij}^2}{\Delta y_{ij}} s\psi_j & -(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j) \\ 0 & -\frac{1}{s\psi_j} & 0 & \frac{\Delta x_{ij}}{s\psi_j} \end{bmatrix}.$$

Exchange R_2 and R_3 to obtain

$$T_4 = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R_{ij}^2 & 0 \\ 0 & -\frac{1}{s\psi_j} & 0 & \frac{\Delta x_{ij}}{s\psi_j} \\ 0 & -\frac{1}{\Delta y_{ij}}(\Delta x_{ij} s\psi_j - \Delta y_{ij} c\psi_j) & \frac{R_{ij}^2}{\Delta y_{ij}} s\psi_j & -(\Delta x_{ij} c\psi_j + \Delta y_{ij} s\psi_j) \end{bmatrix}.$$

Perform elementary operation $R_2 = -R_2 s\psi_j$ and $R_3 = R_3 + R_2 \frac{1}{\Delta y_{ij}}(\Delta x_{ij} s\psi_j - \Delta y_{ij} c\psi_j)$ to obtain

$$T_5 = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R_{ij}^2 & 0 \\ 0 & 1 & 0 & -\Delta x_{ij} \\ 0 & 0 & \frac{R_{ij}^2}{\Delta y_{ij}} s\psi_j & -\frac{R_{ij}^2}{\Delta y_{ij}} s\psi_j \end{bmatrix}.$$

Perform elementary operation $R_3 = fracR_3 \Delta y_{ij} R_{ij}^2 s\psi_j$ to obtain

$$T_6 = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R_{ij}^2 & 0 \\ 0 & 1 & 0 & -\Delta x_{ij} \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Perform elementary row operation $R_1 = R_1 - R_2 \Delta x_{ij}$ to obtain

$$T_7 = \begin{bmatrix} -\Delta y_{ij} & 0 & 0 & -(\Delta y_{ij})^2 \\ 0 & 1 & 0 & -\Delta x_{ij} \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Perform elementary row operation $R_1 = -\frac{R_1}{\Delta y_{ij}}$ to obtain

$$T_8 = \begin{bmatrix} 1 & 0 & 0 & \Delta y_{ij} \\ 0 & 1 & 0 & -\Delta x_{ij} \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Putting back the two columns related to x_j and y_j the observability matrix can be written as

$$\bar{O}_{ij} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & \Delta y_{ij} \\ 0 & 1 & 0 & 0 & -1 & -\Delta x_{ij} \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}. \tag{1.12}$$

It can be verified that the observability matrix has three independent rows, therefore rank of the observability matrix is three.

■

Edge between a robot and a landmark

In this section we derive the observability conditions for an edge η_{il} between a robot and a landmark. We rearrange the nonlinear kinematic equations in the following convenient form for computing Lie derivatives:

$$\dot{X}_i = \underbrace{\begin{bmatrix} c\psi_i \\ s\psi_i \\ 0 \end{bmatrix}}_{f_{v_i}} V_i + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{f_{\omega_i}} \omega_i \tag{1.13}$$

We hereafter compute the necessary Lie derivatives of η_{il} and their gradients.

1) Zeroth-order Lie derivative ($L^0 h$)

$$L^0 h = \eta_{il}$$

with gradient

$$\nabla L^0 h = \begin{bmatrix} -\frac{\Delta y_{il}}{R_{il}^2} & \frac{\Delta x_{il}}{R_{il}^2} & -1 \end{bmatrix}$$

where, $\Delta y_{il} = x_i - x_l$,$\Delta x_{ij} = x_i - x_l$, and $R_{il}^2 = \Delta^2 x_{il} + \Delta^2 y_{il}$.

2) First-order Lie derivatives($L^1_{f_{v_i}} h$, $L^1_{f_{\omega_i}} h$)

$$L^1_{f_{v_i}} = \frac{\Delta x_{il} s\psi_i - \Delta y_{il} c\psi_i}{R_{il}^2}$$
$$L^1_{f_{\omega_i}} = -1$$

To simplify the derivation of higher order Lie derivatives we obtain the the gradient of scaled $L^1_{f_{v_i}} = \Delta x_{il} s\psi_i - \Delta y_{il} c\psi_i$

$$\nabla L^1_{f_{v_i}} = \begin{bmatrix} s\psi_i & -c\psi_i & \Delta x_{il} c\psi_i + \Delta y_{il} s\psi_i \end{bmatrix}$$

3) Second-order Lie derivatives($L^2_{f_{v_i} f_{v_i}} h$, $L^2_{f_{v_i} f_{\omega_i}} h$)

$$L^2_{f_{v_i} f_{v_i}} h = s\psi_i c\psi_i - s\psi_i c\psi_i = 0$$
$$L^2_{f_{v_i} f_{\omega_i}} h = \Delta x_{il} c\psi_i + \Delta y_{il} s\psi_i$$

with gradients

$$L^2_{f_{v_i} f_{\omega_i}} h = \begin{bmatrix} c\psi_i & s\psi_i & \Delta y_{il} c\psi_i - \Delta x_{il} s\psi_i \end{bmatrix}.$$

4) Third-order Lie derivatives ($L^3_{f_{v_i} f_{\omega_i} f_{v_i}}$, $L^3_{f_{v_i} f_{\omega_i} f_{\omega_i}}$)

$$L^3_{f_{v_i} f_{\omega_i} f_{v_i}} = 1$$
$$L^3_{f_{v_i} f_{\omega_i} f_{\omega_i}} = -(\Delta x_{il} s\psi_i - \Delta y_{il} c\psi_i) = -L^1_{f_{v_i}}.$$

It can be verified that all the gradients of third-order Lie derivatives are linearly dependent on lower order Lie derivatives. Therefore, third and higher order Lie derivatives do not contribute any independent row to the observability matrix.

Lemma 3 *[Sufficient and Necessary] Rank of the observability matrix of an edge between a robot and a landmark is two if and only if*

1. $V_i > 0$.
2. $\Delta x_{il} s\psi_i - \Delta y_{il} c\psi_i \neq 0$.

Proof: If $V_i > 0$, $\omega_i \neq 0$ then using gradients of Lie derivatives we can write the observability matrix as

$$\begin{aligned} O_{il} &= \left[R^2_{il} (\nabla L^0 h)^\top \; (\nabla L^1_{f_{v_i}})^\top \; (\nabla L^2_{f_{v_i} f_{\omega_i}})^\top \right]^\top \\ &= \begin{bmatrix} -\Delta y_{il} & \Delta x_{il} & -R^2_{il} \\ s\psi_i & -c\psi_i & \Delta x_{il} c\psi_i + \Delta y_{il} s\psi_i \\ c\psi_i & s\psi_i & \Delta y_{il} c\psi_i - \Delta x_{il} s\psi_i \end{bmatrix}. \end{aligned} \tag{1.14}$$

We perform Gaussian elimination to find out the number of linearly independent rows.

Perform elementary row operations (1) $R_2 = R_2 + \frac{R_1}{\Delta y_{ij}} s\psi_i$ (2) $R_6 = R_6 + \frac{R_1}{\Delta y_{ij}} c\psi_i$ to obtain

$$T_1 = \begin{bmatrix} -\Delta y_{il} & \Delta x_{il} & -R^2_{il} \\ 0 & \frac{1}{\Delta y_{il}} (\Delta x_{il} s\psi_i - \Delta y_{il} c\psi_i) & -\frac{\Delta x_{il}}{\Delta y_{il}} (\Delta x_{il} s\psi_i - \Delta y_{il} c\psi_i) \\ 0 & \frac{1}{\Delta y_{il}} (\Delta x_{il} c\psi_i + \Delta y_{il} s\psi_i) & -\frac{\Delta x_{il}}{\Delta y_{il}} (\Delta x_{il} c\psi_i + \Delta y_{il} s\psi_i) \end{bmatrix}.$$

If $(\Delta x_{il} s\psi_i - \Delta y_{il} c\psi_i) \neq 0$ then we perform elementary row operation $R_3 = R_3 - \frac{R_2(\Delta x_{il} c\psi_i + \Delta y_{il} s\psi_i)}{(\Delta x_{il} s\psi_i - \Delta y_{il} c\psi_i)}$ to obtain

$$T_2 = \begin{bmatrix} -\Delta y_{il} & \Delta x_{il} & -R^2_{il} \\ 0 & \frac{1}{\Delta y_{il}} (\Delta x_{il} s\psi_i - \Delta y_{il} c\psi_i) & -\frac{\Delta x_{il}}{\Delta y_{il}} (\Delta x_{il} s\psi_i - \Delta y_{il} c\psi_i) \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly, the maximum possible rank of the observability matrix of an edge between a landmark and a robot is two.

Perform elementary row operation $R_2 = \frac{R_2 \Delta y_{il}}{(\Delta x_{il} s\psi_i - \Delta y_{il} c\psi_i)}$ to obtain

$$T_3 = \begin{bmatrix} -\Delta y_{il} & \Delta x_{il} & -R^2_{il} \\ 0 & 1 & -\Delta x_{il} \\ 0 & 0 & 0 \end{bmatrix}.$$

Perform elementary row operation $R_1 = R_1 - R_2\Delta x_{il}$ to obtain

$$T_4 = \begin{bmatrix} -\Delta y_{il} & 0 & -(\Delta y_{il})^2 \\ 0 & 1 & -\Delta x_{il} \\ 0 & 0 & 0 \end{bmatrix}$$

Perform elementary row operation $R_1 = -\frac{R_1}{\Delta y_{il}}$ to obtain

$$\bar{O}_{il} = \begin{bmatrix} 1 & 0 & \Delta y_{il} \\ 0 & 1 & -\Delta x_{il} \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.15}$$

It can be verified that the two rows in the above observability matrix are linearly independent and the top two non zero rows correspond to L^0h and $L^1_{f_{v_i}}$, therefore, $V_i > 0$ and $\Delta x_{il}s\psi_i - \Delta y_{il}c\psi_i \neq 0$ is sufficient for rank of the observability matrix being two.

Hereafter we show that if any of the conditions mentioned in the lemma are not satisfied then rank of the observability matrix is two.

1. If $V_i = 0$ then $O_{ij} = \nabla L^0 h$, which implies that the observability matrix is a row vector, therefore, rank of the observability matrix is one.

2. If $V_i > 0$, $\omega_i \neq 0$ then using gradients of Lie derivatives we can write the observability matrix as

$$\begin{aligned} O_{il} &= \left[R^2_{il}(\nabla L^0 h)^\top \ (\nabla L^1_{f_{v_i}})^\top \ (\nabla L^2_{f_{v_i} f_{\omega_i}})^\top\right]^\top \\ &= \begin{bmatrix} -\Delta y_{il} & \Delta x_{il} & -R^2_{il} \\ s\psi_i & -c\psi_i & \Delta x_{il}c\psi_i + \Delta y_{il}s\psi_i \\ c\psi_i & s\psi_i & \Delta y_{il}c\psi_i - \Delta x_{il}s\psi_i \end{bmatrix}. \end{aligned}$$

   To find the number of linearly independent rows we perform Gaussian elimination.
   Perform elementary row operations (1) $R_2 = R_2 + \frac{R_1}{\Delta y_{ij}}s\psi_i$ (2) $R_6 = R_6 + \frac{R_1}{\Delta y_{ij}}c\psi_i$ to obtain

$$T_1 = \begin{bmatrix} -\Delta y_{il} & \Delta x_{il} & -R^2_{il} \\ 0 & \frac{1}{\Delta y_{il}}(\Delta x_{il}s\psi_i - \Delta y_{il}c\psi_i) & -\frac{\Delta x_{il}}{\Delta y_{il}}(\Delta x_{il}s\psi_i - \Delta y_{il}c\psi_i) \\ 0 & \frac{1}{\Delta y_{il}}(\Delta x_{il}c\psi_i + \Delta y_{il}s\psi_i) & -\frac{\Delta x_{il}}{\Delta y_{il}}(\Delta x_{il}c\psi_i + \Delta y_{il}s\psi_i) \end{bmatrix}.$$

   If $(\Delta x_{il}s\psi_i - \Delta y_{il}c\psi_i) = 0$ then $(\Delta x_{il}c\psi_i + \Delta y_{il}s\psi_i) = 0$ and

$$T_1 = \begin{bmatrix} -\Delta y_{il} & \Delta x_{il} & -R^2_{il} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

   Therefore, rank of observability matrix is one if $(\Delta x_{il}s\psi_i - \Delta y_{il}c\psi_i) = 0$.

■

Remark 3 *In Lemma 3 if the vehicle is stationary $V_i = 0$ because landmark is also stationary the bearing measurement remains constant, therefore, it does not provide any information for the state estimation. Next if $\Delta x_{il}s\psi_i - \Delta y_{il}c\psi_i = 0$ then the vehicle moves along the line joining the vehicle and the landmark and the bearing measurement always remains zero and the measurement does not provide new information.*

Definition 2 *An RPMG G^l_n is called a proper RPMG if all the edges between robot nodes satisfy the conditions of Lemma 1 and all the edges between robots and landmarks satisfy Lemma 3.*

Remark 4 *Form observability point an edge between two vehicle nodes in the proper RPMG is bidirectional because each edge contributes three independent rows to the observability matrix.*

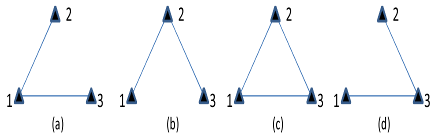

Figure 1.2: The observability conditions between these four possible configurations of a connected, 3-node RPMG are identical.

From (1.11) we can write the basis for the row space of an observability matrix of an edge between two robots in the proper graph G_n^l as,

$$\begin{bmatrix} 0_{3\times 3(i-1)} & \bar{O}_i^{ij} & 0_{3\times(3(j-1)-3i)} & \bar{O}_j^{ij} & 0_{3\times(n-3j)} \end{bmatrix}, \tag{1.16}$$

where $\bar{O}_i^{ij} = \mathbf{I}_3$, and $\bar{O}_j^{ij} = \begin{bmatrix} -1 & 0 & \Delta y_{ij} \\ 0 & -1 & \Delta x_{ij} \\ 0 & 0 & -1 \end{bmatrix}$.

Similarly, from (1.15) we can write the basis for the row space of an observability matrix of an edge between a robot and a landmark in the proper RPMG G_n^l as,

$$[0_{2\times 3(i-1)})\ \bar{O}_i^{il}\ 0_{2\times 3(n-i)})]. \tag{1.17}$$

We can write the observability matrix for a proper graph G_n^l by stacking the observability basis of all the edges.

Remark 5 *The observability matrix obtained using basis of edges given in (1.16) and (1.17) is not the original observability matrix of the graph G_n^l formed using the actual observability matrix of edges given in (1.10) and (1.14). However, the modified observability matrix spans the same observable space as the original matrix.*

1.3 Observability of three nodes

Lemma 4 *If a three node proper RPMG G_3^0 is connected then the rank of observability matrix is six.*

Proof: There are four possible configurations of a connected graph G_3^0, shown as sub-figures (a) through (d) in Fig. 1.2. We can write the observability matrix for these configurations using (1.16) as

$$O_a = \begin{bmatrix} O_1^{12} & O_2^{12} & \mathbf{0} \\ O_1^{13} & \mathbf{0} & O_3^{13} \end{bmatrix} \quad O_b = \begin{bmatrix} O_1^{12} & O_j^{12} & \mathbf{0} \\ \mathbf{0} & O_2^{23} & O_3^{23} \end{bmatrix}$$
$$O_c = \begin{bmatrix} O_1^{12} & O_2^{12} & \mathbf{0} \\ \mathbf{0} & O_2^{23} & O_3^{23} \\ O_1^{13} & \mathbf{0} & O_3^{13} \end{bmatrix} \quad O_d = \begin{bmatrix} O_1^{13} & \mathbf{0} & O_3^{13} \\ \mathbf{0} & O_2^{23} & O_3^{23} \end{bmatrix}.$$

First we perform gaussian elimination on O_a. Plugging values of matrices O_1^{12}, O_2^{12}, O_1^{13}, and O_3^{13} we get,

$$O_a = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & \Delta y_{12} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -\Delta x_{12} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & \Delta y_{13} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -\Delta x_{13} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}. \tag{1.18}$$

By performing following elementary operations $R_4 = R_4 - R_1$, $R_5 = R_5 - R_2$, and $R_6 = R_6 - R_3$ we get

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & \Delta y_{12} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -\Delta x_{12} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\Delta y_{12} & -1 & 0 & \Delta y_{13} \\ 0 & 0 & 0 & 0 & 1 & \Delta x_{12} & 0 & -1 & -\Delta x_{13} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}. \tag{1.19}$$

Performing following elementary operations $R_1 = R_1 + R_4$, $R_2 = R_2 + R_5$, and $R_3 = R_3 + R_1$ we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & \Delta y_{13} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -\Delta y_{13} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -\Delta y_{12} & -1 & 0 & \Delta y_{13} \\ 0 & 0 & 0 & 0 & 1 & \Delta x_{12} & 0 & -1 & -\Delta x_{13} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}. \tag{1.20}$$

Performing following elementary operations $R_4 = R_4 + R_6\Delta y_{12}$ and $R_5 = R_5 - R_6\Delta x_{12}$ we obtain

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & \Delta y_{13} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -\Delta x_{13} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & \Delta y_{23} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -\Delta x_{13} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \tag{1.21}$$

Therefore,

$$\bar{O}_a = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} & O_3^{13} \\ \mathbf{0} & \mathbf{I}_3 & O_3^{23} \end{bmatrix} \tag{1.22}$$

Similarly, we can perform gaussian elimination and elementary ro operations on O_b and O_c and show that $\bar{O}_a = \bar{O}_b = \bar{O}_c = O_d$ and there rank is six. ■

1.4 Observability with landmarks

Consider RPMG (G_2^1) with two robot nodes and one landmark. We can write observability matrix of (G_2^1) where landmark l is connected to node i,

$$\mathcal{O} \;=\; \begin{bmatrix} O_i^{ij} & O_j^{ij} \\ O_{il} & 0_{2\times 3} \end{bmatrix}. \tag{1.23}$$

Plugging values of matrices O_i^{ij}, and O_{il} we get,

$$O = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & \Delta y_{ij} \\ 0 & 1 & 0 & 0 & -1 & -\Delta x_{ij} \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & \Delta y_{il} & 0 & 0 & 0 \\ 0 & 1 & -\Delta x_{il} & 0 & 0 & 0 \end{bmatrix} \tag{1.24}$$

Performing following elementary operations $R_4 = R_4 - R_1$ and $R_5 = R_5 - R_2$ we obtain

$$= \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & \Delta y_{ij} \\ 0 & 1 & 0 & 0 & -1 & -\Delta x_{ij} \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & \Delta y_{il} & 1 & 0 & -\Delta y_{ij}) \\ 0 & 0 & -\Delta x_{il} & 0 & 1 & \Delta x_{ij} \end{bmatrix} \tag{1.25}$$

Performing following elementary operations $R4 = R4 - R3\Delta y_{ik}$ and $R_5 = R_5 + R_3\Delta x_{ik}$ we obtain

$$= \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & \Delta y_{ij} \\ 0 & 1 & 0 & 0 & -1 & -\Delta x_{ij} \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & \Delta y_{jl} \\ 0 & 0 & 0 & 0 & 1 & -\Delta x_{il} \end{bmatrix} \tag{1.26}$$

$$= \begin{bmatrix} O_i^{ij} & O_j^{ij} \\ 0_{2\times 3} & O_j^{jl} \end{bmatrix}. \tag{1.27}$$

This is the observability matrix for landmark l connected to node i is equal to landmark p connected to node j.

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