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The Relationship Between the Minimal Rank of a Tree and the Rank-Spreads of the Vertices and Edges

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THE RELATIONSHIP BETWEEN THE MINIMAL RANK OF A TREE AND THE RANK-SPREADS OF THE VERTICES AND EDGES

by

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GRADUATE COMMITTEE APPROVAL

of a thesis submitted by

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ABSTRACT

THE RELATIONSHIP BETWEEN THE MINIMAL RANK OF A TREE AND THE RANK-SPREADS OF THE VERTICES AND EDGES

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Master of Science

Let F be a field, $G = (V, E)$ be an undirected graph on n vertices, and let $S(F, G)$ be the set of all symmetric $n \times n$ matrices whose nonzero off-diagonal entries occur in exactly the positions corresponding to the edges of G. Let $\mathrm{mr}(F, G)$ be the minimum rank over all matrices in $S(F, G)$. We give a field independent proof of a well-known result that for a tree the sum of its path cover number and minimal rank is equal to the number of vertices in the tree. The rank-spread of a vertex v of G is the difference between the minimal ranks of G and $G - v$, the graph obtained by deleting v and all its incident edges from G. The rank-spread of an edge is defined similarly. We derive a formula that expresses the minimal rank of a tree as the difference of sums of rank-spreads, the first being the sum of the rank-spreads of all the vertices and the second the sum of the rank-spreads of all the edges. We show that this is a special case of a more general inequality for all graphs. In proving the above results we explore how rank-spreads change as graphs are vertex-summed.

ACKNOWLEDGEMENTS

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Chapter 1

Preliminaries

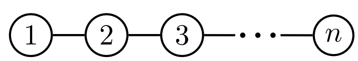
Definition 1

Let F be a field. For any graph $G = (V, E)$ with $V = \{1, 2, \ldots, n\}$ (all graphs in this thesis are considered undirected and simple), let $S(F, G)$ be the set of all symmetric $n \times n$ matrices $A = (a_{i,j})$ with entries in F such that $a_{i,j} \neq 0$, $i \neq j$, if and only if $ij \in E$. There is no restriction on the main diagonal entries of A. Then the *minimal rank of G over F* denoted $\mathrm{mr}(F, G)$ is the minimum rank of all matrices in $S(F, G)$. This may also be written as

$$\mathrm{mr}(F, G) = \min\{\operatorname{rank} A \mid A \in S(F, G)\}.$$

Example 1

Consider the path on $n \geq 2$ vertices, P_n, labeled



If $A \in S(F, P_n)$, then

$$A = \begin{bmatrix} b_1 & a_1 & 0 & \dots & 0 & 0 & 0 \\ a_1 & b_2 & a_2 & \ddots & 0 & 0 & 0 \\ 0 & a_2 & b_3 & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & b_{n-2} & a_{n-2} & 0 \\ 0 & 0 & 0 & \ddots & a_{n-2} & b_{n-1} & a_{n-1} \\ 0 & 0 & 0 & \dots & 0 & a_{n-1} & b_n \end{bmatrix} \text{ with } a_i \neq 0 \text{ for all } i.$$

Then

$$\text{rank } A \geq \text{rank} \begin{bmatrix} a_1 & 0 & \dots & 0 & 0 & 0 \\ b_2 & a_2 & \ddots & 0 & 0 & 0 \\ a_2 & b_3 & \ddots & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & b_{n-2} & a_{n-2} & 0 \\ 0 & 0 & \ddots & a_{n-2} & b_{n-1} & a_{n-1} \end{bmatrix} = n-1.$$

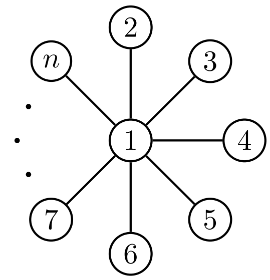
Since

$$\begin{bmatrix} 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 1 & \ddots & 0 & 0 & 0 \\ 0 & 1 & 1 & & 0 & 0 & 0 \\ \vdots & \ddots & & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix} + \dots + \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 \end{bmatrix}$$

$\in S(F, P_n)$ then $\text{mr}(F, P_n) = n - 1$.

Example 2

Consider the star on $n \geq 3$ vertices, S_n, labeled



If $A \in S(F, S_n)$, then

$$A = \begin{bmatrix} b_1 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ a_1 & b_2 & 0 & \dots & 0 & 0 \\ a_2 & 0 & b_3 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n-2} & 0 & 0 & \ddots & b_{n-1} & 0 \\ a_{n-1} & 0 & 0 & \dots & 0 & b_n \end{bmatrix} \text{ with } a_i \neq 0 \text{ for all } i.$$

Since $\begin{vmatrix} b_1 & a_2 \\ a_1 & 0 \end{vmatrix} \neq 0$, then rank $A \geq 2$.

Since

$$\begin{bmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \in S(F, S_n)$$

then $\operatorname{mr}(F, S_n) = 2$.

As all of the results in this thesis are field independent, the field F will be left out and simply $\operatorname{mr}(G)$ and $S(G)$ will be used.

Definition 2

The *path cover number* of a graph G denoted $P(G)$ is the minimum number of vertex disjoint paths, occurring as induced subgraphs of G, which cover all the vertices of G.

Example 3

Certainly P_n can be covered by one path. Thus $P(P_n) = 1$ for all $n \geq 2$.

Consider the graph S_n labeled as in Example 2. Let R be a minimal path cover for S_n. Let p be the path in R which covers vertex 1. Note that p can cover at most three vertices since any more than three would not induce a path. Any vertices not covered by p must necessarily be covered by single vertex covers since all vertices except vertex 1 are pendant. Thus by the minimality of R, p must cover 3 vertices. Therefore $|R| = n-2$, and $P(S_n) = n-2$.

Definition 3

Let G be a graph and $v \in V(G)$. The graph $G-v$ is obtained from G by deleting the vertex v from G and any edges incident to v. In other words $G-v$ is the subgraph induced by all the vertices of G except v.

Definition 4

Let G be a graph and $e \in E(G)$. The graph $G-e$ is obtained from G by deleting the edge e from G.

Definition 5

Let v be a vertex or edge of a graph G. The *rank-spread* of v in G is $\mathrm{mr}(G) - \mathrm{mr}(G-v)$ and is denoted $r_v(G)$.

Observation 1 ([BvdHL04, Observation 1])

$\mathrm{mr}(G) = 1$ if and only if $G = K_m \bigcup K_{n-m}^c, m \geq 2$.

Note that if G is a connected graph on $n \geq 2$ vertices, then $\mathrm{mr}(G) = 1$ if and only if G is complete.

Observation 2 ([Nyl96, Proposition 2.1])

If $G = \displaystyle\bigcup_{i=1}^{k} G_i$, then $\mathrm{mr}(G) = \displaystyle\sum_{i=1}^{k} \mathrm{mr}(G_i)$.

Observation 3

$\mathrm{mr}(K_1) = 0$ and $\mathrm{mr}(nK_1) = 0$

Lemma 4 ([Nyl96, Proposition 2.1])

(1) For any vertex v in G, $\mathrm{mr}(G-v) + 2 \geq \mathrm{mr}(G) \geq \mathrm{mr}(G-v)$.

(2) For any edge e in G, $\mathrm{mr}(G-e) + 1 \geq \mathrm{mr}(G) \geq \mathrm{mr}(G-e) - 1$.

Corollary 5

(1) For any vertex v in G, $2 \geq r_v(G) \geq 0$.

(2) For any edge e in G, $1 \geq r_e(G) \geq -1$.

Proof Using Lemma 4 and subtracting $\mathrm{mr}(G-v)$ and $\mathrm{mr}(G-e)$ in (3) and (4) respectively we have that $2 \geq \mathrm{mr}(G) - \mathrm{mr}(G-v) \geq 0$ and $1 \geq \mathrm{mr}(G) - \mathrm{mr}(G-e) \geq -1$. The result follows from the definition of rank-spread. ∎

Corollary 6

If $p \in V(G)$ is pendant, and $e \in E(G)$ is adjacent to p, then $r_e(G) = r_p(G)$.

Proof Let p be a pendant vertex of G and e the edge adjacent to p. Note that $(G-p)\bigcup K_1 = G - e$. In other words deleting p from G and adding an isolated vertex is equivalent to deleting the edge e. By Observation 2, $\mathrm{mr}((G-p)\bigcup K_1) = \mathrm{mr}(G-p) + \mathrm{mr}(K_1)$. By Observation 3, $\mathrm{mr}(K_1) = 0$. Thus $\mathrm{mr}(G-p) = \mathrm{mr}(G-e)$ and by definition of rank-spread $r_p(G) = r_e(G)$. ∎

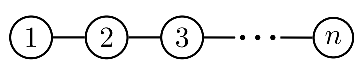
Corollary 7

If p is a pendant vertex of G, then $1 \geq r_p(G) \geq 0$.

Proof By Corollary 6, $r_e(G) = r_p(G)$. The result follows from Corollary 5 parts (1) and (2). ∎

Example 4

Consider the path on $n \geq 2$ vertices, P_n, labeled



Let p be a pendant vertex of P_n. By definition of rank-spread

$$r_p(P_n) = \mathrm{mr}(P_n) - \mathrm{mr}(P_n - p) = \mathrm{mr}(P_n) - \mathrm{mr}(P_{n-1}) = n - 1 - (n-2) = 1.$$

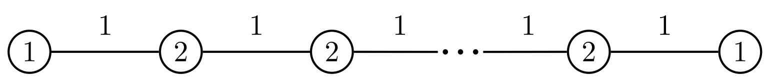
Let v be a non-pendant vertex of P_n. Assume that v is the kth vertex of P_n. Then by definition of rank-spread, $r_v(P_n)$ is equal to

$$\mathrm{mr}(P_n) - \mathrm{mr}(P_n - v) = \mathrm{mr}(P_n) - (\mathrm{mr}(P_{k-1}) + \mathrm{mr}(P_{n-k})) = n - 1 - ((k-2) + (n-k-1)) = 2.$$

Label the edges so that the kth edge is between the kth vertex and the $k+1$st vertex of P_n. Note that $k \in \{1, 2, \ldots, n-1\}$. Let e be the kth edge of P_n. By definition of rank-spread, $r_e(P_n)$ is equal to

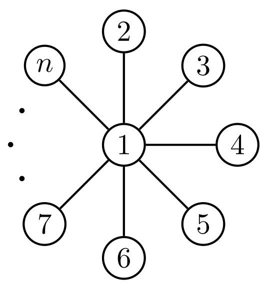
$$\mathrm{mr}(P_n) - \mathrm{mr}(P_n - e) = \mathrm{mr}(P_n) - (\mathrm{mr}(P_k) + \mathrm{mr}(P_{n-k})) = n - 1 - ((k-1) + (n-k-1)) = 1.$$

Relabeling the vertices and edges with their corresponding rank-spreads the graph of P_n is



Example 5

Consider the star on $n \geq 3$ vertices, S_n, labeled



Since $S_3 = P_3$, the rank-spreads on edges and vertices are known from Example 1. So assume $n \geq 4$.

Let v be the dominating vertex of S_n. By definition of rank-spread

$$r_v(S_n) = \mathrm{mr}(S_n) - \mathrm{mr}(S_n - v) = \mathrm{mr}(S_n) - \mathrm{mr}((n-1)K_1).$$

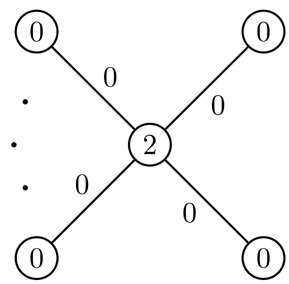
By Observation 3, $\mathrm{mr}((n-1)K_1) = 0$. Thus $r_v(S_n) = 2$.

Let p be a pendant vertex of S_n where $n \geq 4$. By definition of rank-spread

$$r_p(S_n) = \mathrm{mr}(S_n) - \mathrm{mr}(S_n - p) = \mathrm{mr}(S_n) - \mathrm{mr}(S_{n-1}) = 2 - 2 = 0.$$

Note that every edge is adjacent to a pendant vertex in S_n. By Corollary 6, $r_e(S_n) = r_p(S_n)$ for every edge $e \in E(S_n)$.

Relabeling the vertices and edges with their corresponding rank-spreads the graph of S_n, $n \geq 4$ is

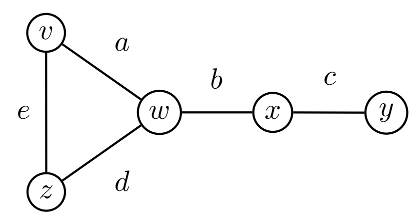


Observation 8 ([BvdHL04, Observation 5])

If H is an induced subgraph of G, then $\mathrm{mr}(G) \geq \mathrm{mr}(H)$.

Example 6

The full range of Corollary 5 is possible. Consider the following graph G



Note that $G - z = P_4$. By Example 1, $\mathrm{mr}(P_4) = 3$. By Observation 8, $\mathrm{mr}(G) \geq \mathrm{mr}(P_4) = 3$. Label the vertices in alphabetical order v, w, x, y, z and let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

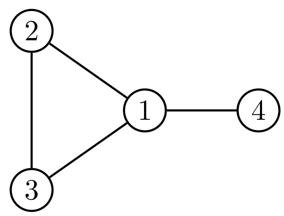
Note that $A \in S(G)$ and that rank $A = 3$. Therefore $\mathrm{mr}(G) = 3$.

Using the definition of rank-spread $r_v(G) = \mathrm{mr}(G) - \mathrm{mr}(P_4) = 3 - 3 = 0$. The same is true for $r_z(G)$.

By Observation 1 and Observation 2, $\mathrm{mr}(2K_2) = \mathrm{mr}(K_2) + \mathrm{mr}(K_2) = 1 + 1 = 2$. Thus $r_w(G) = \mathrm{mr}(G) - \mathrm{mr}(2K_2) = 3 - 2 = 1$.

By Observation 1, $\mathrm{mr}(K_3 \bigcup K_1) = 1$. Thus $r_x(G) = \mathrm{mr}(G) - \mathrm{mr}(K_3 \bigcup K_1) = 2$.

Note that $G-y$ is called the paw. Since P_3 is an induced subgraph of the paw, then by Observation 8, $\mathrm{mr}(\text{paw}) \geq \mathrm{mr}(P_3) = 2$. Label the vertices of the paw as follows



Let

$$B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Note that $B \in S(\text{paw})$ and that $\operatorname{rank} B = 2$. Therefore $\mathrm{mr}(\text{paw}) = 2$.

Thus $r_y(G) = \mathrm{mr}(G) - \mathrm{mr}(G-y) = \mathrm{mr}(G) - \mathrm{mr}(\text{paw}) = 3 - 2 = 1$.

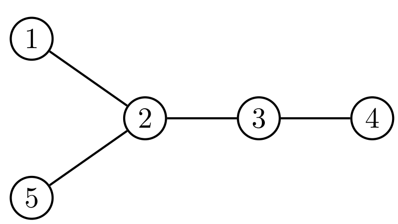
Now consider the rank-spread on the edges of G. By Example 1 $\mathrm{mr}(P_5) = 4$. Thus $r_a(G) = \mathrm{mr}(G) - \mathrm{mr}(P_5) = 3 - 4 = -1$. The same is true for $r_d(G)$ by symmetry.

By Observation 1 and Observation 2, $\mathrm{mr}(K_3 \bigcup K_2) = \mathrm{mr}(K_3) + \mathrm{mr}(K_2) = 1 + 1 = 2$. Thus $r_b(G) = \mathrm{mr}(G) - \mathrm{mr}(G-b) = \mathrm{mr}(G) - \mathrm{mr}(K_3 \bigcup K_2) = 3 - 2 = 1$.

By Observation 2 and Observation 3, $\mathrm{mr}(\text{paw} \bigcup K_1) = \mathrm{mr}(\text{paw}) + \mathrm{mr}(K_1) = 2$. Thus $r_c(G) = \mathrm{mr}(G) - \mathrm{mr}(G-c) = \mathrm{mr}(G) - \mathrm{mr}(\text{paw} \bigcup K_1) = 3 - 2 = 1$.

Since P_4 is an induced subgraph of $G-e$, then by Observation 8, $\mathrm{mr}(G-e) \geq \mathrm{mr}(P_4) = 3$.

Label the vertices of the $G-e$ as follows



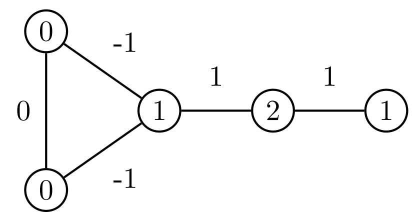
Let

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Note that $C \in S(G-e)$ and that $\operatorname{rank} C = 3$. Therefore $\operatorname{mr}(G-e) = 3$.

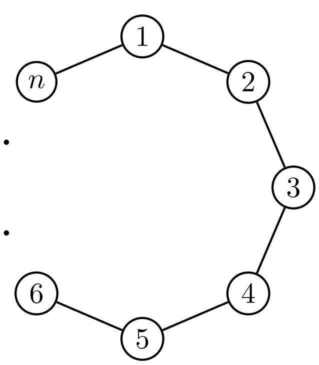
Thus $r_e(G) = \operatorname{mr}(G) - \operatorname{mr}(G-e) = 3 - 3 = 0$.

Relabeling the vertices and edges of G with their corresponding rank-spreads the graph of G is



Example 7

Consider the cycle on $n \geq 3$ vertices, C_n, labeled



Since P_{n-1} is an induced subgraph of C_n, then by Example 1 and Observation 8,

$$\operatorname{mr}(C_n) \geq \operatorname{mr}(P_{n-1}) = n - 2.$$

It will be shown by induction that $\operatorname{mr}(C_n) \leq n - 2$. For $n = 3$, $C_3 = K_3$. By Observation 1, $\operatorname{mr}(K_3) = 1$. Thus $\operatorname{mr}(C_3) = 3$.

Assume that $\operatorname{mr}(C_n) \leq n - 2$.

Let

$$A = \begin{bmatrix} d_1 & c_1 & & & & c_n \\ c_1 & d_2 & c_2 & & & \\ & c_2 & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & c_{n-1} \\ c_n & & & & c_{n-1} & d_n \end{bmatrix} \in S(C_n)$$

such that $\operatorname{rank} A = \operatorname{mr}(C_n)$.

Let

$$\tilde{A} = \begin{bmatrix} 1 & 1 & & & & & -c_n \\ 1 & d_1+1 & c_1 & & & & \\ & c_1 & d_2 & c_2 & & & \\ & & c_2 & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & c_{n-1} \\ -c_n & & & & & c_{n-1} & d_n + c_n^2 \end{bmatrix} \in S(C_{n+1}).$$

Using two row operations on $\tilde{A}$ it is seen that

$$\operatorname{rank} \tilde{A} = \operatorname{rank} \begin{bmatrix} 1 & 1 & & & & & -c_n \\ 0 & d_1 & c_1 & & & & c_n \\ & c_1 & d_2 & c_2 & & & \\ & & c_2 & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & c_{n-1} \\ 0 & c_n & & & & c_{n-1} & d_n \end{bmatrix} = 1 + \operatorname{rank} A = 1 + \operatorname{mr}(C_n) \leq n-1.$$

Therefore $\operatorname{mr}(C_n) = n-2$.

Chapter 2

Minimal Rank of a Tree in Terms of the Path Cover Number

Definition 6

The *maximum multiplicity* of a graph G denoted $M(G)$ is the maximum multiplicity occurring for an eigenvalue of any $A \in S(\mathbb{R}, G)$.

Definition 7

The parameter $\Delta(G)$ is the maximum of $p - q$ such that the deletion of q vertices from G leaves p paths.

There has been much interest in finding the maximum multiplicity of an eigenvalue of a real symmetric matrix whose graph is a given tree. In [JD99], the relationship $M(T) = \Delta(T) = P(T)$ was proven using a string of inequalities beginning and ending with $M(T)$. While this is a beautiful technique, the use of the parameter $\Delta(T)$ is somewhat cumbersome to understanding the proof. In a subsequent paper [BFH04], it was shown that for any graph G, $\Delta(G) \leq P(G)$. They also point out that in [JD99] it was shown that $\Delta(G) \leq M(G)$ for any graph G. The parameter $\Delta(G)$ is somewhat difficult to determine even for graphs of small degree, and can be negative. We will explore an alternative proof of the theorem in [JD99] using induction which relies only on the minimal rank and path cover number. In addition this proof is field independent, while the proof in [JD99] is restricted to the real number field.

Definition 8

Let $G_1, G_2, \dots, G_h$ be disjoint graphs. For each i, we select a vertex $v_i \in V(G_i)$ and join all G_i's by identifying all v_i's as a unique vertex v. The resulting graph is called the *vertex-sum* at v of the graphs $G_1, \dots, G_h$. It will be convenient for the vertex-sum of two graphs at v of G_1 and G_2 to be written $G_1 \bigoplus_v G_2$.

Lemma 9 ([Hsi01, Theorem 10])

Let p be a pendant vertex of G, adjacent to vertex q, then

(1) $\operatorname{mr}(G-q) = \operatorname{mr}(G-p) - 2 \iff \operatorname{mr}(G) = \operatorname{mr}(G-p)$.

(2) $\operatorname{mr}(G-q) = \operatorname{mr}(G-p) - 1$ or $\operatorname{mr}(G-q) = \operatorname{mr}(G-p) \iff \operatorname{mr}(G) = \operatorname{mr}(G-p)+1$.

Or equivalently if v is a vertex of G, then

(1) $\operatorname{mr}(G-v) = \operatorname{mr}(G) - 2 \iff \operatorname{mr}(G \bigoplus_v K_2) = \operatorname{mr}(G)$

(2) $\operatorname{mr}(G-v) = \operatorname{mr}(G) - 1$ or $\operatorname{mr}(G-v) = \operatorname{mr}(G) \iff \operatorname{mr}(G \bigoplus_v K_2) = \operatorname{mr}(G) + 1$

Lemma 10 ([Hsi01, Theorem 16])

If $G = H_1 \bigoplus_p H_2$ is a graph, then

$$\operatorname{mr}(G) = \min\{\operatorname{mr}(H_1) + \operatorname{mr}(H_2), \operatorname{mr}(H_1 - p) + \operatorname{mr}(H_2 - p) + 2\}.$$

Lemma 11 ([Hsi01, Corollary 9])

Let p be a pendant vertex of G, adjacent to vertex q. If $d(q) = 2$, then $\operatorname{mr}(G) = \operatorname{mr}(G-p)+1$.

Lemma 12

Let G be a graph containing a vertex q which is adjacent to at least three pendant vertices. Then if p is any pendant vertex adjacent to q, $\operatorname{mr}(G-p) = \operatorname{mr}(G)$.

PROOF Let p, p', and p'' be three pendant vertices adjacent to q. Consider the graph $G - p = P_3 \bigoplus_q H$ where P_3 is the path $[p', q, p'']$ and $H = G - p - p' - p''$. By Lemma 10, $\operatorname{mr}(G-p) = \min\{\operatorname{mr}(P_3) + \operatorname{mr}(H), \operatorname{mr}(P_3 - q) + \operatorname{mr}(H-q) + 2\}$. Note that $\operatorname{mr}(P_3) = 2$ and $\operatorname{mr}(P_3 - q) = 0$. Thus $\operatorname{mr}(G-p) = \min\{\operatorname{mr}(H)+2, \operatorname{mr}(H-q)+2\}$. By Lemma 4, $\operatorname{mr}(H) \geq \operatorname{mr}(H-q)$. Further $\operatorname{mr}(H)+2 \geq \operatorname{mr}(H-q)+2$. Thus $\operatorname{mr}(G-p) = \operatorname{mr}(H-q)+2$.

Note that since $H = G - p - p' - p''$ and p, p', p'' are all pendant vertices of G, then $\text{mr}(H - q) = \text{mr}(G - q)$. Therefore $\text{mr}(G - p) = \text{mr}(G - q) + 2$. Thus by Lemma 9, $\text{mr}(G) = \text{mr}(G - p)$. ■

Lemma 13

If T is a tree with $k > 2$ vertices, then there exists a vertex v, $d(v) \geq 2$, such that there is at most one non-pendant vertex adjacent to v.

Proof Let T be a tree with $k > 2$ vertices. Since $k > 2$, then $\text{diam}(T) \geq 2$. Let $\text{diam}(T) = m \geq 2$ and $[v_0, v_1, \ldots, v_m]$ be a path of length m in T. Suppose there exists a vertex $u \neq v_2$ adjacent to v_1 which is non-pendant. Since u is not pendant, there exists a vertex $w \neq v_1$ which is adjacent to u. Note that $w \neq v_i$ for any i since such a set of vertices would form a cycle. Now the path $[w, u, v_1, \ldots, v_m]$ is a path whose length is greater than m. Since T is a tree then any path between vertices is unique. Thus $d(w, v_k) = m + 1$, contradicting $\text{diam}(T) = m$. Thus any vertex not equal to v_2 which is adjacent to v_1 must be pendant. Thus v_1 is adjacent to at most one non-pendant vertex. ■

Lemma 14 ([BFH04, Lemma 3.3])

If G is a graph with pendant vertex p, then $P(G) \geq P(G - p) \geq P(G) - 1$.

Proof Consider any minimal path cover of $G - p$. Then that cover along with vertex p is a path cover for G. Thus $P(G - p) + 1 \geq P(G)$.

Consider any minimal path cover $\mathcal{P}$ of G. Since p is pendant, it is a pendant vertex of a path in $\mathcal{P}$. Thus $\mathcal{P}$ with p deleted is a cover of $G - p$ with at most $P(G)$ paths. Thus $P(G) \geq P(G - p)$. ■

Theorem 15 ([JD99, Theorem 1])

If T is a tree with n vertices, then $\text{mr}(T) + P(T) = n$.

Proof Proceeding by induction on the number of vertices n, of T, let

$$S = \{n \mid \text{mr}(T) + P(T) = n \text{ for every tree on } n \text{ vertices}\}.$$

Since K_1 can be covered by one path, then $P(K_1) = 1$. By Observation 3, $\text{mr}(K_1) = 0$. Thus $\text{mr}(K_1) + P(K_1) = 0 + 1 = 1$. Since K_1 is the only graph on 1 vertex, then $1 \in S$.

Now K_2 can be covered by one path and so $P(K_2) = 1$. By Observation 1, $\text{mr}(K_2) = 1$. Thus $\text{mr}(K_2)+P(K_2) = 1+1 = 2$. Note that K_2 is the only tree on 2 vertices.

Assume that $\text{mr}(T) + P(T) = m$ for all trees on $m < k$ vertices.

Let T be a tree on k vertices. We will show that $\text{mr}(T) + P(T) = k$. To do so it suffices to show that for some pendant vertex p, $\text{mr}(T)+P(T) = \text{mr}(T-p)+P(T-p)+1$. For then using the inductive hypothesis we have $\text{mr}(T) + P(T) = (k - 1) + 1 = k$.

Since $k > 2$, then by Lemma 13 there exists a vertex v, $d(v) \geq 2$, such that v is adjacent to at most one non-pendant vertex. Let q be such a vertex of T.

Now if every vertex adjacent to q is pendant then by definition $T = S_n$ for some $n \geq 3$. By Example 2, $\text{mr}(S_n) + P(S_n) = 2 + n - 2 = n$.

Otherwise q is adjacent to exactly one non-pendant vertex.

Case 1 $d(q) = 2$

Let p be the pendant vertex adjacent to q. Since $d(q) = 2$, then by Lemma 11,

$$\text{mr}(T) = \text{mr}(T - p) + 1.$$

Let $\mathcal{P}$ be a minimal path cover for $T - p$. Since q is pendant in $T - p$, then q is a pendant vertex of the path in $\mathcal{P}$ which covers q. Thus the path covering q can be extended to cover p and $P(T) \leq P(T - p)$. Since p is pendant then by Lemma 14, $P(T) \geq P(T - p)$. Thus

$$P(T) = P(T - p).$$

Case 2 $d(q) = 3$

Let p and p' be the two pendant vertices adjacent to q. Consider a minimal path cover $\mathcal{Q}$ for $T - p - p' - q$, then $\mathcal{Q}$ union the path $[p, q, p']$ is a path cover for T. Thus

$$P(T) \leq P(T - p - p' - q) + 1.$$

By repeated application of Lemma 14,

$$P(T) \geq P(T - p) \geq P(T - p - p') \geq P(T - p - p' - q). \tag{2.1}$$

Thus

$$P(T-p-p'-q)+1 \geq P(T) \geq P(T-p-p'-q).$$

Let $\mathcal{R}$ be a minimal path cover for T and suppose that $P(T-p-p'-q) = P(T)$. Note that if $P(T-p-p'-q) = P(T)$, then the above inequalities in (2.1) become equalities. Thus $P(T) = P(T-p)$ and p was not a lone vertex cover in $\mathcal{R}$. Since q is the only vertex adjacent to p, then p and q belong to the same path in $\mathcal{R}$.

By symmetry p' and q belong to the same path in $\mathcal{R}$.

Since p and p' are both pendant vertices in T, then no other vertex can be in the same path as p, p', and q. Thus $[p, q, p']$ is a path in $\mathcal{R}$. Now $\mathcal{R}$ without $[p, q, p']$ is a cover for $T-p-p'-q$, and so $P(T-p-p'-q) < P(T)$, a contradiction. Thus $P(T-p-p'-q) \neq P(T)$ and it must be the case that

$$P(T-p-p'-q)+1 = P(T).$$

By the inductive hypothesis

$$\mathrm{mr}(T-p-p'-q) + P(T-p-p'-q) = k-3.$$

Note that $\mathrm{mr}(T-p-p'-q) = \mathrm{mr}(T-q)$ since the deletion of q from T leaves the vertices p and p' isolated. Thus

$$\mathrm{mr}(T-q) + P(T) = k-2. \tag{2.2}$$

Also by the inductive hypothesis

$$\mathrm{mr}(T-p) + P(T-p) = k-1. \tag{2.3}$$

Now by Lemma 14, either $P(T) = P(T-p)+1$ or $P(T) = P(T-p)$.

Subcase 1 If $P(T) = P(T-p)+1$ then using equations (2.2) and (2.3) above, we have that

$$k-2-\mathrm{mr}(T-q) = P(T) = P(T-p)+1 = k-\mathrm{mr}(T-p)$$

which implies that $\mathrm{mr}(T-q)+2=\mathrm{mr}(T-p)$. Thus by Lemma 9,

$$\mathrm{mr}(T-p)=\mathrm{mr}(T).$$

Thus $P(T)=P(T-p)+1$ and $\mathrm{mr}(T)=\mathrm{mr}(T-p)$.

Subcase 2 If $P(T)=P(T-p)$ then using the equations (2.2) and (2.3), we have that

$$k-2-\mathrm{mr}(T-q)=P(T)=P(T-p)=k-1-\mathrm{mr}(T-p)$$

which implies that $\mathrm{mr}(T-q)+1=\mathrm{mr}(T-p)$. Thus by Lemma 9,

$$\mathrm{mr}(T-p)+1=\mathrm{mr}(T).$$

Thus $P(T)=P(T-p)$ and $\mathrm{mr}(T)=\mathrm{mr}(T-p)+1$.

Case 3 $d(q)\geq 4$

Since $d(q)\geq 4$, then q is adjacent to at least 3 pendant vertices. Let p, p', and p'' be three pendant vertices. Then by Lemma 12,

$$\mathrm{mr}(T)=\mathrm{mr}(T-p).$$

Given a minimal path cover of T, let q be covered by the path P. Now q cannot be a pendant vertex of P since we could then add one of the pendant vertices adjacent to q and create a cover with one less path. Thus P must contain at least one pendant vertex adjacent to q. Without loss of generality, renaming if necessary, let p'' be in P. Since p'' is a pendant vertex of T, and P being a path has only two pendant vertices, then P can contain at most one of either p or p'. In either case there exists a pendant vertex adjacent to q which is not covered by P. Without loss of generality, renaming if necessary, let p be such a vertex. Since p is only adjacent to q and not part of P, then p is covered by a lone vertex path. Thus this path cover of T minus the vertex p is a cover for $T-p$ and $P(T-p)<P(T)$. By Lemma 14,

$$P(T)=P(T-p)+1.$$

Thus in every case there exists a pendant vertex p such that

$$\mathrm{mr}(T) + P(T) = \mathrm{mr}(T - p) + P(T - p) + 1.$$

Therefore having exhausted all the cases, $k \in S$. By the Principle of Mathematical Induction $S = \mathbb{N}$ and for a tree with n vertices, $\mathrm{mr}(T) + P(T) = n$. ■

Chapter 3

Minimal Rank Lemmas for Graphs with a Cut-vertex

Lemma 16 ([BFH04, Theorem 2.3])

Let G be a vertex-sum at v of graphs $G_1, \ldots, G_h$. Then

$$r_v(G) = \min\left\{\sum_{i=1}^{h} r_v(G_i), 2\right\},$$

that is

$$\mathrm{mr}(G) = \sum_{i=1}^{h} \mathrm{mr}(G_i - v) + \min\left\{\sum_{i=1}^{h} r_v(G_i), 2\right\}.$$

It should be noted that in the case that $h = 2$, Lemma 16 is equivalent to Lemma 10. The more general result of Lemma 16 can be proven from Lemma 10 using induction, but this was not the technique used in [BFH04].

In [BFH04] the idea of rank-spreads is developed and the lemma above is a core result. This result seems to be very useful and applicable to a large number of graphs. Unfortunately it only applies to graphs with a cut-vertex. They cite [JD99] and further investigate the relationship between the parameters $\Delta(G)$, $M(G)$, and $P(G)$ for all graphs not just trees.

One natural question that arises from Lemma 16 is how the rank-spreads of vertices and edges change as graphs are vertex summed together. The following lemmas and corollaries somewhat answer that question, while a complete answer is given by Jason Grout in his PhD thesis. Many of the following lemmas and corollaries are very case specific and technical. However they are necessary tools to prove the larger result in Chapter 5.

Corollary 17

Let G be a graph and $v \in V(G)$. If v is adjacent to two or more pendant vertices, then $r_v(G) = 2$.

Proof Let p, q be pendant vertices adjacent to v. Since v is adjacent to p, q then G can be considered as the vertex sum at v of K_2, K_2 and $G - p - q$. Since $r_v(K_2) = 1$ and rank spreads are non-negative, then

$$r_v(K_2) + r_v(K_2) + r_v(G - p - q) \geq 2.$$

By Lemma 16, $r_v(G) = 2$. ■

Corollary 18

Let G be a vertex sum at v of $G_1, G_2, \ldots, G_h$. If $\sum_{i=1}^{h} r_v(G_i) \leq 2$, then $\mathrm{mr}(G) = \sum_{i=1}^{h} \mathrm{mr}(G_i)$.

Proof If $\sum_{i=1}^{h} r_v(G_i) \leq 2$, then by Lemma 16, $\mathrm{mr}(G) = \sum_{i=1}^{h} \mathrm{mr}(G_i - v) + \sum_{i=1}^{h} r_v(G_i)$. The result follows from the definition of rank-spread. ■

Lemma 19

Let G be the vertex-sum at v of graphs $G_1, G_2, \ldots, G_h$ and $w \in V(G_k), w \neq v$. If $\sum_{i=1}^{h} r_v(G_i) \leq 2$ and $\sum_{i=1, i \neq k}^{h} r_v(G_i) + r_v(G_k - w) \leq 2$, then $r_w(G_k) = r_w(G)$.

Proof By hypothesis $\sum_{i=1}^{h} r_v(G_i) \leq 2$, and thus by Corollary 18,

$$\mathrm{mr}(G) = \sum_{i=1}^{h} \mathrm{mr}(G_i).$$

By hypothesis $\sum_{i=1, i \neq k}^{h} r_v(G_i) + r_v(G_k - w) \leq 2$, and thus by Corollary 18,

$$\mathrm{mr}(G - w) = \sum_{i=1, i \neq k}^{h} \mathrm{mr}(G_i) + \mathrm{mr}(G_k - w).$$

By definition of rank-spread, $r_w(G) = \text{mr}(G) - \text{mr}(G - w)$. Using the above two equations we have that $r_w(G) = \text{mr}(G_k) - \text{mr}(G_k - w)$. Again using the definition of rank-spread, $r_w(G) = r_w(G_k)$. ■

3.1 Vertex-sum of Two Graphs at Pendant Vertices

Lemma 20

Let G be a graph with vertices w and v, $w \neq v$, such that v is pendant in G. Then $1 \geq r_v(G - w) \geq 0$.

PROOF If w is adjacent to v, then deleting w will isolate v in G. Thus $r_v(G - w) = 0$. If w is not adjacent to v, then deleting w will not change the fact that v is pendant. By Corollary 7, $1 \geq r_v(G - w) \geq 0$. Thus in either case $1 \geq r_v(G - w) \geq 0$. ■

Lemma 21

Let $G_1 \underset{v}{\oplus} G_2$ be the vertex sum at v of G_1 and G_2 where v is pendant in both G_1 and G_2. If $w \in V(G_i), w \neq v$, then $r_w(G_i) = r_w(G_1 \underset{v}{\oplus} G_2)$.

PROOF Since v is pendant in both G_1 and G_2 then by Corollary 7, $0 \leq r_v(G_i) \leq 1$. Thus $r_v(G_1) + r_v(G_2) \leq 2$.

Without loss of generality let $w \in V(G_1)$. By Lemma 20, $1 \geq r_v(G_1 - w) \geq 0$.

Thus $r_v(G_1 - w) + r_v(G_2) \leq 2$.

By Lemma 19, $r_w(G_1) = r_w(G_1 \underset{v}{\oplus} G_2)$. ■

3.2 Vertex-sum of Graphs at Rank-spread Zero Vertices

Lemma 22

Let $G \underset{v}{\oplus} H$ be the vertex sum at v of G and H. If $r_v(H) = 0$ and $w \in V(G)$, $w \neq v$, then $r_w(G) = r_w(G \underset{v}{\oplus} H)$.

PROOF Since $r_v(H) = 0$ by hypothesis and $r_v(G) \leq 2$ by Corollary 5, then $r_v(H) + r_v(G) \leq 2$.

By Corollary 5, $r_v(G - w) \leq 2$ so $r_v(H) + r_v(G - w) \leq 2$.

By Lemma 19, $r_w(G) = r_w(G \underset{v}{\oplus} H)$. ■

Corollary 23

Let G be the vertex-sum at v of graphs $G_1, G_2, \ldots, G_h$ where $r_v(G_i) = 0$ for every i. If $w \in V(G_i), w \neq v$, then $r_w(G_i) = r_w(G)$.

PROOF Without loss of generality, renaming if necessary let $w \in V(G_1), w \neq v$. Let H be the vertex-sum at v of all G_i, $i \neq 1$. By Lemma 16, $r_v(H) = \sum_{i=2}^{h} r_v(G_i)$. Since $r_v(G_i) = 0$ for every i, then $r_v(H) = 0$. By Lemma 22, $r_w(G_1) = r_w(G_1 \underset{v}{\oplus} H)$. Note that $G = G_1 \underset{v}{\oplus} H$ and thus $r_w(G_1) = r_w(G)$. ■

Corollary 24

Let G be the vertex-sum at v of graphs $G_1, G_2, \ldots, G_h$ where $r_v(G) = 0$. If $w \in V(G_i)$ $w \neq v$, then $r_w(G_i) = r_w(G)$.

PROOF By Corollary 23, it suffices to show that $r_v(G_i) = 0$ for every i. By Lemma 16, $r_v(G) = \min\left\{\sum_{i=1}^{h} r_v(G_i), 2\right\}$. Since $r_v(G) = 0$, then it must be the case that $r_v(G) = \sum_{i=1}^{h} r_v(G_i)$. By Corollary 5, $r_v(G_i) \geq 0$ for every i. Since $r_v(G)$ is the sum of non-negative integers equal to zero, $r_v(G_i) = 0$ for every i. ■

3.3 Vertex-sum of Multiple Graphs at Pendant Vertices

Lemma 25

Let $G \underset{v}{\oplus} H$ be the vertex-sum at v of graphs G and H where $r_v(G) = 0$ and $r_v(H) = 1$. Let $w \in V(G)$, $w \neq v$. If $r_v(G - w) \leq 1$, then $r_w(G) = r_w(G \underset{v}{\oplus} H)$.

PROOF By hypothesis $r_v(G) = 0$ and $r_v(H) = 1$. Thus $r_v(G) + r_v(H) \leq 2$.

By hypothesis $r_v(G - w) \leq 1$ and $r_v(H) = 1$. Thus $r_v(G - w) + r_v(H) \leq 2$.

By Lemma 19, $r_w(G) = r_w(G \underset{v}{\oplus} H)$. ■

Corollary 26

Let $G \underset{v}{\oplus} H$ be the vertex-sum at v of graphs G and H where $r_v(G) = 0, r_v(H) = 1$ and v is pendant in G. If $w \in V(G)$, $w \neq v$, then $r_w(G \underset{v}{\oplus} H) = r_w(G)$.

PROOF Let $w \in V(G)$, $w \neq v$. By Lemma 20, $r_v(G - w) \leq 1$. By Lemma 25, $r_w(G) = r_w(G \underset{v}{\oplus} H)$. ■

Lemma 27

Let G be the vertex-sum at v of graphs $G_1, \ldots, G_h$ where v is pendant in G_i for every i and $r_v(G) = 1$. If $w \in V(G_i)$, $w \neq v$, then $r_w(G_i) = r_w(G)$.

PROOF By Lemma 16, $r_v(G) = \sum_{i=1}^{h} r_v(G_i)$. By Corollary 5, $2 \geq r_v(G_i) \geq 0$. Thus $r_v(G)$ is a sum of non-negative integers adding to 1. Thus there exists k such that $r_v(G_k) = 1$ and $r_v(G_i) = 0$ for all $i \neq k$. Without loss of generality renaming if necessary let $r_v(G_1) = 1$ and $r_v(G_i) = 0$ for all $i \neq 1$.

Case 1 $w \in V(G_1)$, $w \neq v$

Let H_1 be the vertex-sum at v of all G_i, $i \neq 1$. By Lemma 16, $r_v(H_1) = \sum_{i=2}^{h} r_v(G_i)$. Thus $r_v(H_1) = 0$. By Lemma 22, $r_w(G_1) = r_w(G_1 \underset{v}{\oplus} H_1)$. Note that $G_1 \underset{v}{\oplus} H_1 = G$. Thus $r_w(G_1) = r_w(G)$.

Case 2 $w \in V(G_i)$, $w \neq v$, $i \neq 1$

Without loss of generality, renaming if necessary, let $w \in V(G_2)$. Let H_2 be the vertex-sum at v of all G_i, $i \neq 2$. By Lemma 16, $r_v(H_2) = \sum_{i \neq 2} r_v(G_i)$. Thus $r_v(H_2) = 1$. Since $r_v(H_2) = 1$ and $r_v(G_2) = 0$ and v is pendant in G_2, then by Corollary 26, $r_w(G_2) = r_w(G_2 \underset{v}{\oplus} H_2)$. Note that $G_2 \underset{v}{\oplus} H_2 = G$. Thus $r_w(G_2) = r_w(G)$. ■

3.4 Vertex-sum of Two Graphs at a Rank-spread Two Vertex

Lemma 28

Let G and H be graphs where $r_v(H) = 2$. Then $\mathrm{mr}(G \underset{v}{\oplus} H) = \mathrm{mr}(G - v) + \mathrm{mr}(H)$.

PROOF By Lemma 16, $\mathrm{mr}(G \underset{v}{\oplus} H) = \mathrm{mr}(G - v) + \mathrm{mr}(H - v) + \min\{r_v(G) + r_v(H), 2\}$. Since $r_v(H) = 2$, then $r_v(G) + r_v(H) \geq 2$. Thus $\mathrm{mr}(G \underset{v}{\oplus} H) = \mathrm{mr}(G - v) + \mathrm{mr}(H - v) + 2$. Again since $r_v(H) = 2$, we have $\mathrm{mr}(G \underset{v}{\oplus} H) = \mathrm{mr}(G - v) + \mathrm{mr}(H - v) + r_v(H)$. Using the definition of rank-spread we arrive at the result. ■

Lemma 29

Let G and H be graphs such that $r_v(H) = 2$ and $w \in V(G)$, $w \neq v$, then $r_w(G - v) = r_w(G \underset{v}{\oplus} H)$.

PROOF Since $r_v(H) = 2$ and $w \in V(G)$, then by Lemma 28, $\text{mr}(G \underset{v}{\oplus} H) = \text{mr}(G - v) + \text{mr}(H)$ and $\text{mr}((G-w) \underset{v}{\oplus} H) = \text{mr}((G-w)-v) + \text{mr}(H)$.

Note that $(G-w) \underset{v}{\oplus} H = (G \underset{v}{\oplus} H) - w$.

Using the definition of rank-spread and the above results we have that $r_w(G \underset{v}{\oplus} H) = \text{mr}(G \underset{v}{\oplus} H) - \text{mr}((G \underset{v}{\oplus} H) - w) = \text{mr}(G-v) + \text{mr}(H) - (\text{mr}((G-w)-v) + \text{mr}(H)) = \text{mr}(G-v) - \text{mr}((G-w)-v)$.

Also note that $(G-w)-v = (G-v)-w$.

Thus $r_w(G \underset{v}{\oplus} H) = \text{mr}(G-v) - \text{mr}((G-v)-w) = r_w(G-v)$. ■

3.5 Edge-sum of Two Graphs at Rank-spread Two Vertices

Definition 9

Let G_1 and G_2 be disjoint undirected graphs, and let v_1 and v_2 be vertices of G_1 and G_2 respectively. If we connect G_1 and G_2 by adding the edge $e = \{v_1, v_2\}$, the resulting graph G is called the *edge-sum* of G_1 and G_2, and is denoted by $G = G_1 \underset{e}{+} G_2$.

Using the notation established previously for the vertex-sum of two graphs, the edge-sum $G = G_1 \underset{e}{+} G_2$ can be written as $G_1 \underset{v_1}{\oplus} K_2 \underset{v_2}{\oplus} G_2$. Thus an edge-sum can be thought of as a vertex-sum of G_1 and K_2 at v_1 and a vertex-sum of the resulting graph and G_2 at v_2.

Lemma 30

Let G_1 and G_2 be graphs such that $r_{v_i}(G_i) = 2$ for $i = 1, 2$. If $w \in V(G_i)$, then $r_w(G_i) = r_w(G_1 \underset{v_1}{\oplus} K_2 \underset{v_2}{\oplus} G_2)$.

PROOF Without loss of generality let $w \in G_2$. Since $r_{v_1}(G_1) = 2$, then by Lemma 29, $r_{v_2}(G_1 \underset{v_1}{\oplus} K_2) = r_{v_2}(K_2 - v_1)$.

Note that $K_2 - v_1 = K_1$ and by Observation 3, $\text{mr}(K_1) = 0$. Thus $r_{v_2}(K_2 - v_1) = r_{v_2}(K_1) = \text{mr}(K_1) - \text{mr}(\emptyset) = 0 - 0 = 0$. Thus $r_{v_2}(G_1 \underset{v_1}{\oplus} K_2) = 0$.

Now assume that $w \neq v_2$. Since $r_{v_2}(G_1 \underset{v_1}{\oplus} K_2) = 0$, then by Lemma 22, $r_w(G_2) = r_w(G_1 \underset{v_1}{\oplus} K_2 \underset{v_2}{\oplus} G_2)$. Thus for any $w \in V(G_2)$ not equal to v_2 the claim is true. It remains to show that the rank-spread of v_2 is two in $G_1 \underset{v_1}{\oplus} K_2 \underset{v_2}{\oplus} G_2$.

By Lemma 16, $r_{v_2}(G_1 \underset{v_1}{\oplus} K_2 \underset{v_2}{\oplus} G_2) = \min\{r_{v_2}(G_1 \underset{v_1}{\oplus} K_2) + r_{v_2}(G_2), 2\}$.

By hypothesis $r_{v_2}(G_2) = 2$ and we have shown that $r_{v_2}(G_1 \underset{v_1}{\oplus} K_2) = 0$. Thus $r_{v_2}(G_1 \underset{v_1}{\oplus} K_2 \underset{v_2}{\oplus} G_2) = 2 = r_{v_2}(G_2)$. ■

3.6 Rank-spreads of an Edge and the Incident Vertices

Lemma 31

Let $e \in E(G)$ where e is incident to vertices v_1 and v_2. Then $r_{v_i}(G) - r_{v_i}(G-e) = r_e(G)$ for $i = 1, 2$.

Proof By definition $r_{v_i}(G) - r_{v_i}(G-e) = \text{mr}(G) - \text{mr}(G-v_i) - (\text{mr}(G-e) - \text{mr}((G-e)-v_i))$. Since e is incident to v_i, then $G - v_i = (G-e) - v_i$. Thus $\text{mr}(G-v_i) = \text{mr}((G-e)-v_i)$ and $r_{v_i}(G) - r_{v_i}(G-e) = \text{mr}(G) - \text{mr}(G-e) = r_e(G)$. ∎

Chapter 4

Minimal Path Covers, Rank-spreads and Path-spreads of Edges

4.1 How Minimal Path Covers relate to the Rank-spreads of Edges in Trees

Definition 10

Let v be a vertex or edge of a graph G. The *path-spread* of v in G is $P(G) - P(G - v)$ and is denoted $p_v(G)$.

Lemma 32

Let T be a tree and $e \in E(T)$, then $r_e(T) = 0$ if and only if $p_e(T) = 0$.

Proof Let $T = T_1 \underset{e}{+} T_2$, then by Theorem 15

$$|V(T)| = \text{mr}(T) + P(T)$$

and

$$|V(T_i)| = \text{mr}(T_i) + P(T_i) \text{ for } i = 1, 2.$$

Since $|V(T_1)| + |V(T_2)| = |V(T)|$, then using the above formulas we have

$$\text{mr}(T) + P(T) = \text{mr}(T_1) + P(T_1) + \text{mr}(T_2) + P(T_2)$$

Now we have the following equivalences:

$$r_e(T) = 0 \iff \operatorname{mr}(T) - \operatorname{mr}(T - e) = 0 \iff \operatorname{mr}(T) - (\operatorname{mr}(T_1) + \operatorname{mr}(T_2)) = 0 \iff$$

$$\operatorname{mr}(T) = \operatorname{mr}(T_1) + \operatorname{mr}(T_2) \iff P(T) = P(T_1) + P(T_2) \iff$$

$$P(T) - (P(T_1) + P(T_2)) = 0 \iff P(T) - P(T - e) = 0 \iff p_e(T) = 0.$$

Therefore $r_e(T) = 0$ if and only if $p_e(T) = 0$. ■

Note that Lemma 32 is not necessarily true if e is a cut-edge in an arbitrary graph.

Lemma 33

Let G be a graph and $e \in E(G)$ a cut edge. Then $p_e(G) = 0$ if and only if there exists a minimal path cover of G which does not use e.

Proof Let $G = G_1 \underset{e}{+} G_2$.

Forward Implication

$$p_e(G) = 0 \Longrightarrow P(G) - P(G - e) = 0 \Longrightarrow$$

$$P(G) - (P(G_1) + P(G_2)) = 0 \Longrightarrow P(G) = P(G_1) + P(G_2)$$

Let R_1 and R_2 be minimal path covers for G_1 and G_2 respectively. Then $R_1 \bigcup R_2$ is a path cover for G. Since $P(G) = P(G_1) + P(G_2) = |R_1| + |R_2| = |R_1 \bigcup R_2|$ then $R_1 \bigcup R_2$ is a minimal path cover for G. Thus there exists a minimal path cover for G which doesn't use the edge e.

Reverse Implication Let R be a minimal path cover for G which doesn't use e. Then R is a path cover for $G - e$ and $P(G) \geq P(G - e)$

Let R_1 and R_2 be minimal path covers for G_1 and G_2. Then the union of R_1 and R_2 is a path cover for G. Thus

$$P(G) \leq |R_1 \bigcup R_2| = |R_1| + |R_2| = P(G_1) + P(G_2) = P(G - e)$$

Therefore it must be that $P(G) = P(G - e)$, which by definition implies that $p_e(G) = 0$. ■

Lemma 34 ([BFH04, Theorem 2.6])

Let $G = G_1 \underset{e}{+} G_2$, with $e = \{v_1, v_2\}$. Then

$$\mathrm{mr}(G) = \begin{cases} \mathrm{mr}(G_1) + \mathrm{mr}(G_2) & \text{if } r_v(G_i) = 2 \text{ for at least one } i \\ \mathrm{mr}(G_1) + \mathrm{mr}(G_2) + 1 & \text{otherwise.} \end{cases}$$

Lemma 35

Let G be a graph and e a cut-edge of G, then $0 \leq r_e(G) \leq 1$.

PROOF By Lemma 34, if $G = G_1 \underset{e}{+} G_2$, then either $\mathrm{mr}(G) = \mathrm{mr}(G_1) + \mathrm{mr}(G_2)$ or $\mathrm{mr}(G) = \mathrm{mr}(G_1) + \mathrm{mr}(G_2) + 1$. Since $\mathrm{mr}(G - e) = \mathrm{mr}(G_1) + \mathrm{mr}(G_2)$, then $r_e(G) = 0$ or $r_e(G) = 1$. ■

Corollary 36

Let T be a tree, and $e \in E(T)$, then $0 \leq r_e(T) \leq 1$.

PROOF Every edge of a tree is a cut-edge, and thus by Lemma 35, $0 \leq r_e(T) \leq 1$. ■

Theorem 37

Let T be a tree and $e \in E(T)$, then $r_e(T) = 0$ if and only if there exists a minimal path cover for T which does not use e, and $r_e(T) = 1$ if and only if every minimal path cover of T uses e.

PROOF By Lemma 32, $r_e(T) = 0$ if and only if $p_e(T) = 0$. By Lemma 33, $p_e(T) = 0$ if and only if there exists a minimal path cover of T which doesn't use e. By Corollary 36 if $r_e(T) \neq 0$, then $r_e(T) = 1$. Thus $r_e(T) = 1$ if and only if every minimal path cover of T uses e. ■

4.2 Vertex-sums of Trees that Preserve the Rank-spreads of the Edges

Lemma 38

Let $G = G_1 \underset{v}{\oplus} G_2$ with cut-vertex v of degree 2. Then $\text{mr}(G) = \text{mr}(G_1) + \text{mr}(G_2)$.

PROOF Since v is of degree 2 in G, then v is pendant in both G_1 and G_2. By Corollary 7, $r_v(G_1) \leq 1$ and $r_v(G_2) \leq 1$. Thus $r_v(G_1) + r_v(G_2) \leq 2$.

By Corollary 18, $\text{mr}(G) = \text{mr}(G_1) + \text{mr}(G_2)$. ■

Lemma 39

Let G be a vertex-sum at v of the graphs $G_1, G_2, \ldots, G_h$. If $r_v(G) \leq 1$, then $\text{mr}(G) = \sum_{i=1}^{h} \text{mr}(G_i)$.

PROOF By Lemma 16,

$$r_v(G) = \min\left\{\sum_{i=1}^{h} r_v(G_i), 2\right\}.$$

Since $r_v(G) \leq 1$ by hypothesis, then $\sum_{i=1}^{h} r_v(G_i) \leq 1$.

Thus by Corollary 18, $\text{mr}(G) = \sum_{i=1}^{h} \text{mr}(G_i)$. ■

Lemma 40

Let T be the vertex sum at v of trees $T_1, T_2, \ldots, T_h$. If $\text{mr}(T) = \sum_{i=1}^{h} \text{mr}(T_i)$, then

$$P(T) = \sum_{i=1}^{h} P(T_i) - h + 1.$$

PROOF By Theorem 15, we have that

$$\text{mr}(T) + P(T) = |V(T)| \text{ and } \text{mr}(T_i) + P(T_i) = |V(T_i)| \text{ for all } i$$

Since T is the vertex-sum at v of $T_1, T_2, \ldots, T_h$, then $|V(T)| = \sum_{i=1}^{h} |V(T_i)| - h + 1$. Thus

$$P(T) = |V(T)| - \text{mr}(T) = \sum_{i=1}^{h} |V(T_i)| - h + 1 - \sum_{i=1}^{h} \text{mr}(T_i) = \sum_{i=1}^{h} P(T_i) - h + 1$$

as desired. ∎

Definition 11

A vertex v is a *terminal* vertex in G if v is the end point of a path in some minimum path cover of G.

Lemma 41

Let T be a vertex-sum at v of T_1 and T_2 where $v \in V(T)$ has degree 2. Let $e \in E(T_i)$ then $r_e(T_i) = 0$ if and only if $r_e(T) = 0$.

PROOF First we note that since v has degree two then by Lemma 38, $\mathrm{mr}(T) = \mathrm{mr}(T_1) + \mathrm{mr}(T_2)$. Thus by Lemma 40, $P(T) = P(T_1) + P(T_2) - 1$.

Forward Implication Without loss of generality renaming if necessary let $e \in E(T_1)$. Further we assume that $r_e(T_1) = 0$. Then by Theorem 37 there exists a minimal path cover R_1 of T_1 in which no path uses e. Let R_2 be a minimal path cover of T_2.

Since v has degree two in T, then v is pendant in both T_1 and T_2. Trivially pendant vertices are always terminal in any path cover. Let p_i be the path in R_i that covers v in T_i.

Now we will create a path cover R of T by taking all the paths in R_i less p_i for $i = 1, 2$ and then letting p be the vertex-sum at v of p_1 and p_2. Note that p is indeed a path since v was terminal in both p_1 and p_2. Since e was not used in any paths of R_1 and R_2, then R is a path cover of T which does not use the edge e as well.

Now

$$|R| = |R_1 - \{p_1\}| + |R_2 - \{p_2\}| + |\{p\}| = P(T_1) + P(T_2) - 1.$$

Thus R must be minimal.

Since there exists a minimal path cover of T which does not use e, then by Theorem 37, $r_e(T) = 0$.

Reverse Implication Without loss of generality renaming if necessary let $e \in E(T_1)$. Further we assume that $r_e(T) = 0$. By Theorem 37 there exists a minimal path cover R of T which doesn't use e. Let p be the path in R which covers v.

Define p_i to be the part of p which lies in T_i. Note that p_1 and p_2 are nonempty since v lies in both. However it could be that v is the only vertex in either p_1 or p_2.

Define R_i to be the set of paths in $R - \{p\}$ which lie completely in T_i.

Now $R_i \bigcup \{p_i\}$ is a path cover for T_i. Since R is a minimal path cover of T, then

$$P(T) = |R| = |R_1 \bigcup \{p_1\}| + |R_2 \bigcup \{p_2\}| - |\{p_1\}| - |\{p_2\}| + |\{p\}|.$$

But from the beginning of the proof we know that $P(T) = P(T_1) + P(T_2) - 1$. Thus

$$|R_1 \bigcup \{p_1\}| + |R_2 \bigcup \{p_2\}| = P(T_1) + P(T_2).$$

Since $R_i \bigcup \{p_i\}$ is a path cover for T_i, then $|R_i \bigcup \{p_i\}| \geq P(T_i)$. Note that if $|R_i \bigcup \{p_i\}|$ were strictly greater than $P(T_i)$ for some i, then

$$|R_1 \bigcup \{p_1\}| + |R_2 \bigcup \{p_2\}| > P(T_1) + P(T_2).$$

Thus it must be the case that $|R_i \bigcup \{p_i\}| = P(T_i)$ for $i = 1, 2$ and that $R_i \bigcup \{p_i\}$ is a minimal path cover for T_i.

Since e was not used in any of the paths in R, and R_i was constructed from paths in R, then $R_1 \bigcup \{p_1\}$ is a minimal path cover for T_1 which doesn't use e. By Theorem 37, $r_e(T_1) = 0$. ∎

Lemma 42

Let T be a vertex-sum at v of $T_1, T_2, \ldots, T_h$ where $v \in V(T)$ has degree h and $r_v(T) = 0$. Let $e \in E(T_i)$ then $r_e(T_i) = 0$ if and only if $r_e(T) = 0$.

Proof First we note that since $r_v(T) = 0$ then by Lemma 39, $\mathrm{mr}(T) = \sum_{i=1}^{h} \mathrm{mr}(T_i)$. Thus by Lemma 40,

$$P(T) = \sum_{i=1}^{h} P(T_i) - h + 1.$$

Forward Implication Without loss of generality renaming if necessary let $e \in E(T_1)$. Further we assume that $r_e(T_1) = 0$. Since $r_e(T_1) = 0$, then by Theorem 37 there exists a minimal path cover R_1 of T_1 in which no path uses e.

By Lemma 16, $r_v(T) = \min \left\{ \sum_{i=1}^{h} r_v(T_i), 2 \right\}$. By Corollary 5, $r_v(T_i) \geq 0$ for every i. Thus using the fact that $r_v(T) = 0$ it must be the case that $r_v(T_i) = 0$ for every i.

Since v is pendant in T_i then there is an unique edge $e_i \in E(T_i)$ adjacent to v. By Corollary 6, $r_{e_i}(T_i) = r_v(T_i) = 0$ for every i.

Since $r_{e_i}(T_i) = 0$ for every i, then by Theorem 37, for every T_i there exists a minimal path cover of T_i in which no path uses the edge e_i. For each i, $i \neq 1$, choose such a minimal path cover R_i. Since e_i, $i \neq 1$, is not used in any path of R_i, then $\{v\}$ is a single vertex path in R_i, $i \neq 1$.

Now we will create a path cover R of T. Let R be the set of all the paths in R_i for all i except for the single vertex path covering v in each R_i, $i \neq 1$. Note v is covered by a path in R_1.

It remains to be shown that R is minimal.

Now $|R|$ is equal to

$$|R_1| + |R_2 - \{v\}| + \ldots + |R_h - \{v\}| = P(T_1) + P(T_2) - 1 + \ldots + P(T_h) - 1 = \sum_{i=1}^{h} P(T_i) - h + 1.$$

Thus from the conclusion in the beginning of the proof R must be minimal.

Note that since R_1 did not contain a path which used e, then R will not contain a path which uses e. Thus there exists a minimal path cover of T in which no path uses e. Then by Theorem 37, $r_e(T) = 0$.

Reverse Implication Without loss of generality renaming if necessary let $e \in E(T_1)$. Further we assume that $r_e(T) = 0$. By Theorem 37 there exists a minimal path cover R of T which doesn't use e. Let p be the path in R which covers v.

Define p_i to be the part of p which lies in T_i. Note that p_i is nonempty since v lies in T_i for every i. Further there are at most two values of i for which p_i is more than a single vertex cover. In other words p covers vertices not equal to v in at most two of the trees T_i.

Define R_i to be the set of paths in $R - \{p\}$ which lie completely in T_i.

Now $R_i \bigcup \{p_i\}$ is a path cover for T_i. Since R was a minimal path cover of T, then

$$P(T) = |R| = \sum_{i=1}^{h} |R_i \bigcup \{p_i\}| - \sum_{i=1}^{h} |\{p_i\}| + |\{p\}|.$$

But from the beginning of the proof we know that

$$P(T) = \sum_{i=1}^{h} P(T_i) - h + 1.$$

Thus

$$\sum_{i=1}^{h} |R_i \bigcup \{p_i\}| = \sum_{i=1}^{h} P(T_i).$$

Since $R_i \bigcup \{p_i\}$ is a path cover for T_i, then $|R_i \bigcup \{p_i\}| \geq P(T_i)$. Note that if $|R_i \bigcup \{p_i\}|$ were strictly greater than $P(T_i)$ for some i, then

$$\sum_{i=1}^{h} |R_i \bigcup \{p_i\}| > \sum_{i=1}^{h} P(T_i).$$

Thus it must be the case that $|R_i \bigcup \{p_i\}| = P(T_i)$ for every i and that $R_i \bigcup \{p_i\}$ is a minimal path cover for T_i.

Since e was not used in any of the paths in R, and R_i was constructed from paths in R, then $R_1 \bigcup \{p_1\}$ is a minimal path cover for T_1 which doesn't use e. By Theorem 37, $r_e(T_1) = 0$.

Lemma 43

Let T be a vertex-sum at v of $T_1, T_2, \ldots, T_h$ where $v \in V(T)$ has degree h and $r_v(T) = 1$. Let $e \in E(T_i)$ then $r_e(T_i) = 0$ if and only if $r_e(T) = 0$.

Proof First we note that since $r_v(T) = 1$ then by Lemma 39, $\mathrm{mr}(T) = \sum_{i=1}^{h} \mathrm{mr}(T_i)$. Thus by Lemma 40,

$$P(T) = \sum_{i=1}^{h} P(T_i) - h + 1.$$

Forward Implication Without loss of generality renaming if necessary let $e \in E(T_1)$. Further we assume that $r_e(T_1) = 0$. Since $r_e(T_1) = 0$, then by Theorem 37 there exists a minimal path cover R_1 of T_1 in which no path uses e.

By Lemma 16, $r_v(T) = \min \left\{ \sum_{i=1}^{h} r_v(T_i), 2 \right\}$. By Corollary 5, $r_v(T_i) \geq 0$ for every i. Note that for any vertex v and graph G, that $r_v(G) \in \mathbb{N}$. Thus using the fact that $r_v(T) = 1$ it must be the case that $r_v(T_j) = 1$ for exactly one j and $r_v(T_i) = 0$ for all $i \neq j$.

Note that j could be equal to 1. In the case that $j = 1$, then the remainder of the proof is identical to the remainder of the proof of Lemma 42. Thus we will assume that $j \neq 1$.

Since v is pendant in T_i then there is an unique edge $e_i \in E(T_i)$ adjacent to v. By Corollary 6, $r_{e_i}(T_i) = r_v(T_i) = 0$ for every $i \neq j$ and $r_{e_j}(T_j) = r_v(T_j) = 1$.

Since $r_{e_i}(T_i) = 0$ for every $i \neq j$, then by Theorem 37, for every T_i, $i \neq j$ there exists a minimal path cover of T_i in which no path uses the edge e_i. For each i, $i \neq 1, j$ choose such a minimal path cover R_i. Since e_i, $i \neq 1, j$, is not used in any path of R_i, then $\{v\}$ is a single vertex path in R_i, $i \neq 1, j$.

Let R_j be a minimal path cover for T_j. Let p_1 and p_j be the paths in R_1 and R_j respectively which cover v. Let p be the union of p_1 and p_j. Note that p is a path of vertices in $V(T)$ that covers v.

Now we will create a path cover R of T. Let R be the set of all the paths in R_i for all i, except for the paths covering v in each R_i, and the path p.

It remains to be shown that R is minimal.

Now $|R|$ is equal to

$$|R_1 - \{p_1\}| + |R_j - \{p_j\}| + |\{p\}| + \sum_{i=2, i \neq j}^{h} |R_i - \{v\}| = \sum_{i=1}^{h} (P(T_i) - 1) + 1 = \sum_{i=1}^{h} P(T_i) - h + 1.$$

Thus from the conclusion in the beginning of the proof R must be minimal.

Note that since R_1 does not contain a path which uses e, then R does not contain a path which uses e. Thus there exists a minimal path cover of T in which no path uses e. Since there exists a minimal path cover of T which does not use e, then by Theorem 37, $r_e(T) = 0$.

Reverse Implication The proof is the same as in the reverse implication of Lemma 42. No where in the reverse implication does the proof depend on the rank-spread of v in T.

Chapter 5

Minimal Rank of a Tree in Terms of Rank-spreads

Theorem 44

Let T be a tree, then

$$\operatorname{mr}(T) = \sum_{v \in V(T)} r_v(T) - \sum_{e \in E(T)} r_e(T).$$

Proof We will proceed by induction on the number of vertices in T.

The only graph with one vertex is K_1. By Observation 3, $\operatorname{mr}(K_1) = 0$. Since the graph obtained by deleting the only vertex v of K_1 is the empty graph, then $r_v(K_1) = 0$. Also K_1 has no edges. Thus the base case is true.

The only connected graph on two vertices is K_2. By Observation 1, $\operatorname{mr}(K_2) = 1$. Since the graph obtained by deleting any vertex of K_2 is K_1, then $r_{v_i}(K_2) = 1$ for $i = 1, 2$. Deleting the only edge of K_2 leaves $2K_1$. Thus by Observation 3, $r_e(K_2) = 1$. Thus

$$\operatorname{mr}(K_2) = 1 = 1 + 1 - 1 = r_{v_1}(K_2) + r_{v_2}(K_2) - r_e(K_2).$$

Assume that for all trees on $n-1$ vertices or less the conclusion is true.

Case 1 There exists a non-pendant vertex u such that $r_u(T) \leq 1$ or $d_T(u) = 2$.

Let T be a tree on n vertices and u a non-pendant vertex of degree h. Since u has degree h, then T can be considered as the vertex sum at u of trees $T_1, T_2, \ldots, T_h$.

If $h = 2$ or $r_u(T) \leq 1$, then by Lemma 38 or Lemma 39,

$$\mathrm{mr}(T) = \sum_{i=1}^{h} \mathrm{mr}(T_i).$$

By the inductive hypothesis for each i,

$$\mathrm{mr}(T_i) = \sum_{v \in V(T_i)} r_v(T_i) - \sum_{e \in E(T_i)} r_e(T_i).$$

Thus $\mathrm{mr}(T)$ is equal to

$$\sum_{i=1}^{h} \left(\sum_{v \in V(T_i)} r_v(T_i) - \sum_{e \in E(T_i)} r_e(T_i) \right) = \sum_{i=1}^{h} \left(\sum_{v \in V(T_i)} r_v(T_i) \right) - \sum_{i=1}^{h} \left(\sum_{e \in E(T_i)} r_e(T_i) \right).$$

For each of the three subcases it will be shown that:

1. For each vertex $w \in V(T_i)$, $w \neq u$, $r_w(T) = r_w(T_i)$
2. $\sum_{i=1}^{h} r_u(T_i) = r_u(T)$
3. For each edge $e \in E(T)$, $r_e(T) = r_e(T_i)$.

By part 1,

$$\sum_{i=1}^{h} \left(\sum_{v \in V(T_i)} r_v(T_i) \right) = \sum_{i=1}^{h} \left(\sum_{v \in V(T_i), v \neq u} r_v(T) \right) + \sum_{i=1}^{h} r_u(T_i).$$

By part 2,

$$\sum_{i=1}^{h} \left(\sum_{v \in V(T_i), v \neq u} r_v(T) \right) + \sum_{i=1}^{h} r_u(T_i) = \sum_{i=1}^{h} \left(\sum_{v \in V(T_i), v \neq u} r_v(T) \right) + r_u(T).$$

Since for every i, j, $i \neq j$, $V(T_i) \bigcap V(T_j) = \{u\}$, then we have that

$$\sum_{i=1}^{h} \left(\sum_{v \in V(T_i), v \neq u} r_v(T) \right) + r_u(T) = \sum_{v \in V(T)} r_v(T).$$

By part 3,

$$\sum_{i=1}^{h}\left(\sum_{e\in E(T_i)} r_e(T_i)\right) = \sum_{i=1}^{h}\left(\sum_{e\in E(T_i)} r_e(T)\right).$$

Since for every i, j, $i \neq j$, $E(T_i) \bigcap E(T_j) = \emptyset$, then we have that

$$\sum_{i=1}^{h}\left(\sum_{e\in E(T_i)} r_e(T)\right) = \sum_{e\in E(T)} r_e(T).$$

Subcase 1 $h = 2$

Note that since T is the vertex sum at u of T_1 and T_2, then u is pendant in both T_1 and T_2. By Lemma 21, for each $w \in V(T)$, $w \neq u$, $r_w(T) = r_w(T_i)$.

By Lemma 16,

$$r_u(T) = \min\left\{\sum_{i=1}^{2} r_u(T_i), 2\right\}.$$

Since u is pendant in T_1 and T_2 then by Corollary 7, $r_u(T_i) \leq 1$ for all i. Thus

$$r_u(T) = r_u(T_1) + r_u(T_2).$$

By Lemma 41, $r_e(T_i) = 0$ if and only if $r_e(T) = 0$. By Lemma 35, $r_e(T) = 0$ or $r_e(T) = 1$ and similarly for $r_e(T_i)$. Thus $r_e(T_i) = 1$ if and only if $r_e(T) = 1$. Thus for every $e \in E(T)$, $r_e(T) = r_e(T_i)$.

Subcase 2 $r_u(T) = 0$

By Corollary 24, for each $w \in V(T)$, $w \neq u$, $r_w(T) = r_w(T_i)$.

By Lemma 16,

$$r_u(T) = \min\left\{\sum_{i=1}^{h} r_u(T_i), 2\right\}.$$

Since $r_u(T) = 0$, then

$$r_u(T) = \sum_{i=1}^{h} r_u(T_i).$$

By Lemma 42, $r_e(T_i) = 0$ if and only if $r_e(T) = 0$. By Lemma 35, $r_e(T) = 0$ or $r_e(T) = 1$. Thus $r_e(T_i) = 1$ if and only if $r_e(T) = 1$. Thus for every $e \in E(T)$, $r_e(T) = r_e(T_i)$.

Subcase 3 $r_u(T) = 1$

Note that since T is the vertex sum at u of $T_1, T_2, \ldots, T_h$ then u is pendant in each T_i. By Lemma 27, for each $w \in V(T)$, $w \neq u$, $r_w(T) = r_w(T_i)$.

By Lemma 16,

$$r_u(T) = \min\left\{\sum_{i=1}^{h} r_u(T_i), 2\right\}.$$

Since $r_u(T) = 1$, then

$$r_u(T) = \sum_{i=1}^{h} r_u(T_i).$$

By Lemma 43, $r_e(T_i) = 0$ if and only if $r_e(T) = 0$. By Lemma 35, $r_e(T) = 0$ or $r_e(T) = 1$. Thus $r_e(T_i) = 1$ if and only if $r_e(T) = 1$. Thus for every $e \in E(T)$, $r_e(T) = r_e(T_i)$.

Case 2 For every non-pendant vertex u of T, $r_u(T) = 2$ and $d_T(u) \geq 3$.

Subcase 1 There exists exactly one non-pendant vertex u of T.

The only tree satisfying the hypotheses is S_n for some $n \geq 4$. From Example 2, $\mathrm{mr}(S_n) = 2$, $r_u(S_n) = 2$. Further since $n \geq 4$ then for every vertex v, $v \neq u$ $r_v(S_n) = 0$. By Corollary 6, $r_e(S_n) = 0$ for every edge $e \in E(S_n)$. Thus the statement is true for S_n.

Subcase 2 There exists more than one non-pendant vertex in T.

By Lemma 13, there exists a vertex u_1 such that there is at most one non-pendant vertex adjacent to u_1. Since T has more than one non-pendant vertex, then T is not S_n for some n. Thus u_1 is adjacent to exactly one non-pendant vertex u_2.

Now consider T as the edge-sum of T_1 and T_2 between vertices u_1 and u_2. Let f be the edge between u_1 and u_2. Since $d_T(u_1) \geq 3$ and every vertex v, $v \neq u_2$ which is adjacent to u_1 is pendant, then $T_1 = S_n$ for some $n \geq 3$.

From Example 2, $r_{u_1}(T_1) = 2$ and by hypothesis $r_{u_1}(T) = 2$. Note that $r_{u_1}(T_1) = r_{u_1}(T - f)$. By Lemma 31, $r_{u_1}(T) - r_{u_1}(T - f) = r_f(T)$. Thus $r_f(T) = 0$.

By hypothesis $r_{u_2}(T) = 2$. Since $r_{u_2}(T) = 2$ and $r_f(T) = 0$, then by Lemma 31, $r_{u_2}(T_2) = r_{u_2}(T - f) = 2$.

Since $r_{u_i}(T_i) = 2$ for $i = 1, 2$, then by Lemma 30 for every vertex $w \in V(T_i)$ $i = 1, 2$, $r_w(T_i) = r_w(T)$.

It remains to be shown that for every edge $e \in E(T_i)$, $r_e(T) = r_e(T_i)$.

Since $T_1 = S_n$ for some $n \geq 3$ then every edge is adjacent to a pendant vertex. Since $r_w(T_1) = r_w(T)$ for every pendant vertex w of T_1, then by Corollary 6 for every edge $e \in E(T_1)$, $r_e(T) = r_e(T_1)$.

Now for the edges in T_2

Forward Implication Let $e \in E(T_2)$ and assume that $r_e(T_2) = 0$. By Theorem 37, there exists a minimal path cover R_2 of T_2 in which no path uses the edge e. Let R_1 be a minimal path cover for T_1. Note that $R_1 \bigcup R_2$ is a path cover for T.

Since $r_f(T) = 0$ then by Lemma 32, $p_f(T) = 0$. Using the definition of path spread $P(T) - P(T - f) = 0$. This further implies that $P(T) = P(T_1) + P(T_2)$. Thus $R_1 \bigcup R_2$ is a minimal path cover for T. Since there exists a minimal path cover for T in which no path uses the edge e, then by Theorem 37, $r_e(T) = 0$.

Reverse Implication Let $e \in E(T_2)$ and assume that $r_e(T) = 0$.

Since u_1 is adjacent to at least two pendant vertices neither of which is incident to e or f, then u_1 is still adjacent to two pendant vertices in $T-e$ and $(T-e)-f$. By Corollary 17, $r_{u_1}(T - e) = 2$ and $r_{u_1}((T - e) - f) = 2$.

Thus by Lemma 31, $r_f(T - e) = 0$.

Then using definitions and the fact that $r_e(T) = 0$,

$$r_f(T - e) = \mathrm{mr}(T - e) - \mathrm{mr}((T - e) - f) = \mathrm{mr}(T) - \mathrm{mr}((T - f) - e) =$$

$$\mathrm{mr}(T) - \mathrm{mr}(T_1 \bigcup T_2) - e) = \mathrm{mr}(T) - \mathrm{mr}(T_1) - \mathrm{mr}(T_2 - e).$$

By Lemma 34, $\mathrm{mr}(T) = \mathrm{mr}(T_1) + \mathrm{mr}(T_2)$. Continuing with the same line of thought,

$$\mathrm{mr}(T) - \mathrm{mr}(T_1) - \mathrm{mr}(T_2 - e) = \mathrm{mr}(T_2) - \mathrm{mr}(T_2 - e) = r_e(T_2).$$

Thus $r_f(T - e) = r_e(T_2)$ and $r_e(T_2) = 0$.

Therefore for every $e \in E(T_2)$, $r_e(T_2) = 0$ if and only if $r_e(T) = 0$. By Corollary 36, for every $e \in E(T_2)$, $r_e(T_2) = r_e(T)$.

Since $\mathrm{mr}(T) = \mathrm{mr}(T_1) + \mathrm{mr}(T_2)$, then using the inductive hypothesis

$$\mathrm{mr}(T) = \sum_{i=1}^{2} \left(\sum_{v \in V(T_i)} r_v(T_i) - \sum_{e \in E(T_i)} r_e(T_i) \right).$$

It has been shown that for every vertex $w \in V(T_i)$, $r_w(T_i) = r_w(T)$ and for every edge $e \in E(T_i)$, $r_e(T_i) = r_e(T)$. Thus

$$\mathrm{mr}(T) = \sum_{i=1}^{2} \left(\sum_{v \in V(T_i)} r_v(T) - \sum_{e \in E(T_i)} r_e(T) \right).$$

Now $V(T_1) \bigcap V(T_2) = \emptyset$ and $E(T_1) \bigcap E(T_2) = \emptyset$. Thus

$$\mathrm{mr}(T) = \sum_{v \in V(T)} r_v(T) - \sum_{e \in E(T), e \neq f} r_e(T).$$

But $r_f(T) = 0$. Thus

$$\mathrm{mr}(T) = \sum_{v \in V(T)} r_v(T) - \sum_{e \in E(T)} r_e(T).$$

Therefore the statement is true for graphs with n vertices. ■

It should be noted that Theorem 44 does not generalize to graphs other than trees. Consider K_3.

The same lemmas used to prove Theorem 44 can be used to show by induction that the sum of the rank-spreads of the vertices of a tree is even.

Chapter 6

Expansion to All Connected Graphs

Corollary 45

If T is a tree, then $\displaystyle\sum_{e\in E(T)} \mathrm{mr}(T-e) = \sum_{v\in V(T)} \mathrm{mr}(T-v)$.

Proof Let T be a tree on n vertices. By Theorem 44,

$$\mathrm{mr}(T) = \sum_{v\in V(T)} r_v(T) - \sum_{e\in E(T)} r_e(T).$$

Using the definition of rank-spread the above becomes

$$\mathrm{mr}(T) = \sum_{v\in V(T)} (\mathrm{mr}(T) - \mathrm{mr}(T-v)) - \sum_{e\in E(T)} (\mathrm{mr}(T) - \mathrm{mr}(T-e))\,.$$

Since T has n vertices, then T has $n-1$ edges. Thus

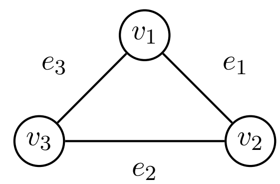
$$\mathrm{mr}(T) = \sum_{i=1}^{n} \mathrm{mr}(T) - \sum_{v\in V(T)} \mathrm{mr}(T-v) - \left(\sum_{i=1}^{n-1} \mathrm{mr}(T) - \sum_{e\in E(T)} \mathrm{mr}(T-e)\right).$$

Simplifying further the desired result is obtained. ■

Lemma 46

If G is a unicyclic connected graph then there exists a bijective function $f : V(G) \to E(G)$ such that for every $v \in V(G)$, $f(v)$ is incident to v.

PROOF Proceed by induction on the number of vertices. There are no unicyclic graphs on 1 or 2 vertices. Consider $n = 3$ for the base case. The only connected unicyclic graph on 3 vertices is K_3. Label the vertices v_1, v_2, and v_3. Let $e_1 = \{v_1v_2\}$, $e_2 = \{v_2v_3\}$, and $e_3 = \{v_3v_1\}$.

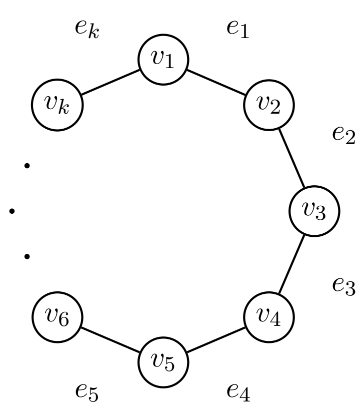


Define $f : V(K_3) \to E(K_3)$ by $f(v_i) = e_i$ for $i = 1, 2, 3$. Clearly f is a bijective function. By construction v_i is incident to e_i for all i. Therefore the base case is true.

Assume that for all unicyclic connected graphs on $k - 1$ vertices there exists such a bijective function.

Let G be a connected unicyclic graph on k vertices.

Case 1 No vertex of G is pendant. Since G is unicyclic and no vertex of G is pendant, then G must be a cycle of length k. Label the vertices of G $v_1, \ldots, v_k$ in a clockwise manner. Let $e_i = \{v_iv_{i+1}\}$ for $i = 1, \ldots, k-1$ and $e_k = \{v_kv_1\}$.



Define $f : V(C_k) \to E(C_k)$ by $f(v_i) = e_i$ for all $i = 1, 2, \ldots, k$. Clearly f is a bijective function. By construction v_i is incident to e_i for all i.

Case 2 There exists a pendant vertex in G. Label a pendant vertex of G, v_k and label the edge incident to v_k, e_k. Label the remaining vertices of G in any order $v_1, \ldots, v_{k-1}$.

Consider $G - v_k$. Since deleting v_k deletes e_k, then $G - v_k$ is a connected unicyclic graph on $k-1$ vertices. By the inductive hypothesis there exists a bijective function g from $V(G - v_k)$ to $E(G - v_k)$ such that for every $v \in V(G - v_k)$, $g(v)$ is incident to v. Let $e_i = g(v_i)$ for all $i = 1, 2, \ldots, k-1$.

Define $f : V(G) \to E(G)$ by $f(v_i) = g(v_i)$ for all $i \neq k$ and $f(v_k) = e_k$. Clearly f is a bijective function by construction. Note that e_i is incident to v_i for every i and $f(v_i) = e_i$ for every i by construction.

Thus for every unicyclic graph G on k vertices there exists a bijective function $f : V(G) \to E(G)$ such that $f(v)$ is incident with v.

Therefore by induction the above statement is true for all connected unicyclic graphs. ■

Theorem 47

If G is a connected graph, then $\sum_{e \in E(G)} \mathrm{mr}(G - e) \geq \sum_{v \in V(G)} \mathrm{mr}(G - v)$. Further, if equality occurs then G is either unicyclic or acyclic. If equality occurs and G is unicyclic, then there exists a bijective function f from $V(G)$ to $E(G)$, such that for every vertex $v \in V(G)$, $f(v)$ is incident to v and $\mathrm{mr}(G - f(v)) = \mathrm{mr}(G - v)$.

Proof Let G be a connected graph.

Case 1 The graph G is a tree. By Corollary 45, $\sum_{e \in E(G)} \mathrm{mr}(G - e) = \sum_{v \in V(G)} \mathrm{mr}(G - v)$.

Case 2 The graph G is not a tree. Since G is not a tree and connected then $|E(G)| \geq |V(G)| \geq 3$. Further G has a unicyclic subgraph H which spans G. By Lemma 46, there exists a bijective function f from $V(H)$ to $E(H)$ such that for every $v \in V(H)$, $f(v)$ is incident to v. Note that since H spans G, then $V(H) = V(G)$.

By Lemma 31, for every $v \in V(G)$, $r_v(G) - r_v(G - f(v)) = r_{f(v)}(G)$. By Corollary 5, $r_v(G - f(v)) \geq 0$. Thus

$$r_v(G) \geq r_{f(v)}(G).$$

Using the definition of rank-spread we have that

$$\mathrm{mr}(G - f(v)) \geq \mathrm{mr}(G - v).$$

Using the above and the fact that f is bijective,

$$\sum_{e\in E(H)} \mathrm{mr}(G-e) \geq \sum_{v\in V(G)} \mathrm{mr}(G-v).$$

Thus

$$\sum_{e\in E(G)} \mathrm{mr}(G-e) = \sum_{e\in E(G)\setminus E(H)} \mathrm{mr}(G-e) + \sum_{e\in E(H)} \mathrm{mr}(G-e) \geq \sum_{v\in V(G)} \mathrm{mr}(G-v).$$

Since $|E(G)| \geq 3$, then $\mathrm{mr}(G-e) > 0$ for any edge e of G. So if $E(G)\setminus E(H) \neq \emptyset$, then

$$\sum_{e\in E(G)} \mathrm{mr}(G-e) > \sum_{v\in V(G)} \mathrm{mr}(G-v).$$

Otherwise G itself is unicyclic, and

$$\sum_{e\in E(G)} \mathrm{mr}(G-e) \geq \sum_{v\in V(G)} \mathrm{mr}(G-v).$$

If equality occurs in the above equation it must be the case that $\mathrm{mr}(G-f(v)) = \mathrm{mr}(G-v)$ for every $v \in V(G)$.

Having exhausted all the cases, the result follows. ■

Definition 12

For any $n \geq 3$, the *n-sun* H_n is the corona graph of an n-cycle, namely, the graph on $2n$ vertices obtained by appending a pendant vertex to each vertex of an n-cycle.

Lemma 48 ([BFH04, Proposition 3.2])

Let H_n be the n-sun on $2n$ vertices. Then

(1) $P(H_n) = \left\lceil \frac{n}{2} \right\rceil, n \geq 3$

(2) $\mathrm{mr}(H_3) = 4$

(3) $\mathrm{mr}(H_n) = 2n - \left\lfloor \frac{n}{2} \right\rfloor, n > 3.$

In particular, if $n > 3$ is odd, $P(H_n) > M(H_n)$.

Definition 13

For any $n \geq 3$, a *partial n-sun* is the graph obtained from C_n by appending a pendant vertex to each vertex in a subset of $V(C_n)$.

Lemma 49 ([BFH05, Corollary 3.5])

Let H be a partial n-sun. Then

(1) $P(H) + \mathrm{mr}(H) = |V(H)| + 1$ if $n > 3$, odd, and H is the n-sun

(2) $P(H) + \mathrm{mr}(H) = |V(H)|$ otherwise

Lemma 50

If G is a partial n-sun, then one of the following must be true.

(1) There exists $e \in E(G)$ such that $r_e(G) \leq 0$

(2) There exists $v \in V(G)$ such that $r_v(G) = 2$.

Proof Let G be a partial n-sun. By Lemma 49, either $P(G) + \mathrm{mr}(G) = |V(G)|$ or $P(G) + \mathrm{mr}(G) = |V(G)| + 1$

Case 1 $P(G) + \mathrm{mr}(G) = |V(G)|$. Let R be a minimal path cover for G. Since R consists of vertex-disjoint paths then not all the edges of the induced cycle can be covered by the paths in R. Let $e \in E(G)$ be such an edge. Since e was not used as part of the minimal path cover, then R is a path cover for $G - e$. Thus $P(G) \geq P(G - e)$.

Consider $G - e$. Since e is an edge of the induced cycle, then $G - e$ is a tree. By Theorem 15, $P(G - e) + \mathrm{mr}(G - e) = |V(G - e)| = |V(G)|$. Thus

$$P(G - e) + \mathrm{mr}(G - e) = |V(G)| = P(G) + \mathrm{mr}(G).$$

Since $P(G) \geq P(G - e)$, then

$$r_e(G) = \mathrm{mr}(G) - \mathrm{mr}(G - e) = P(G - e) - P(G) \leq 0.$$

Case 2 $P(G) + \mathrm{mr}(G) = |V(G)| + 1$ By Lemma 49, $n > 3$, odd and G is an n-sun. Let v be a vertex of the induced cycle. Now $G - v = K_1 \bigcup T$, where T is a tree with $n - 1$ pendant vertices. The tree T can be covered by $(n - 1)/2$ paths and since it has $n-1$ pendant vertices, this path cover is minimal. Thus $P(T) = (n-1)/2$. By Theorem 15, $\text{mr}(T) + P(T) = 2n-2$. Thus $\text{mr}(T) = 3(n-1)/2$. Since $\text{mr}(K_1) = 0$, then $\text{mr}(G-v) = \text{mr}(T)$. Thus by Lemma 48,

$$r_v(G) = 2n - \frac{(n-1)}{2} - 3\frac{(n-1)}{2} = 2.$$

Having exhausted all the cases the result follows. ∎

Lemma 51

If G is a connected graph and $\sum_{e\in E(G)} \text{mr}(G-e) = \sum_{v\in V(G)} \text{mr}(G-v)$ then G is tree.

Proof Let G be a connected graph such that $\sum_{e\in E(G)} \text{mr}(G-e) = \sum_{v\in V(G)} \text{mr}(G-v)$. By Theorem 47, G is either unicyclic or acyclic. Thus it suffices to show that G is not unicyclic.

Suppose by way of contradiction that G is unicyclic. By Theorem 47, there exists a bijective function f from $V(G)$ to $E(G)$, such that for every vertex $v \in V(G)$, $f(v)$ is incident to v and $\text{mr}(G - f(v)) = \text{mr}(G-v)$.

Thus for every $v \in V(G)$, $\text{mr}(G) - \text{mr}(G - f(v)) = \text{mr}(G) - \text{mr}(G-v)$. Using the definition of rank spread $r_{f(v)}(G) = r_v(G)$ for every $v \in V(G)$. By Corollary 5, for any $v \in V(G)$, $2 \geq r_v(G) \geq 0$ and for any $e \in E(G)$, $1 \geq r_e(G) \geq -1$. Thus
$1 \geq r_v(G) = r_{f(v)}(G) \geq 0$.

If p is a pendant vertex adjacent to vertex q, then by Lemma 9 part 1,

$$\text{mr}(G-q) = \text{mr}(G-p) - 2 \iff \text{mr}(G) = \text{mr}(G-p).$$

Using the definition of rank spread $r_q(G) = r_p(G) + 2 \iff r_p(G) = 0$. Thus if $r_p(G) = 0$ then $r_q(G) = 2$. Since no vertex of G has rank-spread 2, then no pendant vertex of G has rank-spread 0. Thus every pendant vertex of G has rank-spread 1. Thus by Corollary 6, every edge incident to a pendant vertex has rank-spread 1.

By Corollary 17, if v is adjacent to two or more pendant vertices, then $r_v(G) = 2$. Since no vertex of G has rank- spread 2, then no vertex of G is adjacent to two or more pendant vertices.

Case 1 The graph G contains no pendant vertices. Since G is unicyclic, then $G = C_n$ for some n. By Example 7, $\text{mr}(G) = \text{mr}(C_n) = n-2$.

Let $e \in E(G)$. Then $\text{mr}(G - e) = \text{mr}(C_n - e) = \text{mr}(P_n) = n - 1$ by Example 1. Thus $r_e(G) = n - 2 - (n - 1) = -1$. This contradicts that $r_e(G) \geq 0$.

Case 2 The graph G contains at least one pendant vertex. By Lemma 31, if $v \in V(G)$ and e is incident to v then, $r_v(G) - r_v(G - e) = r_e(G)$. By Corollary 5, $r_v(G - e) \geq 0$. Thus if $r_v(G) = 0$, then $r_e(G) \neq 1$ for every edge e incident to v. In other words a rank-spread zero vertex cannot be incident to a rank-spread one edge.

Subcase 1 There exists a pendant vertex of G which is not adjacent to any vertex in the induced cycle C of G. Consider a branch B of G which contains such a pendant vertex. Let w be the vertex in the induced cycle of G which is nearest the vertices of the branch B. Let p be a pendant vertex in B which is farthest from w. Let $y \neq w$ be the vertex adjacent to p. Since y is adjacent to p then there are no other pendant vertices adjacent to y. If $d(y) \geq 3$, then there exists a pendant vertex of B whose distance to w is greater than the distance from p to w. Thus $d(y) = 2$. Let z be the other vertex adjacent to y. Note that it is possible that $z = w$.

Since y is incident to an edge with rank-spread 1, then $r_y(G) = 1$. Further because of the bijection the edge between y and z must have rank-spread 1, which in turn forces z to have rank-spread 1.

Now p, y, and z induce P_3. Thus $G = P_3 \underset{z}{\oplus} (G - p - y)$. Since $r_z(P_3) = 1$ and $r_z(G) = 1$, then by Lemma 16, $r_z(G - p - y) = 0$. By Lemma 22,
$r_y(G) = r_y(P_3) = 2$. This contradicts that no vertex in G has rank-spread 2.

Subcase 2 Every pendant vertex of G is adjacent to a vertex of the induced cycle C. Since each vertex of the cycle can be adjacent to at most one pendant vertex, then G is a partial n-sun. Since a rank-spread zero vertex cannot be incident to a rank-spread one edge, then every vertex adjacent to a pendant vertex must have rank-spread 1. Since G has at least one pendant vertex, then there exists a vertex w of the induced cycle such that $r_w(G) = 1$. Since f is a bijection between vertices and edges, such that v and $f(v)$ are incident, then f must map a pendant vertex, to the unique edge incident to it. Thus f maps vertices of the induced cycle to edges on the induced cycle. From above $r_{f(v)}(G) = r_v(G)$ for every $v \in V(G)$. Since $r_w(G) = 1$, then $r_{f(w)}(G) = 1$. Since $r_{f(w)}(G) = 1$, then the other vertex incident to $f(w)$ must have rank-spread one. Thus all the vertices and edges of the induced cycle must have rank-spread 1. This contradicts Lemma 50.

Thus G is not unicyclic. ■

Theorem 52

If G is a connected graph, then $\sum_{e \in E(G)} \mathrm{mr}(G-e) \geq \sum_{v \in V(G)} \mathrm{mr}(G-v)$ with equality if and only if G is a tree.

PROOF The result follows from Corollary 45, Theorem 47, and Lemma 51. ■

The implications of Theorem 52 are still unknown. It does give rise to a non-negative parameter for a connected graph. The parameter is zero if and only if the graph is a tree. The value for K_n, $n \geq 2$ is $n(n-2)$ and the value for C_n is n. It is conjectured that the value for a unicyclic graph on n vertices is less than or equal to n. Over the real field it is also conjectured that for any graph on n vertices the parameter is at most $n(n-2)$. It should also be mentioned that the parameter is not field independent.

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