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PIPE DIAGRAMS FOR THOMPSON'S GROUP F

by

Aaron Peterson

A thesis submitted to the faculty of

Brigham Young University

in partial fulfillment of the requirements for the degree of

Master of Science

Department of Mathematics

Brigham Young University

August 2007

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BRIGHAM YOUNG UNIVERSITY

GRADUATE COMMITTEE APPROVAL

of a thesis submitted by

Aaron Peterson

This thesis has been read by each member of the following graduate committee and by majority vote has been found to be satisfactory.

Date

James Cannon, Chair

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David Cardon

Date

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As chair of the candidate's graduate committee, I have read the thesis of Aaron Peterson in its final form and have found that (1) its format, citations, and bibliographical style are consistent and acceptable and fulfill university and department style requirements; (2) its illustrative materials including figures, tables, and charts are in place; and (3) the final manuscript is satisfactory to the graduate committee and is ready for submission to the university library.

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ABSTRACT

PIPE DIAGRAMS FOR THOMPSON'S GROUP F

Aaron Peterson

Department of Mathematics

Master of Science

We review the definition and standard description of Thompson's Group F . We define the set of *pipe diagrams* and show that this set forms a group isomorphic to F . We use pipe diagrams to prove two theorems about giving a minimal representation for an arbitrary element of F .

ACKNOWLEDGMENTS

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Chapter 1

Preface

Thompson's Group F was first described by Richard Thompson in 1965. It was used to construct finitely presented groups with unsolvable word problems [10]. Thompson's Group also arises in homotopy theory in work on homotopy idempotents. [4], [5], and [7]. The group F has a universal conjugacy idempotent, and is an infinitely iterated HNN extension [7], [2]. Brown and Geoghegan [2] proved that F is FP_∞ , giving the first example of a torsion-free infinite-dimensional FP_∞ group.

An element of F is typically represented by a pair of binary trees [3], [6], [11]. Multiplication is performed by rotating subtrees. Rotation is tedious since each operation requires a new pair of trees to be drawn. In this thesis we introduce, as an alternative, the *pipe diagram*. In pipe diagrams, multiplication by a generator simply requires the lengthening of one pipe. A multiplicative history can be recorded in a single diagram.

A widely studied problem is to find the minimal representation for an element of F in a given presentation. Fordham [6] succeeded in solving this problem for a two generator set by examining the caret types of trees. We study this problem for a three generator presentation by examining pipe diagrams. In this thesis we prove two

theorems concerning the minimal representation problem for pipe diagrams including a local reduction algorithm. We do not yet have a global reduction algorithm.

In chapter 2 we review the definition and standard descriptions of F . In chapter 3 we define the set of pipe diagrams and show that this set is isomorphic to F . In chapters 4 and 5 we examine the problem of minimal reductions, and in chapter 6 we discuss other problems that we are interested in.

Chapter 2

Review of Thompson's Group F

In chapter 2 we review the definition of Thompson's group F and describe how F is equivalent to the set of reduced ordered pairs of rooted ordered binary n -caret trees. We introduce generators for F and give two standard presentations and a third presentation. We also discuss multiplication by a generator and the advantages of using the third presentation. The content of this section is standard in the literature on F and many proofs are not included. For a more detailed description of F see [3].

2.1 Thompson's Group F

Let F be the set of piecewise linear homeomorphisms from the closed unit interval $[0,1]$ to itself that satisfy the following:

- 1) If f is not differentiable at x , then $x \in \{\frac{p}{2^q} \mid p, q \in \mathbb{Z}\}$.
- 2) For $f \in F$, if $f'(x)$ exists, then $f'(x) = 2^k$ for some integer $k \in \mathbb{Z}$.

Since $f \in F$ is a strictly increasing homeomorphism it is clear that $f(0) = 0$ and $f(1) = 1$. Since f is piecewise linear, the set of points at which f is not differentiable must be finite.

It can be checked that a function $f \in F$ maps the set of dyadic rational numbers bijectively onto itself. It follows that F is closed under function composition and inversion. So with the operation of function composition, F forms a group which is Thompson's Group F .

Example 1.1 Three functions in F are the functions A , B and C defined below, and the graphs of A and B are shown in 2.1.

$$A(x) = \begin{cases} \frac{x}{2}, & 0 \leq x \leq \frac{1}{2} \\ x - \frac{1}{4}, & \frac{1}{2} \leq x \leq \frac{3}{4} \\ 2x - 1, & \frac{3}{4} \leq x \leq 1 \end{cases} \quad B(x) = \begin{cases} x, & 0 \leq x \leq \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4}, & \frac{1}{2} \leq x \leq \frac{3}{4} \\ x - \frac{1}{8}, & \frac{3}{4} \leq x \leq \frac{7}{8} \\ 2x - 1, & \frac{7}{8} \leq x \leq 1 \end{cases}$$

$$C(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{8} \\ x + \frac{1}{8}, & \frac{1}{8} \leq x \leq \frac{1}{4} \\ \frac{x}{2} + \frac{1}{4}, & \frac{1}{4} \leq x \leq \frac{1}{2} \\ x, & \frac{1}{2} \leq x \leq 1 \end{cases}$$

Other important examples include the functions

$$X_0 = A, X_1 = B, X_2 = A^{-1}BA, \dots, X_n = A^{-(n-1)}BA^{n-1}.$$

2.2 Binary Trees

Binary trees play an important role in understanding F . Each element $f \in F$ can be realized as a pair of binary trees where multiplication involves rotating subtrees of one of the trees.

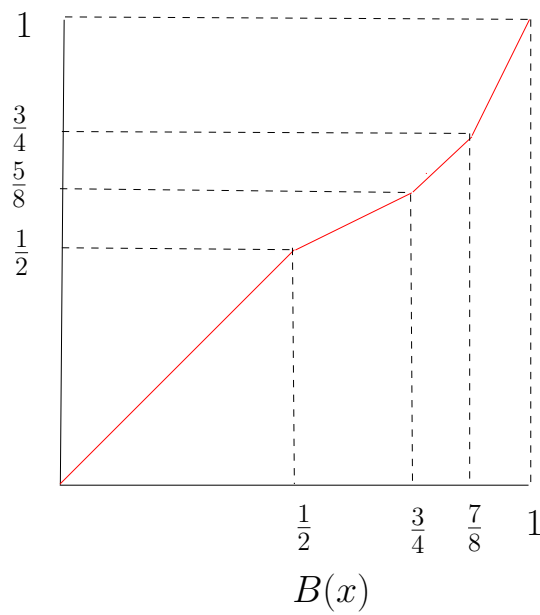
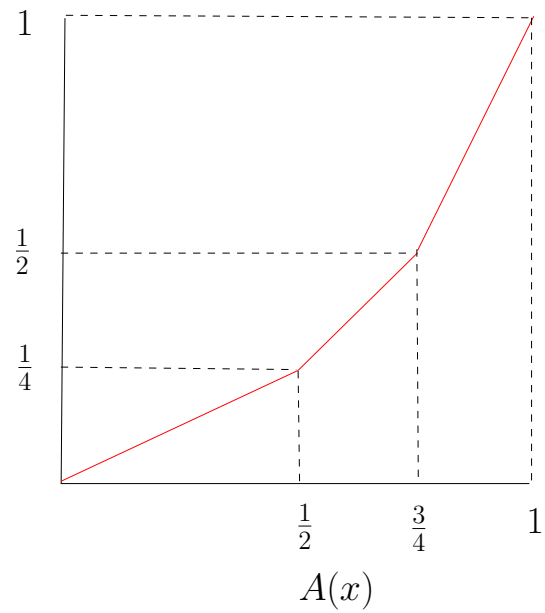


Figure 2.1 The graphs of $A(x)$ and $B(x)$.

Definition 2.2.1. A *rooted ordered binary tree* S is a tree with the following properties.

1. S has a root v_0
2. if S consists of more than v_0 , then v_0 has valence 2
3. if v is a vertex in S with valence greater than 1, then there are exactly two edges $e_{v,L}$, $e_{v,R}$ which contain v but are not contained in the shortest path in S from v_0 to v .

The edge $e_{v,L}$ is the left edge, and the edge $e_{v,R}$ is the right edge. Define a *leaf* to be any vertex of valence 1 or a vertex of valence 0 in the case of the trivial tree. A *caret* consists of a vertex v of valence 3 and the corresponding edges $e_{v,L}$ and $e_{v,R}$. The *left side* of the tree consists of the maximal arc of left edges containing v_0 , and the *right side* is defined analogously. There is a standard numbering for the leaves and the carets of a rooted ordered binary tree. Extend all the leaves so that they lie on a horizontal line, and then number them from left to right. To obtain the numbering on the carets, number the gaps between the leaves from left to right and follow the gap upward to a vertex. The number of the vertex is the same as the number of the gap. In 2.2 we have an example.

Definition 2.2.2. A *standard dyadic interval* in the unit interval $[0,1]$ is an interval of the form $[\frac{a}{2^n}, \frac{a+1}{2^n}]$, with $a, n \geq 0$ and $a \leq 2^n - 1$.

We will now describe the *tree of standard dyadic intervals* \mathcal{T} . The vertices of \mathcal{T} are the standard dyadic intervals in $[0,1]$. An edge of \mathcal{T} is a pair (I, J) of standard dyadic intervals I and J where either I is a left half of J and (I, J) is a left edge, or I is the right half of J and (I, J) is a right edge. A finite portion of the tree of standard dyadic intervals is shown in 2.3.

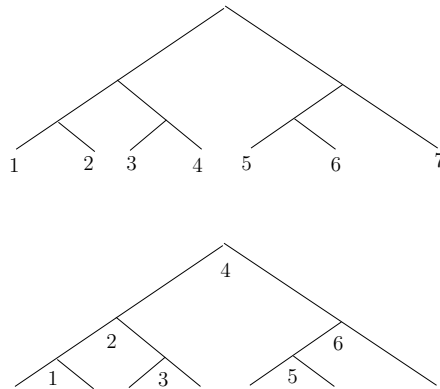


Figure 2.2 An example of an ordered rooted binary tree with the leaves and carets numbered left to right.

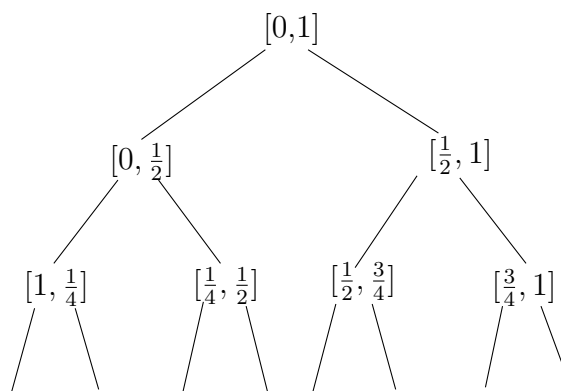


Figure 2.3 The tree of standard dyadic intervals

Definition 2.2.3. A \mathcal{T} -tree is defined to be a finite ordered rooted subtree of \mathcal{T} with root $[0,1]$. From now on \mathcal{T} -trees will be referred to simply as *trees*.

Definition 2.2.4. Define a *tree diagram* to be an ordered pair of trees (D, R) with the same number of leaves. A tree diagram can be thought of as a function

$$D \rightarrow R.$$

In the diagram D is called the *domain tree* and R is called the *range tree*. The leaves of D partition $[0,1]$ into standard dyadic intervals, and the leaves of R do the same. Since a function $f \in F$ maps standard dyadic intervals in $[0,1]$ linearly onto standard dyadic intervals in $[0,1]$ we can construct a function $f \in F$ from a tree diagram (D, R) . Define f to be the function that takes the standard dyadic interval corresponding to the i -th leaf in D linearly onto the standard dyadic interval corresponding to the i -th leaf in R .

Given a function $f \in F$ it is not difficult to construct a tree diagram (D, R) which represents f , but this diagram is not unique. Another tree diagram can be constructed by adding carets in a way which we will now describe. Let I and J be the n -th leaves of D and R respectively. Let I_1 and I_2 be the leaves in order of the caret C at vertex I , and let J_1 and J_2 be the leaves in order of the caret K at vertex J . Since f maps I linearly onto J , $f(I_1) = J_1$ and $f(I_2) = J_2$. So (D', R') is a tree diagram for f where $D' = D \cup C$ and $R' = R \cup K$. An example is shown in 2.4.

To obtain a bijective correspondence between the set of tree diagrams and the set of elements of Thompson's Group F define an equivalence relation on tree diagrams as follows. If for an integer n the n -th and $(n+1)$ -st leaves of D are leaves of a caret C and the n -th and $(n+1)$ -st leaves of R are leaves of a caret K , then by removing the carets C and K except for the roots we obtain a new tree diagram for f . If there are no such carets then the diagram is said to be *reduced*. There is exactly one reduced

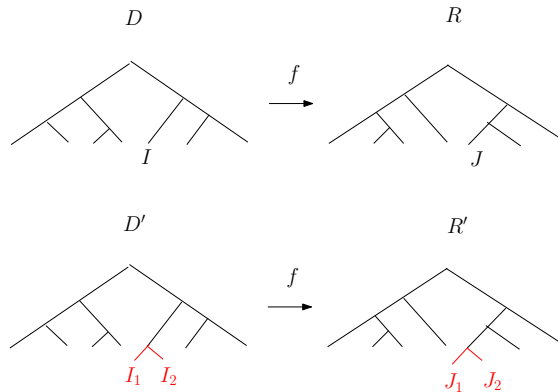


Figure 2.4 In D' and R' , the 5th and 6th leaves are leaves of the same caret in each tree. This is an unreduced tree diagram for f . In D and R those carets are removed. This is the reduced diagram for f .

tree diagram associated to a function $f \in F$.

In 2.4, the 5th and 6th leaves of D' are leaves of a caret C and the 5th and 6th leaves of R' are leaves of a caret K . If we remove these carets except for their roots, we obtain the tree diagram (D, R) . After removing these carets there are no more carets of this type, so (D, R) is the reduced tree diagram for the function f .

Theorem 2.2.1. *There is a bijection from the set of reduced tree diagrams to Thompson's Group F .*

The bijection between F and the set of tree diagrams induces a multiplication structure on the set of tree diagrams. Given two tree diagrams (D, R) and (R, S) , their product is simply (D, S) . This corresponds to function composition. Since R is the range of the first function and the domain of the second, we can compose these two functions. The function representing (D, S) will take the standard dyadic intervals of D linearly onto the standard dyadic intervals of S . Given two tree diagrams (D, R) and (R', S) where $R \neq R'$, we add carets where necessary to make R match up with R' . An example is shown in 2.5.

Theorem 2.2.2. *The set of reduced tree diagrams with the multiplication operation*

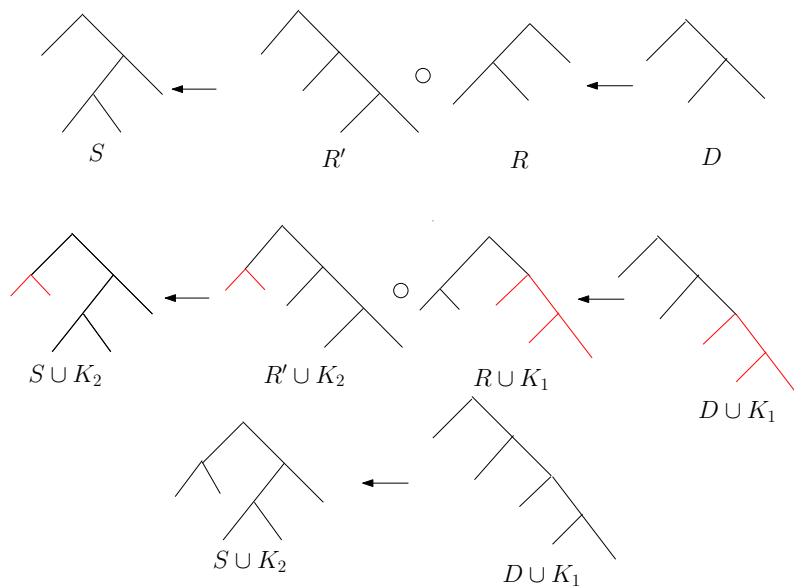


Figure 2.5 Two trees are multiplied by hanging carets on appropriate vertices until the domain tree of one is identical to the range tree of the other.

described above forms a group isomorphic to Thompson's Group F .

2.3 Presentations for F

Of great importance are the elements A , B , and C as shown in the previous chapter. From now on we will refer to these elements as a , b , and c . In 2.6 we show the tree diagrams for the elements a , b , and c . Multiplication on the right by a is shown in 2.7. Multiplication on the right by b and c is similar to multiplication on the right by a . These elements generate F [3].

There are two standard presentations for F .

$$F_2 = \langle a, b : [ab^{-1}, a^{-1}ba], [ab^{-1}, a^{-2}ba^2] \rangle$$

where $[x, y] = xyx^{-1}y^{-1}$ and

$$F_\infty = \langle X_0, X_1, X_2, \dots : X_k^{-1}X_nX_k = X_{n+1} \text{ for } k < n \rangle.$$

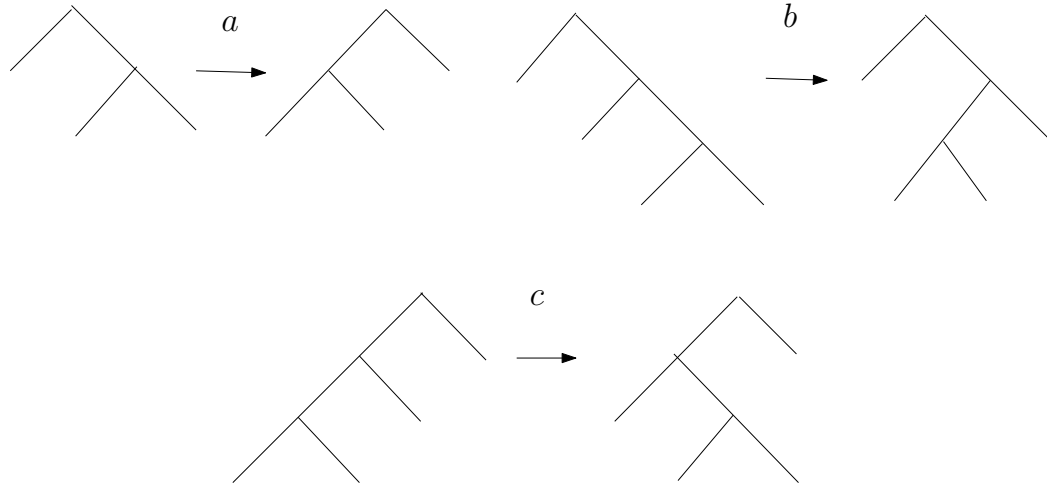


Figure 2.6 The tree diagrams for the generators $a, b,$ and $c.$

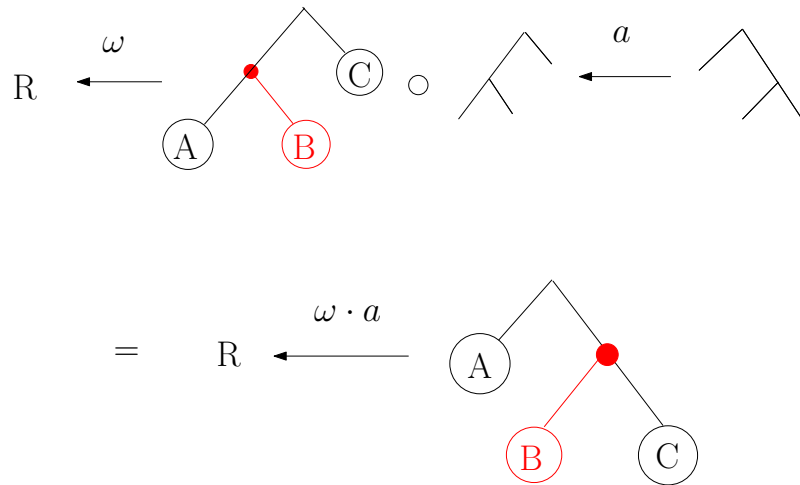


Figure 2.7 Right multiplication by a essentially rotates the subtree B to the right side of the diagram.

In these presentations, the symbol X_k corresponds to the element of F previously defined using this symbol. It is not difficult to determine $c = aba^{-2}$. This yields a third presentation for F

$$F_3 = \langle a, b, c : [ab^{-1}, a^{-1}ba], [ab^{-1}, a^{-2}ba^2], c^{-1}aba^{-2} \rangle.$$

Chapter 3

Pipe Diagrams

In chapter 3 we introduce a new diagram which we will call a pipe diagram. Then we show that the set of equivalence classes of pipe diagrams is equivalent to the set of reduced ordered pairs of rooted ordered n -caret trees. We define multiplication on the set of equivalence classes of pipe diagrams and show that, with this operation, the set forms a group isomorphic to Thompson's Group F .

3.1 Definition of a Pipe Diagram

Definition 3.1.1. A *pipe diagram* consists of a horizontal line segment with a finite number of vertical line segments crossing perpendicular to the horizontal segment. The vertical line segments are called *pipes*.

We require that no two pipes can see each other. This means that if two pipes are the same length on top (on bottom), then there is a pipe that is longer on top (respectively on bottom). An example is shown in 3.1.

A pipe diagram is completely characterized by the number of pipes and the relative heights of the pipes on either side of the horizontal line segment in a way which we

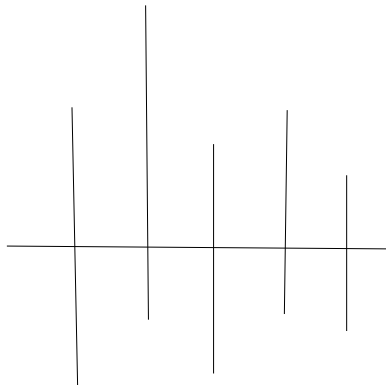


Figure 3.1 An example of a pipe diagram

will now describe.

To facilitate the description we will assign a top number and a bottom number to each pipe. The tallest pipe above the horizontal line segment is given top number 1. The tallest pipe on either side of 1 is given top number 2. The tallest pipe on either side of each 2 lying on the same side of the 1 as the 2 is given top number 3. We continue in this manner until all of the pipes are labeled. Bottom numbers are assigned similarly. We say that two pipe diagrams are equivalent if numbers appear in the same order. The set of equivalence classes of pipe diagrams will simply be called the set of pipe diagrams. For convenience, the top number of a pipe will be called the *level* of the pipe. We illustrate the numbering system in 3.2.

3.2 Correspondence between pipe diagrams and F

Theorem 3.2.1. *The set of pipe diagrams is in bijective correspondence with the set of ordered pairs of rooted ordered n -caret trees.*

Proof. Given an ordered pair of rooted ordered n -caret trees (D, R) , we will construct the corresponding pipe diagram. Draw a horizontal line segment. Beginning with D

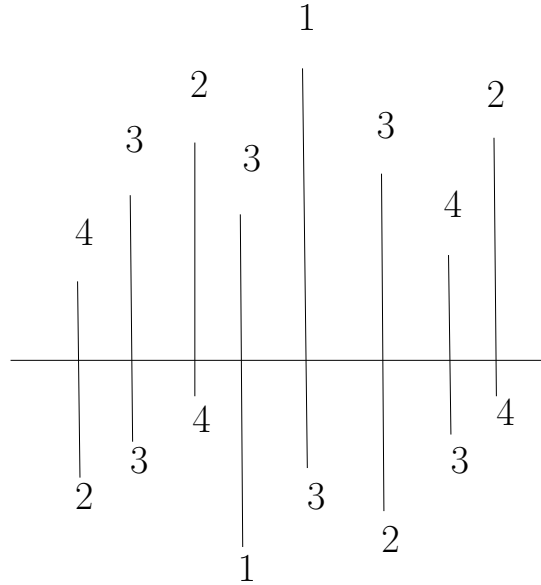


Figure 3.2 The tallest pipe is number 1, the tallest pipe on each side of 1 is 2, and so forth.

label the initial caret 1. Inductively, a caret is labeled $i + 1$ if it is a child of i . Corresponding to caret 1 draw a vertical line segment extending upward from the horizontal line segment which will be the level 1 pipe. Corresponding to each caret 2 draw a vertical line segment extending upward from the horizontal line segment on the appropriate side of pipe 1. Each of these vertical line segments will be a level 2 pipe. We proceed in this manner until we have the upper half of a pipe diagram which corresponds to D . Then we construct the lower half of a pipe diagram which corresponds to R by proceeding in the same way as before and then reflecting our diagram about the horizontal line segment. Then we match the two diagrams together so that the horizontal line segments coincide and all of the pipes line up. Our construction yields the bijective correspondence. An example can be seen in 3.3.

Definition 3.2.1. We say a pipe is a *trivial pipe* if it is shorter than any adjacent pipe on both sides of the horizontal line. A pipe is also trivial if it becomes trivial upon removal of a pipe already known to be trivial. A pipe is *lower-trivial* if it is

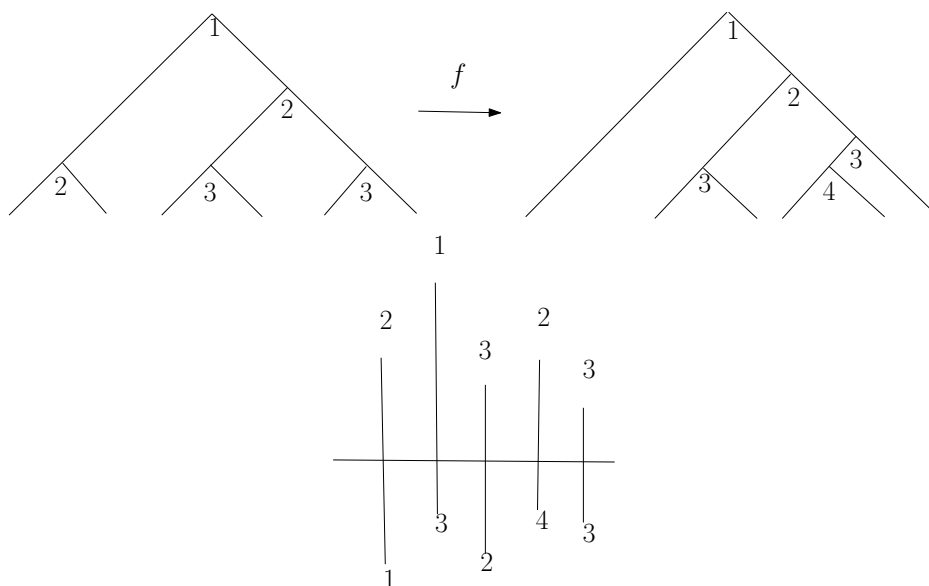


Figure 3.3 Number the vertices in a tree diagram according to the distance from the root vertex. Each vertex then corresponds to a pipe with the same number.

locally shortest on bottom.

On the set of pipe diagrams we will define an equivalence relation as follows: Two pipe diagrams are equivalent if one can be obtained from the other by inserting or deleting trivial pipes. In each class there is a unique pipe diagram that contains no trivial pipes. We will call this representative a *reduced pipe diagram*.

Theorem 3.2.2. *The set of reduced pipe diagrams is in bijective correspondence with the set of reduced ordered pairs of rooted ordered n -caret trees.*

Proof. We only need to show that a pipe diagram is reduced if and only if the corresponding pair of rooted ordered caret trees is reduced. Let (D, R) be a reduced pair of ordered caret trees, and let P be the corresponding pipe diagram. We will renumber the pipes in P as follows: the pipe on the far left is given number 1, and inductively, the pipe immediately to the right of pipe i is given number $i + 1$. This numbering corresponds to the left to right numbering of carets on an n -caret tree.

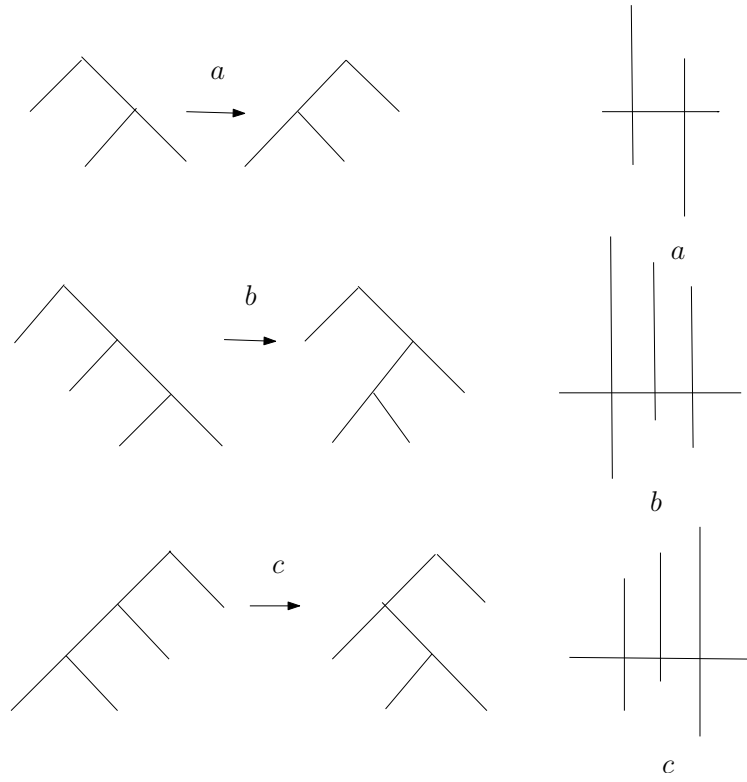


Figure 3.4 The generators a, b , and c as pipe diagrams.

Assume pipe j is trivial. Since pipe j is locally shortest on both sides, both children of caret j are leaves in both R and D . Since this does not occur we conclude that there are no trivial pipes in P . This completes the proof.

We have shown that the elements of Thompson's Group F are in bijective correspondence with the set of pipe diagrams. In 3.4 we show the generators a, b , and c as pipe diagrams.

Definition 3.2.2. If a single pipe p is lengthened from level n to level $n - 1$, we say that p is *promoted*.

The multiplication in F viewed as pairs of trees induces a multiplication on the set of pipe diagrams. We will interpret this multiplication in terms of multiplication by a generator. In 3.5 we show how multiplication by a acts on the domain tree and

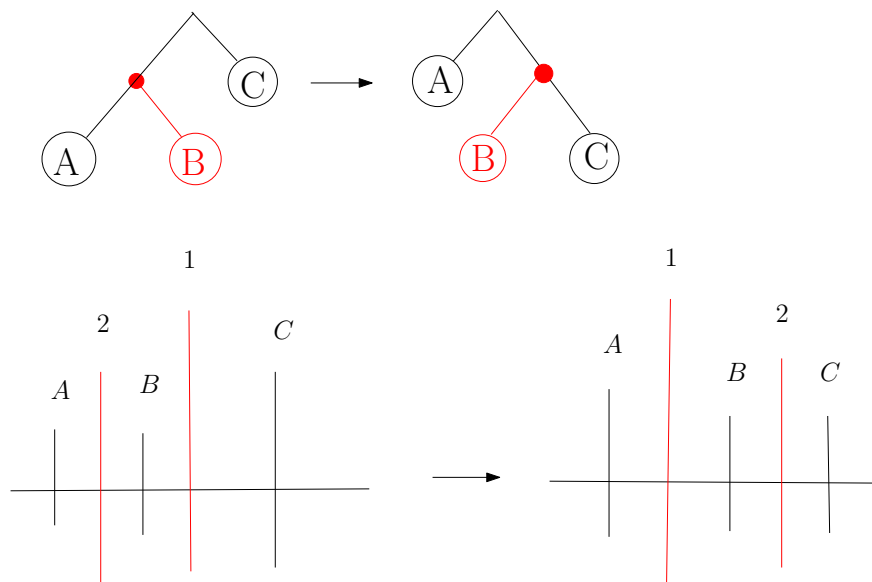


Figure 3.5 The generator a rotates the right subtree of the level 2 caret to the right side of the diagram. In the pipe diagram this corresponds to promoting the level 2 pipe to level 1.

the corresponding action on the pipe diagram.

Here the pipes A , B , and C represent the tallest pipes among a number of pipes. We see that multiplication by a essentially promotes the left level 2 pipe to level 1. A similar analysis shows that multiplication by b promotes the right left level 3 pipe to level 2, and multiplication by c promotes the left right level 3 pipe to level 2. This gives us the following theorem.

Theorem 3.2.3. *The set of pipe diagrams with the multiplication operation described above forms a group isomorphic to Thompson's Group F .*

Chapter 4

The Trivial Pipe Theorem

Given a generating set for F , it would be ideal to know how to reduce an arbitrary pipe diagram to the trivial diagram in the shortest number of steps. In chapter 4 we show that with certain generating sets which include generating sets from the standard presentations there is no need to insert a trivial pipe in a minimal reduction. In fact, if a trivial pipe is promoted, the reduction is not minimal. This is the *Trivial Pipe Theorem*.

4.1 The Trivial Pipe Theorem

Definition 4.1.1. A *saturated* generating set for F is a generating set $S_n = \{x_1, x_2, \dots\}$ which satisfies

1. each generator $x_i \in S_n$ promotes a particular pipe exactly one level
2. for every pipe at level $k \leq n$ there exists a generator $x_i \in S_n$ that promotes the pipe.

We now state and prove the Trivial Pipe Theorem.

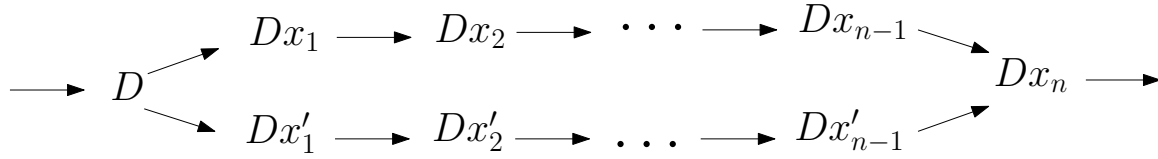


Figure 4.1 When the trivial pipe is raised we have two paths. The lower path is the shadow reduction.

Theorem 4.1.1. The Trivial Pipe Theorem. Let $F = \langle x_1, x_2, x_3, \dots : r_1, r_2, r_3, \dots \rangle$ be a presentation for F where the generating set is a saturated generating set. Then a reduction for an element of F in which a trivial pipe is promoted is not a minimal reduction. In particular, using the presentation F_3 , a reduction for an element of F in which a trivial pipe is promoted is not a minimal reduction.

Proof. This proof will involve several cases which we present using tree diagrams, although the proof was discovered using pipe diagrams. Promotions will sometimes be referred to as *moves*. Suppose we have a pipe diagram and a reduction of this diagram in which a trivial pipe is promoted. We consider the last trivial pipe to be promoted and assume that after this pipe is promoted, all other trivial pipes are removed from the diagram. Beginning with the move in which the trivial pipe is promoted, we will construct an alternate reduction for this diagram which will be shorter than the original reduction. The alternate reduction will be called the *shadow reduction* and the moves in the shadow reduction will be called *shadow moves*. This is illustrated in 4.1.

Before the trivial pipe is promoted, the diagram is D . The move x_1 promotes the trivial pipe, and the move x'_1 leaves the diagram unchanged. Following this move, for each move x_i we define exactly one move x'_i so that eventually the paths meet again. The moves x'_i are the shadow moves. Since there is exactly one shadow move for each move in the reduction and at least one of the shadow moves is trivial, the shadow reduction is shorter. Here we describe how to construct the shadow reduction.

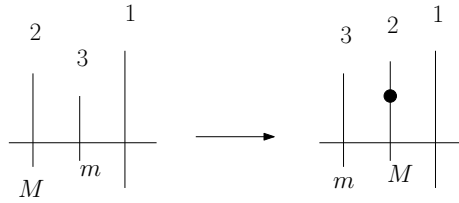


Figure 4.2 The move in which the trivial pipe is promoted

Since the pipe is trivial, it must lie between two pipes. Both of these pipes are longer on top and bottom of the diagram than the trivial pipe. One of these two adjacent pipes must be taller than the other, and by symmetry we will assume that the shorter of the two pipes lies to the left of the trivial pipes. In Figure 4.2 we show the three pipes and the move in which the trivial pipe is promoted.

Considering only the trivial pipe and the pipe immediately to the left of this pipe it is clear that at any time during the reduction process one of these pipes is taller than the other. We will denote the taller of the two by M , and the shorter of the two by m . At any stage in the reduction process we have a pipe diagram, and corresponding to that pipe diagram we have a tree diagram representing the domain tree. Since these two pipes must always lie next to each other, there are only two possible tree diagrams. If m is on the left of M , we know that the pipe M will be at a higher level than m , the pipe m must lie in the left subtree of M , and since m is to the immediate left of M , m can have no right subtree. If m is on the right of M , we know that the pipe M will be at a higher level than m , the pipe m must lie in the right subtree of M , and since m is to the immediate right of M , m can have no left subtree. Figure 4.3 shows the two diagrams here where A , B , C , and D are subtrees of the tree.

For each of these two diagrams there is a corresponding diagram which we call the *shadow diagram*. The shadow diagrams will be used in the construction of the

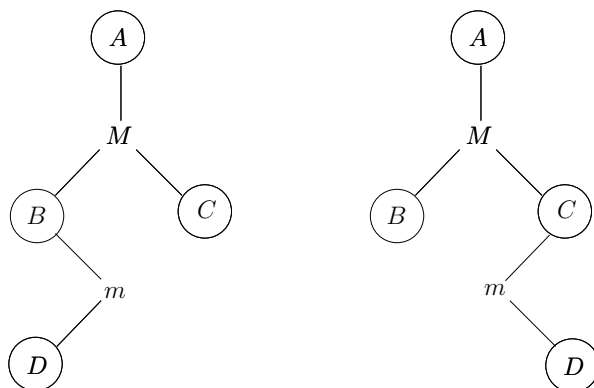


Figure 4.3 The diagram on the left is where m is on the left of M , and the diagram on the right is where m is on the right of M .

shadow reduction in which m is always on the right of M and m has no subtree. Let T be the transformation that takes a diagram to its shadow diagram. In the case where m is on the left of M the shadow diagram is obtained by raising m to the level of M , and lowering M so that it is trivial. In the case where m is on the right of M the shadow diagram is obtained by lowering m so that it is trivial. In either case the shadow diagram is a diagram that can be obtained by never promoting the trivial pipe. The shadow diagrams are shown in 4.4.

We will show that if $x_i \in S$ transforms a diagram D to a diagram Dx_i then there exists $x'_i \in S$ such that x'_i transforms $T(D)$ to $T(Dx_i)$. This is depicted in 4.5.

In other words, given a move in the reduction that takes a diagram to its image, there is a corresponding move in the shadow reduction which takes the shadow diagram to the shadow diagram of the image. From this we will see that at some point in the reduction the shadow diagram becomes identical to the actual diagram and that it remains the same for the rest of the reduction. Moreover, we will see that a trivial pipe is never promoted in the shadow reduction and that there are fewer moves in the shadow reduction than in the actual reduction.

There are ten cases. We will show each case by a diagram. In these diagrams

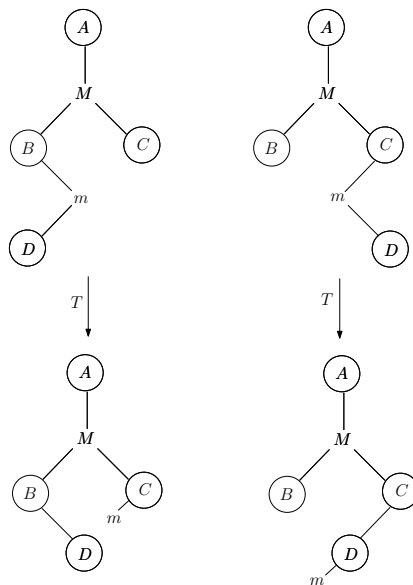


Figure 4.4 The two shadow diagrams are seen here.

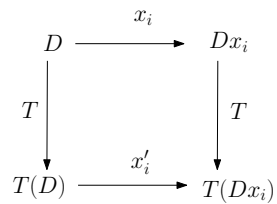


Figure 4.5 The generator x_i transforms D to Dx_i , and the generator x'_i transforms $T(D)$ to $T(Dx_i)$.

A_L, B_L, C_L , and D_L denote the left subtrees of A, B, C , and D respectively, and A_R, B_R, C_R , and D_R denote the right subtrees of A, B, C , and D respectively. Any letter without a circle will denote an individual pipe, usually the tallest or shortest pipe of a subtree. In the first five cases we assume that we will be starting with m on the left of M , and in the last five we will assume the opposite. The first five are shown in figures 4.6 through 4.10, and the last five are shown in figures 4.11 through 4.15.

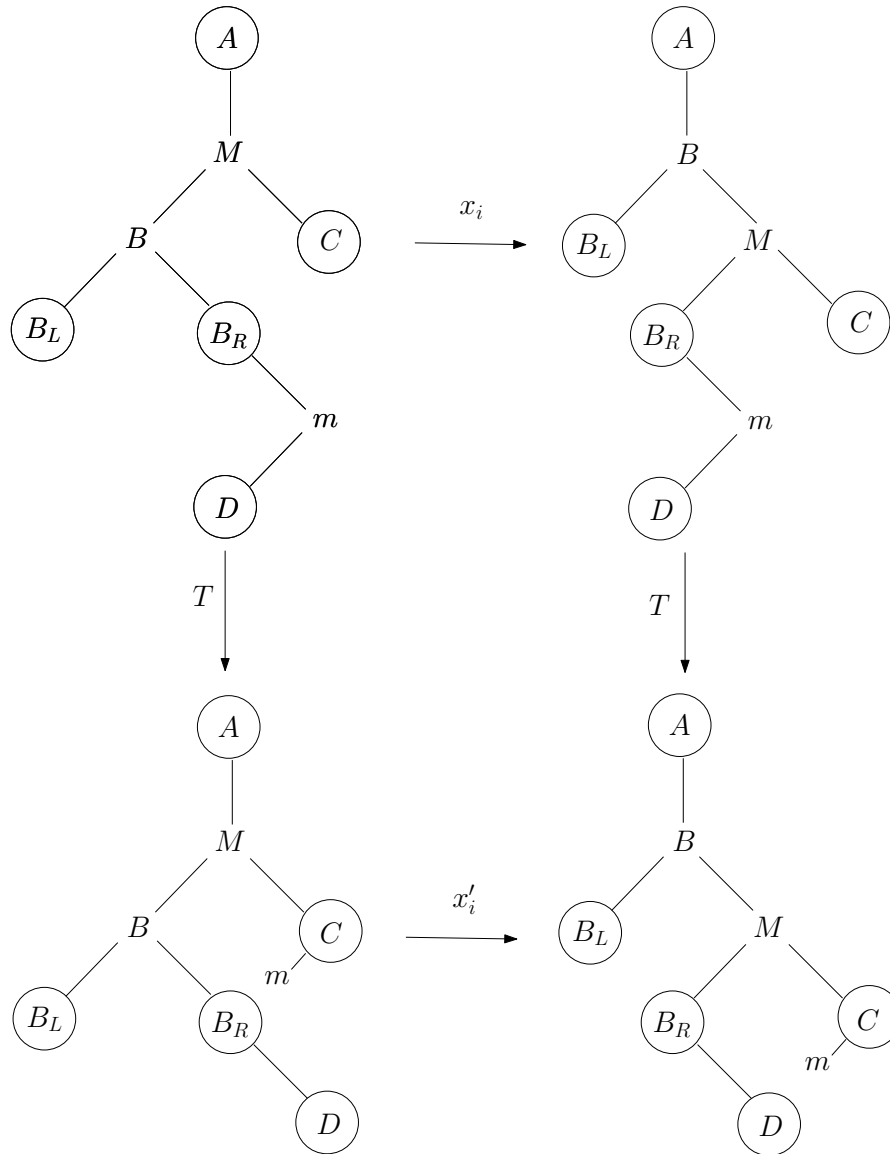


Figure 4.6 Case 1: Raising the tallest pipe in B past M means that M will now be in the right subtree of B . The pipes that form the subtree B_R are now in the left subtree of M , and the right subtree of M remains unchanged. In the shadow diagram we see the result of the move is the same. In fact, B is at the same level and on the same side so the move can be accomplished through multiplication by the same generator. The move takes the shadow diagram to the shadow diagram of its image.

If the move takes place entirely in one of the subtrees A , B , or C then the move in the shadow diagram will be similar. If the move takes place entirely in the subtree D then the move may be at a higher level and on a different side in the shadow diagram.

We have shown that whenever M is promoted past a pipe or a pipe is promoted past M , the same generator takes the shadow diagram to the shadow diagram of the image. We have also shown that whenever m is promoted past a pipe or a pipe is promoted past m , we do the trivial move in the shadow reduction. If the move takes place entirely in a subtree, then there is a corresponding move at least as high in the shadow reduction.

At some point in the reduction process the trivial pipe that was raised will become trivial again, say after multiplication by x_n . We have shown that $Dx'_1 \cdots x'_n$ is the shadow diagram of $Dx_1 \cdots x_n$. But when the pipe is trivial the diagram is identical to its shadow diagram. So after this point the reduction processes are the same. At least one of the moves x'_i is trivial, so the shadow reduction process is shorter than the original process. This completes the proof.

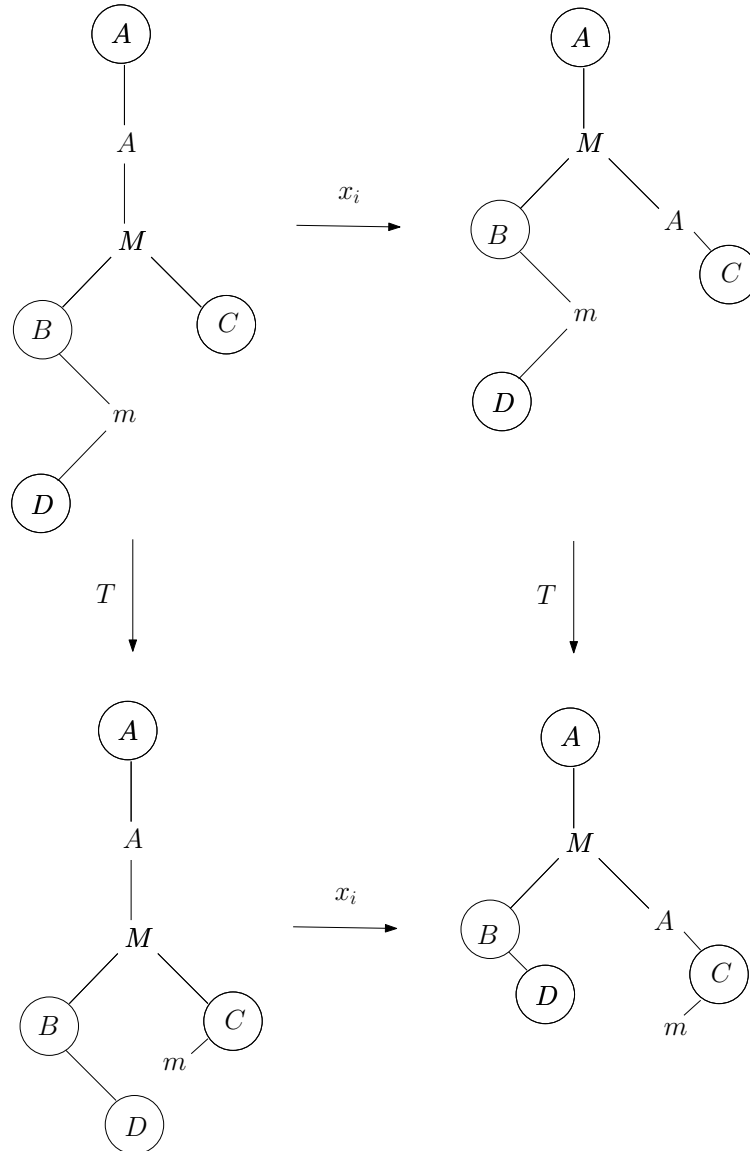


Figure 4.7 Case 2: Here we assume that A lies to the right of M . If A is on the left of M then the case will be similar. Since M is moved past A , A now lies in the right subtree of M . The subtree C will still lie below A . The rest of the diagram remains unchanged. In the shadow diagram the move occurs at the same level, so the same generator is used. The pipe A is still on the right of M , so it lies in the right subtree of M .

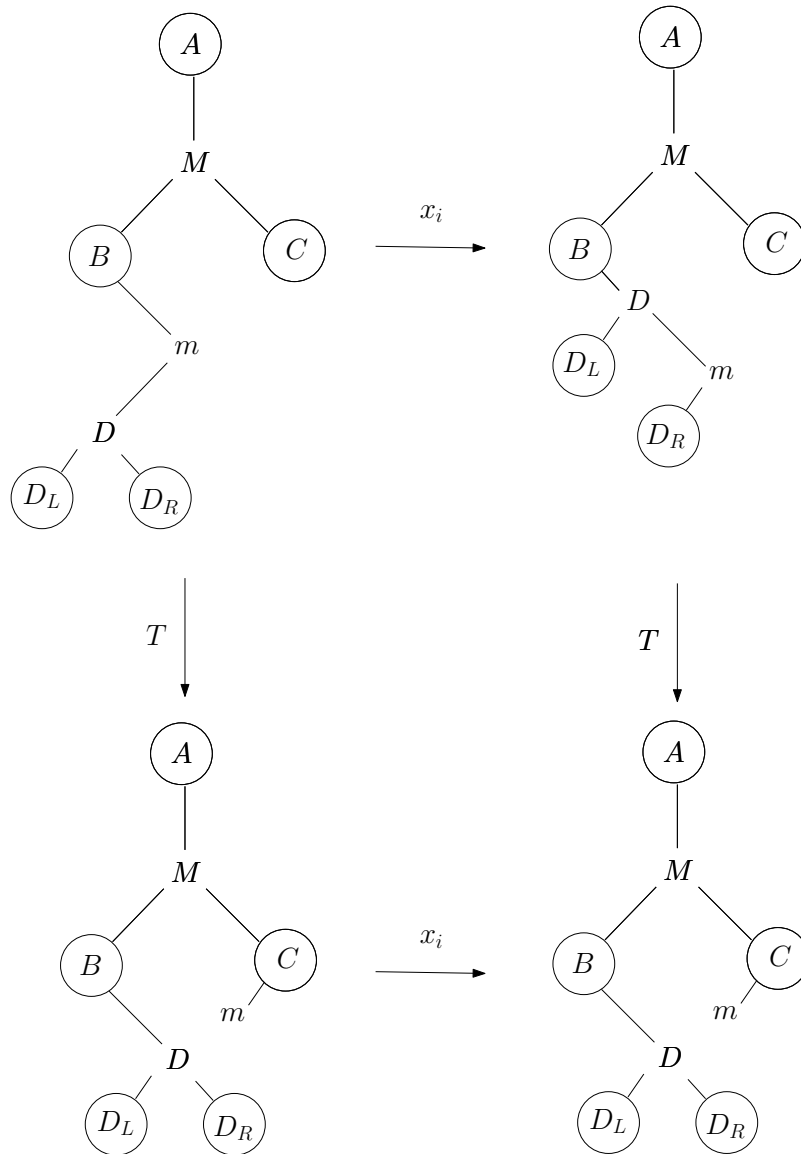


Figure 4.8 Case 3: After the promotion m is now in the right subtree of D . The subtree D_R is still lower than m and lies to the left of m , so it is the left subtree of m . The rest of the diagram remains unchanged. The shadow diagrams are identical, so the move required in the shadow reduction is the trivial move.

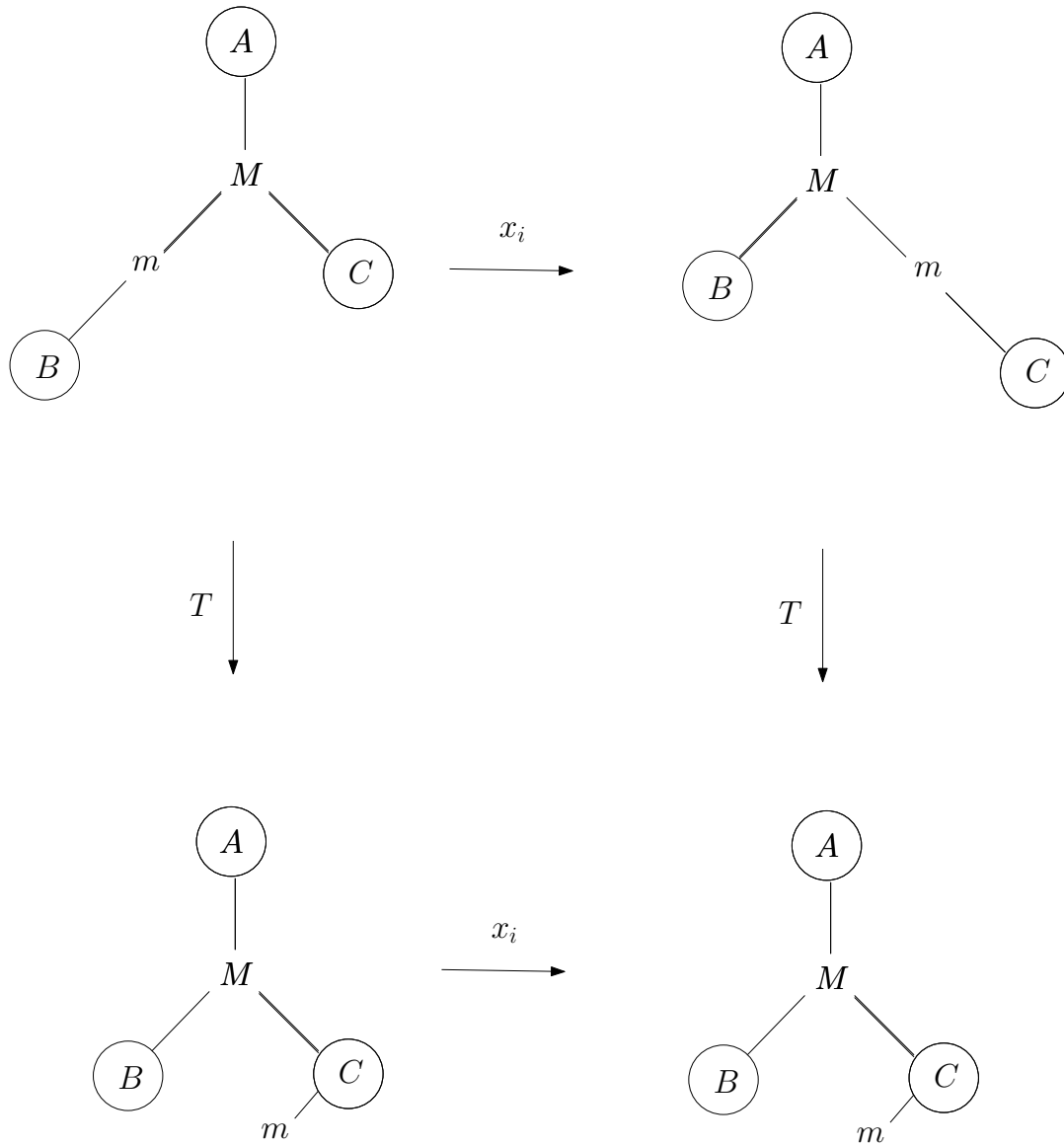


Figure 4.9 Case 4: Moving m past M , m now lies on the right of M . Since C is still below m , C is the right subtree of m . B is now the left subtree of M .

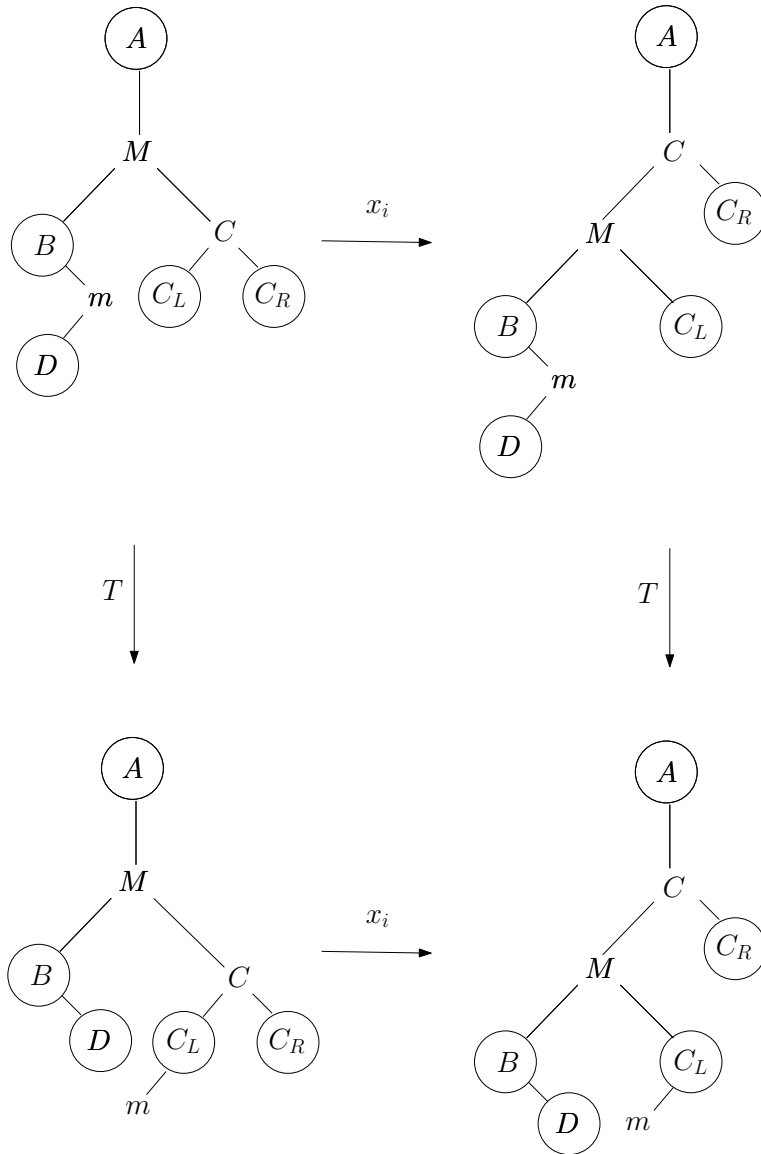


Figure 4.10 Case 5: This case is similar to the case where we promote the tallest pipe in B past M . In the shadow reduction we get the same result through multiplication by the same generator.

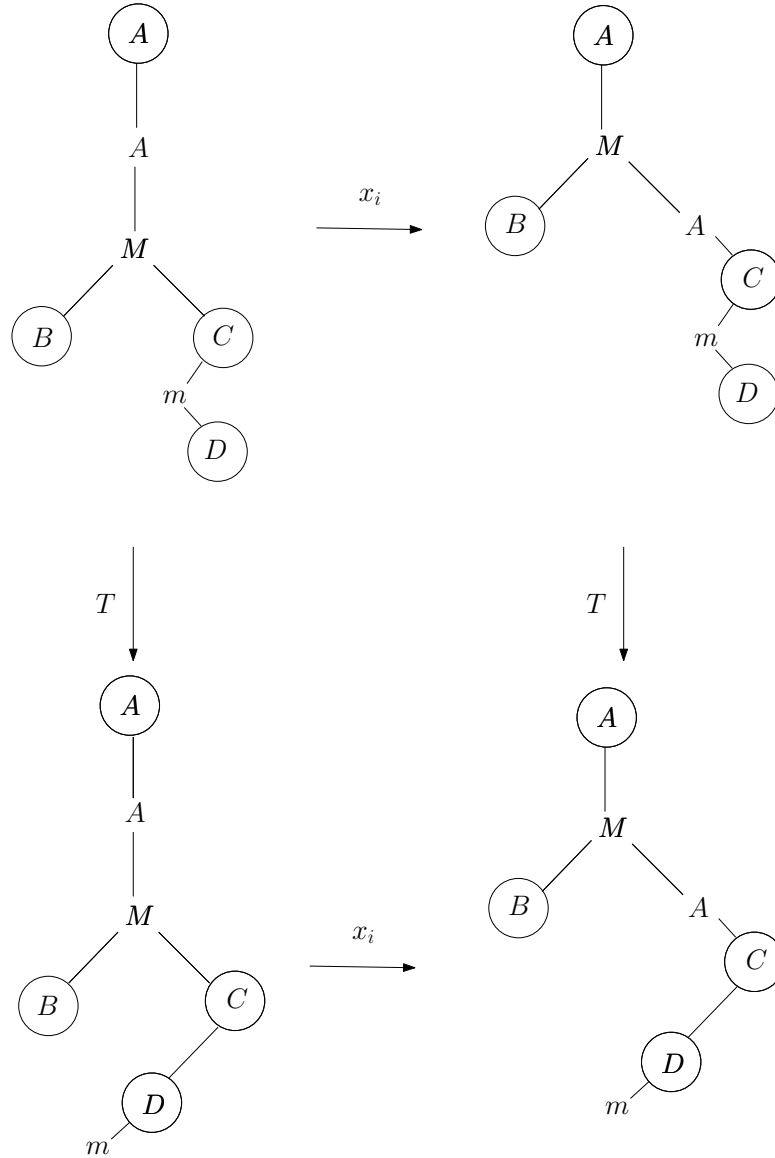


Figure 4.11 Case 6: This case is similar to Case 2. Multiplying the shadow diagram by the same generator gives the desired result.

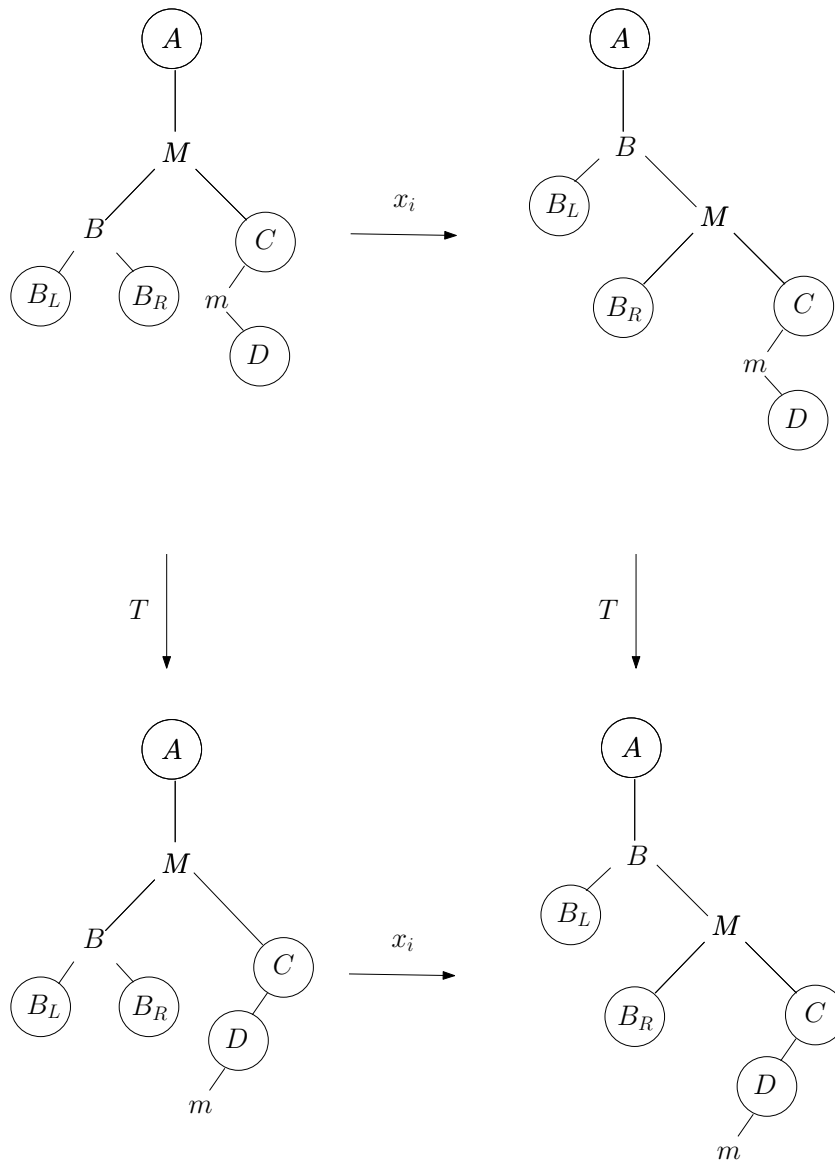


Figure 4.12 Case 7: This case is similar to case 5. In the shadow diagram we use the same generator to promote the tallest pipe in B past M .

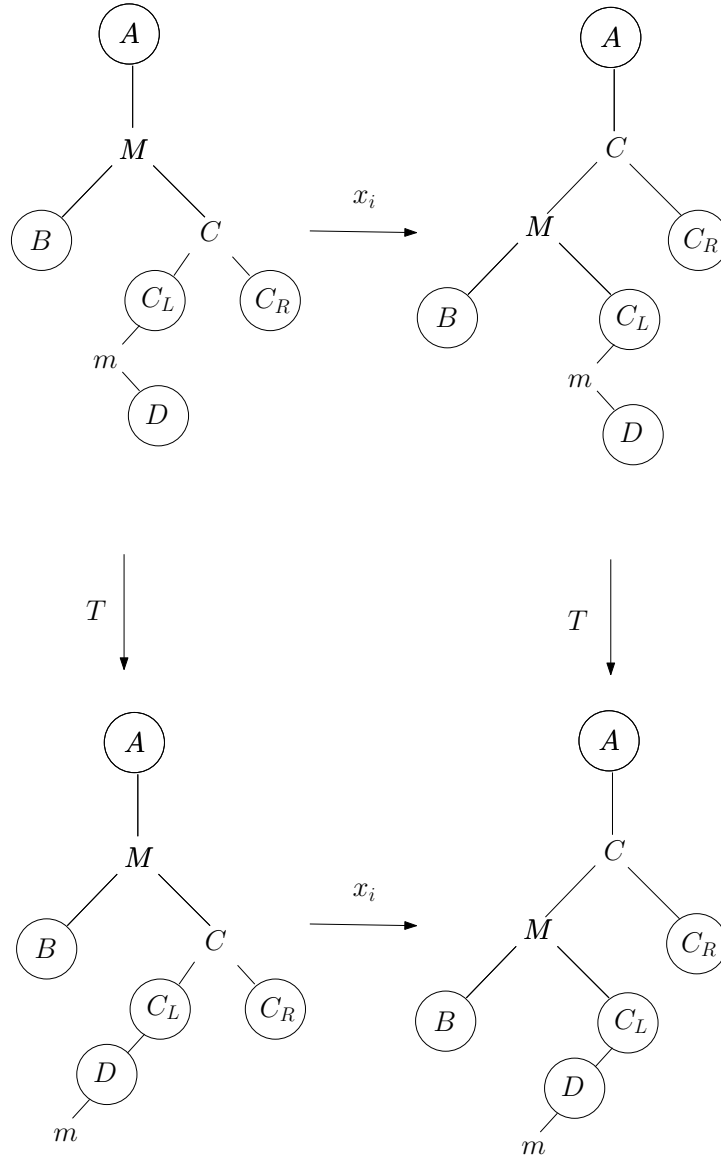


Figure 4.13 Case 8: Multiplying the shadow diagram by the same generator gives the desired result.

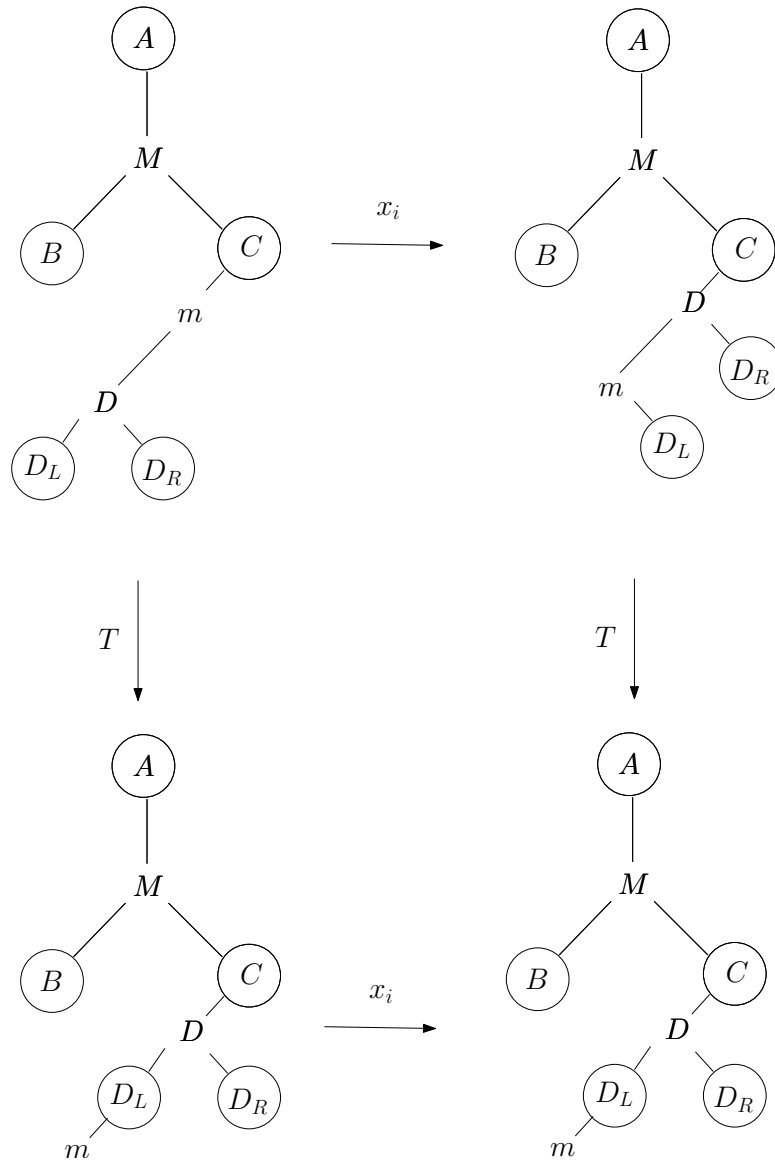


Figure 4.14 Case 9: This case is similar to case 3. The necessary move in the shadow reduction is the trivial move.

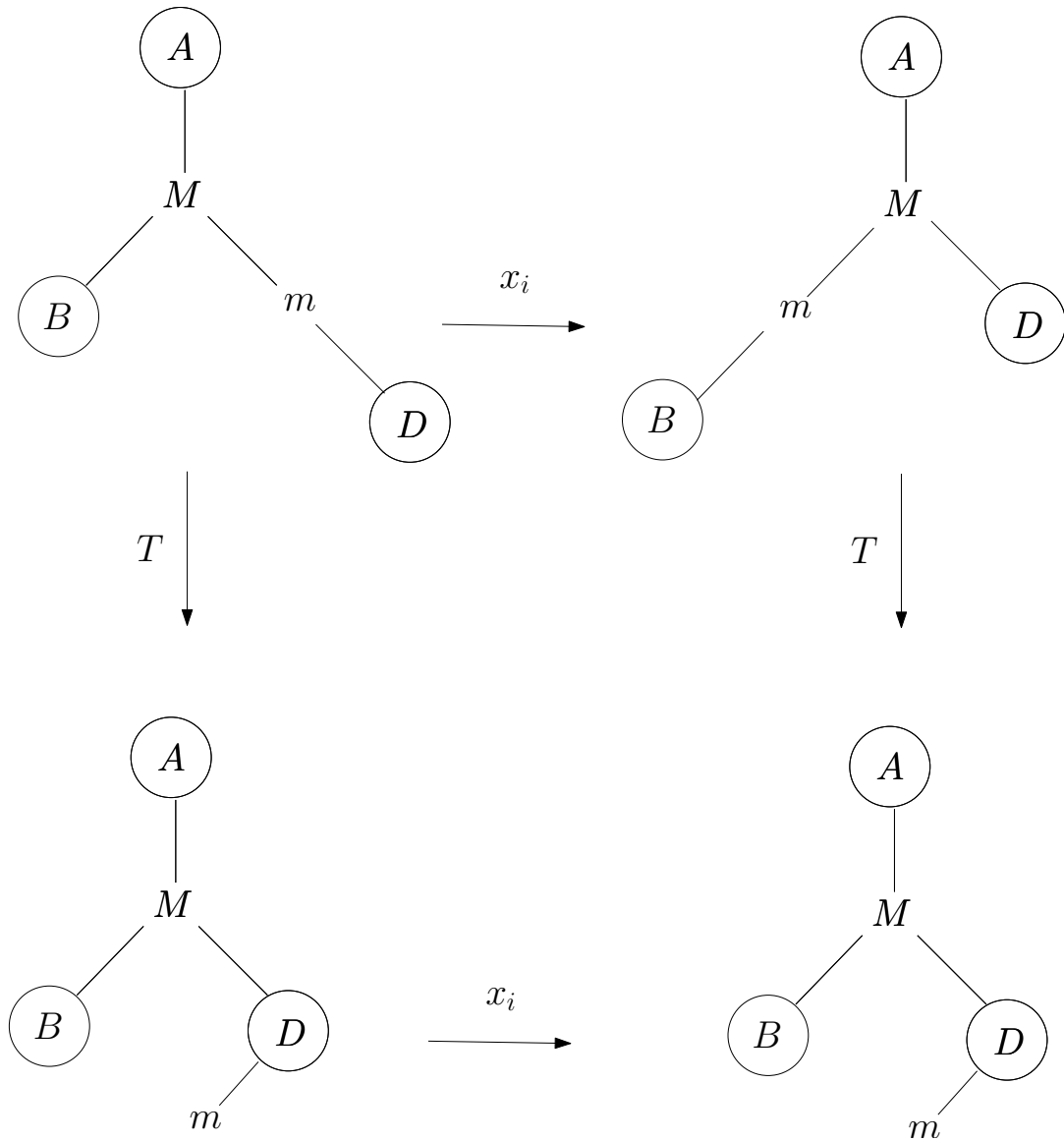


Figure 4.15 Case 10: This case is similar to case 4. The necessary move in the shadow reduction is the trivial move.

Chapter 5

The First Reduction Theorem

Given a pipe diagram it would be ideal to know how to reduce the diagram to the trivial diagram in the minimal number of steps. In chapter 5 we develop an algorithm to eliminate one pipe in a pipe diagram, and we show that this algorithm eliminates the pipe in the minimal number of steps.

Fordham [6] gives a linear-time algorithm which takes as input an element of F in the form of its reduced tree diagram and gives as output the minimal length of the element in the generators $\{a, b\}$. Other algorithms for this generating set were given by Guba [8] and Belk-Bux [1]. In [9] an alternative algorithm is given for certain generating sets including the generating set $\{X_0, X_1, \dots, X_n\}$ which is a finite version of the standard infinite generating set. Woodruff [11] unsuccessfully investigated the problem of generalizing Fordham's algorithm to the generating set $\{a, b, c\}$. In the setting of pipe diagrams this generating set is especially advantageous since it allows pipes to be raised on both sides of the diagram.

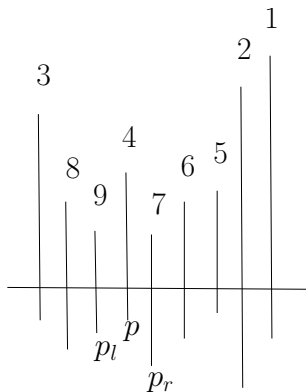
We now consider the problem of reducing a pipe diagram to the trivial diagram. Given a pipe diagram, choose a lower-trivial pipe. We will give an algorithm to eliminate this pipe, and since the diagram is trivial when all the pipes are trivial, we

repeat this process for each lower-trivial pipe until all are eliminated.

Define the set

$$\mathcal{D} = \{(D, p) \mid D \text{ is a pipe diagram and } p \text{ is a pipe that is lower-trivial in } D\}.$$

We will categorize each pair (D, p) into nine cases. For each case we will give an algorithm to eliminate p , and we will give the number of steps required by the algorithm to do so. We claim that this number is the minimal number of moves that will eliminate p . In the pair (D, p) let p_l and p_r be the pipes to the left and right of p respectively. Let n be the level of p and let j_l and j_r be the levels of p_l and p_r respectively. The assigned number will be a function of these parameters and another parameter which we will describe later. In 5.1 we give an example.



5.1 Nine Cases

Definition 5.1.1. A pair $(D, p) \in \mathcal{D}$ is *right (left) cascading* if the pipe p lies on the right (left) side of the diagram and there is no pipe of higher level to the right (left) of p .

The first six cases are cases in which the pair is cascading, where the difference in each case arises in the different subtrees p may have. We will assume that the pairs

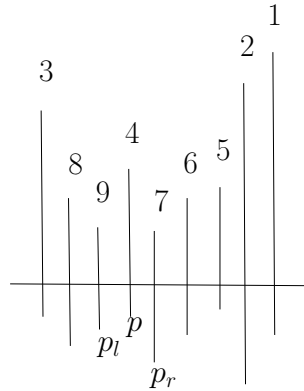


Figure 5.1 Here $n = 4$, $j_l = 9$, and $j_r = 7$

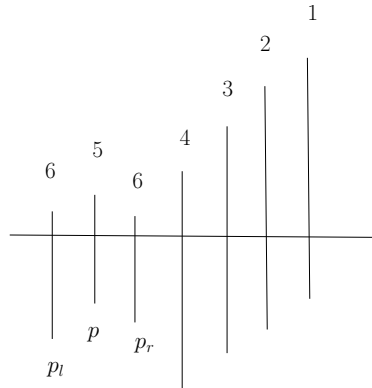


Figure 5.2 A Left Cascading Diagram

are right cascading, and the case where they are left cascading is symmetric. A left cascading diagram is shown in 5.2.

Case I: Right Cascading where $j_r = n + 1$ and $j_l < n$ or $j_l = n + 1$ and there is no pipe p_r .

Case II: Right Cascading and $j_r = j_l = n + 1$.

In the first two cases we need to promote p_l and p_r so that they are taller than p . The idea is to promote p to level 2 and then promote p_l and p_r . An example is shown in 5.3. We use the following procedure.

1. If p is at level 2 proceed to (3). Otherwise go to (2).

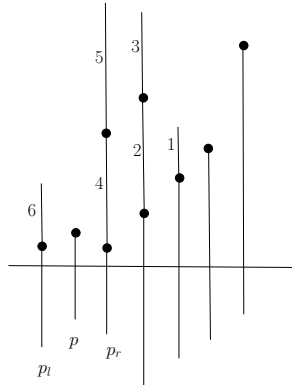


Figure 5.3 An example of a cascading reduction, where the numbers indicate the order in which the moves are performed.

2. If p is at level 3, promote the level two pipe to level 1 and go to (1). Otherwise promote the pipe which is at level 3 to level 2 and repeat (2).
3. Promote p_r to level 2 and go to (4).
4. If p is trivial stop the process. Otherwise promote p_r to level 1 and go to (5).
5. Promote p_l to level 2.

Case III: Right Cascading with no pipe p_r and $j_l = n + 2$. This case is similar to the first two cases.

1. If p_l is at level 3 promote p_l . Otherwise go to (2).
2. If p is at level 3, promote the level 2 pipe to level 1. Otherwise promote the level 3 pipe to level 2. If p_l is at level 3 go to (1), otherwise repeat (2).

Case IV: Right Cascading with no pipe p_r and $j_l > n + 2$.

1. If p is at level 1 go to (4). Otherwise go to (2).

2. If p is at level 2, promote p to level 1 and go to (4). Otherwise go to (3).
3. If p is at level 3, promote the level 2 pipe to level 1 and go to (2). Otherwise promote the level 3 pipe to level 2 and repeat (3).
4. If p_l is at level 2, promote p_l . Otherwise promote the level 3 pipe to level 2 and repeat (4).

Case V: Right Cascading with $j_l < n$ and $j_r > n + 1$.

1. If p is at level 1 go to (4). Otherwise go to (2).
2. If p is at level 2, promote p to level 1 and go to (4). Otherwise go to (3).
3. If p is at level 3, promote the level 2 pipe to level 1 and go to (2). Otherwise promote the level 3 pipe to level 2 and repeat (3).
4. If p_r is at level 2, promote p_r and go to (5). Otherwise promote the level 3 pipe to level 2 and repeat (4).
5. Promote p_l .

Case VI: Right Cascading with $j_r > n + 1$ and $j_l \geq n + 1$ or $j_l > n + 1$ and $j_r \geq n + 1$.

1. If p is at level 2, promote p to level 1 and go to 4. Otherwise go to (2).
2. If p is at level 3, promote the level 2 pipe to level 1. Otherwise go to (3).
3. Promote the level 3 pipe to level 2 and go to (1).
4. If p_r is at level 2 go to (6), otherwise promote the left right level 3 pipe to level 2 and repeat (4).

5. If p_l is at level 2 go to (6), otherwise promote the left left level 3 pipe to level 2 and repeat (5).
6. Promote j_r to level 1 and then promote j_l to level 2.

Case VII: The pipe p is at level $n = 1$.

1. If p_l is at level 2, go to (3). Otherwise promote the left right level 3 pipe to level 2 and repeat (1).
2. If p_r is at level 2, go to (3). Otherwise promote the right left level 3 pipe to level 2 and repeat (2).
3. Promote p_l to level 1 and go to (4).
4. Promote p_r to level 2.

Case VIII: The pair (D, p) is non-cascading with $n \geq 3$ and $\min j_l, j_r > n$. We will eliminate p by first promoting it to level 1, and then promoting the pipes on both sides of p until p_l and p_r are at level 2. Then we will promote p_l past p and then p_r past p . Before describing the procedure we must define the five possible situations that could arise. We call these situations *types*. The types are determined by the locations of p , the level one pipe, the left level 2 pipe, the level 3 pipe which lies on the same side of the left level 2 pipe as p , and the level 4 pipe which lies on the same side of the level 3 pipe as p . The types are shown in 5.4.

1. If p is at level 2, promote p to level 1 and go to (4). Otherwise go to (2).
2. If p is at level 3, promote p to level 2 and go to (1). Otherwise go to (3).
3. If we have Type I, promote the level 3 pipe to level 2. If we have Type II, promote the level 2 pipe to level 1. If we have Type III, promote the level 3

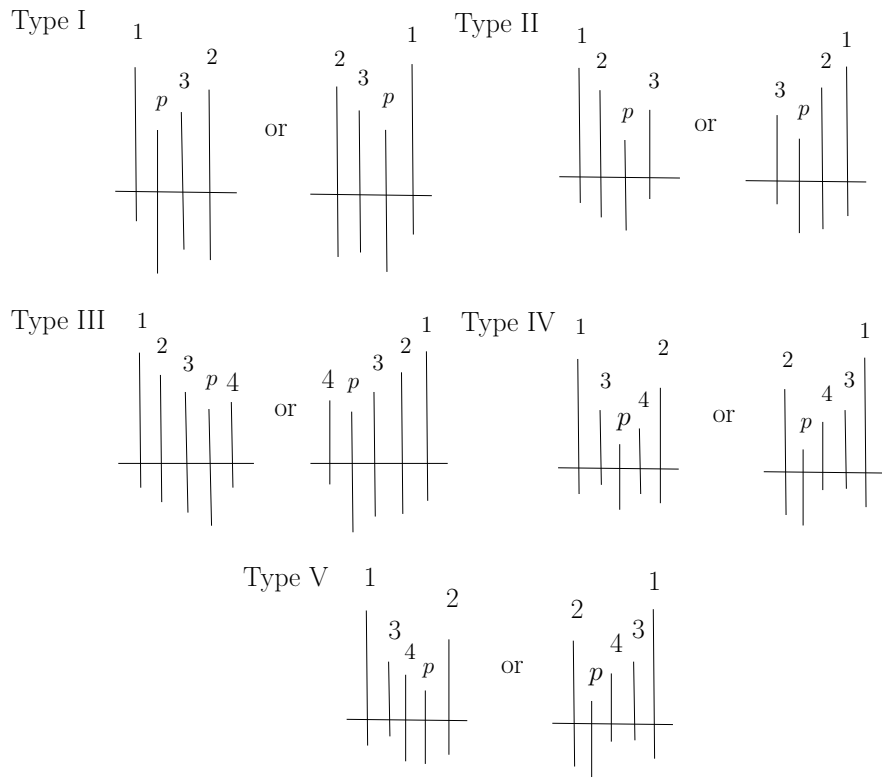


Figure 5.4 The five types of diagrams.

pipe to level 2. If we have Type IV, promote the level 3 pipe to level 2. If we have Type V, promote the level 2 pipe to level 1. Return to (1).

4. If p_l is at level 2, go to (5). Otherwise promote the left right level 3 pipe to level 2 and repeat (4).
5. If p_r is at level 2, go to (6). Otherwise promote the right left level 3 pipe to level 2 and repeat (5).
6. Promote p_l to level 1 and go to (7).
7. Promote p_r to level 2.

Case IX: The pipe p is at level $n \geq 3$ and $\min j_l, j_r < n$. In this case we proceed as in case VIII until p is at level 3. Then we promote p . This gives us a cascading diagram and we proceed as directed on those cases.

5.2 Number of moves

Given a pair (D, p) , we wish to calculate the number of moves that are required by the algorithm to eliminate p . We consider each case.

Case I: The pair (D, p) is left cascading with no pipe p_l . In this case the pipe p_r starts at level j_r , and is always promoted one level at each step until it ends at level 2. The number of moves required to achieve this is $j_r - 2$.

Case II: The pair (D, p) is left cascading and there is a pipe p_l . Here, two additional moves are required to promote p_l past p . In this case, the total number of moves required is $(j_r - 2) + 2 = j_r$.

Case III: Through similar analysis we determine that the number of moves required to eliminate p is $j_l - 1$.

Case IV: Through similar analysis we determine that the number of moves required to eliminate p is j_l .

Case V: Through similar analysis we determine that the number of moves required to eliminate p is j_r .

Case VI: Through similar analysis we determine that the number of moves required to eliminate p is $j_r + j_l - n$.

Case VII: Here it takes $j_l + j_r - 2$ moves to eliminate p if there are pipes p_r and p_l . It takes $j_l - 1$ or $j_r - 1$ if there is only a pipe p_l or p_r respectively.

Case VIII: The pair (D, p) is non-cascading with $n \geq 3$ and $\min j_l, j_r > n$. In this case the pipe p is first promoted to level 1. Examining Step 3 of the algorithm we note that the Type IV and Type V moves do not raise the level of p , but all other moves raise the level of p by one. So if k is the total number of Type IV and Type V moves, the total number of moves required to promote p to level 1 is $n - 1 + k$. The difficulty is in calculating the number of Type IV and Type V moves which we will discuss shortly. Each time the level of p is raised by one the levels of p_l and p_r are each raised by one. So when p is at level 3, the pipes p_l and p_r are at levels $j_l - (n - 3)$ and $j_r - (n - 3)$. Promoting p the final two levels to level 1 either the level of one of the pipes p_l or p_r is raised by 2 and the level of the other is not raised, or the level of each pipe is raised by 1. In step 4, p_l is promoted to level 2 by a sequence of moves which each raise the level of p_l by one. So the number of moves required to perform this step is 2 less than the level of p_l when p is at level 1. The same is true for p_r in step 5. So the total number of moves required to perform steps 4 and 5 is

$$(j_l - (n - 3) - 2) + (j_r - (n - 3) - 2) - 2 = j_l + j_r - 2n.$$

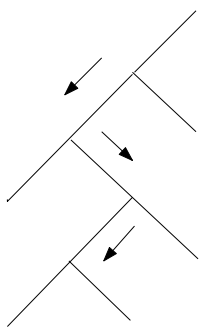
Since two moves are required to complete Steps 6 and 7, we have a total of

$$(j_l + j_r - 2n) + (n - 1 + k) + 2 = j_l + j_r - n + k + 1$$

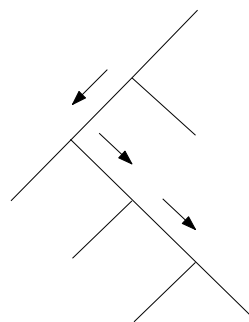
moves to eliminate p where k is the total number of Type IV and Type V moves.

Case IX: There are two subcases. If p is on the right side of the level 1 pipe, $j_r < n$, $j_l = n + 1$, and the number of Type V moves is even then the number of required moves is $j_l + k$. If p is on the right side of the level 1 pipe, $j_l < n$, $j_r = n + 1$, and the number of Type V moves is odd then the number of moves is $j_r + k$. In all other cases, the number is $\max(j_l, j_r) + k + 1$.

We will now describe how to calculate the number of Type IV and Type V moves. We will assume that p initially lies on the right side of the diagram. It is helpful to study the tree diagram of the domain tree. Traverse the diagram by moving along the geodesic between the root and the vertex representing p . Since the diagram is not cascading at some point the geodesic will turn to the left. As we descend to the left, we can change direction in two ways. In one way we turn to the right for one edge length, and then we turn back to the left. This corresponds to a Type IV move since in a Type IV move the 3 is promoted to a 2 and the p still lies on the same side of the diagram. In the other way we turn to the right and continue in this direction for at least two edge lengths. This corresponds to a Type V move since in a Type V move the 2 is promoted to a 1 and now p lies on the opposite side of the diagram.



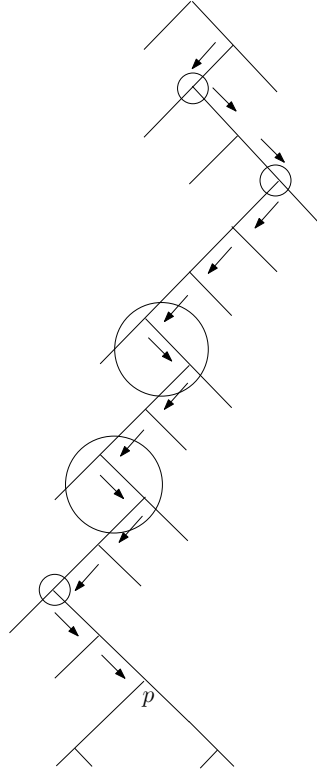
Type IV



Type V

The mirror images of these cases also yields Type IV and Type V moves. Here we draw a geodesic from the root to a vertex p and count the number of Type IV and

Type V moves. The Type IV moves are circled with the bigger circles, and the Type V moves are circled by the smaller circles.



5.3 Minimal elimination

We now have a function $\phi : \mathcal{D} \rightarrow \{0, 1, 2, \dots\}$ with $\phi((D, p))$ taking the value determined by the case in which it lies. The function ϕ gives the number of moves needed to eliminate a pipe p in a diagram D using the algorithm. Now we will show that the algorithm is the most efficient algorithm. We first make this notion precise.

Let S be the generating set of F_3 . We define the elimination function on \mathcal{D} with respect to S

$$\mathcal{E} : \mathcal{D} \rightarrow \{0, 1, 2, \dots\}$$

to be the function which takes a pair (D, p) to the fewest number of generators

required to make p a trivial pipe. In particular, $\mathcal{E}(D, p) = n$ if

1. there exists generators $x_1, x_2, \dots, x_n \in S$ such that p is trivial in $Dx_1x_2 \cdots x_n$ where the generators act on D by raising pipes one level at a time, and
2. for any generators $x_1, x_2, \dots, x_m \in S$ with $m < n$, p is not trivial in $Dx_1x_2 \cdots x_m$.

Fordham [6] calculated minimal length elements for Thompson's Group F . He constructed a function from the elements of F to the nonnegative integers and then used a lemma to show that his function was the length function. The following lemma will allow us to check if a function from \mathcal{D} to the nonnegative integers is the same as the elimination function. The lemma is analogous to the one used by Fordham and the proof will be similar.

Lemma 5.3.1. *Given a function $\phi : \mathcal{D} \rightarrow \{0, 1, 2, \dots\}$, if ϕ has the properties*

1. $\phi((D, p)) = 0$ if and only if p is a trivial pipe in D ,
2. if (D, p) is an element of \mathcal{D} and x is a generator of F then $\phi((D, p)) - 1 \leq \phi((Dx, p))$, and
3. for any $(D, p) \in \mathcal{D}$ with p nontrivial, there is at least one generator x such that $\phi((Dx, p)) = \phi((D, p)) - 1$,

then $\phi((D, p)) = \mathcal{E}((D, p))$ for all $(D, p) \in \mathcal{D}$.

Proof. Assume that the product of generators $x_1x_2 \cdots x_n$ is a minimal eliminator p . Then $\mathcal{E}((Dx_1 \cdots x_i, p)) = n - i$ for $1 \leq i \leq n$, and $\phi((Dx_1 \cdots x_i, p)) \geq \phi((D, p)) - i$ by property (ii). When $i = n$, we have $0 = \phi((Dx_1 \cdots x_n, p)) \geq \phi((D, p)) - n$. So

$$\phi((D, p)) \leq n = \mathcal{E}((D, p)).$$

Now assume that $\phi((D, p)) = n > 0$. By property (iii), there exists generators x_1, x_2, \dots, x_n such that $\phi((Dx_1x_2 \cdots x_n, p)) = 0$. By property (i), p is trivial in $Dx_1 \cdots x_n$. So $x_1 \cdots x_n$ is an eliminator, but not necessarily a minimal eliminator, of p . Therefore $\mathcal{E}((D, p)) \leq n = \phi((D, p))$. In conclusion $\mathcal{E}((D, p)) = \phi((D, p))$.

5.4 The First Reduction Theorem

Theorem 5.4.1. *The First Reduction Theorem.* *Given a pipe diagram D and a lower trivial pipe p , the described algorithm will eliminate p in the fewest possible number of steps.*

Proof. To prove this result we simply need to show that the function ϕ satisfies the properties listed in Lemma 5.3.1 where our generating set is S , the generating set of F_3 . In proving this we will see that the moves prescribed in the algorithm are moves which satisfy property (iii) in Lemma 5.3.1. In particular, they reduce the elimination number by one each time.

By symmetry we will only need to consider the diagrams where p is initially on the right side. We do not need to consider the generators which raise level 2 pipes to level 3 pipes on the left side of the diagram since they have no affect on the parameters that define the proposed number of moves required. In addition we do not consider the generator which promotes the left level 2 pipe to level 1 since it will always raise the required number of moves. The only generators we need to consider are a^{-1} , b , and b^{-1} . Recall that a^{-1} promotes the right level 2 pipe to level 1, b promotes the right left level 3 pipe to level 2, and b^{-1} promotes the right right level 3 pipe to level 2.

In checking these cases, the procedure is simple. First, draw the diagram, determine the case of the diagram, and calculate the number of required moves. Second,

multiply by a generator, determine the case of the new diagram, and calculate the new number of required moves. Verify that the necessary conditions are satisfied for each case. Some diagrams will be included, but in most cases the results are simply stated and can be verified by following the described procedure.

We first check the first six cases for $n > 3$. These cases are all cascading, so multiplication by one of the three generators results in another cascading diagram. In the first five cases the number of moves required is measured in terms of j_l or j_r . No move can change the level of any pipe by more than one, so there is no move that reduces the length by more than one. In the sixth case the length is measured by $j_l + j_r - n$. Any move either raises each of these by one or lowers each of these by one. So the total number is either raised by one or lowered by one. In all six cases, multiplication by b^{-1} lowers the number of required moves by one. Now we need to check the six cases individually for $n \leq 3$.

Case I: The number of moves required is $j_l - 2$ or $j_r - 2$ which are both equal to 2. For $n = 3$ multiplication by a^{-1} lowers $j_r - 2$ or $j_l - 2$ to 1. Multiplication by b raises $j_r - 2$ or $j_l - 2$ to 3. If there is a pipe p_r multiplication by b^{-1} lowers the level of $j_r - 2$ to 1. If there is no pipe p_r , then multiplication by b^{-1} gives us a case III diagram in which the number of moves is now $j_l - 1$ which is increased.

If $n = 2$ the number of moves required is clearly 1 which matches the given number.

Case II: For $n = 3$ the number of moves required is 4. Multiplication by a^{-1} lowers the number of moves to 3. Multiplication by b gives us a case VI diagram in which the number of moves is 5.

For $n = 2$ the number of moves required is 3. Multiplication by a^{-1} gives us a case VII diagram in which the number of moves required is 3. Multiplication by b gives us a case I diagram in which the number of moves required is 2. Multiplication

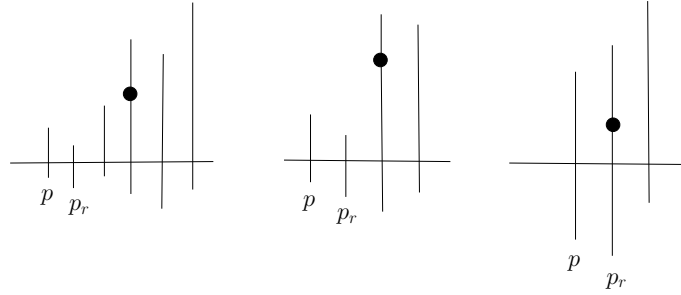


Figure 5.5 In the diagram on the left $n = 5$. The level 3 pipe is promoted and the level of p_r changes from 6 to 5. In the middle diagram $n = 3$. The level 2 pipe is promoted, and the level of p_r changes from 4 to 3. In the case on the right $n = 2$. The pipe p_r is promoted from level 3 to level 2.

by b^{-1} gives a case IX diagram in which the number of moves required is 4.

Case III: For $n = 3$ the number of moves is 4. Multiplication by a^{-1} lowers this number to 3. Multiplication by b raises this number to 5. Multiplication by b^{-1} gives a case IV diagram in which the number of moves is 5.

For $n = 2$ the number of moves required is 3. Multiplication by a^{-1} gives a case VII diagram in which the number of moves is 3. Multiplication by b gives a case I diagram in which the number of moves is 2. Multiplication by b^{-1} gives us a case IX diagram in which the number of moves is 6.

Case IV: For $n = 3$ the number of moves is j_l . After multiplication by a^{-1} we still have a case IV diagram and the number of moves is $j_l - 1$. If we multiply by b^{-1} we still have a case IV diagram and the number of moves is j_l . If we multiply by b we have a case IV diagram and the number of moves is $j_l + 1$.

For $n = 2$ the number of moves is j_l . After multiplication by a^{-1} we have a case VII diagram, and the number of moves is $j_l - 1$. If we multiply by b there are two possibilities. In one case we still have a case IV diagram and the number of moves is j_l . This occurs if $j_l \geq 6$. If $j_l = 5$ then we have a case III diagram and the number of moves is 4. If we multiply by b^{-1} we have a case IX diagram in which the number

of moves is $j_l + 2$.

Case V: For $n = 3$ the number of moves is j_r . After multiplication by a^{-1} we still have a case V diagram and the number of moves is $j_r - 1$. If we multiply by b^{-1} , we have a case V diagram and the number of moves is $j_r - 1$. Multiplying by b^{-1} gives a case V diagram and the number of moves is $j_r + 1$.

For $n = 2$ the number of moves is j_r . If we multiply by a^{-1} we have a case VII diagram and the number of moves is $2 + (j_r - 1) - 2 = j_r$. After multiplying by b we have a case V diagram and the number of moves is $j_r + 1$. After multiplying by b^{-1} we have a case IX diagram. Here the number of moves is $j_r + 2$.

Case VI: For $n = 3$ the number of moves is $j_l + j_r - 3$. If we multiply by a^{-1} the number becomes $(j_l - 1) + (j_r - 1) - 2 = j_l + j_r - 4$. After multiplying by b we have a case VI diagram and the number becomes $(j_l + 1) + (j_r + 1) - 4 = j_l + j_r - 2$. If we multiply by b^{-1} we still have a case VI diagram, and the number becomes $j_l + (j_r - 1) - 2 = j_l + j_r - 3$.

For $n = 2$ the number of required moves is $j_l + j_r - 2$. Multiplying by a^{-1} gives a case VII diagram in which the number of moves is $j_l + (j_r - 1) - 2 = j_l + j_r - 3$. Multiplying by b gives four possibilities. The first possibility arises if $j_r = 3$ and $j_l = 4$. The initial number of moves in this case is 5. After multiplying we get a case II diagram and the number of moves is 4. The second possibility arises if $j_l \geq 4$ and $j_r \geq 4$. Here the initial number of moves is $j_r + j_l - 2$. After multiplication we have a case VI diagram and the number of moves is $j_l + (j_r + 1) - 3 = j_l + j_r - 2$. The next possibility is $j_l = 3$. Here the initial number of moves is $j_r + 1$. After multiplication we have a case V diagram and the number of moves is $j_r + 1$. The fourth possibility is if $j_r = 3$ and $j_l > 4$. Here the initial number of moves is $j_l + 1$. After multiplication we have a case VI diagram and the number of moves is $j_l + 4 - 3 = j_l + 1$. If we multiply by b^{-1} then the total number of moves there are two cases. The first is if

$j_r \geq 4$. The initial number of moves is $j_l + 2$. Here multiplication gives us a case VIII diagram and the number of moves is $(j_l + 1) + 4 - 3 + 1 = j_l + 3$. If $j_r = 3$ then the initial number is $j_l + 1$. After multiplication we have a case IX diagram and the number of moves is $j_l + 2$.

Case VII: If there is no pipe p_r then the number of moves is $j_l - 1$. If $j_l = 2$ then we promote p_l . If $j_l > 2$ then we promote the level 3 pipe to a level 2 pipe and this clearly reduces the number of moves by one. There is no move that changes j_l by more than one, so this case is complete. If there is a pipe p_r the argument is the same.

Case VIII: We will first check this case for $n > 3$. There are five possibilities given by the five types of diagrams that arise in the description of the algorithm. The algorithm was discovered using pipe diagrams, but we will use the tree diagram in the proof since it will enable us to determine the number k before and after each move. In the tree diagram the subtree of p contains both p_l and p_r . For each diagram we will see what happens under the action of each generator.

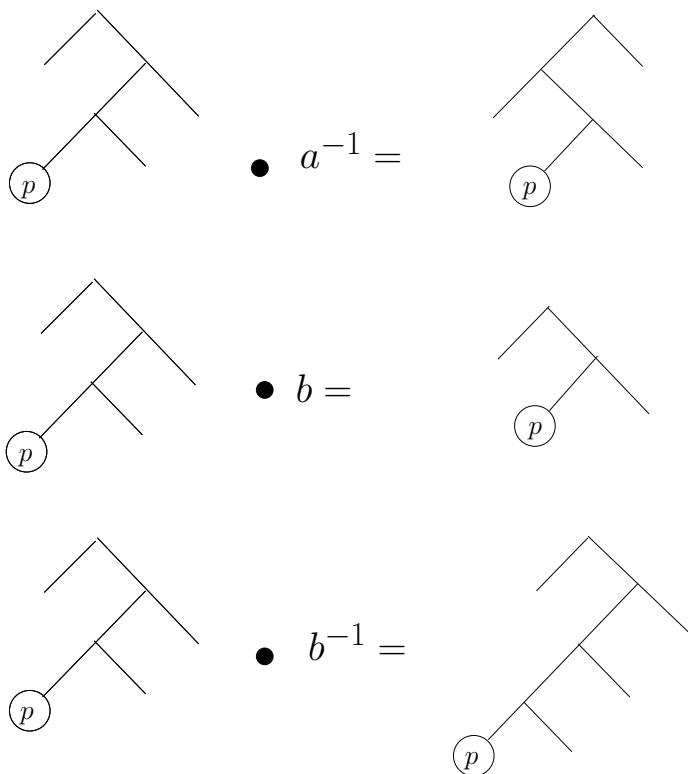


Figure 5.6 Type I: When multiplied by a^{-1} , the numbers n , j_l , and j_r remain unchanged, but a Type IV or Type V move is added. So k is increased by one, so this move does not decrease the length. When multiplied by b , the numbers n , j_l , and j_r are all lowered by one, and k remains unchanged. So the length is decreased by one. When multiplied by b^{-1} the numbers k , n , j_l , and j_r are all increased by one. This move increases the length.

Now we consider the case where $n = 3$, The pipe p must lie between the level 1 pipe and the level 2 pipe, otherwise, the diagram would be cascading. So $\phi((D, p)) = j_l + j_r - 2$. If we multiply by a^{-1} , then the level 2 pipe is promoted and $\phi((D \cdot a^{-1}, p)) = j_l + j_r - 2$ which is the same. If we multiply by b then p is promoted, p_l is raised one level, and we have a case VI diagram. So

$$\phi((D \cdot b, p)) = (j_l - 1) + j_r - 2 = j_l + j_r - 3$$

which is reduced by one. If we multiply by b^{-1} then the right right level 3 pipe is

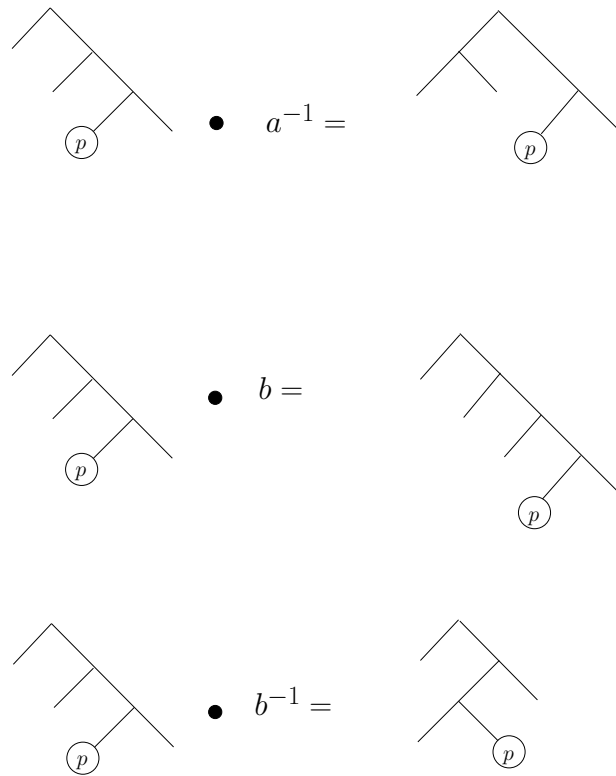


Figure 5.7 Type II: When multiplied by a^{-1} the numbers n , j_l , and j_r are all decreased by one and k remains unchanged. This move decreased the length by one. When multiplied by b , the numbers n , j_l , and j_r are all increased by one and k remains unchanged. This move increases the length. When multiplied by b^{-1} the numbers n , j_l , and j_r remain unchanged, but k is increased by one. This move increases the length.

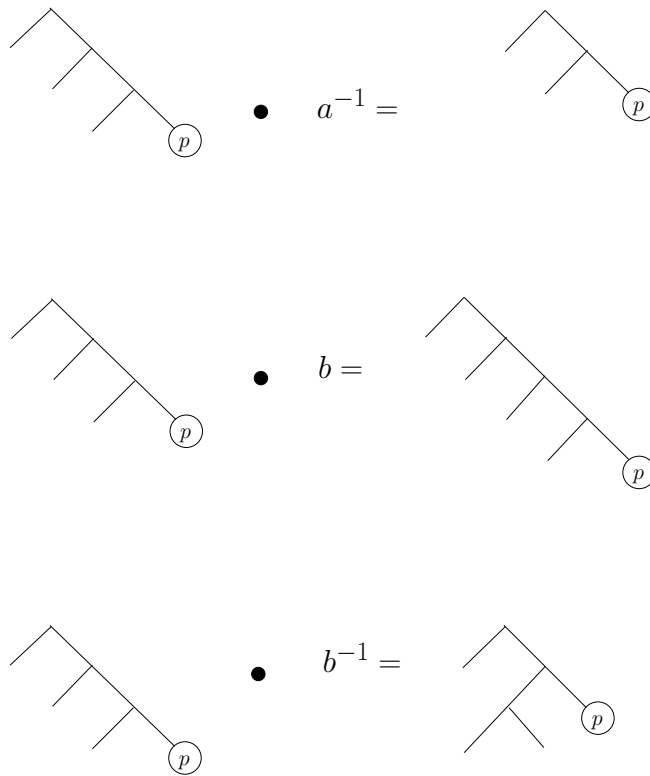


Figure 5.8 Type III: When multiplied by a^{-1} the numbers n , j_l , and j_r are all decreased by one and k remains unchanged. This move decreases the length by one. When multiplied by b , the numbers n , j_l , and j_r are all increased by one and k remains unchanged. This move increases the length. When multiplied by b^{-1} the numbers n , j_l , and j_r are all decreased by one, but k remains unchanged. This move decreases the length by one.

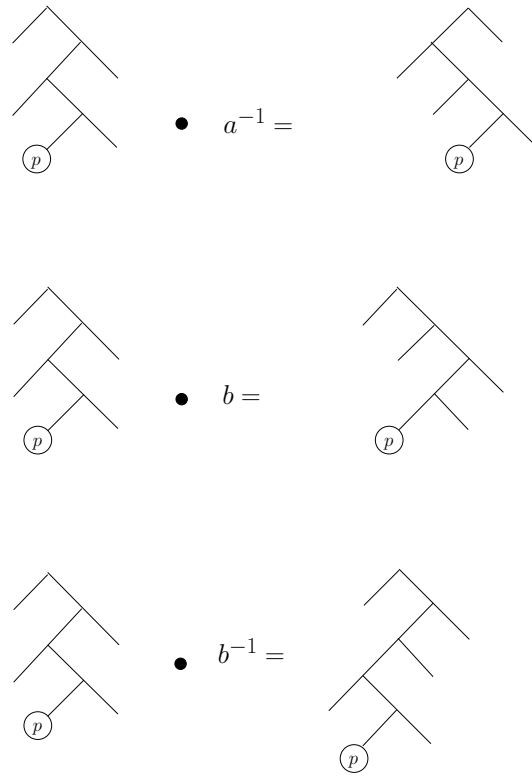


Figure 5.9 Type IV: When multiplied by a^{-1} the numbers n , j_l , and j_r remain unchanged and k is decreased by one. This move lowers the length by one. When multiplied by b , the numbers n , j_l , and j_r remain unchanged and k is decreased by one. This move decreases the length by one. When multiplied by b^{-1} the numbers n , j_l , and j_r are all increased by one, and k remains unchanged. This move increases the length.

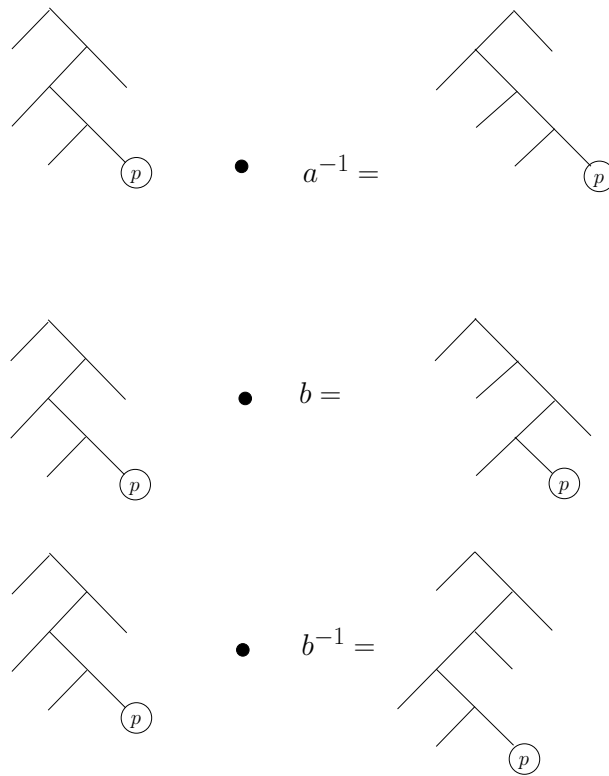


Figure 5.10 Type V: When multiplied by a^{-1} the numbers n , j_l , and j_r remain unchanged and k is decreased by one. This move decreases the length by one. When multiplied by b , the numbers n , j_l , and j_r remain unchanged and k is also unchanged. This move does not affect the length. When multiplied by b^{-1} the numbers n , j_l , and j_r are all increased by one, and k does not change. This move does not change the length.

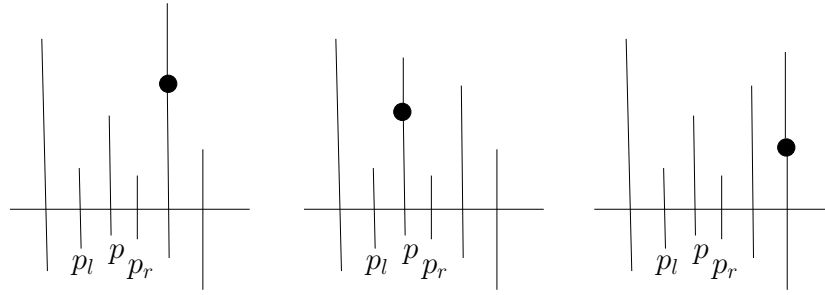


Figure 5.11 In the diagram on the left the level 2 pipe is promoted. In the middle diagram p is promoted. In the diagram on the right the right right 3 is promoted.

promoted and

$$\phi((D \cdot b^{-1}, p)) = (j_l + 1) + (j_r + 1) - 3 + 1 = j_l + j_r.$$

which is raised. We illustrate this in 5.11.

Case IX: We first consider the latter case, which is when the number is equal to $\max j_l, j_r + k$. For $n > 3$ the diagrams for the five types in this case are the same as those in case VIII except that now only one of the pipes p_l or p_r lies in the subtree of p . Without loss of generality we may assume that p_r lies in the right subtree of p . Then $\phi((D, p)) = j_r + k + 1$. The value of k changes in the same way as in case VIII. When the levels of p, p_l , and p_r are changed by one in case VIII, the value of p_r is changed by one here. The result is a change in the value of ϕ by one.

Again we must check the case where $n = 3$. There are two possible diagrams for this case. The first is done in 5.12 and the second is done in 5.13.

In the first diagram has two possibilities. If $j_l = 4$ then the total number of moves is 4. If we multiply by b then we have a case II diagram and the number of moves is 3. If we multiply by a^{-1} we have another case IX diagram and the number is 5. If we multiply by b^{-1} we have a case IX diagram and the number of moves is 5. The second possibility is if $j_l > 4$ in which case the total number of moves is $j_l + 1$. In this case, multiplication by b gives a case VI diagram in which the number of moves

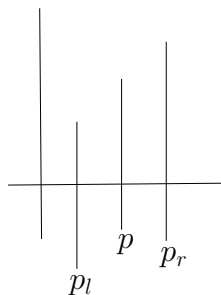


Figure 5.12 The first diagram for Case IX when $n = 3$

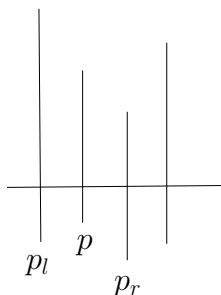


Figure 5.13 The second diagram for Case IX when $n = 3$

is j_l . If we multiply by a^{-1} , then we have a case IX diagram in which the number of moves is $j_l + 1$. If we multiply by b^{-1} we have another case IX diagram in which the number of moves is $j_l + 2$.

In this case the arguments are similar to those of the previous case. Now we must consider the first subcase of case IX. For this we assume that p is on the right side of the diagram, $j_l = n + 1$, $j_r < n$, and the number of Type V moves is odd. Here the number is $j_l + k$. If $n > 3$ we assume consider the same diagrams as we considered in Case VIII. First we note that if k is increased, then the number in question is either raised or unchanged. Second, note that j_l can never be lowered by more than one at a time. Taking this into account it is clear that in the first three types the number is never lowered by more than one. We also note that b , a^{-1} , and b^{-1} all lower the numbers for Type I, Type II, and Type III respectively.

In Type IV, if we multiply by a^{-1} then j_l is unchanged. The number k remains the same, but the number of Type V moves is increased to an odd number. This does not affect the number since p is now on the left side of the diagram. So the number remains the same. If we multiply by b , the number j_l is unchanged, but k is decreased by one as a Type IV move is removed. So the new number is $j_l + (k - 1)$ which is lower. If we multiply by b^{-1} it is clear that the number will be increased.

In Type V, if we multiply by a^{-1} then p is now on the left side of the diagram, k is decreased by one as the number of Type V moves is decreased, and j_l is unchanged. So the new number is $j_l + (k - 1)$ which is decreased. When multiplied by b all the numbers remain unchanged, and when multiplied by b^{-1} , j_l is increased, and k does not change. Now if $n = 3$ the cases are as before since $k = 0$.

We have shown that the generator specified in the algorithm decreases the value of ϕ by one, and no generator decreases the value of ϕ by more than one. So ϕ satisfies the conditions in the lemma and the algorithm is minimal.

Chapter 6

Conclusion

We have defined the pipe diagram as an alternate description of F , and we have investigated the problem of finding a minimal representative for an element of F in the presentation F_3 . We have shown that a minimal reduction does not require the insertion of a trivial pipe, and we have given an algorithm that eliminates a lower-trivial pipe in the fewest possible number of steps. Pipe diagrams were essential in discovering these theorems.

Some open problems that remain of interest are

1. finding an algorithm that reduces an element of F to the trivial element in the fewest number of steps in the presentation F_3 ,
2. finding a minimal representation for an element of F in the presentations with saturated generating sets, and ultimately
3. determining the amenability of F .

We hope that the minimal elimination of a given pipe will provide insight on how to construct the algorithm for the presentation F_3 . The problem then would be to choose which lower-trivial pipe to eliminate first in each diagram.

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