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A Note on k-Commutative Matrices

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Let A and B be square matrices over a field in which the minimum polynomial of A is completely reducible. It is shown that A is k commutative with respect to B for some non-negative integer k if and only if B commutes with every principal idempotent of A. The proof is brief, simplifying much of the previous study of k-commutative matrices. The result is also used to generalize some well-known theorems on finite matrix commutators that involve a complex matrix and its transposed complex conjugate.

INTRODUCTION

THE study of matrix commutators of higher order has received attention from several authors.[1] In particular, W. E. Roth[2] considered what he called k-commutative matrices. The main purpose of this note is to prove briefly a useful characterization of these matrices.

Let A and B be n by n matrices over a field F in which the minimum polynomial $\prod_\alpha (x-\alpha)^{s_\alpha}$ of A is completely reducible. If $[A,B]=AB-BA$ denotes the commutator of A and B, then $[_{(k)}A,B]$ is defined recursively by

$$[_{(0)}A,B]=B \quad \text{and} \quad [_{(k)}A,B]=[A,[_{(k-1)}A,B]]$$

for $k>0$.[3] A is said to be k commutative with respect to B if and only if $[_{(k)}A,B]=0$, and $[_{(j)}A,B]=0$ implies $j\geqq k$. Clearly, A is k commutative with respect to B for at most one non-negative integer k.

THEOREM

A is k commutative with respect to B for some non-negative integer k if and only if B commutes with every principal idempotent[4] of A.

Preliminary to the proof of this theorem, the following lemma is demonstrated.

LEMMA

If E_α is a principal idempotent of A and $[A,B]$ commutes with E_α, then B commutes with E_α.

To prove this, let $E_\alpha'=I-E_\alpha$. Since E_α commutes with A,

$$[A,E_\alpha BE_\alpha']=E_\alpha[A,B]E_\alpha'.$$

Thus, since $E_\alpha E_\alpha'=0$, under the hypothesis of the lemma, A commutes with $E_\alpha BE_\alpha'$. But since E_α is a polynomial in A, E_α also commutes with $E_\alpha BE_\alpha'$. Hence, $E_\alpha BE_\alpha'=0$. By a similar argument, $E_\alpha'BE_\alpha=0$. Finally, $E_\alpha B=E_\alpha BE_\alpha=BE_\alpha$.

The proof of the theorem is now given.

NECESSITY

The result is obvious in case $k=0$. Thus, suppose that A is k commutative with respect to B for some $k>0$, and let E_α be any principal idempotent of A. Since A commutes with $[_{(k-1)}A,B]$, and E_α is a polynomial in A, then E_α commutes with $[_{(k-1)}A,B]$. Hence, by repeated use of the lemma above, E_α commutes with $[_{(j)}A,B]$, $j=k-1, k-2, \cdots, 0$. In particular, E_α commutes with B.[5]

SUFFICIENCY

Let $A_\alpha=(A-\alpha I)E_\alpha$ for α any characteristic value of A with associated principal idempotent E_α. Thus, if B commutes with every principal idempotent of A, it follows by induction on $k\geqq 1$ that

$$[_{(k)}A,B]=\sum_\alpha \sum_{j=0}^{k} (-1)^j \binom{k}{j} A_\alpha^{k-j}BA_\alpha^{j},$$

where the first sum is taken over all of the distinct characteristic values of A, and where the usual notation for the binomial coefficient is used. Since A_α is nilpotent, by choosing k sufficiently large, the sum on the right is zero, and the desired conclusion is obtained.

Moreover, since A_α is nilpotent of order equal to the index s_α of α, the following result due to Roth[6] is also a consequence of the preceding equation.

COROLLARY 1

Let m be the largest of the indices associated with the distinct characteristic values of A. If A is k commutative with respect to B, then $k<2m$.

Furthermore, for any scalar polynomial $\phi(x)$, since the index of the characteristic value $\phi(\alpha)$ of $\phi(A)$ is at most m, and the principal idempotent of $\phi(A)$ associated with $\phi(\alpha)$ is the sum $\sum E_\beta$ over all of the distinct characteristic values β of A such that $\phi(\beta)=\phi(\alpha)$, the following result is also immediate.

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[1] For a survey of these results see O. Taussky, Am. Math. Monthly **64**, 229 (1957).

[2] W. E. Roth, Trans. Am. Math. Soc. **39**, 483 (1936).

[3] See also, M. Marcus and N. A. Khan, J. Research Natl. Bur. Standards **64B**, 51 (1960), and M. F. Smiley, Am. Math. Soc. Notices **7**, 927 (1960).

[4] For the definition and properties of the principal idempotents of a matrix, see, for example, N. Jacobson, *Lectures in Abstract Algebra* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1953), Vol. II, pp. 130–132.

[5] See also, W. E. Roth, Trans. Am. Math. Soc. **39**, 483 (1936), Theorem 9.

[6] Reference 5, Theorem 5.

COROLLARY 2

Let m be defined as in Corollary 1, and let $\phi(x)$ and $\theta(x)$ be polynomials over F. If A is k commutative with respect to B for some k, then $\phi(A)$ is j commutative with respect to $\theta(B)$ for some $j<2m$.

Roth[7] considered only the case $\theta(x)=x$, and showed under this condition that $j\leqq k$. However, this stronger inequality is not in general valid for every polynomial $\theta(x)$.

As an application of the preceding results, some remarks are now given concerning commutators that involve a complex matrix A and its transposed complex conjugate A^*.

First, as is well known, the principal idempotents of a normal matrix are Hermitian. More generally, the following is now demonstrated.

COROLLARY 3

Any complex matrix A is k commutative with respect to A^* for some non-negative integer k if and only if the principal idempotents of A are Hermitian.

To prove this, it is first observed that the principal idempotents of A^* are the transposed complex conjugates of the principal idempotents E_α of A. Thus, since $E_\alpha{}^*$ commutes with A^*, if $E_\alpha=E_\alpha{}^*$, then by the theorem above A is k commutative with respect to A^* for some non-negative integer k. Conversely, if E_α commutes with A^*, then it also commutes with $E_\alpha{}^*$. But any normal idempotent matrix is Hermitian.

Finally, as an application of Corollary 1 above, a well-known theorem is generalized.

COROLLARY 4[8]

The commutator $[A,A^*]$ is k commutative with respect to A for some non-negative integer k if and only if A is normal.

It is only necessary to prove that $C=[A,A^*]$ and $[_{(k)}C,A]=0$, for some positive integer k, implies $C=0$. But, since C is diagonable, applying Corollary 1 with $m=1$, $[C,A]=0$. Thus, by a theorem of Jacobson,[9] C is nilpotent. But any diagonable nilpotent matrix is necessarily zero.

[7] Reference 5, Theorem 3.

[8] For a proof of this corollary, in case either $k=1$ or $k=2$, see also T. Kato and O. Taussky, J. Wash. Acad. Sci. **46**, 38 (1956).

[9] N. Jacobson, Ann. Math. **36**, 877 (1935).

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New Possibilities for a Unified Field Theory

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The introduction of nonsymmetric g_{ik} in unified field theories of the Einstein-Schrödinger type is open to the objection, on group-theoretical grounds, that the symmetric and antisymmetric parts transform independently. This objection does not apply to the use of nonsymmetric $\Gamma_{ik}{}^\mu$, since these quantities are irreducible under the "extended group," consisting of the point transformations and the Einstein λ transformations.

We consider a theory based on symmetric g_{ik} and nonsymmetric $\Gamma_{ik}{}^\mu$. The Lagrangian L is assumed to depend only on g_{ik} and the contracted curvature tensor R_{ik} (this insures the λ invariance and transposition invariance of the theory). For simplicity, we suppose further that L involves R_{ik} rationally and, at most, quadratically.

The resulting theory is able to account satisfactorily for the main feature of gravitation, electromagnetism, and their interaction. In particular, the theory yields the correct equations of motion for charged masses. The electromagnetic tensor is associated with the skew part of R_{ik}, and the λ transformations correspond roughly to the gauge transformations of electrodynamics.

1. INTRODUCTION

An important feature of the unified field theory of Einstein and Schrödinger is the property of "λ invariance," i.e., invariance of the field equations under the group of transformations

$$'\Gamma_{ik}{}^\mu=\Gamma_{ik}{}^\mu+\delta_i{}^\mu\frac{\partial\lambda}{\partial x^k},\quad 'g_{ik}=g_{ik},\qquad (1.1)$$

where the function $\lambda(x)$ is arbitrary. This invariance property depends essentially on the invariance of the curvature tensor

$$R_{.ijk}{}^h\equiv\Gamma_{ik,j}{}^h-\Gamma_{ij,k}{}^h+\Gamma_{\alpha j}{}^h\Gamma_{ik}{}^\alpha-\Gamma_{\alpha k}{}^h\Gamma_{ij}{}^\alpha$$

and its contractions under the transformations (1.1). Quite generally, let us consider theories whose field equations are derived from a variational principle

$$\delta\int\mathfrak{L}(g,\Gamma)d^4x=0\qquad (1.2)$$

by independently varying the g_{ik} and the $\Gamma_{ik}{}^\mu$ (Palatini