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College of Physical and Mathematical Sciences

From Pebbles to Commutators

Donald W. Robinson

Upon review of the list of annual Sigma Xi lectures given on this campus, I noted that this marks the first time that a mathematician has been invited to participate. Since mathematics is often referred to as the "Queen of the Sciences," this seemed at first an affront to her Majesty. But, upon further reflection, I considered the possibility that the selection committees for the preceding lectures had been moved by the words of St. Augustine: "The good Christian should beware of mathematicians and all those who make empty prophecies. The danger already exists that mathematicians have made a covenant with the devil to darken the spirit and confine man in the bonds of Hell." Since the selection committee for this lecture has obviously failed to heed this warning, I accept this opportunity to plead innocent of any such charge. Indeed, although admitting the fact that the Queen readily rejects many who wish to know her, my purpose is not to "cast you into outer darkness," but rather to illuminate some of the facets of her remarkable life.

In order to avoid the customary effects of frustration and disappointment that are suffered by an audience before the Queen, we will clear her court of the mass of grubby little details and technical paraphernalia that surround her. On the other hand, in order properly to acquaint you with her, it will be necessary to speak her language much of the time. Moreover, since on an occasion such as this the topic of conversation is somewhat arbitrary, it should be stated that my personal tastes and interests alone have dictated the choice. Thus, it is hoped that you may sense through and beyond this particular message to a better understanding of and an appreciation for the Queen herself.

To most of us the word "mathematics" is associated with some form of the word "calculation." Thus, it seems appropriate to begin

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with the observation that the word is derived from the word *calculi*. The Romans used it as the name of a counting device that was made of stone. Indeed, the latin noun *calculus* means “pebble” and is the derivative of *calx*, a piece of limestone. Thus, to calculate means literally to pebble, and a calculator is one who works with pebbles.

Our work will be with some pebbles. First, we shall present in brief allegory the birth and early development of the science of numbers and computation. Second, we shall recall some historical facts and identify some of the significant mathematical pebbles of the past. Finally, in order to meet the requirements of this lecture, which is to speak on some aspect of my own research, we shall investigate one particular pebble. Although I do not intend to present the complete details of this one pebble, it is hoped that you will be sufficiently motivated at least to perceive why I wanted to “crack it.”

Imagine a shepherd who is tending his sheep. In the morning he allows them to leave the safety of their enclosure and go off to graze, and in the evening he gathers them back into the fold. Most shepherds know their sheep. But this one has a short memory, and as such looks for a machine to help him keep track of his flock. He collects some pebbles and procures a pouch. As each sheep leaves in the morning he places a pebble in the pouch. In the evening he removes one pebble for each sheep as it returns. If he has pebbles left over, then he goes out in search of his lost sheep; if he has extra sheep, then—well—he lets the other shepherds worry. The shepherd has solved his problem by simply matching the pebbles with the sheep.

One day the shepherd decides to speed up the process of sending off his sheep. He assigns his son to one gate, and he takes another. As the sheep depart, each assembles a collection of pebbles as before, which are later all put into one bag. He seems to sense that the matching is still all right, and in fact that it does not matter whether he or his son first place the pebbles in the bag.

Let us now bring out the mathematician in the shepherd. He recognizes in his flock a collection of objects. His matching in a one-to-one way leads him to the process of counting and eventually to the concept of number. The combination of two piles of pebbles introduces the notion of an operation—addition. Finally, his observation of the fact that it does not matter which comes first suggests some kind of a principle—commutativity.

Of course, the shepherd does not stop here. More sophisticated problems prompt him to improve his newly discovered tool. He en-

larges his concept of number to include other objects such as negatives and fractions. His facility to compute increases, and he introduces new operations and techniques. He recognizes other principles and begins to abstract their formal content.

A system finally emerges which consists of first, the collection R of real numbers; second, the operations of addition ($+$) and multiplication (\cdot); and third, a list of basic properties: for example, commutativity of addition

$$\alpha + \beta = \beta + \alpha$$

and of multiplication

$$\alpha \cdot \beta = \beta \cdot \alpha$$

are included. This system $[R, +, \cdot]$ is called the *real number system*, and the end result of the shepherd's labors is *symbolic arithmetic*—the science of the real number system.

Thus, by application of his ability to reason about some elementary observations, the shepherd places at his disposal a useful, flexible, and powerful tool. He is now in a position to look up from his pile of pebbles and glimpse, ever so faintly, the vast expanse of a new ocean—algebra.

Unfortunately, the very mention of this word strikes fear in the minds of some of us. Thus, before proceeding, let me offer you some assurance. We have, in fact, reached the point where we can stand by Sir Isaac Newton when he said:

“I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.”

Newton sensed that something of great magnitude lay just beyond him. What was it? In particular, if he had been free to explore the ocean of algebra, what would he have found? First, he would have marveled at the rich display and great variety of algebraic structures. Second, as a scientist, he would have plunged into the task of classifying these systems into their genera and species. He would, for example, find that the system of real numbers is only one of several algebraic structures that have the same basic features. Each system $[F, +, \cdot]$, called a field, of this one genus consists of a collection of objects, F , two operations, $+$ and \cdot , and the same basic list of postulates that describe the real numbers. He would discover further, however, that this genus had several species. These species would be identified by such properties as the following:

Order. In the real number field $[R, +, \cdot]$ the collection R contains an infinite number of elements. Some fields, however, have only a finite number of elements. The *order* of the field $[F, +, \cdot]$ is defined to be the number of elements in F . In particular, the order of a given field may be either some finite counting number or infinite.

Characteristic. Every field contains subsystems which are themselves fields. If the order of the *smallest* such subsystem of a given field is a finite number p , then it can be shown that p must be one of the prime numbers $[2, 3, 5, 7, 11, \dots]$; in this case the given field is said to be of *characteristic prime* p . Otherwise, the field is said to be of *characteristic zero*. Thus, a given field is classified according as its characteristic is zero or some prime number p .

Algebraic closure. Although some equations such as

$$1 - 3x + 2x^2 = 0$$

have real solutions for x , some equations such as

$$1 + x + x^2 = 0$$

cannot be solved in the real number field. This is because the polynomial

$$1 - 3x + 2x^2 = 2(1/2 - x)(1 - x)$$

can be factored, whereas $1 + x + x^2$ cannot be factored over the real numbers. In general, if every polynomial over a given field $[F, +, \cdot]$ is completely factorable in the sense that

$$\alpha_0 + \alpha_1x + \dots + \alpha_nx^n = \beta_0(\beta_1 - x)(\beta_2 - x)\dots(\beta_n - x)$$

for some $\beta_0, \beta_1, \dots, \beta_n$ in F , then the field is said to be *algebraically closed*. Otherwise it is not. Thus, a given field is or is not algebraically closed.

These and other properties identify a multitude of species in the genus field. But Newton was not even aware of these different species; in fact he did not even get his "feet wet" in this vast ocean. The time of exploration had not yet arrived; it was still over a century in the future. The story is as follows:

Mathematics entered into its modern phase when René Descartes published his analytic geometry in 1637. Here, for the first time, the tool of symbolic arithmetic was applied to geometry. The idea was to give a "name" to each point on a line. Each name was a real number, and the assignment was made in a precise, orderly way. (A crude approximation of this concept of number line is provided by

the number scale on a ruler.) The extension to the plane was a brilliant, yet simple device. Name the points of the plane by a *pair* (α_1, α_2) of real numbers! The elements of this pair were obtained by "projecting" the point onto two (perpendicular) number lines, and the collection of all such pairs was denoted by R^2 . The next step was obvious. Name the points in three-space by the collection R^3 of ordered triplets $(\alpha_1, \alpha_2, \alpha_3)$, and in general n -space by the collection R^n of n -tuples $(\alpha_1, \alpha_2, \dots, \alpha_n)$.

Thus, the points of space correspond to some algebraic objects. What algebraic structure is applicable? In the first case, clearly the entire field $[R, +, \cdot]$ is at our disposal. But it took the genius of Carl Friedrich Gauss in 1831 to give the first coherent interpretation of the second case. By the proper definition of addition and multiplication, he recognized $[R^2, +, \cdot]$ as the heretofore mysterious complex number field.

The next question was a natural one. If n is greater than 2, then how can the operations $+$ and \cdot be given to provide a field $[R, +, \cdot]$? The answer was far from obvious. But one mathematician—an Irishman named William Rowan Hamilton—became obsessed with the problem. For nearly fifteen years it haunted him. Finally, on October 16, 1843, as he was out taking a walk, a partial solution came to him. Then and there he scribbled the answer onto the nearest object at hand—a stone in the bridge upon which he stood at the time. Later that same day, he requested permission to present his idea before the Royal Irish Academy, which he did a few weeks later (on November 13, 1843; see also [2] and [7]).

The idea was a bold one. It was not a field that he sought, but a "skew" field—an algebraic structure that possessed all of the properties of a field except one—commutativity of multiplication!

Reject the principle of commutativity? But could this be done? And why not? Suppose the shepherd had matched his sheep with cannon balls, and his son had matched his with peaches. Would we not be aware of some difference in who first placed their respective objects into the bag? Does anyone who has tried to put on his socks after his shoes were on wish to argue that this is the same as the normal procedure? The same idea is also suggested by the following verse, which is found on the Game Board of Algebra (I.B.M., Pacific Science Center, Seattle):

4 pills 365 times a year can cure you
365 pills 4 times a year can kill you.

Finally, the usefulness of this concept of noncommutativity may be

sensed in the words of Dr. P. A. M. Dirac, who wrote in one of his early papers on quantum theory [5]:

“For the purpose of atomic physics it has been found convenient to introduce the idea of quantities that do not in general satisfy the commutative law of multiplication, but satisfy all the other laws of ordinary algebra.”

Yes, commutativity may be just the thing we do *not* want. In mathematics, Hamilton was the first to recognize this fact, and once again, a most difficult, yet simple step had been taken. A step so significant, that it may be ranked with the construction of the first non-Euclidean geometries. It was not that Hamilton's particular system, which he called quaternions, was all-important. It was not. Hamilton's discovery was significant because it showed that the commutative law of multiplication was not necessary for a self-consistent algebraic system. Algebra was no longer just symbolic arithmetic: it now recognized a variety of algebraic structures. Thus, the step transcended the traditions of centuries and gave algebra its freedom. Although Newton had not been free to do so, the mathematicians of the mid-nineteenth century were now free to explore the vast ocean of algebra.

One of the first algebraic structures to be investigated in this new era was introduced by Arthur Cayley in 1858. It included, in particular, the complex numbers of Gauss and the quaternions of Hamilton. The objects of its collection M were square “arrays” of elements from a field. These arrays were called *matrices*. Operations of addition and multiplication of matrices were defined and most of the principles of ordinary algebra were found to be satisfied. However, there was one exception; in general, for matrices a and b ,

$$a \cdot b \neq b \cdot a$$

That is, the algebra $[M, +, \cdot]$ of matrices was a noncommutative system.

Today, over a century later, the algebra of matrices is still one of the prized tools in the mathematician's briefcase, and the non-commutativity of the system is still one of its fascinating features.

We now conclude this lecture with an examination of one particular pebble of this discipline. Early in the study of matrices it was observed that some pairs of matrices do commute. In fact, some commute with every matrix: the collection Z of such matrices is called the *center*. Furthermore, it was noted that a given matrix a commutes with every matrix of the form

$$a_0 + a_1 \cdot a + a_2 \cdot a^2 + \dots + a_n \cdot a^n,$$

where a_0, a_1, \dots, a_n belong to the center. Such a matrix is said to be *generated* by a over the center. In 1875, W. K. Clifford [3] attempted to prove the converse of this observation; namely, every matrix that commutes with a given matrix is generated by that matrix. In 1878, G. Frobenius [6] proved this result for a very special case, but in 1884, J. J. Sylvester [15] showed that in general the conjecture was false. In other words, if $K(a)$, which is called the *centralizer* of a , is the collection of all matrices that commute with a , and $Z[a]$ is the collection of matrices that are generated by a over the center, then every matrix in $Z[a]$ is in $K(a)$, but not vice-versa: symbolically $Z[a] \subset K(a)$, but in general, $K(a) \neq Z[a]$.

Thus, the question remained: what "nice" necessary and sufficient condition guarantees that a given matrix is generated by another? This question was answered in 1910 by L. Autonne [1]. (See also [8].) He reasoned as follows: if b is generated by a , then b not only commutes with a but also with every matrix that commutes with a . He then demonstrated the converse, which is now called the "double-centralizer" theorem. Specifically, any matrix which commutes with every matrix which commutes with a must be generated by a . In other words, if $K[K(a)]$ denotes the collection of all matrices that commute with each matrix in the centralizer $K(a)$, then

$$K[K(a)] = Z[a].$$

Consequently, the class of all matrices that are generated by a given matrix is here completely characterized by the apparently superficial notions of commutativity.¹

It is very often the case in the study of matrices that commutativity is a sufficient but not necessary condition for some conclusion (see [16]). This leads one to seek a "weaker" condition, which gives rise to the same result, thus producing a "stronger" theorem. One such device is suggested by the following observation. The matrix b commutes with the matrix a if and only if the difference $b \cdot a - a \cdot b$ is zero. This difference is called the (additive) commutator of b and a , and is denoted by

$$b\delta_a = b \cdot a - a \cdot b.$$

¹Proofs of this result may be found in references [10] and [11]. One important application was made by P.A.M. Dirac: specifically, this concept provided the starting-point of his function theory for quantum algebra [5].

In a heuristic sense, the commutator measures how much b and a fail to commute.

Now, if b does not commute with a , it may happen that the commutator of the commutator is zero; that is $(b\delta_a)\delta_a = 0$ even though $b\delta_a \neq 0$. In other words, although b does not commute with a in the ordinary sense, it does "commute" in a higher order sense. The extension to still higher order commutators is immediate. The collection $K_m(a)$ of all matrices b such that

$$(\dots((b\delta_a)\delta_a)\dots)\delta_a = 0,$$

where δ_a is repeated m times, is called the m -centralizer of a . Finally, $K_m[K_m(a)]$ denotes the collection of matrices that are in the m -centralizer of every matrix in the m -centralizer of a .

In 1960, M. Marcus and N. A. Khan [12], (both of the University of California at Santa Barbara) published the following modification of the double-centralizer theorem. In the algebra of matrices over a field which is (1) of characteristic zero, and (2) algebraically closed,

$$K_2[K_2(a)] \subset Z[a].$$

In 1961, M. F. Smiley ([14] of the University of California at Riverside) generalized this result. Under essentially the same two restrictions on the field, but for any positive integer m , he showed that

$$K_m[K_m(a)] \subset Z[a].$$

In other words, if the matrix b is in the m -centralizer of every matrix in the m -centralizer of a , then b is generated by a over the center.

This result leaves two glaring questions unanswered. First, are the restrictions (1) and (2) on the field really necessary? That is, does the result fail for other species of fields or is it valid for the entire genus? Second, can it be shown that the inclusion is actually an equality? Or (as was shown in [12] for the case $m = 2$) does the left-hand member describe a particular subcollection of matrices generated by a ?

As in the case of Hamilton, these questions began to haunt me. They also haunted others, and in 1963, Dr. Olga Taussky Todd [17] of California Institute of Technology formally posed (in part) these questions to the American Mathematical Society as a research problem. Fortunately for me (my wife and children) the haunting ended the following year. In 1964, I was able to discover the key to the solution, and in 1965 the results were published (see [13]).

In final form this is the theorem. Let a be a matrix over a field F and let m be a positive integer.

If F is of characteristic prime p , and f is the integer defined by the inequalities of $p^{f-1} < m \leq p^f$, then

$$K_m[K_m(a)] = Z[a^{p^f}] \subset Z[a].$$

If F is of characteristic zero, and s is the semisimple part² of a , then

$$K_1[K_1(a)] = Z[a].$$

and

$$K_m[K_m(a)] = Z[s] \subset Z[a].$$

for $m > 1$.

This theorem completely answers the above questions. But, as is the case with all research, it asks many more of its own. Some have recently been answered: Professor Willes Werner, of our own staff, and I have just extended this theorem to matrices over the skew field of quaternions (see also [4].) Some questions have yet to be answered: is the theorem valid in some sense over any skew field?

With this question we come to the end (for now) of one thread of mathematical research. Although this thread has brought us to commutators in particular, as was promised in the title of this lecture, hopefully it has revealed along the way a few general propositions. First, mathematics is based on the simple faith that man can pebble. Second, its function is to identify and abstract the kernel of this experience. Third, its resources are extensive and are abundant in overwhelming variety, yet it refuses to give to man its wealth without extracting a price. Fourth, although in part it is a deductive science, in the large it is a creative science; not only does the mathematician draw necessary conclusions, but he also decides what to prove and discovers how to establish the proof. Fifth, it evolves with the changing times; in the words of R. H. Bing, "mathematics is an alive and growing subject." In summary, we may say as Warren Weaver did of science in general, mathematics is "an adventure of the human spirit."

I leave you now to ponder this lofty conclusion, while I go off to play with another pile of pebbles.

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²The semisimple part of a is a special matrix generated by a (see, for example, [9]).

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