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Unification of Ernst-equation Bäcklund transformations using a modified Wahlquist–Estabrook technique[a)]

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The three known Bäcklund transformations for the Ernst equation are derived using a modification of the Wahlquist–Estabrook prolongation procedure. The modification requires that the equation to be studied be cast into a set of differential forms and their exterior derivatives, such that all coefficients are constant (a "CC ideal"). Analysis of the resulting equations produces 16 solutions composed of the three basic transformations combined with identity and other essentially trivial transformations. The group structure of the transformations is discussed. A Bäcklund transformation (already known) for the Ernst–Maxwell equations can be found by the same method. Promising generalizations are mentioned.

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1. INTRODUCTION

In 1978–1979 three Bäcklund transformations (BT) for the Ernst equation,[1]

$$(\mathrm{Re}\, E)\nabla^2 E = (\nabla E)^2, \tag{1}$$

which is the fundamental equation for vacuum stationary axially symmetric space times and which also occurs in nonlinear field theories,[2] were found—one by the present author[3] and two by G. Neugebauer.[4] (These references will be denoted by I and II, respectively.) The present paper shows how to derive all three BT's in a unified manner, using a modified Wahlquist–Estabrook (MWE) approach, and it is noted, as has been done before, that these BT's are elements of a group. In addition to providing this information about Ernst-equation BT's, this paper demonstrates the general use of the MWE method, which is suitable for systems of equations which can be cast into an ideal of differential forms with constant coefficients (CC ideal). The Ernst–Maxwell equations are also explored. (The term "Bäcklund transformation" is used in this paper to mean a Bäcklund transformation from solutions of an equation to solutions of the same equation, or "auto-Bäcklund transformation.")

2. FORMULATION OF THE EQUATIONS

We write the metric as in I, but with T replaced by f and Q by ω in order to conform to more common usage:

$$\begin{aligned} ds^2 = {} & \lambda f(dx^1 + \omega\, dx^2)^2 + S^2 f^{-1}(dx^2)^2 \\ & + e^{2\gamma} f^{-1}((dx^3)^2 - \lambda\,(dx^4)^2), \end{aligned} \tag{2}$$

where $\lambda = \pm 1$ and S, f, ω, and γ are functions of x^3 and x^4 only. $\lambda = 1$ corresponds to cylindrical waves, $\lambda = -1$ to axially symmetric fields. We write $k = \sqrt{\lambda}$ ($= 1$ or i), $x = \frac{1}{2}(x^3 + kx^4)$, and $y = \frac{1}{2}(x^3 - kx^4)$. We define a linear Hodge star operator by $*dx^3 = dx^4$, $*dx^4 = \lambda dx^3$, yielding $*dx = k^{-1}\, dx$, $*dy = -k^{-1}\, dy$. Then $** = \lambda$. Note that $\lambda = -1$ in II.

The field equation for ω may be formally satisfied by defining a potential ϕ such that

$$*d\phi = S^{-1} f^2\, d\omega. \tag{3}$$

$d\phi$ is exact; closure ($dd\phi = 0$) yields the ω field equation. (Alternately: the ω field equation shows that $*S^{-1}f^2\, d\omega$ is closed, so we write it as an exact form $\lambda\, d\phi$.) (Note: our E is Neugebauer's f, and our S is his V.) We write $E = f + i\phi$. Then the field equations for E and S may be written, where subscripts denote differentiation, as

$$S_{xy} = 0, \tag{4}$$

$$E_{xy} + \tfrac{1}{2} S^{-1}(S_x E_y + S_y E_x) = f^{-1} E_x E_y \tag{5}$$

($f = \mathrm{Re}\, E$). Equation (5) is the Ernst equation, in which no specialization of S has yet taken place. An alternate way of writing Eqs. (4) and (5) is

$$d(*dS) = 0, \tag{4'}$$

$$d(Sf^{-1} *df) + Sf^{-2} d\phi \wedge *d\phi = 0, \tag{5'}$$

$$d(Sf^{-2} *d\phi) = 0. \tag{3'}$$

It is usual to satisfy Eq. (4) by choosing $S = x + y = x^3$. In fact, this can always be done for the axially symmetric case, and this choice was made in I. However, this limits the possible Bäcklund transformations to that one which exhibits $S' = S$, i.e., the one found in I. Paper II did not make this restriction, apparently requiring $S' \neq S$, and found the other two BT's. For the purposes of the present paper, therefore, we make no restriction on S beyond Eq. (4).

We now redefine the variables t, u, v, and w of I:

$$\begin{aligned} t &= f^{-1}E_x - S^{-1}S_x, \quad u = f^{-1}E_y - S^{-1}S_y, \\ v &= f^{-1}\bar{E}_x - S^{-1}S_x, \quad w = f^{-1}\bar{E}_y - S^{-1}S_y. \end{aligned} \tag{6}$$

To compare with Neugebauer (II), we first note that his x^1 and x^2 are given in terms of our variables x and y by

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$x^1 = 2x$ and $x^2 = 2y$. Then his M_i and N_i, in our notation, are

$$M_1 = \tfrac{1}{4} f^{-1} E_x, \quad N_1 = \tfrac{1}{4} f^{-1} \overline{E}_y,$$
$$M_2 = \tfrac{1}{4} f^{-1} \overline{E}_x, \quad N_2 = \tfrac{1}{4} f^{-1} E_y, \tag{7}$$
$$M_3 = \tfrac{1}{2} S^{-1} S_x, \quad N_3 = \tfrac{1}{2} S^{-1} S_y,$$

so that we have

$$t = 4M_1 - 2M_3, \quad u = 4N_2 - 2N_3, \tag{8}$$
$$v = 4M_2 - 2M_3, \quad w = 4N_1 - 2N_3.$$

Cosgrove[5] also made this observation; his paper will be denoted by III.

Equation (5) and its complex conjugate may now be written

$$\begin{aligned} t_y &= \tfrac{1}{2} t(u-w) - \tfrac{1}{2} S^{-1}(wS_x + tS_y), \\ u_x &= \tfrac{1}{2} u(t-v) - \tfrac{1}{2} S^{-1}(uS_x + vS_y), \\ v_y &= \tfrac{1}{2} v(w-u) - \tfrac{1}{2} S^{-1}(uS_x + vS_y), \\ w_x &= \tfrac{1}{2} w(v-t) - \tfrac{1}{2} S^{-1}(wS_x + tS_y). \end{aligned} \tag{9}$$

We define potentials η and R by

$$^*df = S^{-1} f (d\eta + \omega\, d\phi), \tag{10}$$
$$^*dS = dR \tag{11}$$

whose existence is guaranteed by Eqs. (4) and (5), and we define 1-forms as in I:

$$\xi_1 = f^{-1} d\phi, \quad \xi_2 = S^{-1} f\, d\omega,$$
$$\xi_3 = S^{-1}(d\eta + \omega\, d\phi), \quad \xi_4 = f^{-1} df, \tag{12}$$
$$\xi_5 = S^{-1} dS, \quad \xi_6 = S^{-1} dR.$$

These then satisfy a set of 2-form equations [the differential form versions of Eqs. (4) and (9)], which are given in I and elsewhere.[6–8] [Strictly speaking, the exterior derivatives of the ξ_i may be given in terms of wedge products of the ξ_i themselves, with constant coefficients; and there exists a set of 2-forms, with constant coefficients, in the space of variables ϕ, ω, η, f, S, and R, which—when annulled—represent two-dimensional solution manifolds of Eqs. (4) and (9).]

It is convenient to define new 1-forms η_i, $i = 1$–6, which are self-dual or anti-self-dual up to a factor k, as follows, where the subscript in parentheses goes with the lower sign:

$$\eta_{1(2)} = \xi_4 \pm i\xi_1 + k\xi_3 \pm ik\xi_2,$$
$$\eta_{3(4)} = \xi_4 \mp i\xi_1 - k\xi_3 \pm ik\xi_2, \tag{13}$$
$$\eta_{5(6)} = \xi_5 \pm k\xi_6.$$

These satisfy

$$^*\eta_i = k^{-1}\eta_i, \quad i = 1,2,5,$$
$$^*\eta_i = -k^{-1}\eta_i, \quad i = 3,4,6. \tag{14}$$

We also have, in the notation of I,

$$\eta_1 - \eta_5 = 2t\, dx, \quad \eta_4 - \eta_6 = 2u\, dy,$$
$$\eta_2 - \eta_5 = 2v\, dx, \quad \eta_3 - \eta_6 = 2w\, dy, \tag{15}$$
$$\eta_5 = 2S^{-1} S_x\, dx, \quad \eta_6 = 2S^{-1} S_y\, dy,$$

and, in the notation of II,

$$\eta_1 = 8M_1\, dx, \quad \eta_3 = 8N_1\, dx,$$
$$\eta_2 = 8M_2\, dx, \quad \eta_4 = 8N_2\, dy, \tag{16}$$
$$\eta_5 = 4M_3\, dx, \quad \eta_6 = 4N_3\, dy.$$

Finally, we note that the exterior derivatives of the η_i are identically:

$$\begin{aligned} 4d\eta_1 &= \eta_1 \wedge (\eta_3 + \eta_6 - \eta_4) - \eta_4 \wedge \eta_5, \\ 4d\eta_2 &= \eta_2 \wedge (\eta_4 + \eta_6 - \eta_3) - \eta_3 \wedge \eta_5, \\ 4d\eta_3 &= \eta_3 \wedge (\eta_1 + \eta_5 - \eta_2) - \eta_2 \wedge \eta_6, \\ 4d\eta_4 &= \eta_4 \wedge (\eta_2 + \eta_5 - \eta_1) - \eta_1 \wedge \eta_6, \\ 2d\eta_5 &= -2d\eta_6 = \eta_5 \wedge \eta_6, \end{aligned} \tag{17}$$

and the following (monomial!) 2-forms vanish:

$$\eta_1 \wedge \eta_2, \quad \eta_1 \wedge \eta_5, \quad \eta_2 \wedge \eta_5 \quad (=0),$$
$$\eta_3 \wedge \eta_4, \quad \eta_3 \wedge \eta_6, \quad \eta_4 \wedge \eta_6 \quad (=0). \tag{18}$$

The content of the field equations appears in the possibility of the definition of the potentials ϕ, η, and R.

3. MODIFIED WAHLQUIST–ESTABROOK APPROACH

We pursue the search for BT's by a modified Wahlquist–Estabrook (MWE) method. The original Wahlquist–Estabrook method[9] consists of (1) the search for a pseudopotential and (2) the search for BT's. The modification given here is suitable for use with a constant coefficient, or CC, ideal, here denoted by C. The basic essentials of the MWE method are given in Eqs. (19) and (20) [or Eqs. (22) and (23)] for the pseudopotential and Eq. (45) for the BT.

A CC ideal, by (limited) definition here, consists of two sets of 2-forms, built from a number of 1-forms τ_i. The first set expresses the exterior derivatives $d\tau_i$, in terms of sums of wedge products of the τ_i. The second set is composed of sums of wedge products of the τ_i, which are to be annulled to obtain the solution manifold. We see that Eqs. (17) and (18) are a CC ideal. CC ideals have been discussed recently by Estabrook.[10] Estabrook has suggested a useful alternate title for a CC ideal: an "invariant Pfaffian system," or IPS.[11] By putting certain ("invariant") 1-forms equal to zero in a set of Cartan–Maurer equations it may be possible to produce an IPS; see Ref. 10.

4. SEARCH FOR A PSEUDOPOTENTIAL

We require that there exist a "pseudopotential" q and a 1-form θ

$$\theta = -dq + F^i(q)\tau_i \quad (\text{sum on } i), \tag{19}$$

where the functions F^i depend only on q. We require that

$$d\theta = 0 \quad (\text{mod } \theta, C); \tag{20}$$

i.e., we replace dq, where it occurs, by $F^i\tau_i$, obtained from $\theta = 0$; replace the $d\tau_i$ by their values as given in C; and use the remaining 2-forms in C to further simplify the equation $d\theta = 0$ by treating them as equations among the various monomial 2-forms. The coefficients are then set equal to zero. We obtain in this way a set of ordinary (nonlinear) differential equations for the F^i.

It is often convenient to use an alternate approach. We consider a column vector Q of pseudopotentials q^α and a column vector Θ of 1-forms θ^α, assumed to be linear in the q^α. We write a linear representation:

$$\theta^\alpha = -dq^\alpha + B^{i\alpha}{}_\beta q^\beta \tau_i \quad (\text{sum on } i,\beta), \tag{21}$$

where the $B^{i\alpha}{}_\beta$ are constant. The index i is summed over the number of 1-forms τ_i; the range of α and β is the dimension of the representation, as yet unspecified (but possibly even infinite). In matrix form we have, where the matrices $B^i = [B^{i\alpha}{}_\beta]$,

$$\Theta = -dQ + (B^i\tau_i)Q. \tag{22}$$

The equation

$$d\Theta = 0 \pmod{\Theta, C} \tag{23}$$

becomes, after substitution for dQ and dropping the vector Q,

$$B^i d\tau_i - B^i B^k \tau_i \wedge \tau_k \pmod{C} = 0. \tag{24}$$

Setting the coefficients of this equation equal to zero yields (generally) an incomplete Lie algebra for the matrices B^i, the "prolongation structure" of the problem. If a two-dimensional representation for the B^i is found, a single pseudopotential q may be defined as q^2/q^1.

The latter approach is suitable for the Ernst equation CC ideal, Eqs. (17)–(18), if one makes one generalization: The matrices B^i must be taken to be functions of the variable

$$\zeta = \{[k(R+l)-S][k(R+l)+S]^{-1}\}^{1/2}, \tag{25}$$

where l is a parameter. If we take $S = x^3$ and $R = x^4$, we see that ζ may be written as

$$\zeta = [(kl-y)(kl+x)^{-1}]^{1/2}, \tag{26}$$

an invariant or similarity variable for the Ernst equation (5). [The scale transformation $x^{3\prime} = \alpha x^3$, $x^{4\prime} = \alpha x^4$, and the translation, $x^{4\prime} = x^4 + \beta$, α and β constant, leave both ζ and Eq. (5) invariant.] We note, from Eqs. (25), (11), and (15), that

$$d\zeta = g\eta_5 + h\eta_6 \tag{27}$$

where $g = \frac{1}{4}\zeta(\zeta^2 - 1)$ and $h = \frac{1}{4}\zeta^{-1}(\zeta^2 - 1)$. (Thus ζ itself is actually a pseudopotential; this point is clear in paper II.)

We now formulate our pseudopotential equations. We write, with the η_i for τ_i,

$$\Theta = -dQ + B^i(\zeta)\eta_i Q \tag{28}$$

and

$$d\Theta = 0 \text{ [mod Eqs. (28), (17), (18)]}. \tag{29}$$

Equation (29) becomes, after expansion and use of Eqs. (27) and (28),

$$0 = B^{i\prime}(g\eta_5 + h\eta_6) \wedge \eta_i - B^i B^k \eta_i \wedge \eta_k + B^i d\eta_i \quad \text{[mod (17), (18)]}, \tag{30}$$

where $' = d/d\zeta$. (We choose $B^5 = B^6 = 0$ without loss of generality, since we may reasonably expect $B^5\eta_5 + B^6\eta_6$ to be proportional to $d\zeta$, and then we may eliminate these terms entirely by definition of a new pseudopotential as a function of ζ and the old one.)

Expansion of Eq. (30) now gives

$$B^4 - B^1 = 4gB^{4\prime} = -4hB^{1\prime},$$
$$B^3 - B^2 = 4gB^{3\prime} = -4hB^{2\prime}, \tag{31}$$

$$B^3 - B^1 = 4[B^3, B^1], \quad B^4 - B^1 = 4[B^1, B^4],$$
$$B^3 - B^2 = 4[B^2, B^3], \quad B^4 - B^2 = 4[B^4, B^2]. \tag{32}$$

Solution of Eq. (31) yields

$$B^1 = a\zeta + b, \quad B^2 = c\zeta + d,$$
$$B^3 = c\zeta^{-1} + d, \quad B^4 = a\zeta^{-1} + b, \tag{33}$$

where a, b, c, and d are constant matrices.

Equation (32) now gives

$$4[a,d] = 4[b,a] = a, \tag{34}$$

$$4[c,b] = 4[d,c] = c, \tag{35}$$

$$4[a,c] + 4[b,d] = b - d, \tag{36}$$

the prolongation structure. If we define

$$e = 4[b,d] \tag{37}$$

and use the Jacobi identity on all commutators, we get, in addition to Eqs. (34), (35), and (37),

$$b - d - e = 4[a,c],$$
$$[a,e] = [c,e] = 0, \tag{38}$$
$$4[e,b] = 4[e,d] = e.$$

We can specialize to a homomorphic image of the algebra of Eqs. (34)–(38) by taking $e = 0$, $4a = \tau - i\nu$, $4c = \tau + i\nu$, and $4d = -4b = i\theta$. We get

$$[\tau,\theta] = -\nu, \quad [\nu,\theta] = \tau, \quad \text{and} \quad [\nu,\tau] = \theta, \tag{39a}$$

the sl(2, R) algebra, as obtained before.[3,7]

A particular representation of Eq. (39a) is

$$\tau = -\sigma_z, \quad \nu = \sigma_x, \quad \theta = i\sigma_y, \tag{39b}$$

where the σ_i are the Pauli spin matrices. With this representation, we define $u = q_2/q_1$; u is related to the pseudopotential q defined in I by[7]

$$u = -i(\zeta - 1)(\zeta + 1)^{-1}(q - 1)(q + 1)^{-1}. \tag{40}$$

We now write Eq. (19) $\theta = 0$, the definition of q, in terms of the η_i [this is merely Eq. (8) of I].

$$4dq = q(1 + q\zeta)\eta_1 - (q + \zeta)\eta_2 + \zeta^{-1}q(q + \zeta)\eta_3 - \zeta^{-1}(1 + q\zeta)\eta_4 + (1 - q^2)(\zeta\eta_5 + \zeta^{-1}\eta_6). \tag{41}$$

We will write further equations in terms of q.

We note the following relations of ζ and q to the variables γ and α in II:

$$\gamma = \zeta^2, \tag{42}$$

$$\alpha = \zeta(1 + \zeta q)(q + \zeta)^{-1}. \tag{43}$$

We also note that the pseudopotential q ($= q_c$, say) used in III, Eq. (4.22), is related to q in this paper by

$$q_c = -(1+\zeta q)(q+\zeta)^{-1} = -\zeta^{-1}\alpha. \tag{44}$$

5. SEARCH FOR A BÄCKLUND TRANSFORMATION (BT)

Once a single pseudopotential q is found from Eqs. (19) and (20) [or (22) and (23)], we may assume that a BT exists with the following form:

$$\tau_i' = A_i^{\ j}(q)\tau_j \text{ (sum on } j). \tag{45}$$

This relates the exterior derivatives of the new (primed) variables to those of the old (unprimed) variables, providing the relation between first derivatives of the variables which is typical of BT's. (Estabrook has already published an example of this type of transformation of the KdV equation; see Eq. (23) in Ref. 10.) For the Ernst equation problem, we must generalize by letting the $A_i^{\ j}$ be functions of ζ as well.

[The reader may ask what happens if *both* q_1 and q_2 from Eq. (21), when the B^i are given by Eq. (33) and the 2×2 representation (39b), are used as arguments in Eq. (45) instead of just q. This calculation has been performed; the results are the same.]

Since there are six η_i, we would expect there to be 36 $A_i^{\ j}$—a nearly unmanageable number. However, we defined the η_i to be self- or anti-self-dual [Eq. (14)]—indeed, the current need for simplification was precisely the reason for making the definitions (13). This enables us to treat the two sets of η_i separately, so that we have only $2\times 9 = 18$ coefficients $A_i^{\ j}$. Indeed, we can go further; we assume that the fields S and R—or the independent variables—transform only among themselves (as is true in Refs. I and II), so that η_5' and η_6' are proportional to η_5 and η_6, respectively. The final assumed form is now the following, where as before we must assume the coefficients $a,b,\ldots,v$ to be functions of q and ζ:

$$\begin{aligned} \eta_1' &= a\eta_1 + b\eta_2 + c\eta_5, \quad \eta_3' = h\eta_3 + m\eta_4 + n\eta_6,\\ \eta_2' &= e\eta_1 + f\eta_2 + g\eta_5, \quad \eta_4' = p\eta_3 + r\eta_4 + t\eta_6,\\ \eta_5' &= u\eta_5, \quad \eta_6' = v\eta_6. \end{aligned} \tag{46}$$

(*Note*: t, u, v are not the earlier t, u, v.) We require for nondegeneracy that $af - be \neq 0$ and $hr - mp \neq 0$.

The η_i' are to satisfy Eqs. (17) and (18) (primed), as did the η_i. We see that the primed Eqs. (18) are satisfied automatically by virtue of the old Eqs. (18).

We remark parenthetically that we can express Eq. (46) in Neugebauer's matrix notation (II). We write column vector 1-forms as follows:

$$\psi = \begin{bmatrix} \eta_1 \\ \eta_2 \\ 2\eta_5 \end{bmatrix} = 8\begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} dx, \tag{47}$$

$$\sigma = \begin{bmatrix} \eta_3 \\ \eta_4 \\ 2\eta_6 \end{bmatrix} = 8\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} dy, \tag{48}$$

with

$$\psi' = M\psi, \quad \sigma' = N\sigma. \tag{49}$$

Then

$$M = \begin{bmatrix} a & b & \frac{1}{2}c \\ e & f & \frac{1}{2}g \\ 0 & 0 & u \end{bmatrix} \tag{50}$$

and

$$N = \begin{bmatrix} h & m & \frac{1}{2}n \\ p & r & \frac{1}{2}t \\ 0 & 0 & v \end{bmatrix}. \tag{51}$$

The solution for the coefficients is rather complicated, but a few details will be provided here to show the procedure. We first note that we expect [from (17)']

$$2d\eta_5' = \eta_5' \wedge \eta_6'. \tag{52}$$

Equation (46) now gives

$$2du \wedge \eta_5 + 2u\, d\eta_5 = uv\eta_5 \wedge \eta_6. \tag{53}$$

But $du = u_q\, dq + u_\zeta\, d\zeta$, where dq and $d\zeta$ are supplied from Eqs. (41) and (27) and subscripts indicate derivatives, so that

$$\begin{aligned} 4du \wedge \eta_5 &= u_\zeta \zeta^{-1}(\zeta^2-1)\eta_6 \wedge \eta_5 \\ &\quad + u_q\zeta^{-1}[q(q+\zeta)\eta_3 - (1+q\zeta)\eta_4 \\ &\quad + (1-q^2)\eta_6] \wedge \eta_5 \\ &= 2uv\eta_5 \wedge \eta_6 - 2u\eta_5 \wedge \eta_6, \end{aligned} \tag{54}$$

where we have used Eqs. (17) and (18). Equating coefficients, we see that $u_q = 0$, so that u is a function of ζ only, and ($' = d/d\zeta$)

$$\zeta^{-1}(\zeta^2-1)u' = 2u(1-v). \tag{55}$$

In a similar manner, $v = v(\zeta)$ and

$$\zeta(\zeta^2-1)v' = 2v(1-u). \tag{56}$$

It may be shown from the other equations that

$$uv = 1. \tag{57}$$

These three equations result in two cases:

$$u = v = 1 \tag{58}$$

or

$$u = \zeta^2, \quad v = \zeta^{-2}. \tag{59}$$

Equation (58) is the choice used in I, equivalent to choosing $S' = S$ and $R' = R$. Equation (59) is that used in II, equivalent to choosing

$$S' \pm kR' = A\,[S \pm k(R+2l\,]^{-1} \pm B, \tag{60}$$

where A and B are parameters.

The $\eta_1 \wedge \eta_3$ coefficient from the $d\eta_1'$ equation and the $\eta_2 \wedge \eta_3$ coefficient from the $d\eta_2'$ equation are, respectively,

$$a(h-p) = -\zeta^{-1}q(q+\zeta)a_q + a, \tag{61}$$

$$f(p-h) = -\zeta^{-1}q(q+\zeta)f_q - f. \tag{62}$$

Elimination of $h-p$ yields $(af)_q = 0$, so that

$$af = A(\zeta). \tag{63}$$

Similar considerations give

TABLE I. Solutions for BT's, Case I. $b = e = m = p = 0$.

·	a	c	f	g	h	n	r	t	u	v
E	1	0	1	0	1	0	1	0	1	1
J	$\zeta\beta$	0	$\zeta\beta^{-1}$	0	$\zeta^{-1}\beta^{-1}$	0	$\zeta^{-1}\beta$	0	ζ^2	ζ^{-2}
H	$-q\beta$	μ	$-q^{-1}\beta^{-1}$	$q^{-1}\nu$	$-q\beta^{-1}$	$\zeta^{-1}\nu$	$-q^{-1}\beta$	$\zeta^{-1}q^{-1}\mu$	1	1
K	$-\zeta q$	$\zeta\nu$	$-\zeta q^{-1}$	$\zeta q^{-1}\mu$	$-\zeta^{-1}q$	$\zeta^{-2}\mu$	$-\zeta^{-1}q^{-1}$	$\zeta^{-2}q^{-1}\nu$	ζ^2	ζ^{-2}

$$be = B(\zeta), \tag{64}$$

$$hr = C(\zeta), \tag{65}$$

$$mp = D(\zeta). \tag{66}$$

Further combinations show that $A = A_0 u$, $B = B_0 u$, $C = C_0 v$, and $D = D_0 v$, where $A_0,...,D_0$ are constants. Other combinations show that if $p \neq 0$, then we must have $r = 0$. Furthermore, $m \neq 0$ implies $h = 0$, $e \neq 0$ implies $f \neq 0$, and $b \neq 0$ implies $a = 0$. The nondegeneracy conditions [following Eq. (46)] play a key role here. Continuing in this vein yields four cases:

$$\begin{aligned} &\text{I. } b = e = m = p = 0,\ afhr \neq 0;\\ &\text{II. } b = e = h = r = 0,\ afmp \neq 0;\\ &\text{III. } a = f = m = p = 0,\ behr \neq 0;\\ &\text{IV. } a = f = h = r = 0,\ bemp \neq 0. \end{aligned} \tag{67}$$

The remaining work is just algebra. It is facilitated by noting that in cases I and II, for example, we get two separate equations for a_q, enabling an algebraic relation among the variables to be found. Equations (63)–(66) are very helpful. In each case the relation promised above, Eq. (57), is found to hold. We find eventually that each of the cases above produces four cases, so that we have 16 final solutions.

6. SOLUTIONS FOR COEFFICIENTS IN EQ. (46)

The solutions in the four cases given in Eq. (67) are listed in Tables I–IV, each table giving the four subcases. We denote each of the 16 solutions by a different English capital letter, listed in the left-hand column in each table. Those coefficients, from Eq. (46), which are nonzero are given across the top of each table, and their expressions, for each case, are given in the body of the table. We abbreviate certain expressions as follows [$\beta = -q_c$, Eq. (44)]:

$$\mu = 1 + \zeta q, \quad \nu = q + \zeta, \quad \beta = \mu\nu^{-1}. \tag{68}$$

Certain of these transformations may be easily identified. E is the identity. J and K are the two Neugebauer transformations, I_1 and I_2, respectively, from Ref. II. H is the Harrison transformation (from I). A is the Neugebauer–Kramer transformation[12]

$$f' = Sf^{-1}, \quad \omega' = -k^{-1}\phi, \quad \phi' = -k\omega, \tag{69}$$

denoted I by Cosgrove in III; change the sign of k for B. (Cosgrove's $I^* = I^{-1}$ and Neugebauer's S.) C is a simple sign change:

$$\omega' = -\omega, \quad \phi' = -\phi. \tag{70}$$

(Various sign conventions may make differences in the statement of some of these transformations.) It will be seen later that the other BT's are combinations of these basic ones.

The matrices M and N, as defined by Eqs. (50) and (51), clearly may be constructed for any of these transformations.

7. GROUP STRUCTURE OF THE SET OF BÄCKLUND TRANSFORMATIONS

The set of 16 Bäcklund transformations, given above, forms a group, with group composition defined as successive transformation. Several obvious subgroups exist. (Some of this material was given by Neugebauer and Cosgrove in II and III; the current treatment presents a more unified, if brief, view.)

The method of successive transformation, to form the group composition, needs to be carefully defined. We first require that the parameter l, occurring in ζ [Eq. (26)], keep the same value from transformation to transformation. This group structure is to be distinguished from the group struc-

TABLE II. Solutions for BT's, Case II. $b = e = h = r = 0$.

·	a	c	f	g	m	n	p	t	u	v
A	-1	1	-1	1	-1	1	-1	1	1	1
M	ζq	$-\zeta q$	ζq^{-1}	$-\zeta q^{-1}$	$\zeta^{-1}q^{-1}$	$-\zeta^{-1}q^{-1}$	$\zeta^{-1}q$	$-\zeta^{-1}q$	ζ^2	ζ^{-2}
L	$q\beta$	$-\zeta q$	$q^{-1}\beta^{-1}$	$-\zeta q^{-1}$	$q^{-1}\beta$	$-\zeta^{-1}q^{-1}$	$q\beta^{-1}$	$-\zeta^{-1}q$	1	1
N	$-\zeta\beta$	ζ^2	$-\zeta\beta^{-1}$	ζ^2	$-\zeta^{-1}\beta$	ζ^{-2}	$-\zeta^{-1}\beta^{-1}$	ζ^{-2}	ζ^2	ζ^{-2}

TABLE III. Solutions for BT's, Case III. $a = f = m = p = 0$.

	b	c	e	g	h	n	r	t	u	v
B	-1	1	-1	1	-1	1	-1	1	1	1
U	ζq^{-1}	$-\zeta q^{-1}$	ζq	$-\zeta q$	$\zeta^{-1}q$	$-\zeta^{-1}q$	$\zeta^{-1}q^{-1}$	$-\zeta^{-1}q^{-1}$	ζ^2	ζ^{-2}
T	$q^{-1}\beta^{-1}$	$-\zeta q^{-1}$	$q\beta$	$-\zeta q$	$q\beta^{-1}$	$-\zeta^{-1}q$	$q^{-1}\beta$	$-\zeta^{-1}q^{-1}$	1	1
V	$-\zeta\beta^{-1}$	ζ^2	$-\zeta\beta$	ζ^2	$-\zeta^{-1}\beta^{-1}$	ζ^{-2}	$-\zeta^{-1}\beta$	ζ^{-2}	ζ^2	ζ^{-2}

ture discussed in III, Eq. (4.13) and following, in which all of the group elements are the same type of transformation, but with differing parameters. Second, we point out that, after one transformation, with given ζ and q, has been performed, there is no reason to expect ζ and q for the second transformation to be the same; in general, they will be different. (For example, ζ may be replaced by ζ^{-1}.) How do we determine these new quantities, which we denote by ζ' and q'? We outline a method as follows.

From Eqs. (27), (41), (47), and (48), we see that we can write

$$d\zeta = j\psi + s\sigma, \tag{71}$$

$$dq = w\psi + z\sigma, \tag{72}$$

where j, s, w, and z are row vectors:

$$j = \tfrac{1}{8}\zeta(\zeta^2 - 1)[0 \quad 0 \quad 1], \tag{73}$$

$$S = \tfrac{1}{8}\zeta^{-1}(\zeta^2 - 1)[0 \quad 0 \quad 1], \tag{74}$$

$$w = \tfrac{1}{4}[q\mu \quad -\nu \quad \tfrac{1}{2}\zeta(1 - q^2)], \tag{75}$$

$$z = \tfrac{1}{4}\zeta^{-1}[q\nu \quad -\mu \quad \tfrac{1}{2}(1 - q^2)], \tag{76}$$

where μ and ν are defined in Eq. (68). We then note that

$$d\zeta' = j'\psi' + s'\sigma' = j'M\psi + s'N\sigma \tag{77a}$$

and

$$dq' = w'\psi' + z'\sigma' = w'M\psi + z'N\sigma, \tag{77b}$$

where we have used Eq. (49) and where primes on j, etc., mean replacement of ζ by ζ' and q by q'. If we assume $\zeta' = F(\zeta, q)$ and $q' = G(\zeta, q)$, substitute into Eq. (77), and equate coefficients of ψ and σ, we get the row vector equations:

$$j'M = F_\zeta j + F_q w, \tag{78a}$$

$$s'N = F_\zeta s + F_q z, \tag{78b}$$

$$w'M = G_\zeta j + G_q w, \tag{79a}$$

$$z'N = G_\zeta s + G_q z. \tag{79b}$$

When solved for F and G, these equations give ζ' and q', which are to be used in the second transformations of any group composition. The matrices M and N of Eq. (49), used above, are those associated with the first transformation.

Equations (78) are easy to solve. We note from Eqs. (50), (51), (73), and (74) that the left-hand sides of Eq. (78) have only a third component. This implies, since w and z have nonzero first and second components, that $F_q = 0$. The remaining equations give ($\zeta' = F$)

$$F(F^2 - 1)u = \zeta(\zeta^2 - 1)F_\zeta, \tag{80a}$$

$$F^{-1}(F^2 - 1)v = \zeta^{-1}(\zeta^2 - 1)F_\zeta, \tag{80b}$$

where we have used $u = M_{33}$ and $v = N_{33}$. It is easily seen that if $u = v = 1$, then $F = \zeta$; if $u = \zeta^2$ and $v = \zeta^{-2}$, then $F = \zeta^{-1}$.

Equations (79) are

$$[G(1 + FG) \quad -(F + G) \quad \tfrac{1}{2}F(1 - G^2)]M = \tfrac{1}{2}\zeta(\zeta^2 - 1)G_\zeta[0 \quad 0 \quad 1] + G_q[q(1 + q\zeta) \quad -(q + \zeta) \quad \tfrac{1}{2}\zeta(1 - q^2)] \tag{81}$$

and

$$F^{-1}[G(F + G) \quad -(1 + FG) \quad \tfrac{1}{2}(1 - G^2)]N = \tfrac{1}{2}\zeta^{-1}(\zeta^2 - 1)G_\zeta[0 \quad 0 \quad 1] + \zeta^{-1}[q(q + \zeta) \quad -(1 + q\zeta) \quad \tfrac{1}{2}(1 - q^2)]. \tag{82}$$

We solve them by choosing one of the 16 transformations, finding M and N and substituting for them, and solving the separate equations for G. All equations are found to be self consistent. We summarize the results in Table V.

TABLE IV. Solutions for BT's, Case IV. $a = f = h = r = 0$.

·	b	c	e	g	m	n	p	t	u	v
C	1	0	1	0	1	0	1	0	1	1
X	$\zeta\beta^{-1}$	0	$\zeta\beta$	0	$\zeta^{-1}\beta$	0	$\zeta^{-1}\beta^{-1}$	0	ζ^2	ζ^{-2}
W	$-q^{-1}\beta^{-1}$	$q^{-1}\nu$	$-q\beta$	μ	$-q^{-1}\beta$	$\zeta^{-1}q^{-1}\mu$	$-q\beta^{-1}$	$\zeta^{-1}\nu$	1	1
Y	$-\zeta q^{-1}$	$\zeta q^{-1}\mu$	$-\zeta q$	$\zeta\nu$	$-\zeta^{-1}q^{-1}$	$\zeta^{-2}q^{-1}\nu$	$-\zeta^{-1}q$	$\zeta^{-2}\mu$	ζ^2	ζ^{-2}

TABLE V. Transformation of q and ζ for various BT's.

Transformation	ζ'	q'
E	ζ	q
J	ζ^{-1}	q
H	ζ	q^{-1}
K	ζ^{-1}	q^{-1}
A	ζ	$-\beta$
M	ζ^{-1}	$-\beta$
L	ζ	$-\beta^{-1}$
N	ζ^{-1}	$-\beta^{-1}$
B	ζ	$-\beta^{-1}$
U	ζ^{-1}	$-\beta^{-1}$
T	ζ	$-\beta$
V	ζ^{-1}	$-\beta$
C	ζ	q^{-1}
X	ζ^{-1}	q^{-1}
W	ζ	q
Y	ζ^{-1}	q

TABLE VII. The 16 BT's in terms of generators.

E	A	C	AC
MA	M	MAC	MC
M^2	M^2A	M^2C	M^2AC
M^3A	M^3	M^3AC	M^3C

We are now in a position to explore the composition of successive BT's. If we wish to explore the effect of BT P followed by BT Q, we first construct the matrices M_P, M_Q, N_P, and N_Q. If we apply M_P to ψ, we get $\psi' = M_P\psi$. Before applying M_Q, we must recognize that ζ and q will now be different; so we look up ζ' and q' in Table V for BT P and replace ζ and q in M_Q by ζ' and q'. Call the new matrix M'_Q. We then have $\psi'' = M'_Q\psi' = M'_Q M_P\psi$. Thus the matrix for the composition is $M'_Q M_P$. We repeat for N_P and N_Q. In this way we can demonstrate that we always get one of the other sets (M, N), so that composition is closed.

It is now a simple matter to construct the multiplication table, Table VI, for BT's. We see that the four cases I, II, IV, and III (the order listed in Table VI) transform separately; the subcases in each case transform among themselves.

If we attempt to analyze the group by means of basic generators, we note that E is of order 1; M, N, U, V are of order 4; all other elements are of order 2. It is convenient to choose E, A, C, and M as generators. Then we have $A^2 = C^2 = E$, $M^4 = E$, $AM = M^3A$, $AM^3 = MA$, $AM^2 = M^2A$, $CA = AC$, and $CM = MC$, as the defining equations for the group, which may be denoted[13] $G_{16}{}^9 = G_8{}^4 \otimes G_2{}^1$. We identify the other elements in terms of the generators: $J = MA$, $K = M^3A$, $H = M^2$, $N = M^3$, $L = M^2A$, $T = M^2AC$, $U = MC$, $V = M^3C$, $B = AC$, $X = MAC$, $W = M^2C$, and $Y = M^3AC$. We write the 16 transformations in order, in a square array, in Table VII, in terms of the generators. We see that the first column is composed of the identity E, the two Neugebauer transformations (MA, M^3A), and the Harrison transformation (M^2). The succeeding columns are simply the first column postmultiplied by A (the Neugebauer–Kramer transformation), C (the sign change), and AC ($= B$). Thus it is clear that the basic three BT's and their transformations are the only nontrivial BT's which can be derived with the given assumptions.

It is more transparent, however, to write the 16 BT's as in Table VIII,[14] which clearly shows the effect of B, A, and $C = BA$ on the three basic BT's, J, H, and K.

We use, in the following discussion, Neugebauer's symbols γ and α (paper II). These are defined in Eqs. (42) and (43) above.

Neugebauer states that $I_2 = SI_1S$; in our notation, this is $K = BJB$. We see this easily from Table VI ($JB = U$, $BU = K$; or $BJ = V$, $VB = K$.) This procedure is consistent with Cosgrove's Eqs. (4.8)–(4.12). We consider B acting first; then we multiply by J to get JB, but replacing q in J with $-\beta^{-1}$ (see Table V, entry for B). ζ remains the same. If

TABLE VI. Composition (multiplication) of BT's. The BT listed on the top row is the first transformation, followed by that in the left column. (Thus, for example, $AJ = N$.)

	E	J	H	K	A	M	L	N	C	X	W	Y	B	U	T	V
E	E	J	H	K	A	M	L	N	C	X	W	Y	B	U	T	V
J	J	E	K	H	M	A	N	L	X	C	Y	W	U	B	V	T
H	H	K	E	J	L	N	A	M	W	Y	C	X	T	V	B	U
K	K	H	J	E	N	L	M	A	Y	W	X	C	V	T	U	B
A	A	N	L	M	E	K	H	J	B	V	T	U	C	Y	W	X
M	M	L	N	A	J	H	K	E	U	T	V	B	X	W	Y	C
L	L	M	A	N	H	J	E	K	T	U	B	V	W	X	C	Y
N	N	A	M	L	K	E	J	H	V	B	U	T	Y	C	X	W
C	C	X	W	Y	B	U	T	V	E	J	H	K	A	M	L	N
X	X	C	Y	W	U	B	V	T	J	E	K	H	M	A	N	L
W	W	Y	C	X	T	V	B	U	H	K	E	J	L	N	A	M
Y	Y	W	X	C	V	T	U	B	K	H	J	E	N	L	M	A
B	B	V	T	U	C	Y	W	X	A	N	L	M	E	K	H	J
U	U	T	V	B	X	W	Y	C	M	L	N	A	J	H	K	E
T	T	U	B	V	W	X	C	Y	L	M	A	N	H	J	E	K
V	V	B	U	T	Y	C	X	W	N	A	M	L	K	E	J	H

TABLE VIII. The 16 BT's in terms of J, H, K, B, A.

E	B	A	C		E	B	A	BA
J	V	N	X		J	BJ	AJ	BAJ
				=				
H	T	L	W		H	BH	AH	BAH
K	U	M	Y		K	BK	AK	BAK

$q' = -\beta^{-1}$, we get

$$\begin{aligned}\alpha' &= \zeta'\beta' = \zeta\beta' = \zeta(1+q'\zeta)(q'+\zeta)^{-1}\\ &= \zeta(1-\zeta\beta^{-1})(\zeta-\beta^{-1})^{-1}\\ &= \zeta(\beta-\zeta)(\zeta\beta-1)^{-1} = -\zeta q^{-1};\end{aligned} \tag{83}$$

α' can be shown to be the α which occurs in Cosgrove's Eqs. (4.11) and (4.12). Since the third factor, another B, has constant entries, there is no further replacement of ζ and q. Matrix multiplication now gives the proper matrices for K.

Cosgrove notes, in his Eq. (4.19), that a Harrison transformation may be given in terms of I_2 and I_1 by

$$H(\alpha,\gamma) = I_2((\alpha-\gamma)/\gamma(\gamma-1),\gamma^{-1})I_1(\alpha,\gamma). \tag{84}$$

To understand this, we note that the first argument in I_2 is the expression to be substituted for α in Eq. (4.12) for I_2 in III. But this α is the $\alpha' = -\zeta q^{-1}$ given in our Eq. (83) above. Thus we see that the proper procedure for proving Eq. (84) is: (1) Replace α in Cosgrove's Eq. (4.12) by $-\zeta q^{-1}$; this gives, in fact, exactly the matrices M and N found for $K\ (=I_2)$ in Table I above; (2) now make the change $q \to q$, $\zeta \to \zeta^{-1}$ in these matrices [this is the change prescribed in Table V when there is an initial transformation $J\ (=I_1)$]; (3) perform the M, N matrix multiplications for I_2I_1. One obtains exactly H. In fact, we see this from Table VI: $H = KJ$ $(=JK)$, or $H = BJBJ = JBJB$. As noted above, the transformations E, H, J, K form a subgroup.

8. ERNST–MAXWELL EQUATIONS

The Ernst–Maxwell equations[15]

$$f(\nabla^2 G + S^{-1}\nabla S\cdot \Delta G) = \nabla G\cdot(\nabla G - 2\lambda \overline{H}\nabla H), \tag{85a}$$

$$f(\nabla^2 H + S^{-1}\nabla S\cdot \nabla H) = \nabla H\cdot(\nabla G - 2\lambda \overline{H}\nabla H), \tag{85b}$$

$$\nabla^2 S = 0, \tag{85c}$$

where $G = f + i\phi + \lambda H\overline{H}$ and H is a complex electromagnetic potential, may be written in a manner similar to Eqs. (3'), (4'), and (5'):

$$d(*dS) = 0, \tag{86a}$$

$$d(Sf^{-1}*df) + Sf^{-2}\Omega\wedge *\Omega + \lambda Sf^{-1}(dH\wedge *d\overline{H} + d\overline{H}\wedge *dH) = 0, \tag{86b}$$

$$d(S*\Omega) - 2Sf^{-1}\,df\wedge *\Omega = 0, \tag{86c}$$

$$d(S*dH) - Sf^{-1}(df + i\Omega)\wedge *dH = 0, \tag{86d}$$

where

$$\Omega = d\phi + i\lambda(\overline{H}dH - Hd\overline{H}). \tag{86e}$$

We define

$$\begin{aligned}
\xi_1 &= f^{-1}\Omega,\\
\xi_2 &= S^{-1}f\,d\omega,\\
\xi_3 &= S^{-1}(d\eta + \omega\,d\phi + L\,d\overline{H} + \overline{L}\,dH),\\
\xi_4 &= f^{-1}\,df,\\
\xi_5 &= S^{-1}\,dS,\\
\xi_6 &= S^{-1}\,dR,\\
\xi_7 &= S^{-1}f^{1/2}(dK - i\omega\,dH),\\
\xi_8 &= f^{-1/2}\,dH,\\
\xi_9 &= \overline{\xi}_7 = S^{-1}f^{1/2}(d\overline{K} + i\omega\,d\overline{H}),\\
\xi_{10} &= \overline{\xi}_8 = f^{-1/2}\,d\overline{H},
\end{aligned} \tag{87}$$

where

$$L = \lambda(K - i\omega H) \tag{88}$$

and

$$dK = Sf^{-1}*dH + i\omega\,dH \tag{89}$$

defines a new potential K. Then we can write a CC ideal for the Ernst–Maxwell equations. The exterior derivatives are

$$\begin{aligned}
&d\xi_1 = \xi_1\wedge\xi_4 - 2i\lambda\xi_8\wedge\xi_{10},\quad d\xi_2 = \xi_2\wedge(\xi_5-\xi_4),\\
&d\xi_3 = \xi_3\wedge\xi_5 - \xi_1\wedge\xi_2 + \lambda(\xi_7\wedge\xi_{10} - \xi_8\wedge\xi_9),\\
&d\xi_4 = 0,\quad d\xi_5 = 0,\quad d\xi_6 = -\xi_5\wedge\xi_6,\\
&d\xi_7 = \xi_7\wedge(\xi_5 - \tfrac12\xi_4) - i\xi_2\wedge\xi_8,\\
&d\xi_8 = -\tfrac12\xi_4\wedge\xi_8,\quad d\xi_9 = \xi_9\wedge(\xi_5-\tfrac12\xi_4) + i\xi_2\wedge\xi_{10},\\
&d\xi_{10} = \tfrac12\xi_4\wedge\xi_{10},
\end{aligned} \tag{90}$$

while the remaining annulled 2-forms are

$$\begin{aligned}
&\xi_3\wedge\xi_1 - \xi_2\wedge\xi_4 = 0, && \xi_3\wedge\xi_8 - \xi_7\wedge\xi_4 = 0,\\
&\xi_3\wedge\xi_2 - \lambda\xi_1\wedge\xi_4 = 0, && \xi_3\wedge\xi_7 - \lambda\xi_8\wedge\xi_4 = 0,\\
&\xi_5\wedge\xi_2 - \xi_1\wedge\xi_6 = 0, && \xi_3\wedge\xi_{10} - \xi_9\wedge\xi_4 = 0,\\
&\xi_5\wedge\xi_1 - \lambda\xi_2\wedge\xi_6 = 0, && \xi_3\wedge\xi_9 - \lambda\xi_{10}\wedge\xi_4 = 0,\\
&\xi_5\wedge\xi_3 - \xi_4\wedge\xi_6 = 0, && \xi_5\wedge\xi_8 - \lambda\xi_7\wedge\xi_6 = 0,\\
&\xi_4\wedge\xi_5 + \lambda\xi_3\wedge\xi_6 = 0, && \xi_5\wedge\xi_7 - \xi_8\wedge\xi_6 = 0,\\
&\xi_1\wedge\xi_8 - \lambda\xi_7\wedge\xi_2 = 0, && \xi_5\wedge\xi_{10} - \lambda\xi_9\wedge\xi_6 = 0,\\
&\xi_1\wedge\xi_7 - \xi_8\wedge\xi_2 = 0, && \xi_5\wedge\xi_9 - \xi_{10}\wedge\xi_6 = 0,\\
&\xi_1\wedge\xi_{10} - \lambda\xi_9\wedge\xi_2 = 0, && \xi_7\wedge\xi_{10} - \xi_9\wedge\xi_8 = 0,\\
&\xi_1\wedge\xi_9 - \xi_{10}\wedge\xi_2 = 0, && \xi_7\wedge\xi_9 - \lambda\xi_{10}\wedge\xi_8 = 0.
\end{aligned} \tag{91}$$

We write a prolongation form in terms of the ξ_i, for variety, instead of the η_i as before:

$$\kappa = -dq + (C^i\xi_i)q, \tag{92}$$

where κ and q are column vectors and C^i are matrix functions of ζ. The prolongation equation becomes

$$C^i d\xi_i - [C^i, C^j]\xi_i\wedge\xi_j + \tfrac14\zeta^{-1}(\zeta^2-1)C''\times[(\zeta^2+1)\xi_5 + k(\zeta^2-1)\xi_6]\wedge\xi_i = 0. \tag{93}$$

We set $C^5 = C^6 = 0$ as before. Expansion of Eq. (93) yields a set of differential equations for the C^i plus a set of equations for their commutation relations. Solution of the differential equations yields

$$C^1=\tfrac12\zeta^{-1}(\zeta^2+1)\nu+\theta,\quad C^3=\tfrac12 k\zeta^{-1}(\zeta^2-1)\tau,$$
$$C^2=\tfrac12 k\zeta^{-1}(\zeta^2-1)\nu,\quad C^4=\tfrac12\zeta^{-1}(\zeta^2+1)\tau+\phi,$$
$$C^7=\tfrac12 k\zeta^{-1}(\zeta^2-1)\alpha,\quad C^9=\tfrac12 k\zeta^{-1}(\zeta^2-1)\beta,$$
$$C^8=\tfrac12\zeta^{-1}(\zeta^2+1)\alpha+\gamma,\quad C^{10}=\tfrac12\zeta^{-1}(\zeta^2+1)\beta+\delta, \tag{94}$$

where $\nu,\theta,\tau,\phi,\alpha,\beta,\gamma$, and δ are constant matrices. τ,θ, and ν play the same role as their counterparts, defined above following Eq. (38). These matrices satisfy

$$[\nu,\theta]=\tau,\quad \nu=[\theta,\tau],\quad [\tau,\gamma]=-\tfrac12\alpha,$$
$$\theta=[\theta,\phi]+[\nu,\tau],\quad [\tau,\alpha]+[\phi,\gamma]=-\tfrac12\gamma,$$
$$[\nu,\phi]=[\tau,\phi]=0,\quad [\beta,\gamma]=\lambda(\tau+i\nu),$$
$$[\nu,\gamma]=[\alpha,\theta]=-\tfrac12 i\alpha,\quad [\alpha,\delta]=\lambda(\tau-i\nu),$$
$$0=[\nu,\alpha]+[\theta,\gamma],\quad [\alpha,\beta]+[\gamma,\delta]=-2i\gamma\theta,$$
$$[\alpha,\gamma]=[\alpha,\phi]=0. \tag{95}$$

These equations can be satisfied only by a representation of size 3×3 or greater (reminiscent of the 3×3 matrices required for Ernst–Maxwell in the Ernst–Hauser[16] approach to this problem.) If we make the ansatz that the upper left corners of τ,ν, and θ are given by the spin matrices as before, we find *two* representations, as follows (subscripts indicate the representation label; k_1 and k_2 are constants):

$$\phi_1=\phi_2=0,\quad \tau_1=\tau_2=-\tfrac12\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix},$$
$$\nu_1=\nu_2=\tfrac12\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix},\quad \theta_1=\theta_2=\tfrac12\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix},$$
$$\alpha_1=k_1\begin{bmatrix}0&0&0\\0&0&0\\1&i&0\end{bmatrix},\quad \beta_1=\frac{\lambda}{2k_1}\begin{bmatrix}0&0&1\\0&0&-i\\0&0&0\end{bmatrix}, \tag{96}$$
$$\gamma_1=k_1\begin{bmatrix}0&0&0\\0&0&0\\-1&i&0\end{bmatrix},\quad \delta_1=\frac{\lambda}{2k_1}\begin{bmatrix}0&0&1\\0&0&i\\0&0&0\end{bmatrix},$$
$$\alpha_2=k_2\begin{bmatrix}0&0&1\\0&0&i\\0&0&0\end{bmatrix},\quad \beta_2=\frac{\lambda}{2k_2}\begin{bmatrix}0&0&0\\0&0&0\\1&-i&0\end{bmatrix},$$
$$\gamma_2=k_2\begin{bmatrix}0&0&1\\0&0&-i\\0&0&0\end{bmatrix},\quad \delta_2=-\frac{\lambda}{2k_2}\begin{bmatrix}0&0&0\\0&0&0\\1&i&0\end{bmatrix}.$$

We now define the η_i, in the same way as before (the number in parentheses corresponds to the lower sign):

$$\eta_{1(2)}=\xi_4\pm i\xi_1+k\xi_3\pm ik\xi_2,$$
$$\eta_{3(4)}=\xi_4\mp i\xi_1-k\xi_3\pm ik\xi_2, \tag{97}$$
$$\eta_{5(6)}=\xi_5\pm k\xi_6,\quad \eta_{7(9)}=\xi_{10}\pm k\xi_9,\quad \eta_{8(10)}=\xi_8\pm k\xi_7.$$

The η_i divide into self-dual η_i ($i=1,2,5,7,8$; ${}^*\eta_i=k^{-1}\eta_i$) and anti-self-dual η_i ($i=3,4,6,9,10$; ${}^*\eta_i=-k^{-1}\eta_i$). We can show that $\eta_k\wedge\eta_l=0$ if η_k and η_l are either both self-dual or both anti-self-dual. Equations (90) and (97) enable us to derive expressions for the $d\eta_k$.

Kramer and Neugebauer[17] (denoted by IV) define a set of quantities $A_i,\ldots,E_i$, $i=1,2$. They are related to the η_i by (remember $\lambda=-1$ in IV)

$$\eta_1=4A_1\,dx,\quad \eta_6=2C_2\,dy,$$
$$\eta_2=4B_1\,dx,\quad \eta_7=-2iD_1\,dx,$$
$$\eta_3=4B_2\,dy,\quad \eta_8=-2iE_1\,dx, \tag{98}$$
$$\eta_4=4A_2\,dy,\quad \eta_9=-2iD_2\,dy,$$
$$\eta_5=2C_1\,dx,\quad \eta_{10}=-2iE_2\,dy.$$

Expansion of Eq. (92) ($\kappa=0$) for the first representation in Eq. (96) yields, for the pseudopotentials $u(v)=q_1\pm iq_2$, $w=q_3$:

$$du=-\tfrac18(\eta_1-\eta_2-\eta_3+\eta_4)u-\tfrac14(\zeta\eta_2+\zeta^{-1}\eta_3)v+\tfrac12 ik_1^{-1}\lambda(\zeta\eta_7+\zeta^{-1}\eta_9)w,$$
$$dv=\tfrac18(\eta_1-\eta_2-\eta_3+\eta_4)v-\tfrac14(\zeta\eta_1+\zeta^{-1}\eta_4)u+\tfrac12 k_1^{-1}\lambda(\eta_7+\eta_9)w, \tag{99}$$
$$dw=\tfrac12 k_1[(\zeta\eta_8+\zeta^{-1}\eta_{10})u-(\eta_8+\eta_{10})v].$$

Similar equations result for the second representation (with analogous pseudopotentials u',v',w'). Comparison with IV shows that λ in that paper is our ζ, and we also have the following relations for the quantities $\psi,\cdots$ in IV:

$$\psi=\tfrac12 iak_2^{-1}\sqrt f\,u',\quad \tilde\psi=-ibk_1\sqrt f\,v,$$
$$\chi=-\tfrac12 iak_2^{-1}\sqrt f\,v',\quad \tilde\chi=ibk_1\sqrt f\,u, \tag{100}$$
$$\sigma=a\sqrt f\,w',\quad \tilde\sigma=b\sqrt f\,w,$$

where a and b are arbitrary constants. In general (whether $\lambda=\pm1$, $k=1$ or i, and whether or not complex conjugation is taken on k if $k=i$), we have

$$\bar u=cv',\quad \bar v=cu',\quad \bar w=2\lambda ck_2\bar k_1 w', \tag{101}$$

where c is an arbitrary constant.

It is convenient to consider five variables made of the ratios of these variables (uw^{-1}, vw^{-1}, $\bar u\bar w^{-1}$, $\bar v\bar w^{-1}$, $\bar w w^{-1}$). The function ζ makes a sixth variable. Denote the variables as α_A, $A=1$–6. The exterior derivatives of the α_A are linear combinations of the η_i, the coefficients being functions of the α_A:

$$d\alpha_A=A_A{}^K(\alpha_B)\eta_k. \tag{102}$$

An arbitrary function, $f(\alpha_A)$, has exterior derivative

$$df=(\partial f/\partial\alpha_A)d\alpha_A=(\partial f/\partial\alpha_A)A_A{}^k\eta_k=B^k(f)\eta_k, \tag{103}$$

where the $B^k=A_A{}^k\partial/\partial\alpha_A$ are linear operators.

To search for BT's, we assume a linear transformation like Eq. (45), as before—except that now, instead of 14 functions of two variables, we have 42 functions of six variables!

$$\eta'_k=C_k{}^l(\alpha_A)\eta_l \tag{104}$$

where the families of self-dual and anti-self-dual η_i transform among themselves. The problem is made tractable only by the shorthand use of the B_k, which operate on the $C_k{}^l$. Even then, progress can be made only by making an ansatz, typically that several of the $C_k{}^l$ vanish. For example, one may try $\eta_1' = F\eta_1$, or $\eta_1' = F\eta_1 + G\eta_5$. It appears likely that only two or three coefficients, in the expression for any η_k', are nonzero. However, even this approach does not work until we make the additional ansatz, made in IV, that

$$\psi\tilde{\psi} + \sigma\tilde{\sigma} - \chi\tilde{\chi} = 0. \tag{105}$$

This reduces the number of independent variables from six to five. Fortunately, the equations—expressed in terms of the operators B^k—keep the same form. Only the B^k change.

The ansatz which gives results is: In η_1', include either η_1 or η_2, but not both; either η_7 or η_8, but not both; and η_5. It is clear that we have four possible choices (η_1 and η_7, η_1 and η_8, etc). Setting certain of the coefficients to zero then induces other coefficients to be zero. One continues in a similar manner. The problem becomes possible to solve, though still very complicated.

Solutions found to date after pursuit of this approach all reduce to the single BT reported by Kramer and Neugebauer in IV.

The possible relation of this BT approach to electrovac space-times to that of Cosgrove[18] is yet to be explored.

9. FINAL REMARKS

It is interesting that the Ernst equation admits three BT's found by the method in this paper, while the Ernst–Maxwell equations admit only one (at least, known to date)! It may be that the extra equations force enough additional structure to preclude more than one BT.

The MWE approach to finding BT's appears to have promise. It suggests that it would be desirable to find a canonical set of 1-forms for any equation, or set of equations, such that a CC ideal of forms can be constructed. It has been demonstrated, for example, that CC ideals exist for the sine–Gordon and Korteweg–de Vries equations and that they can be used to derive the associated BT's.[10,19]

It is possible to formulate more complicated problems in elegant ways by using differential forms. As an example, the vacuum Einstein equations, with only one nonnull Killing vector, may be cast into an ideal of 1-, 2-, and 3-forms with most coefficients constant.[8,20] Investigation into an inverse scattering or BT formulation for this problem, using techniques due to Morris,[21] has been done—without definite results to date.[20] The formulation of the full vacuum equations in differential forms, due to Israel,[22] might be suitable for application of these methods.

The MWE approach also serves to suggest new generalizations. If the equations being studied admit similarity or invariant variables μ_α, their exterior derivatives can be written in the form

$$d\mu_\alpha = F_\alpha{}^k(\mu_\beta)\eta_k. \tag{106}$$

ζ is such a variable for the Ernst or Ernst–Maxwell equations, as noted above, and it was convenient (even necessary!) to include it as a variable in the coefficients in Eqs. (28) and (45). But others may exist. Such a generalization is currently being tried by the author and is already yielding new BT's to the (vacuum) Ernst equation,[8] although it is not yet known whether these BT's provide new solutions to the equation. These results will be reported later as they are completed.

Note added in proof: It now appears that these new BT's are combinations of Ehlers transformations and known BT's.

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