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Some aspects of the isogroup of the self-dual Yang–Mills system

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A generalized isovector formalism is used to derive the isovectors and isogroup of the self-dual Yang–Mills (SDYM) equation in the so-called J formulation. In particular, the infinitesimal “hidden symmetry” transformation, a linear system, and a well-known Bäcklund transformation of the SDYM equation are derived in the process. Thus symmetry and integrability aspects of the SDYM system appear in natural relationship to each other within the framework of the isovector approach.

I. INTRODUCTION

In a recent paper¹ the authors discussed the application of isovector techniques^{2,3} to systems of partial differential equations corresponding to exterior equations for vector-valued (and, in particular, matrix-valued) differential forms. It was seen that the application of the Lie derivative operator on vector-valued one-forms presents some technical difficulties, and for this reason an *internal exterior derivative* (i.e., an exterior derivative that acts on the fields but not on the variables of the solution manifold) was introduced by the formula

$$\bar{d}F(x^\mu, \psi^i) \equiv dF - \partial_\mu F dx^\mu, \quad (1.1)$$

where F is any function of the scalar variables x^μ of the solution manifold and the vector-valued fields ψ^i . If the system of partial differential equations (PDE's) is of order 2 or higher, the variables ψ^i will comprise the dependent variables u^a of the PDE's and the derivatives, up to a certain degree, of the u^a with respect to the x^μ . Given that, in the absence of specific restrictions on the exterior differential forms that represent the system, the variables ψ^i are considered independent of each other (and of the x^μ), we conclude that the problem can be naturally formulated on a jet space with “mixed” (i.e., both scalar- and vector-valued) coordinates.

In the present paper the formalism developed in Ref. 1 is applied to the self-dual Yang–Mills (SDYM) equation in the so-called J formulation.⁴ It is seen that the isovector method provides a natural framework for the unification of such distinct concepts as symmetry and integrability. The independence of the coordinates of the underlying jetlike space is important in this context, as the reader will realize. In Sec. II we calculate the isovectors of the SDYM system. These vector fields can be used to construct infinitesimal symmetries (both geometrical and internal) of the system, as discussed in Ref. 1. The above-mentioned independence of coordinates is used in Sec. III to rewrite certain symmetries in a form equivalent to the parametric infinitesimal transformation introduced in Ref. 5. (This transformation is related to the so-called hidden symmetry of the SDYM field.⁶) Remarkably, the process also yields a pair of linear “inverse scattering” equations, the integrability of which is equivalent to the SDYM equation, and the parameter of which is identical to that of the infinitesimal transformation mentioned above. Finally, the results of Secs. I–III are used

in Sec. IV to derive Bäcklund transformations for the SDYM system. In particular, the process gives the parametric Bäcklund transformation proposed in Ref. 7.

II. ISOVECTORS OF THE SDYM SYSTEM

The SDYM equation in the J formulation is written as^{4–7}

$$\partial_{\bar{y}}(J^{-1} \partial_y J) + \partial_{\bar{z}}(J^{-1} \partial_z J) = 0. \quad (2.1)$$

The complex coordinates $y, z, \bar{y}$, and $\bar{z}$ are related to the coordinates x_1, x_2, x_3 , and x_4 of complexified Euclidean space by

$$\begin{aligned} 2^{1/2}y &= x_1 + ix_2, & 2^{1/2}z &= x_3 - ix_4, \\ 2^{1/2}\bar{y} &= x_1 - ix_2, & 2^{1/2}\bar{z} &= x_3 + ix_4. \end{aligned} \quad (2.2)$$

[Note that the pairs $(y, \bar{y})$ and $(z, \bar{z})$ involve elements that are complex-conjugately related in *real* Euclidean space.] For our purposes, J is assumed to be a nonsingular element of the algebra $\text{gl}(N, \mathbb{C})$ in its defining representation.

Equation (2.1) can be rewritten as a set of first-order PDE's:

$$B_{\bar{y}}^1 + B_{\bar{z}}^2 = 0, \quad B^1 = J^{-1} J_y, \quad B^2 = J^{-1} J_z, \quad (2.3)$$

where a standard notation for partial derivatives has been used. We are thus led, in the spirit of Ref. 1, to define the following set of four-forms in seven variables:

$$\begin{aligned} \gamma_1 &= dy dz dB^1 d\bar{z} + dy dz d\bar{y} dB^2, \\ \gamma_2 &= dJ dz d\bar{y} d\bar{z} - JB^1 dy dz d\bar{y} d\bar{z}, \\ \gamma_3 &= dy dJ d\bar{y} d\bar{z} - JB^2 dy dz d\bar{y} d\bar{z}. \end{aligned} \quad (2.4)$$

It is easily seen that the $d\gamma_k$ are in the ideal of the γ_k ; thus this ideal is closed.

We now proceed to find the isovectors of the system. For this purpose we must expand the Lie derivative of each γ_i into a “linear” combination of all three γ_k . The expansion must be made consistently with the requirement that the Lie derivative preserve the tensorial character of each γ_i separately.

Now, from Eqs. (2.4) it can be seen that the four-forms γ_k have values in $\text{gl}(N, \mathbb{C})$, which is closed under both addition and multiplication. This observation suggests the following expansion:

$$\mathbb{L}\gamma_i = b_i^k \gamma_k + \Lambda_i^k \gamma_k + \gamma_k M_i^k, \quad (2.5)$$

where the b_i^k are scalars, whereas the zero-forms Λ_i^k and M_i^k have values in $\text{gl}(N, C)$.

The vector field V is defined on a jetlike space with “coordinates” $y, z, \bar{y}, \bar{z}, J, B^1$, and B^2 . As argued in Ref. 1, V will have a formal representation,

$$V = D^1 \frac{\partial}{\partial y} + D^2 \frac{\partial}{\partial z} + D^3 \frac{\partial}{\partial \bar{y}} + D^4 \frac{\partial}{\partial \bar{z}} + G \frac{\partial}{\partial J} + A^1 \frac{\partial}{\partial B^1} + A^2 \frac{\partial}{\partial B^2}, \quad (2.6)$$

where the $D^1, \dots, D^4$ are assumed to be scalar functions of $y, z, \bar{y}, \bar{z}$, while the G, A^1, A^2 are $\text{gl}(N, C)$ -valued functions of the above four variables and J, B^1 , and B^2 . As in Ref. 1, we seek vector fields V for which the coefficients of expansion in Eq. (2.5) depend only on $y, z, \bar{y}$, and $\bar{z}$.

Substituting Eqs. (2.4) and (2.6) into Eq. (2.5), and using Eq. (1.1) to write

$$\mathbb{L} dJ = dG = G_{,\mu} dy^\mu + \bar{d}G, \\ \mathbb{L} dB^k = dA^k = A^k_{,\mu} dy^\mu + \bar{d}A^k \quad (k = 1, 2),$$

where the y^μ ($\mu = 1, \dots, 4$) denote the $y, \dots, \bar{z}$, we obtain a set of three exterior equations for four-forms. By equating the coefficients of $dy dz d\bar{y} d\bar{z}$ on both sides of each exterior equation, the following set of PDE's is derived:

$$A^1_{\bar{y}} + A^2_{\bar{z}} = -(b^2_1 + \Lambda^2_1)JB^1 - (b^3_1 + \Lambda^3_1)JB^2 \\ - JB^1 M^2_1 - JB^2 M^3_1, \\ G_y - GB^1 - JA^1 - D^1_{,\mu} JB^1 \\ = -(b^2_2 + \Lambda^2_2)JB^1 - (b^3_2 + \Lambda^3_2)JB^2 \\ - JB^1 M^2_2 - JB^2 M^3_2, \quad (2.7) \\ G_z - GB^2 - JA^2 - D^2_{,\mu} JB^2 \\ = -(b^2_3 + \Lambda^2_3)JB^1 - (b^3_3 + \Lambda^3_3)JB^2 \\ - JB^1 M^2_3 - JB^2 M^3_3,$$

where $D^\mu \equiv D^1, \dots, D^4$.

We now put

$$A^i = \alpha^{ik}(y^\mu)B^k + \beta^i(y^\mu)J + \bar{A}^i(y^\mu, B^k, J), \\ G = \delta^k(y^\mu)B^k + \epsilon(y^\mu)J + \bar{G}(y^\mu, B^k, J), \quad (2.8)$$

where the α^{ik} , β^i , δ^k , and ϵ are scalars. Then

$$\bar{d}A^i = \alpha^{ik} dB^k + \beta^i dJ + \bar{d}\bar{A}^i, \\ \bar{d}G = \delta^k dB^k + \epsilon dJ + \bar{d}\bar{G}.$$

We substitute these expressions into the expansion of Eq. (2.5) and equate coefficients of terms that are *scalar* multiples of similar $\text{gl}(N, C)$ -valued basis four-forms. There are 12 such basis four-forms; therefore we obtain a set of 36 equations (eight of which are trivial identities). These results can be summarized as follows:

$$\beta^1 = \beta^2 = \delta^1 = \delta^2 = 0; \quad \alpha^{12} = D^3_{\bar{z}}, \quad \alpha^{21} = D^4_{\bar{y}}; \\ D^1_{\bar{y}} = D^1_{\bar{z}} = 0, \quad D^2_{\bar{y}} = D^2_{\bar{z}} = 0, \\ D^3_{\bar{y}} = D^3_{\bar{z}} = 0, \quad D^4_{\bar{y}} = D^4_{\bar{z}} = 0; \\ b^1_1 = D^1_y + D^2_z + D^3_{\bar{y}} + \alpha^{22} = D^1_y + D^2_z + D^4_{\bar{z}} + \alpha^{11},$$

$$b^2_2 = D^2_z + D^3_{\bar{y}} + D^4_{\bar{z}} + \epsilon, \quad b^3_3 = D^1_y + D^3_{\bar{y}} + D^4_{\bar{z}} + \epsilon, \\ b^3_2 = -D^2_y, \quad b^2_3 = -D^1_z, \quad b^1_1 = b^3_1 = b^1_2 = b^1_3 = 0.$$

We notice, in particular, that the D^1 and D^2 depend only on y and z , while the D^3 and D^4 depend only on $\bar{y}$ and $\bar{z}$.

The remaining terms in the expansion of Eq. (2.5) are those that cannot be expressed as *scalar* multiples of basis four-forms [in the sense that the coefficients in these terms do not commute with the $\text{gl}(N, C)$ -valued basis four-forms]. Terms of this type can be divided into four kinds according to their dependence on the basis three-forms $dy dz d\bar{y}$, $dy dz d\bar{z}$, $dy d\bar{y} d\bar{z}$, or $dz d\bar{y} d\bar{z}$. The $\text{gl}(N, C)$ -valued coefficients of each of these basis three-forms must be equated in each of the three exterior equations; this process yields a set of 12 equations which can be divided into two general types:

$$\Lambda_i^k dY + (dY)M_i^k = 0, \quad i \neq k, \quad (2.9)$$

and

$$\bar{d}\bar{H} = \Lambda_k^k dY + (dY)M_k^k, \quad (2.10)$$

where $Y \equiv B^1, B^2, J$ and $H \equiv A^1, A^2, G$. The variable Y , by assumption, does not commute with Λ_i^k and M_i^k . Thus Eq. (2.9) is satisfied only if $\Lambda_i^k = M_i^k = 0, i \neq k$. Also, given that, by definition of the internal exterior derivative and by assumption about the Λ_i^k and M_i^k , $dY = \bar{d}Y$, $\bar{d}\Lambda_k^k(y^\mu) = 0$, $\bar{d}M_k^k(y^\mu) = 0$, Eq. (2.10) can be integrated immediately:

$$\bar{H} = \Lambda_k^k Y + YM_k^k + h(y^\mu),$$

where $h(y^\mu)$ is an arbitrary function. Our results are explicitly stated as follows:

$$\Lambda^1_1 \equiv \Lambda^1(y^\mu), \quad M^1_1 \equiv M^1(y^\mu), \\ \Lambda^2_2 = \Lambda^3_3 \equiv \Lambda^2(y^\mu), \quad M^2_2 = M^3_3 \equiv M^2(y^\mu), \\ \Lambda^k_i = M^k_i = 0, \quad \text{for } i \neq k; \\ \bar{A}^1 = \Lambda^1 B^1 + B^1 M^1 + h^1(y^\mu), \\ \bar{A}^2 = \Lambda^1 B^2 + B^2 M^1 + h^2(y^\mu), \\ \bar{G} = \Lambda^2 J + JM^2 + g(y^\mu),$$

where the h^1, h^2 , and g are arbitrary $\text{gl}(N, C)$ -valued functions.

Appropriate substitutions into Eqs. (2.8) will now give expressions for A^i and G , which can be substituted back into Eqs. (2.7). By using previous results, the coefficients b_i^k can be eliminated in favor of other quantities, while certain replacements can also be made with regard to the Λ_i^k and M_i^k . The result is a set of equalities between some kind of generalized “polynomial” expressions in the variables B^1, B^2 , and J , with y^μ -dependent coefficients. The “constant” term in such a “polynomial” is a matrix function $F(y^\mu)$, while the other terms are of the following kinds: $qB^k, qJ, qJB^k, QB^k, B^k Q, QJ, JQ, QJB^k, JQB^k$, and $JB^k Q$, where $q(y^\mu)$ is a scalar function and $Q(y^\mu)$ is a $\text{gl}(N, C)$ -valued function. Equating coefficients of similar terms we obtain a set of partial differential and algebraic equations, which are not hard to solve. In particular, we find

$$-\Lambda^1 = M^1 = M^2 \equiv M(y, z), \quad \Lambda^2 \equiv \Lambda(\bar{y}, \bar{z}), \\ h^1(y, z) = M_y, \quad h^2(y, z) = M_z, \quad g(y^\mu) = 0.$$

Equations (2.11) give the complete solution for the components of the isovector field V :

$$\begin{aligned} D^1 &= c_1 y + k_1 z + \alpha_1, & D^2 &= k_2 y + c_2 z + \alpha_2, \\ D^3 &= (c_2 - c) \bar{y} - k_2 \bar{z} + \alpha_3, \\ D^4 &= -k_1 \bar{y} + (c_1 - c) \bar{z} + \alpha_4, \\ A^1 &= -c_1 B^1 - k_2 B^2 - [M(y, z), B^1] + M_y, \\ A^2 &= -k_1 B^1 - c_2 B^2 - [M(y, z), B^2] + M_z, \\ G &= \epsilon(\bar{y}, \bar{z}) J + \Lambda(\bar{y}, \bar{z}) J + JM(y, z), \end{aligned} \quad (2.11)$$

where $c_1, c_2, k_1, k_2, c, \alpha_1, \dots, \alpha_4$ are nine complex parameters, $\epsilon(\bar{y}, \bar{z})$ is a scalar function, and $M(y, z)$ and $\Lambda(\bar{y}, \bar{z})$ are $\text{gl}(N, \mathbb{C})$ -valued functions. From Eqs. (2.11) we can read off the infinitesimal operators P_k corresponding to the nine complex parameters (cf. Ref. 1) and we can show that they form the basis of a Lie algebra. In particular, the operators

$$P_{\alpha_\mu} = \frac{\partial}{\partial y^\mu}$$

and

$$P_{c_i} + P_{c_i} = y^\mu \frac{\partial}{\partial y^\mu} - B^k \frac{\partial}{\partial B^k}$$

represent translations and dilatations, respectively.

Following the discussion in Ref. 1, from Eq. (2.11) we can construct the following infinitesimal internal symmetry transformations:

$$\begin{aligned} B^{1'} &\simeq B^1 + [M(y, z), B^1] - M_y, \\ B^{2'} &\simeq B^2 + [M(y, z), B^2] - M_z, \\ J' &\simeq J - \epsilon(\bar{y}, \bar{z}) J - \Lambda(\bar{y}, \bar{z}) J - JM(y, z), \end{aligned} \quad (2.12)$$

where the ϵ, M , and Λ are infinitesimal. The corresponding finite transformations are

$$\begin{aligned} B^{1'} &= UB^1 U^{-1} + U \partial_y U^{-1}, \\ B^{2'} &= UB^2 U^{-1} + U \partial_z U^{-1}, \\ J' &= \beta \bar{U} J U, \end{aligned} \quad (2.13)$$

where

$$\begin{aligned} U(y, z) &= \exp\{-M(y, z)\}, \\ \bar{U}(\bar{y}, \bar{z}) &= \exp\{-\Lambda(\bar{y}, \bar{z})\}, \end{aligned}$$

and

$$\beta(\bar{y}, \bar{z}) = \exp\{-\epsilon(\bar{y}, \bar{z})\}.$$

These are, of course, familiar symmetries of the SDYM system.

III. PARAMETRIC INFINITESIMAL TRANSFORMATION AND LINEAR SYSTEM

If we define a new function

$$\xi(y, z, \bar{y}, \bar{z}) \equiv M(y, z) + \epsilon(\bar{y}, \bar{z}) 1_N, \quad (3.1)$$

where 1_N denotes the N -dimensional unit matrix, then the infinitesimal transformations of Eq. (2.12) with $\Lambda(\bar{y}, \bar{z}) = 0$ can be rewritten as

$$\begin{aligned} \delta B^1 &= [\xi(y^\mu), B^1] - \xi_y, \\ \delta B^2 &= [\xi(y^\mu), B^2] - \xi_z, & \delta J &= -J\xi(y^\mu), \end{aligned} \quad (3.2)$$

where $\delta B^k \simeq B^{k'} - B^k$ and $\delta J \simeq J' - J$. We wish to rewrite

these symmetries without the restriction (3.1). It turns out that this is possible due to the independence of the coordinates of the underlying jetlike space. Of course, there is a price to be paid for such an adjustment. But this "price" is a most welcome one: Restriction (3.1) is replaced by a set of linear PDE's which, in the case of actual SDYM fields, lead to a linear system for the SDYM equation.

From Eq. (3.1) it is seen that $\xi(y^\mu)$ satisfies the PDE,

$$[\xi_{\bar{y}}, B^1] + [\xi_{\bar{z}}, B^2] - \xi_{y\bar{y}} - \xi_{z\bar{z}} = 0.$$

Given the independence of the y^μ and the B^k (this is the case as long as no restriction on the solution manifold is imposed), the above equation may be written as

$$\partial_{\bar{y}}([\xi, B^1] - \xi_y) + \partial_{\bar{z}}([\xi, B^2] - \xi_z) = 0. \quad (3.3)$$

This is satisfied if there exists a "potential" $\psi(y^\mu, B^k)$ such that

$$[\xi, B^1] - \xi_y = \lambda \psi_{\bar{z}}, \quad [\xi, B^2] - \xi_z = -\lambda \psi_{\bar{y}}, \quad (3.4)$$

where λ is an arbitrary complex parameter. We thus replace the system of Eqs. (3.1) and (3.2) by the following alternate one:

$$\delta B^1 = \lambda \psi_{\bar{z}}, \quad \delta B^2 = -\lambda \psi_{\bar{y}}, \quad \delta J = -J\xi(y^\mu), \quad (3.5)$$

where ψ and ξ satisfy the linear system (3.4). Note that Eqs. (3.4) and (3.5) become independent of Eqs. (3.1) and (3.2) upon restriction to the solution manifold, i.e., for actual SDYM fields.

Let us explore further the significance of Eqs. (3.4) for actual SDYM fields (in which case the B^k are dependent upon the y^μ). In particular, let us examine the ansatz $\psi(y^\mu) = \xi(y^\mu)$, all y^μ :

$$[\xi, B^1] - \xi_y = \lambda \xi_{\bar{z}}, \quad [\xi, B^2] - \xi_z = -\lambda \xi_{\bar{y}}. \quad (3.6)$$

The integrability criterion $\xi_{\bar{y}\bar{z}} - \xi_{z\bar{y}} = 0$ yields Eq. (3.3), which, in combination with Eq. (3.6), gives

$$[\xi, B_y^2 - B_z^1 + [B^1, B^2] + \lambda(B_{\bar{y}}^1 + B_{\bar{z}}^2)] = 0.$$

We seek conditions for B^1 and B^2 in order that the above equality holds for all λ and independently of ξ . The following pair of PDE's must therefore be satisfied:

$$\partial_y B^2 - \partial_z B^1 + [B^1, B^2] = 0, \quad (3.7)$$

$$\partial_{\bar{y}} B^1 + \partial_{\bar{z}} B^2 = 0. \quad (3.8)$$

Equation (3.7) is a condition for zero curvature and implies that the B^1 and B^2 are pure gauges:

$$B^1 = J^{-1} \partial_y J, \quad B^2 = J^{-1} \partial_z J, \quad (3.9)$$

where J is a nonsingular $\text{gl}(N, \mathbb{C})$ matrix. Then Eq. (3.8) becomes identical to the SDYM equation (2.1), of which Eq. (3.6) is seen to be a linear system.

We remark that our results are in agreement with those of Ref. 5 (although they are given in a slightly different form). The thing to notice is that these results were actually derived here, in a rather straightforward manner, by using the isovector technique.

IV. CONNECTION WITH BÄCKLUND TRANSFORMATIONS

By using the original definitions of B^1 and B^2 as given in Eqs. (2.3), the infinitesimal transformations of these quanti-

ties may be written, according to Eqs. (3.5) as

$$J'^{-1}J'_y - J^{-1}J_y = \lambda\psi_{\bar{z}}, \quad (4.1a)$$

$$J'^{-1}J'_z - J^{-1}J_z = -\lambda\psi_{\bar{y}}. \quad (4.1b)$$

Clearly, as J' approaches J , the $\psi_{\bar{y}}$ and $\psi_{\bar{z}}$ must approach zero. One way to achieve this is to put

$$\psi = \xi = 1 - J^{-1}J'. \quad (4.2)$$

Now, if the left-hand sides of Eqs. (4.1a) and (4.1b) are considered as *finite*, rather than infinitesimal differences, then Eqs. (4.1) and (4.2) constitute one possible form of the Bäcklund transformation (BT) proposed in Ref. 7. Alternatively, the infinitesimal parametric transformation (4.1) and (4.2) is also an infinitesimal BT. This was observed in Ref. 5, but we include it in the present discussion due to its direct (and quite interesting) relevance to the isovector method.

Incidentally, the transformation (3.1) and (3.2) is also an infinitesimal BT, with Eq. (3.1) being a sort of algebraic constraint. Indeed, putting $\xi = 1 - J^{-1}J'$ and introducing an arbitrary complex parameter μ , we write

$$J'^{-1}J'_y - J^{-1}J_y = \mu\{[J^{-1}J', J^{-1}J_y] - \partial_y(J^{-1}J')\}, \quad (4.3a)$$

$$J'^{-1}J'_z - J^{-1}J_z = \mu\{[J^{-1}J', J^{-1}J_z] - \partial_z(J^{-1}J')\}, \quad (4.3b)$$

$$J^{-1}J' = M(y, z) + \epsilon(\bar{y}, \bar{z})1_N, \quad (4.3c)$$

where $M(y, z)$ is $\mathfrak{gl}(N, \mathbb{C})$ valued and $\epsilon(\bar{y}, \bar{z})$ is a scalar. Tak-

ing $\partial_{\bar{y}}(4.3a) + \partial_{\bar{z}}(4.3b)$ and using (4.3c), we find

$$\begin{aligned} & \{\partial_{\bar{y}}(J'^{-1}J'_y) + \partial_{\bar{z}}(J'^{-1}J'_z)\} \\ & - \{\partial_{\bar{y}}(J^{-1}J_y) + \partial_{\bar{z}}(J^{-1}J_z)\} \\ & = \mu[J^{-1}J', \partial_{\bar{y}}(J^{-1}J_y) + \partial_{\bar{z}}(J^{-1}J_z)], \end{aligned}$$

according to which J' satisfies the SDYM equation (2.1) if J does. Note that the BT was constructed so as to yield the trivial solution $J' = J$ as a particular solution [this corresponds to $M = 0$ and $\epsilon = 1$ in the algebraic constraint (4.3c)].

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