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Nonlocal symmetries and Bäcklund transformations for the self-dual Yang–Mills system

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The observation is made that generalized evolutionary isovectors of the self-dual Yang–Mills equation, obtained by “verticalization” of the geometrical isovectors derived in a previous paper [J. Math. Phys. **28**, 1261 (1987)], generate Bäcklund transformations for the self-dual system. In particular, new Bäcklund transformations are obtained by “verticalizing” the generators of point transformations on the solution manifold. A geometric ansatz for the derivation of such (generally nonlocal) symmetries is proposed.

I. INTRODUCTION

In previous papers^{1,2} the authors have discussed isovector³ techniques for partial differential equations (PDE's) associated with vector-valued differential forms. It was mentioned² that such a PDE (or system of PDE's) defines, through its solutions, sections of a vector bundle over the solution manifold. This manifold serves as a base space, while the fibers are isomorphic to some vector space (or Lie algebra).

In Ref. 1 the isovector approach was employed to derive point symmetries for the self-dual Yang–Mills (SDYM) equation in its so-called J formulation⁴ (this is mathematically different from the usual formulation in which the self-duality condition is directly written in covariant form). The system was represented by three $\mathfrak{gl}(N, C)$ -valued four-forms in seven variables. Since these forms generated a differential ideal by themselves, we did not include the integrability condition of the system in the ideal (indeed, such an inclusion can be seen to be superfluous for the purpose of deriving point transformations).

Calculation of the isovectors gave a nine parameter group of transformations on the base space, together with a set of infinitesimal internal transformations in the fiber space in which the SDYM fields have values. It was then observed that the internal symmetries were related to parametric Bäcklund transformations (BT's) for the SDYM equation. In particular, a well-known^{5,6} BT was recovered.

With the observation that internal symmetries are generated by *evolutionary*⁷ (i.e., “vertical”) vector fields (EVF's), it is natural to inquire for other EVF's that may generate BT's for the SDYM system. Such fields cannot be sought, of course, among the “geometrical” symmetries found in Ref. 1. The most accessible nongeometrical (i.e., nonlocal) symmetries at our disposal are those generated by the *evolutionary representatives*⁷ of the nine generators of coordinate transformations mentioned previously. In Sec. III we establish the generalized isovector property of these EVF's by examining the effect of the corresponding Lie derivatives on the original ideal of the three four-forms. It is found that these Lie derivatives map this ideal into a larger ideal comprising the original system, its integrability conditions, and certain prolongations⁸ of all of the above. The

emergence of a prolonged ideal was to be expected since we are now dealing with Lie–Bäcklund-type symmetries. On the other hand, the appearance of the integrability conditions as an inseparable part of the system is quite interesting, considering the passive role these conditions played in the derivation of point symmetries.

In Sec. IV we construct the infinitesimal parametric nonlocal transformations generated by the aforementioned nine EVF's. It is seen that, by letting the transformation parameters be considered finite, rather than infinitesimal, the transformations of the prolongation⁸ variables become BT's for the SDYM equation. This observation constitutes a further indication of the intimate connection between symmetry and integrability aspects⁹ of nonlinear systems, and, in particular, of the SDYM system.^{1,10}

To make the paper as self-contained as possible, we review in the next section some of the results of Ref. 1 that will be needed for the present treatment.

II. GEOMETRICAL SYMMETRIES OF THE SDYM SYSTEM

As in Ref. 1, we write the SDYM equation (a second-order nonlinear PDE) as a set of first-order PDE's:

$$B_y^1 + B_z^2 = 0, \quad (2.1a)$$

$$B^1 = J^{-1}J_y, \quad (2.1b)$$

$$B^2 = J^{-1}J_z, \quad (2.1c)$$

where the subscripts denote partial differentiation (partial derivatives will occasionally be used). The $y, z, \bar{y}, \bar{z}$, collectively denoted by x^μ ($\mu = 1, 2, 3, 4$), are four complex coordinates,^{1,4,5} while J is assumed to have values in $\mathfrak{gl}(N, C)$. Loosely speaking, the B^1 and B^2 are “prolongation variables” for the system. The integrability condition $J_{yz} = J_{zy}$ yields

$$B_y^2 - B_z^1 + [B^1, B^2] = 0. \quad (2.2)$$

The system (2.1) can be represented by a set of three $\mathfrak{gl}(N, C)$ -valued four-forms:

$$\begin{aligned} \gamma_1 &= dy dz dB^1 d\bar{z} + dy dz d\bar{y} dB^2, \\ \gamma_2 &= dJ dz d\bar{y} d\bar{z} - JB^1 dy dz d\bar{y} d\bar{z}, \\ \gamma_3 &= dy dJ d\bar{y} d\bar{z} - JB^2 dy dz d\bar{y} d\bar{z}. \end{aligned} \quad (2.3)$$

These forms generate a differential ideal. Indeed,

$$\begin{aligned} d\gamma_1 &= 0, \quad d\gamma_2 = \gamma_2 dy B^1 + J d\bar{y} \gamma_1, \\ d\gamma_3 &= \gamma_3 dz B^2 + J d\bar{z} \gamma_1. \end{aligned}$$

The geometrical (local) symmetries of the system are generated by vector fields of the form¹

$$V = \xi^\mu(x^\nu) \frac{\partial}{\partial x^\mu} + G(x^\nu, J) \frac{\partial}{\partial J} + A^i(x^\nu, B^k) \frac{\partial}{\partial B^i}, \quad (2.4)$$

where the usual summation convention is assumed over repeated indices. The ξ^μ ($\mu = 1, \dots, 4$) are scalars, whereas the G and A^i ($i = 1, 2$) are $\text{gl}(N, C)$ valued. The $\partial/\partial J$ and $\partial/\partial B^i$ are formal operators only—i.e., no differentiations are actually performed. The symmetry property is expressed by the requirement that the Lie derivative with respect to V leave the ideal of the γ_k invariant. Formally,

$$\mathcal{L}_V \gamma_i = b_i^k \gamma_k + \Lambda_i^k \gamma_k + \gamma_k M_i^k \quad (2.5)$$

($i = 1, 2, 3$), where the b_i^k are scalars, whereas the Λ_i^k and M_i^k are $\text{gl}(N, C)$ -valued zero-forms. The calculation of V from Eq. (2.5) becomes possible if we make the ansatz that the coefficients of expansion in this equation depend only on the x^μ . As we will see shortly, this condition is violated in the case of nongeometrical symmetries.

As was seen in Ref. 1, the vector V is parametrized by nine (complex) parameters, and depends on three arbitrary functions. The nine parameters correspond to transformations on the base space. These transformations are generated by the following independent vector fields, which are written here in unprolonged form (i.e., without the terms in $\partial/\partial B^i$):

$$\begin{aligned} V_\mu &= -\frac{\partial}{\partial x^\mu}, \quad \mu = 1, 2, 3, 4, \quad V_5 = -y \frac{\partial}{\partial y} - \bar{z} \frac{\partial}{\partial \bar{z}}, \\ V_6 &= -z \frac{\partial}{\partial z} - \bar{y} \frac{\partial}{\partial \bar{y}}, \quad V_7 = -z \frac{\partial}{\partial y} + \bar{y} \frac{\partial}{\partial \bar{z}}, \\ V_8 &= -y \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{y}}, \quad V_9 = -\bar{y} \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{y}}. \end{aligned} \quad (2.6)$$

Internal symmetries are generated by EVF's with components

$$\begin{aligned} G &= \epsilon(\bar{y}, \bar{z})J + \Lambda(\bar{y}, \bar{z})J + JM(y, z), \\ A^1 &= -[M(y, z), B^1] + M_y, \\ A^2 &= -[M(y, z), B^2] + M_z, \end{aligned}$$

where ϵ is a scalar function, while Λ and M are $\text{gl}(N, C)$ -valued functions. The symmetries in ϵ and M combine to give BT's for SDYM.¹ The symmetry in Λ yields a BT that is less interesting, since it merely consists of $\delta(J^{-1}J_y) = 0$, $\delta(J^{-1}J_z) = 0$ (i.e., $J'^{-1}J'_y = J^{-1}J_y$, etc.).

III. EVOLUTIONARY ISOVECTORS FOR FIRST-ORDER GENERALIZED SYMMETRIES

A. Nongeometrical vectors and prolongation forms

We now relax the geometrical requirement and seek generalized⁷ symmetries of SDYM. We confine our attention to first-order symmetries generated by *evolutionary* vector fields (i.e., vector fields with vanishing projection on the base space) of the form

$$V = Q \frac{\partial}{\partial J} + R^1 \frac{\partial}{\partial B^1} + R^2 \frac{\partial}{\partial B^2}, \quad (3.1)$$

where Q may depend on x^μ , J , and $J_\mu \equiv \partial_\mu J$. With the definitions

$$J_y = JB^1, \quad J_z = JB^2, \quad (3.2a)$$

$$J_{\bar{y}} = E^1, \quad J_{\bar{z}} = E^2, \quad (3.2b)$$

we have that

$$Q = Q(x^\mu, J, B^1, B^2, E^1, E^2). \quad (3.3)$$

[The reader may be concerned with the appearance of several noncommuting variables in the functional dependence of Q . However, as Q is calculated (see below), no ambiguity in the order of these variables arises.]

It is easily seen [see Eqs. (3.11) and (3.12)] that the R^1 and R^2 depend, collectively, on the additional variables $B_\mu^1 \equiv \partial_\mu B^1$ and $B_\mu^2 \equiv \partial_\mu B^2$. The variables $B_{\bar{z}}^2$ and B_z^1 can be eliminated from the problem by using the field equation (2.1a) and the integrability condition (2.2), respectively:

$$B_{\bar{z}}^2 = -B_{\bar{y}}^1, \quad (3.4)$$

$$B_z^1 = B_y^2 + [B^1, B^2]. \quad (3.5)$$

Thus we are left with the variables

$$\begin{aligned} B_y^1 &= C^1, \quad B_{\bar{y}}^1 = C^2, \quad B_{\bar{z}}^1 = C^3, \\ B_y^2 &= C^4, \quad B_z^2 = C^5, \quad B_{\bar{y}}^2 = C^6. \end{aligned} \quad (3.6)$$

Equations (3.2) and (3.6) each admit six integrability conditions, thus a total of 12 such conditions can be written [including the one given by Eq. (2.2)].

Let us consider the basic system (2.1), together with its integrability condition (2.2). We prolong Eqs. (2.1a) and (2.2) in the usual way by taking the derivatives with respect to the x^μ . By convention, only those prolongations defined *within the variables at our disposal* are considered. Specifically, we can construct the y , z , and $\bar{y}$ prolongations of Eq. (2.1a), and the y and $\bar{y}$ prolongations of Eq. (2.2):

$$C_y^1 + C_z^4 = 0, \quad (3.7a)$$

$$C_z^2 + C_{\bar{z}}^5 = 0, \quad (3.7b)$$

$$C_{\bar{y}}^2 + C_{\bar{z}}^6 = 0; \quad (3.7c)$$

$$C_y^4 - C_z^1 + [C^1, B^2] + [B^1, C^4] = 0, \quad (3.8a)$$

$$C_y^6 - C_z^2 + [C^2, B^2] + [B^1, C^6] = 0. \quad (3.8b)$$

[Note that, in a sense, the prolongations of Eqs. (2.1b) and (2.1c) are contained in Eq. (3.6).]

We now express our equations in terms of differential forms. Thus we define 28 four-forms corresponding, successively, to Eqs. (2.1), (2.2), (3.7), (3.8), (3.6) and its integrability conditions, and Eq. (3.2b), and the five remaining integrability conditions of (3.2) (we put $\tilde{\omega} \equiv dy dz d\bar{y} d\bar{z}$):

$$\gamma_1 = dy dz dB^1 d\bar{z} + dy dz d\bar{y} dB^2,$$

$$\gamma_2 = dJ dz d\bar{y} d\bar{z} - JB^1 \tilde{\omega},$$

$$\gamma_3 = dy dJ d\bar{y} d\bar{z} - JB^2 \tilde{\omega},$$

$$\gamma_4 = dB^2 dz d\bar{y} d\bar{z} - dy dB^1 d\bar{y} d\bar{z} + [B^1, B^2] \tilde{\omega},$$

$$\gamma_5 = dy dz dC^1 d\bar{z} + dy dz d\bar{y} dC^4,$$

$$\gamma_6 = dy dC^2 d\bar{y} d\bar{z} + dy dz d\bar{y} dC^5,$$

$$\begin{aligned}
\gamma_7 &= dy dz dC^2 d\bar{z} + dy dz d\bar{y} dC^6, \\
\gamma_8 &= dC^4 dz d\bar{y} d\bar{z} - dy dC^1 d\bar{y} d\bar{z} \\
&\quad + ([C^1, B^2] + [B^1, C^4])\bar{\omega}, \\
\gamma_9 &= dC^6 dz d\bar{y} d\bar{z} - dy dC^2 d\bar{y} d\bar{z} \\
&\quad + ([C^2, B^2] + [B^1, C^6])\bar{\omega}, \\
\gamma_{10} &= dB^1 dz d\bar{y} d\bar{z} - C^1\bar{\omega}, \\
\gamma_{11} &= dy dz dB^1 d\bar{z} - C^2\bar{\omega}, \\
\gamma_{12} &= dy dz d\bar{y} dB^1 - C^3\bar{\omega}, \\
\gamma_{13} &= dB^2 dz d\bar{y} d\bar{z} - C^4\bar{\omega}, \\
\gamma_{14} &= dy dB^2 d\bar{y} d\bar{z} - C^5\bar{\omega}, \\
\gamma_{15} &= dy dz dB^2 d\bar{z} - C^6\bar{\omega}, \\
\gamma_{16} &= dC^2 dz d\bar{y} d\bar{z} - dy dz dC^1 d\bar{z}, \\
\gamma_{17} &= dC^3 dz d\bar{y} d\bar{z} - dy dz d\bar{y} dC^1, \\
\gamma_{18} &= dy dz dC^3 d\bar{z} - dy dz d\bar{y} dC^2, \\
\gamma_{19} &= dC^5 dz d\bar{y} d\bar{z} - dy dC^4 d\bar{y} d\bar{z}, \\
\gamma_{20} &= dC^6 dz d\bar{y} d\bar{z} - dy dz dC^4 d\bar{z}, \\
\gamma_{21} &= dy dC^6 d\bar{y} d\bar{z} - dy dz dC^5 d\bar{z}, \\
\gamma_{22} &= dy dz dJ d\bar{z} - E^1\bar{\omega}, \\
\gamma_{23} &= dy dz d\bar{y} dJ - E^2\bar{\omega}, \\
\gamma_{24} &= dy dz dE^2 d\bar{z} - dy dz d\bar{y} dE^1, \\
\gamma_{25} &= dE^1 dz d\bar{y} d\bar{z} - dy dz d(JB^1) d\bar{z}, \\
\gamma_{26} &= dE^2 dz d\bar{y} d\bar{z} - dy dz d\bar{y} d(JB^1), \\
\gamma_{27} &= dy dE^1 d\bar{y} d\bar{z} - dy dz d(JB^2) d\bar{z}, \\
\gamma_{28} &= dy dE^2 d\bar{y} d\bar{z} - dy dz d\bar{y} d(JB^2).
\end{aligned} \tag{3.9}$$

B. Generalized isovectors

The postulate (2.5), used to derive symmetries of the system, can now be generalized¹¹ by requiring that the Lie derivative with respect to a generalized EVF of the form (3.1) map the original ideal $\{\gamma_1, \gamma_2, \gamma_3\}$ into the prolonged ideal $\{\gamma_1, \dots, \gamma_{28}\}$. [We can see that such a generalization is necessary because the Lie derivative of the γ_i ($i = 1, 2, 3$) will yield variables which appear only in the prolonged forms γ_α .] Formally,

$$\mathcal{L}_V \gamma_i = b_i^\alpha \gamma_\alpha + \Lambda_i^\alpha \gamma_\alpha + \gamma_\alpha M_i^\alpha, \tag{3.10}$$

where $i = 1, 2, 3$, as before, but now the index α runs from 1 to 28. The coefficients Λ_i^α and M_i^α are no longer required to be independent of the internal variables J, B^1 , and B^2 .

One would like, of course, to solve Eq. (3.10) for the components Q, R^1 , and R^2 of V and thus obtain independent first-order generalized symmetries of SDYM. The computations, however, are now of great complexity due to the presence of a very large set of variables and forms. Rather than solving Eq. (3.10) directly, we will instead construct certain solutions by utilizing the coordinate (point) symmetries of SDYM, as these are expressed by the vector fields of Eq. (2.6).

First of all, it is seen from Eqs. (3.9) and (3.10) that some of the forms of the prolonged ideal will not occur in the

expansion of the Lie derivative in Eq. (3.10). Indeed, by comparing similar terms and taking into account Eq. (3.3), it can be shown that the forms $\gamma_8, \gamma_{16}, \gamma_{17}, \gamma_{19}$, and γ_{21} make no contribution and thus can be eliminated. This leaves us with a somewhat smaller ideal of 23 forms.

Second, since the B^1 and B^2 are prolongation variables, the R^1 and R^2 of Eq. (3.1) are expressible in terms of Q . Again, this can be done by comparing similar terms in Eq. (3.10), using the “internal exterior derivative” (introduced in Refs. 1 and 2) whenever necessary.

There is, however, an easier way to do this: Let us recall that the vector field V , in the form (3.1), defines an infinitesimal “motion” in the space of J, B^1 , and B^2 . Thus, if δt is an infinitesimal parameter, $\delta J = Q\delta t$. Furthermore, if we regard t as a variable parametrizing an integral curve of V , then $Q = \delta J / \delta t$ and

$$\begin{aligned}
R^1 &= \frac{\delta B^1}{\delta t} = -J^{-1} \frac{\delta J}{\delta t} J^{-1} D_y J + J^{-1} \frac{\delta}{\delta t} (D_y J) \\
&= -J^{-1} Q B^1 + J^{-1} D_y Q
\end{aligned} \tag{3.11}$$

and similarly

$$R^2 = \frac{\delta B^2}{\delta t} = -J^{-1} Q B^2 + J^{-1} D_z Q, \tag{3.12}$$

where the D_y and D_z are total derivatives.¹² [In general, total derivatives must be defined consistently with Eqs. (3.2) and (3.4)–(3.6). Thus, on functions of the form (3.3),

$$\begin{aligned}
D_y &= \frac{\partial}{\partial y} + JB^1 \frac{\partial}{\partial J} + C^1 \frac{\partial}{\partial B^1} + C^4 \frac{\partial}{\partial B^2} \\
&\quad + (E^1 B^1 + JC^2) \frac{\partial}{\partial E^1} + (E^2 B^1 + JC^3) \frac{\partial}{\partial E^2}, \\
D_z &= \frac{\partial}{\partial z} + JB^2 \frac{\partial}{\partial J} + (C^4 + [B^1, B^2]) \frac{\partial}{\partial B^1} + C^5 \frac{\partial}{\partial B^2} \\
&\quad + (E^1 B^2 + JC^6) \frac{\partial}{\partial E^1} + (E^2 B^2 - JC^2) \frac{\partial}{\partial E^2}.
\end{aligned}$$

Note that D_μ , like ∂_μ , is a *derivation*, i.e., it satisfies the Leibniz rule.]

Finally, we need to define the *evolutionary representative*⁷ (or verticalization) of a “horizontal” vector field. The following definition is pertinent to the SDYM case but can be easily generalized: Given a vector field of the (unprolonged) form

$$V_h = \xi^\mu(x^\nu) \frac{\partial}{\partial x^\mu}, \tag{3.13}$$

one can construct an EVF (in prolonged form)

$$V_q = Q \frac{\partial}{\partial J} + R^1 \frac{\partial}{\partial B^1} + R^2 \frac{\partial}{\partial B^2} \tag{3.14}$$

such that

$$Q = -\xi^\mu D_\mu J, \tag{3.15}$$

where $D_y J = JB^1$, $D_z J = JB^2$, $D_{\bar{y}} J = E^1$, $D_{\bar{z}} J = E^2$, and where the components R^1 and R^2 are related to Q as in Eqs. (3.11) and (3.12). The nongeometrical vector field V_q is called the evolutionary representative of the geometrical

field V_h .¹³ The reason for this particular definition is that, if V_h is a symmetry, it can be shown⁷ that V_q is also a symmetry.

We are now in a position to construct the (prolonged) evolutionary representatives of the nine vector fields given in Eq. (2.6). The symbol $\bar{V}_k$ will denote the representative of V_k :

$$\begin{aligned}\bar{V}_1 &= JB^1 \frac{\partial}{\partial J} + C^1 \frac{\partial}{\partial B^1} + C^4 \frac{\partial}{\partial B^2}, \\ \bar{V}_2 &= JB^2 \frac{\partial}{\partial J} + (C^4 + [B^1, B^2]) \frac{\partial}{\partial B^1} + C^5 \frac{\partial}{\partial B^2}, \\ \bar{V}_3 &= E^1 \frac{\partial}{\partial J} + C^2 \frac{\partial}{\partial B^1} + C^6 \frac{\partial}{\partial B^2}, \\ \bar{V}_4 &= E^2 \frac{\partial}{\partial J} + C^3 \frac{\partial}{\partial B^1} - C^2 \frac{\partial}{\partial B^2}, \\ \bar{V}_5 &= (yJB^1 + \bar{z}E^2) \frac{\partial}{\partial J} + (B^1 + yC^1 + \bar{z}C^3) \frac{\partial}{\partial B^1} \\ &\quad + (yC^4 - \bar{z}C^2) \frac{\partial}{\partial B^2}, \\ \bar{V}_6 &= (zJB^2 + \bar{y}E^1) \frac{\partial}{\partial J} + (zC^4 + z[B^1, B^2] + \bar{y}C^2) \\ &\quad \times \frac{\partial}{\partial B^1} + (B^2 + zC^5 + \bar{y}C^6) \frac{\partial}{\partial B^2}, \\ \bar{V}_7 &= (zJB^1 - \bar{y}E^2) \frac{\partial}{\partial J} + (zC^1 - \bar{y}C^3) \frac{\partial}{\partial B^1} \\ &\quad + (B^1 + zC^4 + \bar{y}C^2) \frac{\partial}{\partial B^2}, \\ \bar{V}_8 &= (yJB^2 - \bar{z}E^1) \frac{\partial}{\partial J} \\ &\quad + (B^2 + yC^4 + y[B^1, B^2] - \bar{z}C^2) \frac{\partial}{\partial B^1} \\ &\quad + (yC^5 - \bar{z}C^6) \frac{\partial}{\partial B^2}, \\ \bar{V}_9 &= (\bar{y}E^1 + \bar{z}E^2) \frac{\partial}{\partial J} + (\bar{y}C^2 + \bar{z}C^3) \frac{\partial}{\partial B^1} \\ &\quad + (\bar{y}C^6 - \bar{z}C^2) \frac{\partial}{\partial B^2}.\end{aligned}\quad (3.16)$$

The consistency of these expressions with the geometrical derivation of symmetries, as this is expressed in the prescription (3.10), may now be seen. Direct substitution into Eq. (3.10) shows that the above EVF's are generalized iso-vectors for the SDYM system (details are deferred to Appendix A). Our search for other generalized symmetries, of the same order or higher, has not been successful so far. In particular, one can see that the (generalized) Lie brackets⁷ of the known symmetries do not produce new symmetries, contrary to what one might have hoped (this point is more easily verified by using true jet-space variables x^μ , J , J_μ , and $J_{\mu\nu}$).

IV. INFINITESIMAL NONLOCAL SYMMETRIES AND BÄCKLUND TRANSFORMATIONS

By using the EVF's (3.16) and the definitions (3.2), the following infinitesimal parametric nonlocal transformations of the J function are constructed⁷ (λ is an infinitesimal parameter):

$$\begin{aligned}\delta_\mu J &= \lambda \partial_\mu J \equiv \lambda J_\mu, \quad \mu = 1, 2, 3, 4, \quad \delta_5 J = \lambda (yJ_y + \bar{z}J_z), \\ \delta_6 J &= \lambda (zJ_z + \bar{y}J_{\bar{y}}), \quad \delta_7 J = \lambda (zJ_y - \bar{y}J_z), \\ \delta_8 J &= \lambda (yJ_z - \bar{z}J_{\bar{y}}), \quad \delta_9 J = \lambda (\bar{y}J_{\bar{y}} + \bar{z}J_z).\end{aligned}\quad (4.1)$$

We can also construct the corresponding transformations for the variables $B^1 = J^{-1}J_y$ and $B^2 = J^{-1}J_z$. We will initially regard these transformations as general nonlocal symmetries generated by the EVF's (3.16) and independent of the SDYM equation. This means that one is allowed to make replacements in the components of the EVF's according to the definitions (3.2) and (3.6) and the integrability condition (3.5), but one may not use the equation of motion (3.4) (thus the apparent asymmetry of the equations below). The transformation equations are

$$\begin{aligned}\delta_1(J^{-1}J_y) &= \lambda \partial_y(J^{-1}J_y), \\ \delta_1(J^{-1}J_z) &= \lambda \partial_y(J^{-1}J_z), \\ \delta_2(J^{-1}J_y) &= \lambda \partial_z(J^{-1}J_y), \\ \delta_2(J^{-1}J_z) &= \lambda \partial_z(J^{-1}J_z), \\ \delta_3(J^{-1}J_y) &= \lambda \partial_{\bar{y}}(J^{-1}J_y), \\ \delta_3(J^{-1}J_z) &= \lambda \partial_{\bar{y}}(J^{-1}J_z), \\ \delta_4(J^{-1}J_y) &= \lambda \partial_{\bar{z}}(J^{-1}J_y), \\ \delta_4(J^{-1}J_z) &= -\lambda \partial_{\bar{y}}(J^{-1}J_y), \\ \delta_5(J^{-1}J_y) &= \lambda [J^{-1}J_y + y \partial_y(J^{-1}J_y) + \bar{z} \partial_z(J^{-1}J_y)], \\ \delta_5(J^{-1}J_z) &= \lambda [y \partial_y(J^{-1}J_z) - \bar{z} \partial_{\bar{y}}(J^{-1}J_y)], \\ \delta_6(J^{-1}J_y) &= \lambda [z \partial_z(J^{-1}J_y) + \bar{y} \partial_{\bar{y}}(J^{-1}J_y)], \\ \delta_6(J^{-1}J_z) &= \lambda [J^{-1}J_z + z \partial_z(J^{-1}J_z) + \bar{y} \partial_{\bar{y}}(J^{-1}J_z)], \\ \delta_7(J^{-1}J_y) &= \lambda [z \partial_y(J^{-1}J_y) - \bar{y} \partial_{\bar{z}}(J^{-1}J_y)], \\ \delta_7(J^{-1}J_z) &= \lambda [J^{-1}J_y + z \partial_y(J^{-1}J_z) + \bar{y} \partial_{\bar{y}}(J^{-1}J_y)], \\ \delta_8(J^{-1}J_y) &= \lambda [J^{-1}J_z + y \partial_z(J^{-1}J_y) - \bar{z} \partial_{\bar{y}}(J^{-1}J_y)], \\ \delta_8(J^{-1}J_z) &= \lambda [y \partial_z(J^{-1}J_z) - \bar{z} \partial_{\bar{y}}(J^{-1}J_z)], \\ \delta_9(J^{-1}J_y) &= \lambda [\bar{y} \partial_{\bar{y}}(J^{-1}J_y) + \bar{z} \partial_{\bar{z}}(J^{-1}J_y)], \\ \delta_9(J^{-1}J_z) &= \lambda [\bar{y} \partial_{\bar{y}}(J^{-1}J_z) - \bar{z} \partial_{\bar{y}}(J^{-1}J_y)].\end{aligned}\quad (4.2)$$

We now observe that the above nonlocal transformations of the prolongation variables bear an interesting property: Suppose that we let the parameters λ become *finite* in each case (λ stands for nine different independent parameters). In this case the left-hand sides of Eq. (4.2) become *finite* differences:

$$\begin{aligned}\delta_k(J^{-1}J_y) &\equiv J'^{-1}J'_y - J^{-1}J_y, \\ \delta_k(J^{-1}J_z) &\equiv J'^{-1}J'_z - J^{-1}J_z.\end{aligned}\quad (4.3)$$

We thus obtain nine pairs of independent parametric equations. Cross-differentiation of each pair with respect to $\bar{y}$ and $\bar{z}$, and use of the various integrability conditions, will then reveal that all nine pairs are parametric Bäcklund transfor-

mations for the SDYM equation, which J is now required to satisfy. Specifically, if J is a solution of

$$\partial_{\bar{y}}(J^{-1}J_y) + \partial_{\bar{z}}(J^{-1}J_z) = 0, \quad (4.4)$$

then so is J' , which is related to J by Eqs. (4.2) and (4.3). A more detailed proof of this is given in Appendix B.

We remark that one could obtain a more symmetric set of equations than Eq. (4.2) by allowing the components of the EVF's (3.16) to be evaluated on solutions of the SDYM equation, i.e., by using Eq. (3.4) to reintroduce $B_{\bar{z}}^2$ into the problem. The reader is invited to construct this alternate set of BT's.

In order for the BT's described in Eqs. (4.2) and (4.3) to be valid, one must also require the integrability condition $(J'_y)_z = (J'_z)_y$. If L represents any one of the differential operators in Eq. (2.6) [which operators appear on the right-hand side of Eq. (4.2)], this condition can be shown to require $[H_y, H_z] = 0$, where $H = (LJ)J^{-1}$. Thus this is a condition on the original solution J . Many solutions satisfy this condition, so that it is not excessively restrictive. This condition will be explored in future publications.

V. CONCLUSIONS AND SUMMARY

Let us summarize our main conclusions.

(1) The study of first-order Lie-Bäcklund type symmetries of the SDYM system requires the construction of a prolonged ideal of four-forms, which is many times larger than the ideal used for the derivation of point transformations. A noteworthy feature of the expanded ideal is the presence of the integrability conditions of the system.

(2) Starting with the (point) symmetries of SDYM on the base manifold, one can construct nine evolutionary vector fields that are generalized isovectors of the system. Nine nonlocal symmetries of SDYM are thus obtained. No further generalized symmetries can be generated by simply taking the Lie brackets of the nine EVF's.

(3) It is our conclusion that all evolutionary representatives of the corresponding point symmetries of SDYM yield Bäcklund transformations for the system. These BT's are "weak," in the sense that they relate two functions, of which the second is a solution of SDYM provided that the first one is. The physical implications of these transformations are not yet totally clear to us.

APPENDIX A: PROOF OF ISOVECTOR PROPERTY

We display the expansions of the Lie derivatives of the forms $\gamma_1, \gamma_2, \gamma_3$, with respect to the prolonged EVF's $\bar{V}_s$ ($s = 1, \dots, 9$) of Eq. (3.16). We will use the notation

$$\gamma_i^{(s)} \equiv \frac{L}{\bar{V}_s} \gamma_i \quad (i = 1, 2, 3).$$

Analytically,

$$\begin{aligned} \gamma_1^{(1)} &= \gamma_5, \\ \gamma_2^{(1)} &= \gamma_2 B^1 + J\gamma_{10}, \\ \gamma_3^{(1)} &= \gamma_3 B^1 - J\gamma_4 + J\gamma_{13}, \\ \gamma_1^{(2)} &= \gamma_6 + \gamma_9 + [\gamma_{11}, B^2] + [B^1, \gamma_{15}] - \gamma_{20}, \\ \gamma_2^{(2)} &= \gamma_2 B^2 + J\gamma_{13}, \end{aligned}$$

$$\begin{aligned} \gamma_3^{(2)} &= \gamma_3 B^2 + J\gamma_{14}, \\ \gamma_1^{(3)} &= \gamma_7, \\ \gamma_2^{(3)} &= J\gamma_{11} + \gamma_{22} B^1 + \gamma_{25}, \\ \gamma_3^{(3)} &= J\gamma_{15} + \gamma_{22} B^2 + \gamma_{27}, \\ \gamma_1^{(4)} &= \gamma_{18}, \\ \gamma_2^{(4)} &= J\gamma_{12} + \gamma_{23} B^1 + \gamma_{26}, \\ \gamma_3^{(4)} &= J\gamma_1 - J\gamma_{11} + \gamma_{23} B^2 + \gamma_{28}, \\ \gamma_1^{(5)} &= y\gamma_5 + \gamma_{11} + \bar{z}\gamma_{18}, \\ \gamma_2^{(5)} &= \gamma_2 y B^1 + yJ\gamma_{10} + \bar{z}J\gamma_{12} + \gamma_{23} \bar{z} B^1 + \bar{z}\gamma_{26}, \\ \gamma_3^{(5)} &= \bar{z}J\gamma_1 + \gamma_3 y B^1 - yJ\gamma_4 - \bar{z}J\gamma_{11} \\ &\quad + yJ\gamma_{13} + \gamma_{23} \bar{z} B^2 + \bar{z}\gamma_{28}, \\ \gamma_1^{(6)} &= \gamma_1 + z\gamma_6 + \bar{y}\gamma_7 + z\gamma_9 - \gamma_{11} \\ &\quad + [\gamma_{11}, z B^2] + [z B^1, \gamma_{15}] - z\gamma_{20}, \\ \gamma_2^{(6)} &= \gamma_2 z B^2 + \bar{y}J\gamma_{11} + zJ\gamma_{13} + \gamma_{22} \bar{y} B^1 + \bar{y}\gamma_{25}, \\ \gamma_3^{(6)} &= \gamma_3 z B^2 + zJ\gamma_{14} + \bar{y}J\gamma_{15} + \gamma_{22} \bar{y} B^2 + \bar{y}\gamma_{27}, \\ \gamma_1^{(7)} &= z\gamma_5 + \gamma_{12} - \bar{y}\gamma_{18}, \\ \gamma_2^{(7)} &= \gamma_2 z B^1 + zJ\gamma_{10} - \bar{y}J\gamma_{12} - \gamma_{23} \bar{y} B^1 - \bar{y}\gamma_{26}, \\ \gamma_3^{(7)} &= -\bar{y}J\gamma_1 + \gamma_3 z B^1 - zJ\gamma_4 + \bar{y}J\gamma_{11} \\ &\quad + zJ\gamma_{13} - \gamma_{23} \bar{y} B^2 - \bar{y}\gamma_{28}, \\ \gamma_1^{(8)} &= y\gamma_6 - \bar{z}\gamma_7 + y\gamma_9 + [\gamma_{11}, y B^2] \\ &\quad + [y B^1, \gamma_{15}] + \gamma_{15} - y\gamma_{20}, \\ \gamma_2^{(8)} &= \gamma_2 y B^2 - \bar{z}J\gamma_{11} + yJ\gamma_{13} - \gamma_{22} \bar{z} B^1 - \bar{z}\gamma_{25}, \\ \gamma_3^{(8)} &= \gamma_3 y B^2 + yJ\gamma_{14} - \bar{z}J\gamma_{15} - \gamma_{22} \bar{z} B^2 - \bar{z}\gamma_{27}, \\ \gamma_1^{(9)} &= \bar{y}\gamma_7 + \bar{z}\gamma_{18}, \\ \gamma_2^{(9)} &= \bar{y}J\gamma_{11} + \bar{z}J\gamma_{12} + \gamma_{22} \bar{y} B^1 \\ &\quad + \gamma_{23} \bar{z} B^1 + \bar{y}\gamma_{25} + \bar{z}\gamma_{26}, \\ \gamma_3^{(9)} &= \bar{z}J\gamma_1 - \bar{z}J\gamma_{11} + \bar{y}J\gamma_{15} + \gamma_{22} \bar{y} B^2 \\ &\quad + \gamma_{23} \bar{z} B^2 + \bar{y}\gamma_{27} + \bar{z}\gamma_{28}. \end{aligned}$$

APPENDIX B: PROOF OF BÄCKLUND TRANSFORMATIONS

Consider the nine pairs of parametric equations defined by Eqs. (4.2) and (4.3) (each pair share a common subscript k in δ_k). We show that each of these pairs is a (finite) BT for SDYM. For this purpose we cross-differentiate with respect to $\bar{y}$ and $\bar{z}$, and then add by terms, assuming that all integrability conditions are satisfied. We will use the notation

$$F[J] \equiv \partial_{\bar{y}}(J^{-1}J_y) + \partial_{\bar{z}}(J^{-1}J_z),$$

so that $F[J] = 0$ implies the SDYM equation. From the nine pairs of equations we thus obtain, respectively,

$$\begin{aligned} F[J'] - F[J] &= \lambda \partial_y F[J], \\ F[J'] - F[J] &= \lambda \partial_z F[J], \\ F[J'] - F[J] &= \lambda \partial_{\bar{y}} F[J], \\ F[J'] - F[J] &= 0, \\ F[J'] - F[J] &= \lambda y \partial_y F[J], \end{aligned}$$

$$F[J'] - F[J] = \lambda(1 + z \partial_z + \bar{y} \partial_{\bar{y}})F[J],$$

$$F[J'] - F[J] = \lambda z \partial_y F[J],$$

$$F[J'] - F[J] = \lambda(y \partial_z - \bar{z} \partial_{\bar{y}})F[J],$$

$$F[J'] - F[J] = \lambda \bar{y} \partial_{\bar{y}} F[J].$$

Thus $F[J] = 0$ implies $F[J'] = 0$ in each case, which establishes the Bäcklund transformation property.

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⁸We emphasize that the term "prolongation" is used in a very liberal sense in this article, and it may apply to variables, vector fields, differential equations, or ideals. Its precise meaning will always be clear from the context.

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¹¹This type of generalization can be supported by rigorous, general arguments that are not presented here. See, for example, P. H. M. Kersten, Ph.D. thesis, Twente University of Technology, The Netherlands, 1985.

¹²We write $B^1 = J^{-1}D_y J$ and $B^2 = J^{-1}D_z J$, where the x^μ and J are treated as independent variables at this stage (whence the use of total derivatives). Here, $D_y J$ and $D_z J$ are names for prolongation variables in the jet space, which reduce, upon projection to the solution manifold, simply to J_y and J_z .

¹³More generally, if $V = \xi^\mu \partial / \partial x^\mu + A \partial / \partial J$, then the evolutionary representative (3.14) of V is constructed by taking $Q = A - \xi^\mu D_\mu J$.