2006-07-14

Knots Not for Naught

Sharleen Adrienne Roberts
Brigham Young University - Provo

KNOTS NOT FOR NAUGHT

by

Sharleen Adrienne Roberts

A project submitted to the faculty of

Brigham Young University

in partial fulfillment of the requirements for the degree of

Master of Science

Department of Mathematics

Brigham Young University

Advisor: David Wright

August 2006

1 Introduction

A knot is a closed, piecewise-smooth curve of finite length in $\Re^3$ that does not intersect itself. A link is made up of one or more knots, which are pairwise disjoint. Informally, two knots are equivalent if one can be formed from the other by moving it around without allowing one piece of the knot to pass through another [2]. A formal definition for the equivalence of two knots can be found in Godsil and Royle's text *Algebraic Graph Theory*.

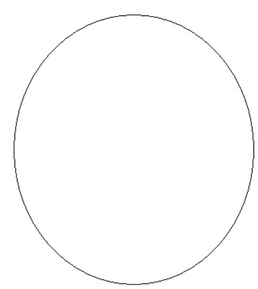
One of the most interesting problems in knot theory is deciphering whether two links are equivalent. Much progress has been made with polynomial link invariants. A polynomial link invariant assigns each link a polynomial that is the same for all equivalent links. If two links result in two distinct polynomials, then we can conclude that the links are distinct. However, if two links have the same polynomial, then our results are inconclusive; we cannot say whether they are equivalent or not.

Two famous polynomial link invariants are the Jones polynomial and the Alexander polynomial. The Alexander polynomial was discovered in 1923 and the Jones in 1983. More recently is the discovery of the Homfly polynomial. Its name is derived from the first letter of each of the co-discoverers' last names. The Homfly polynomial incorporates the information of both the Alexander and Jones polynomials into a single polynomial.

The way in which we will illustrate links is to use link diagrams. For a given link, the link diagram is obtained by projecting the knot onto $\Re^2$ along with information at the crossings to inform us of which piece lies under which. One way to think of the link diagram is to picture "the shadow that a link would cast onto a wall if a light were shone through it"[1].

If we view a knot as a single piece of string with its ends glued together, then for two links to be equivalent, we must be able “to get from one [link] to the other by a series of slides and distortions of the string which do not tear it, and do not pass one segment of string through another” [2]. The links are different if such a sequence cannot be found. Determining whether two links are equivalent, by proving that there is no such sequence, is difficult and cumbersome. This is why polynomial link invariants, such as the Homfly polynomial, are so powerful. Rather than attempting every sequence of allowable distortions to the string, the Homfly polynomial is calculated for each of the two links, and the polynomials are then compared.

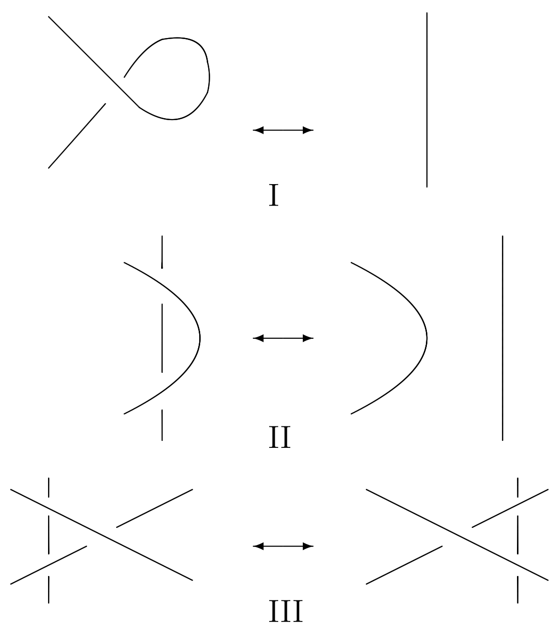
Determining whether two links are equivalent is not a trivial task. For example, one difficult problem in knot theory is determining whether a link is the unknot (equivalent to a circle lying in $\mathfrak{R}^2$) or not. It is not obvious by looking at a link diagram whether it is equivalent to the unknot. For example, the two pictures below are both the unknot, although they are represented quite differently [3].



The Homfly polynomial is useful in determining if a link is not equivalent to the unknot. This is done by showing that its polynomial is different from that of the polynomial of the unknot.

As mentioned earlier, sometimes the polynomial link invariant fails to provide new information about the link. For example, the two knots above have the same Homfly polynomial and are also equivalent, but the two knots below have the same Homfly polynomial, and yet are not equivalent to each other. The two knots below were obtained from a handout posted online by Greg McNulty. It can be shown that the Homfly polynomials for these two knots are the same and that their Kauffman brackets (a partial link invariant) lead to the conclusion that they are different knots.

There are three important operations that can be performed on links which do not actually change the link. These three moves are called the Reidemeister moves and are shown in the illustration below.



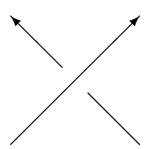
A theorem in knot theory states that two links are equivalent if and only if one link can be obtained from the other through a series of Reidemeister moves and planar isotopies [1]. This fact allows us to change the way a link is represented without actually changing the link, and consequently the Homfly polynomial. Also, it gives us a way to verify that the Homfly polynomial is in fact a polynomial link invariant. Since the Homfly polynomial is unchanged by each of the Reidemeister moves, then by the above theorem, the Homfly polynomial is a polynomial link invariant. The Reidemeister moves are useful in changing links into their simplified equivalents, and are useful as a checklist for proving a polynomial is a polynomial link invariant.

2 Calculating the Homfly Polynomial

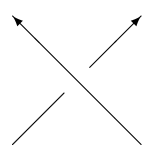
2.1 The Mechanics

The Homfly polynomial is not an ordinary polynomial; it is a 2-variable Laurent polynomial in the variables l and m. Thus, the polynomials in this paper will be a linear combination, using integer coefficients, of terms of the form: $l^k m^n$ where k and n are integers and l and m are the variables.

In order to calculate the Homfly polynomial for a link, we must first orient the closed curve(s) making up the link. We then select one of the crossings on the link. Each crossing is one of two kinds: $\{+\}$ and $\{-\}$.

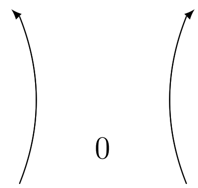


$+$



$-$

Once a crossing is selected we will then form two new links by reversing the crossing type or by deleting the crossing altogether. Suppose we selected a crossing of type $\{+\}$, then we let $L_+ = L$ and we form two knots, L_- and L_0 from L_+. L_- is obtained by changing the selected crossing to a $\{-\}$ type crossing. L_0 is achieved by removing the selected crossing so that it now has the $\{0\}$ form, which is shown below.



If we had started with a crossing of type $\{-\}$, then we would have set $L_- = L$ and we would have formed L_+ and L_0 from L_- in a manner analogous to that described above.

The Homfly polynomial is defined using skein relations, which are given below.

$$P(\text{unknot}) = 1$$

$$lP(L_+) + l^{-1}P(L_-) + mP(L_0) = 0$$

And for two knots, L_1 and L_2 that are not intertwined:

$$P(L_1 \cup L_2) = -(l + l^{-1})m^{-1}P(L_1)P(L_2).$$

With a crossing already selected, we then check which type of crossing it is and solve the second skein relation for that particular type of crossing. That is, if we have selected a crossing of type $\{+\}$, then we would solve the second skein relationship for $P(L_+)$. The result is:

$$(1) \qquad P(L+) = -l^{-2}(P(L_-) + lmP(L_0)).$$

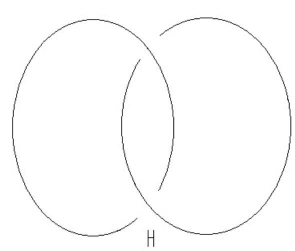
Similarly, we find that:

$$(2) \qquad P(L_-) = -l(lP(L_+) + mP(L_0)).$$

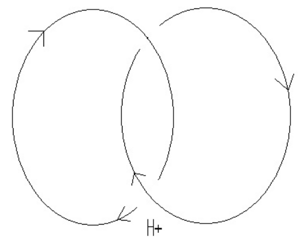
If we have selected a crossing of type $\{+\}$, then we would use Equation (1). Next we would find $P(L_-)$ by redrawing the knot with the $\{+\}$ crossing changed to a $\{-\}$ crossing. The new link is called L_-. Its polynomial is found by either choosing a crossing and repeating the process or if L_- has no crossings, then its polynomial can be found directly by using the first and third skein relations. We then would do the same to find $P(L_0)$. First we would redraw $L_+ = L$ without the $\{+\}$ crossing in order to obtain L_0. $P(L_0)$ is then found by either picking a crossing and repeating the process or, if L_0 has no crossings, by calculating its polynomial directly through the use of the first and third skein relations. In both cases we would continue the same process until we reach the unknot, or a set of non-crossing unknots; the Homfly polynomial for such knots can be found directly through the first and third skein relations. Once this is done, we plug everything back into the skein relations and find $P(L_+)$,which gives us the Homfly polynomial of our knot.

2.2 The Hopf Link Example

To illustrate this algorithm, we will demonstrate the calculation of the Homfly polynomial for the one of the orientations of the well-known Hopf link. We will call the Hopf link H.



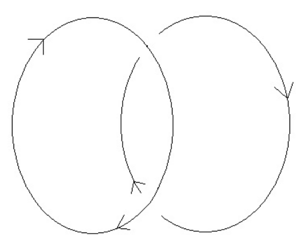
Let us note that the Homfly polynomial of a knot does not depend on the orientation, but for a link it does. In this example we will be calculating the Homfly polynomial of the orientation of H shown below. We will call this oriented link H^+ since its crossings are all of the $\{+\}$ type. To calculate the Homfly polynomial, we will start by using a skein relation involving a selected crossing.



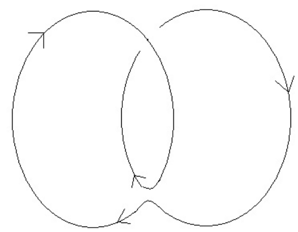
The selection is arbitrary. For this example we will select the bottom crossing which is a $\{+\}$ crossing. Since we are labeling the link by H^+, we will use the skein relationship:

$$P({H^+}_+) = -l^{-2}(P({H^+}_-) + lmP({H^+}_0))$$

When we redraw H^+ by reversing the selected crossing , that is we change the strand crossing over to crossing under, we get ${H^+}_-$:



By performing a Reidemeister II move, we can redraw this link equivalently as two independent unknots. By using the skein relationship for links that are not intertwined, we see that $P({H^+}_-) = -(l + l^{-1})m^{-1}P(\text{unknot})P(\text{unknot}) = -lm^{-1} - l^{-1}m^{-1}$. When we redraw our H^+ by removing the selected crossing, the resulting knot is ${H^+}_0$.



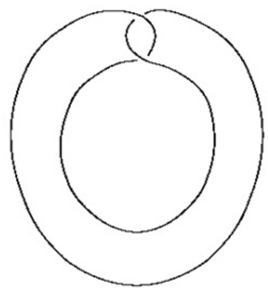
By performing a Reidemeister I move, we can redraw this link equivalently as the unknot. Thus, $P(H^{+}{}_{0}) = 1$. Now we plug our results into:$P(H^{+}{}_{+}) = -l^{-2}(P(H^{+}{}_{-}) + lmP(H^{+}{}_{0}))$ and we get

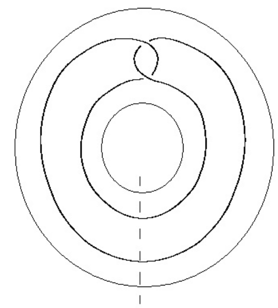
$$P(H^{+}{}_{+}) = -l^{-2}(-lm^{-1} - l^{-1}m^{-1} + lm) = l^{-1}m^{-1} + l^{-3}m^{-1} - l^{-1}m$$

This is the Homfly polynomial for our particular orientation of the Hopf link, H^{+}.

3 Statement of the Problem

The goal of this paper is to find the Homfly polynomial for each knot in a specific family of knots. This family of knots is generated from placing the Whitehead link (see picture below) into a solid torus, slicing the torus at a spot where the Whitehead has no crossings and then twisting the torus 360° in either direction an integral number of times. Let $L(n)$ denote the knot obtained by twisting the torus 360°, n times. Note that n is an integer. Let the twists be towards the center of the torus for positive n and away from the center for negative n. Below are some pictures to help illustrate this idea:





4 Motivation for the Problem

A paper by Masaharu Kuono, Kimihiko Motegi, and Tetsuo Shibuya entitled *Twisting and Knot Types*, has the following abstract.

> Let K be a trivial or nontrivial knot lying in a solid torus V, knotted or unknotted in S^3. This paper studies whether the isotopy type and the type of K are changed by twisting homeomorphisms of V. Denote a twisting homeomorphism on V of n twists by h_n....Finally, it is proved that if K is a trivial knot in an unknotted solid torus V with [wrapping number] $W_v(K) \geq 2$, then for every integer $n \neq 0$, $h_n(K)$ is nontrivial except when K is homotopic in V to a (2,1)-curve on the boundary of V, and in which case $h_n(K)$ is trivial only when $n = 0$ or $n = 1$.

In this paper we will show that when the Whitehead link is lying in a solid torus, then $h_n(\text{Whitehead}) = L(n)$ is nontrivial for all $n \neq 0$ and that each $L(n)$ is distinct. This will be shown by calculating the Homfly polynomial for $L(n)$. This goes beyond what can be concluded from the above abstract. Since the Whitehead link is clearly a trivial knot, not a (2,1)-curve on the boundary of the solid torus, and has wrapping number $W_V(\text{Whitehead}) = 2$, then by the work in *Twisting and Knot Types*, we can conclude that $h_n(\text{Whitehead}) = L(n)$ is nontrivial for all $n \neq 0$.

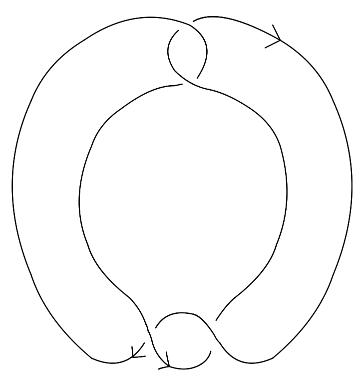
This paper, through the use of the Homfly polynomial, will prove this and more. This paper will, in addition, give a general formula for the Homfly polynomial of $L(n)$ as well as prove that each knot $L(n)$ is distinct.

5 Solution to the Problem

5.1 Twisting Toward the Center of the Torus

We will first begin by calculating the Homfly polynomial for the first few twists (twisting towards the center of the torus). The knot resulting from one twist is shown below. We will call this knot $L(1)$ and we will start off by finding the Homfly polynomial for this knot.

We must first orient the curve, and then pick a crossing. Below is a picture illustrating which crossing we picked and the orientation.



$L(1)_{-}$

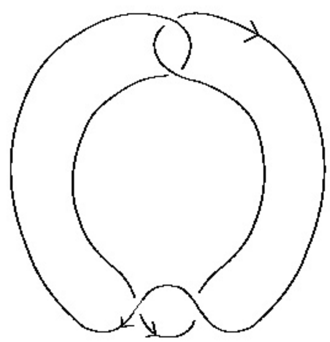
Because of the type of crossing we picked,$\{-\}$, we will be using the skein relationship:

$$P(L(1)_{-}) = -l(lP(L(1)_{+}) + mP(L(1)_{0}))$$

to find the Homfly polynomial of $L(1)$.

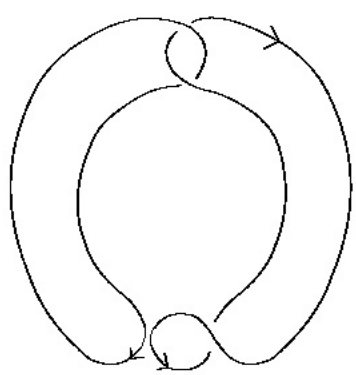
One thing that we should note is that the Whitehead link is equivalent to the unknot. This can be shown through a series of Reidemeister moves.

Let us first examine the knot obtained by changing our chosen crossing so that the strand crossing over is now crossing under. This knot is $L(1)_+$.



By a Reidemeister II move, we see that $L(1)_+$ is equivalent to the Whitehead link, which, in turn, is equivalent to the unknot and thus $P(L(1)_+) = 1$.

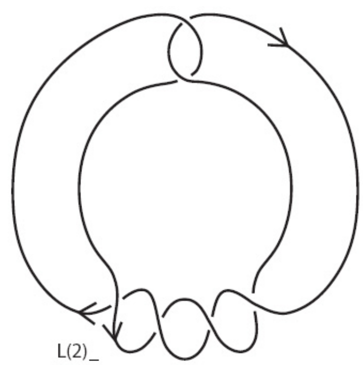
Next, we will look at $L(1)_0$.



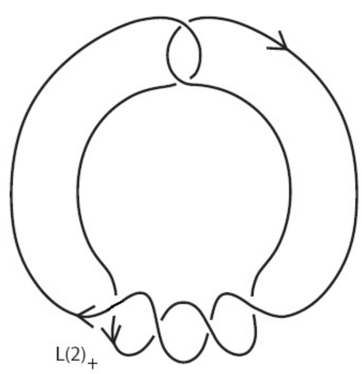
In this case, a Reidemeister I move changes the link to the link H^+. Since we have already calculated the Homfly polynomial for the H^+, we get the result: $P(L(1)_0) = l^{-1}m^{-1} + l^{-3}m^{-1} - l^{-1}m$. Now we will put together our findings for $P(L(1)_+)$ and $P(L(1)_0)$ into the skein relationship $P(L(1)_-) = -l(lP(L(1)_+) + mP(L(1)_0))$. The result is:$P(L(1)_-) = -l(l(1) + m(l^{-1}m^{-1} + l^{-3}m^{-1} - l^{-1}m))$. When simplified, we get:

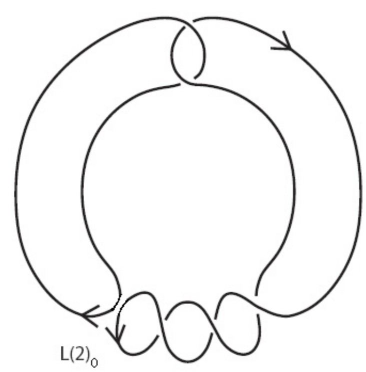
$$P(L(1)) = m^2 - l^2 - 1 - l^{-2}.$$

Let $L(2)$ be the link generated, as described above, by twisting towards the center 360°, two times. Again, we will first orient the knot and then select a crossing. The crossing we have selected is illustrated in the picture below.



Since the crossing has type $-$, we will, again, be using the skein relationship:$P(L(2)_-) = -l(lP(L(2)_+) + mP(L(2)_0))$. Below are drawings representing $L(2)_+$ and $L(2)_0$ respectively.



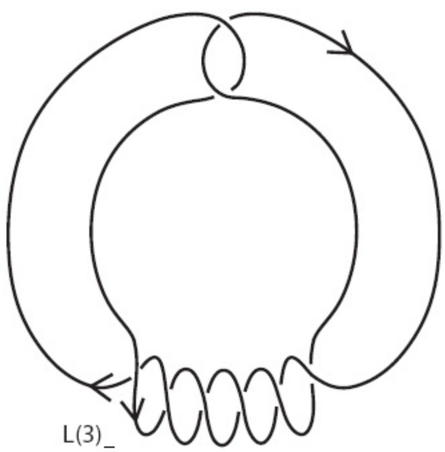


Note that by a Reidemeister II move, $L(2)_+$ can be transformed into $L(1)$ and that by three Reidemeister I moves, $L(2)_0$ can be transformed into H^+. Thus,

$P(L(2)) = -l(lP(L(1)) + mP(H^+)) = -l(l(m^2 - l^2 - 1 - l^{-2}) + m(l^{-1}m^{-1} + l^{-3}m^{-1} - l^{-1}m))$, and simplifying this expression, we get the result:

$$P(L(2)) = m^2 - l^2m^2 + l^2 + l^4 - l^{-2}.$$

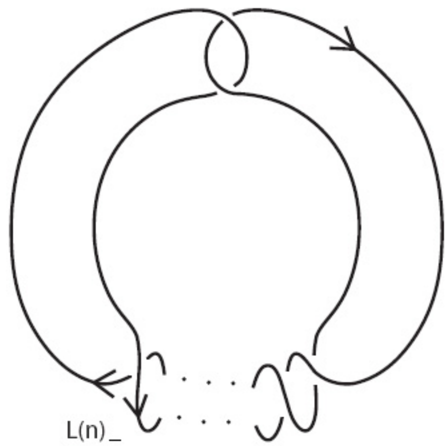
Now let us look at $L(3)$ (obtained from 3 twists). Again, we will choose a crossing and the one chosen is illustrated below.



Because of the type of crossing it is, we will use the skein relationship: $P(L(3)_-) = -l(lP(L(3)_+) + mP(L(3)_0))$. It can be shown that $L(3)_+$ is equivalent to $L(2)$ through Reidemeister II moves and that $L(3)_0$ is equivalent to H^+ through Reidemeister I moves. Thus,$P(L(3)_-) = -l(lP(L(2)) + mP(H^+)) = -l(l(m^2 - l^2m^2 + l^2 + l^4 - l^{-2}) + m(l^{-1}m^{-1} + l^{-3}m^{-1} - l^{-1}m))$. Simplifying this expression gives us the result:

$$P(L(3)) = m^2 + l^4m^2 - l^2m^2 - l^6 - l^4 - l^{-2}.$$

Let us now look at , the knot obtained from n twists. We will follow the pattern set forth already and select the bottom crossing furthest to the left, as is illustrated below.



Because of the type of crossing it is, we will use the skein relationship: $P(L(n)_-) = -l(lP(L(n)_+) + mP(L(n)_0))$. From our previous work, it should be clear that a Reidemeister II move will transform $L(n)_+$ into $L(n-1)$ and $2n-1$ Reidemeister I moves will transform $L(n)_0$ into H^+. Noting this fact we can rewrite our Homfly polynomial expression for $L(n)$ as: $P(L(n)) = -l(lP(L(n-1)) + mP(H^+)) = -l(lP(L(n-1)) + m(l^{-1}m^{-1} + l^{-3}m^{-1} - l^{-1}m)))$. That is,

$$P(L(n)) = m^2 - l^2 P(L(n-1)) - 1 - l^{-2}.$$

Now we have a recursive formula for the Homfly polynomial of $L(n)$. From this and our result for $L(3)$, we get that $P(L(4)) = m^2 - l^6m^2 + l^4m^2 - l^2m^2 + l^8 + l^6 - l^{-2}$. This suggests the following formula for $L(n)$.

$$P(L(n)) = m^2((-1)^{n-1}l^{2(n-1)} + (-1)^{n-2}l^{2(n-2)} + \cdots$$

$$\cdots + (-1)^2l^{2(2)} + (-1)^1l^{2(1)} + (-1)^0l^{2(0)}) + (-1)^n(l^{2n} + l^{2(n-1)}) - l^{-2}.$$

We will prove this is the correct formula for $L(n)$ by the method of mathematical induction. First we must show that our formula is correct for a specific n. Note that when $n = 0$, then $P(L(n)) = (-1)^0(l^{2(0)} + l^{2(0-1)}) - l^{-2} = 1 + l^{-2} - l^{-2} = 1$.

This is correct, since we know that the Whitehead link is equivalent to the unknot and its polynomial is 1. Thus we have completed our first step of the induction.

Next, let us assume that our formula is true for $n = k$ where k is a nonnegative integer. We will show that the formula also holds for $n = k + 1$.

From our recursive formula we have that $P(L(k+1)) = m^2 - l^2 P(L(k)) - 1 - l^{-2}$. Since we are assuming that $P(L(k)) = m^2((-1)^{k-1} l^{2(k-1)} + (-1)^{k-2} l^{2(k-2)} + ... + (-1)^2 l^{2(2)} + (-1)^1 l^{2(1)} + (-1)^0 l^{2(0)}) + (-1)^k (l^{2k} + l^{2(k-1)}) - l^{-2}$, then we have:

$$P(L(k+1)) = m^2 - l^2(m^2((-1)^{k-1} l^{2(k-1)} + (-1)^{k-2} l^{2(k-2)} + \cdots$$

$$\cdots + (-1)^1 l^{2(1)} + (-1)^0 l^{2(0)}) + (-1)^k (l^{2k} + l^{2(k-1)}) - l^{-2}) - 1 - l^{-2}.$$

Some simplification gives us:

$$P(L(k+1)) = m^2((-1)^k l^{2(k)} + (-1)^{k-1} l^{2(k-1)} + ... \cdots$$

$$\cdots + (-1)^0 l^{2(0)}) + (-1)^{k+1} (l^{2(k+1)} + l^{2(k)}) - l^{-2}.$$

Which is what we would get if we plugged in $k + 1$ into our proposed formula for $L(n)$. Thus, by mathematical induction, for any nonnegative integer n, it is the case that

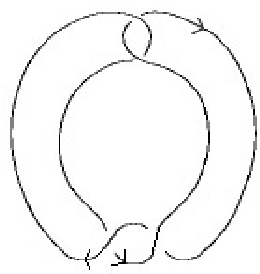
$$P(L(n)) = m^2((-1)^{n-1} l^{2(n-1)} + (-1)^{n-2} l^{2(n-2)} + ... + \cdots$$

$$\cdots + (-1)^1 l^{2(1)} + (-1)^0 l^{2(0)}) + (-1)^n (l^{2n} + l^{2(n-1)}) - l^{-2}.$$

Note that whenever $n \neq 0$ then $P(L(n)) \neq 1$ and hence is nontrivial. Also, note that for any two distinct positive integers m and n, $P(L(m)) \neq P(L(n))$. Thus, each twist gives a new, and distinctive, nontrivial knot.

5.2 Twisting in the Other Direction

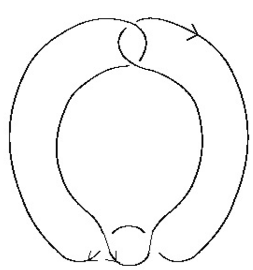
We expect similar results when we twist the torus in the opposite direction. One twist in this direction yields the knot below:

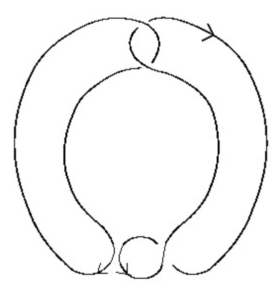


The orientation and selected crossing are shown in the picture above. Notice that no matter how we orient the curve, every crossing is of the $\{+\}$ type. Thus we must use the relation:

$$P(L(-1)_+) = -l^{-2}(P(L(-1)_-) + lmP(L(-1)_0)).$$

Below are the knots $L(-1)_-$ and $L(-1)_0$, respectively.

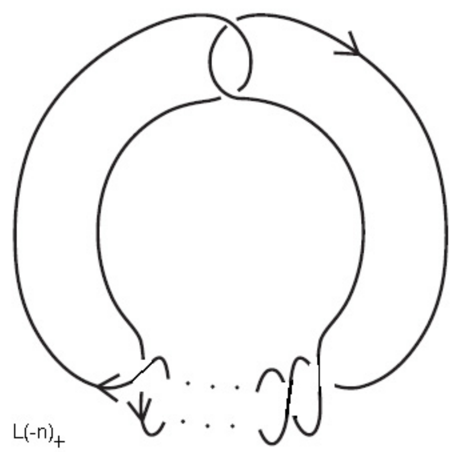




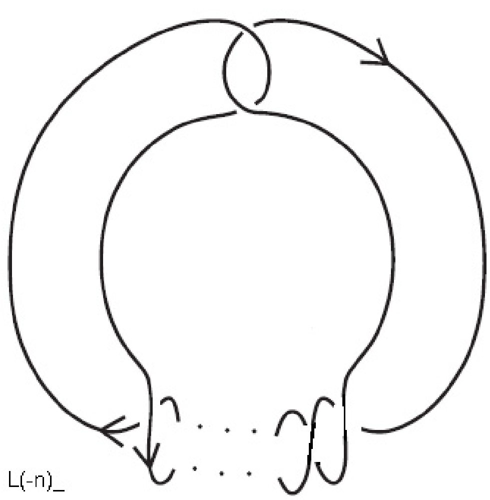
By a Reidemeister II move, $L(-1)_-$ becomes the Whitehead link and by a Reidemeister I move, $L(-1)_0$ becomes H^+, our orientation of the Hopf link. Thus, $P(L(-1)_-) = 1$ and $P(L(-1)_0)) = l^{-1}m^{-1} + l^{-3}m^{-1} - l^{-1}m$. Using the above mentioned skein relation, we get:

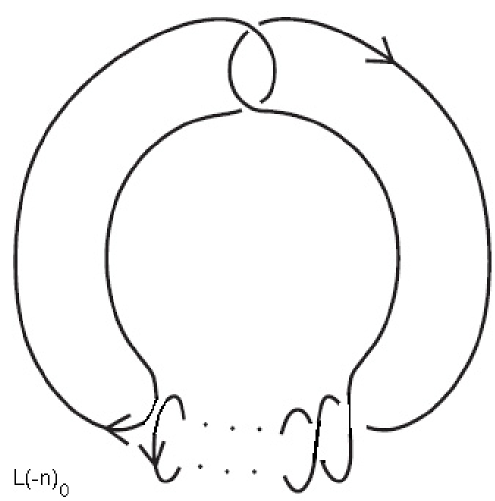
$$P(L(-1)_+) = -2l^{-2} - l^{-4} + l^{-2}m^2.$$

Similar to what happened when we twisted towards the center of the torus, we can find a recurrence relation for $L(-n)$ where n is a nonnegative integer. We will orient $L(-n)$, as shown below, and use the crossing that is labeled to find the Homfly polynomial of $L(-n)$.



Below are the pictures of $L(-n)_-$ and $L(-n)_0$.





Clearly $L(-n)_-$ is equivalent to $L(-n+1)$ by a Reidemeister II move and that $L(-n)_0$ is equivalent to H^+ by $2n-1$ Reidemeister I moves. Thus $P(L(-n)_-) = P(L(-n+1))$ and $P(L(-n)_0) = P(H^+)$. This gives us the recurrence relation:

$$P(L(-n)_+) = -l^{-2}(P(L(-n+1))+lm(P(H^+)) = -l^{-2}P(L(-n+1))-l^{-2}-l^{-4}+m^2l^{-2}.$$

Using this recurrence relation, it can be shown that $P(L(-2)) = m^2(-l^{-4} + l^{-4}) + l^{-6} + l^{-4} - l^{-2}$ and $P(L(-3)) = m^2(l^{-6} - l^{-4} + l^{-2}) - l^{-6} - l^{-8} - l^{-2}$.

This suggests the formula:

$$P(L(-n)) = (-1)^n(l^{-2n} + l^{-2(n-1)}) + m^2((-1)^{-n+1}l^{2(-n)} + (-1)^{-n+2}l^{2(-n+1)} + \cdots$$

$$\cdots + (-1)^2l^{2(-1)}) - l^{-2}.$$

This can be proven to be the exact formula for $P(L(-n))$ where n is a nonnegative integer. Its proof is similar to the proof for $P(L(n))$ where n is a nonnegative integer. The proof consists of using the recurrence relation involving $P(L(-n))$ along with the method of mathematical induction.

From the formula for $P(L(-n))$ we can see that each twist gives a unique knot and that each twist provides a nontrivial knot. This is as we expected.

In conclusion, we find from our proven formula for $L(n)$ that if $n \neq 0$ then $P(L(n))$ is nontrivial. In addition, we see that if m and n are distinct integers, then $P(L(m)) \neq P(L(n))$. Thus, each knot in our family of knots is distinct.

References

[1] Chris Godsil and Gordon Royle, *Algebraic Graph Theory*, Springer-Verlag, New York, Inc., 2001.

[2] Wikipedia, http://en.wikipedia.org/wiki/Knot_theory.

[3] Wikipedia, http://en.wikipedia.org/wiki/Unknot.

[4] Greg McNulty, http://www.math.ucla.edu/∼radko/191.1.05w/greg_handout.pdf.

[5] ThinkQuest, http://library.thinkquest.org/12295/data/Invariants/Articles/HOMFLY.html.

[6] WolframMathWorld, http://mathworld.wolfram.com/HOMFLYPolynomial.html.

[7] Kuono, Mafaharu; Motegi, Kinihiko; Shibuya, Tetsuo, "Twisting and Knot Types", J. Math. Soc. Japan, 44(1992), 2, 199-16.