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CONJUGACY CLASSES OF THE PIECEWISE LINEAR GROUP

by

Matthew L. Housley

A thesis submitted to the faculty of

Brigham Young University

in partial fulfillment of the requirements for the degree of

Master of Science

Department of Mathematics

Brigham Young University

August 2006

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BRIGHAM YOUNG UNIVERSITY

GRADUATE COMMITTEE APPROVAL

of a thesis submitted by

Matthew L. Housley

This thesis has been read by each member of the following graduate committee and by majority vote has been found to be satisfactory.

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ABSTRACT

CONJUGACY CLASSES OF THE PIECEWISE LINEAR GROUP

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Master of Science

The piecewise linear group is the set of all piecewise linear orientation preserving homeomorphisms from the interval to itself under the operation of composition. We present here a complete set of invariants to classify the conjugacy classes of this group. Our approach to this problem relies on the factorization of elements into elements having only a single breakpoint.

ACKNOWLEDGMENTS

I would like to thank my advisor Jim Cannon for his patience with my sometimes meandering efforts in mathematics research. His insights and encouragement have been crucial to the completion of this work.

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Chapter 1

Introduction

Definition 1.1. Let $PL_0(I)$ be the group of orientation preserving piecewise linear homeomorphisms from the interval $[0, 1]$ to itself (Figure 1.1). Multiplication is given by composition. We'll refer to this as the piecewise linear group.

Interest in $PL_0(I)$ within the mathematical community has been primarily motivated by Thompson's group F , which may be treated as a group of piecewise linear functions on $[0, 1]$ with power of 2 slopes and dyadic breakpoints and thus as a subgroup of $PL_0(I)$. It was with a desire to better understand the structure of F that this work was commenced. The conjugacy problem in Thompson's group was solved by Victor Guba and Mark Sapir[3], but $PL_0(I)$ continues to be a rich source of interesting mathematical structure[1].

Many of the results of this paper were reported previously by Matthew Brin and Craig Squier[2]. However, the theorems and proofs were developed independently, with the exception of Theorem 2.21. Our approach differs from theirs in its heavier reliance on breakpoint cancellation and the factorization of elements into simple generators. These methods provide a nice geometric intuition for working with $PL_0(I)$.

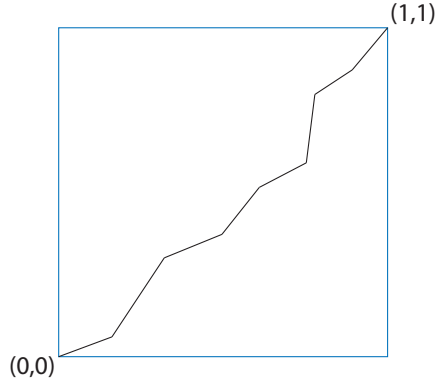


Figure 1.1: An Element of $PL_0(I)$

1.1 Simple Elements

Definition 1.2. We say that $\phi \in PL_0(I)$ is *simple* if ϕ has exactly 1 breakpoint (Figure 1.2). Let $\phi_{a,b}$ denote a simple element with breakpoint (a, b) . Then,

$$f(x) = \begin{cases} \frac{b}{a}x, & \text{if } x \leq a, \\ b + \frac{b-1}{a-1}(x-a), & \text{if } x > a. \end{cases} \quad (1.1)$$

(Note that $a \neq b$. If $a = b$ then (a, b) is not a breakpoint of the graph.)

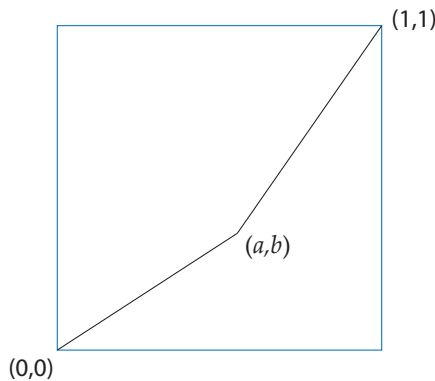


Figure 1.2: A Simple Element

Multiplication by a simple element has a nice geometric interpretation. Multiplying an element γ on the right by $\phi_{a,b}$ is equivalent to drawing the line $x = b$

through the graph of γ and sliding it to the line $x = a$ while linearly rescaling each side of the graph (Figure 1.3). Likewise, left multiplication by $\phi_{a,b}$ is equivalent to

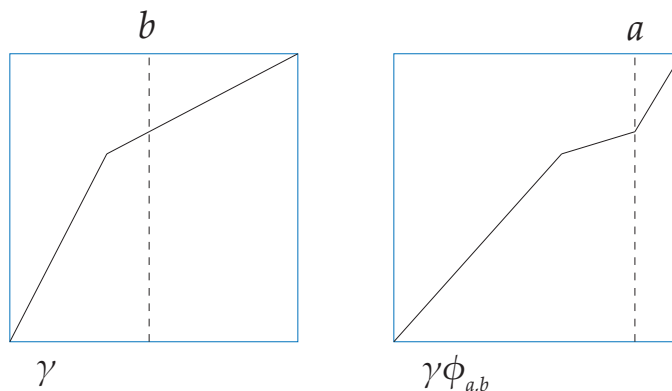


Figure 1.3: Right Multiplication

drawing the line $y = a$ through the graph of γ and sliding it to the line $y = b$ (Figure 1.4). In the product $\phi\gamma$, we may view ϕ as an action that moves points in the graph of γ to points in the graph of $\phi\gamma$. In particular, left multiplication by ϕ sends $(x, \gamma(x))$ to $(x, \phi \circ \gamma(x))$ so that left multiplication is a map that modifies the y -coordinates of points in the graph of γ .

For right multiplication, observe that $\gamma\phi = (\phi^{-1}\gamma^{-1})^{-1}$. Start with the point $(x, \gamma(x))$ in the graph of γ . This becomes $(\gamma(x), x)$ in the graph of γ^{-1} . Left multiplying by ϕ^{-1} moves this point to $(\gamma(x), \phi^{-1}(x))$. Applying the inverse again gives us back $\gamma\phi$ and the point $(\phi^{-1}(x), \gamma(x))$. Thus, right multiplication sends $(x, \gamma(x))$ to $(\phi^{-1}(x), \gamma(x))$ and is an action that modifies the x -coordinates of graph points. These interpretations of left and right multiplication fit with our earlier geometric views of left and right simple multiplication in that left and right multiplication rescale y and x -coordinates respectively.

We will say that $(x, \phi \circ \gamma(x))$ *corresponds* to $(x, \gamma(x))$ under left multiplication by ϕ and that $(\phi^{-1}(x), \gamma(x))$ *corresponds* to $(x, \gamma(x))$ under right multiplication by ϕ .

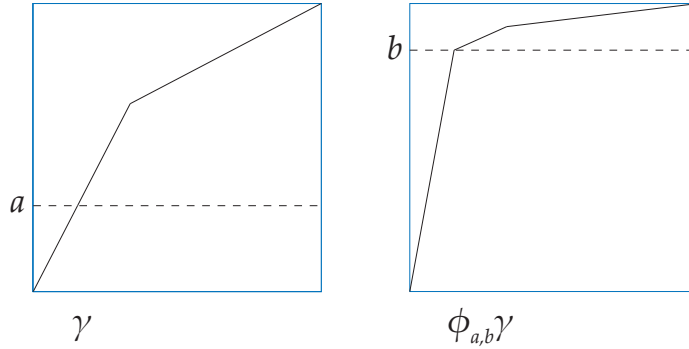


Figure 1.4: Left Multiplication

For the sake of economy, we will often refer to points of a graph by their x -coordinates. In this case, $\phi^{-1}(x)$ will correspond to x under right multiplication by ϕ and we will call ϕ^{-1} the *correspondence map* for right multiplication by ϕ . Left multiplication does not change x -coordinates, so such a map will not be necessary in this case.

1.2 Generating the Group

Definition 1.3. We define the grade of $\phi \in PL_0(I)$ as the number of breakpoints of ϕ .

It will often be useful to factor an element into elements of lower grade. Geometrically, we can visualize multiplying an element γ by an appropriate simple element to remove one of its breakpoints. (Figure 1.5). We will use the following definition extensively in working with breakpoints.

Definition 1.4. Given $\phi \in PL_0(I)$ and a point $x \in (0, 1)$, define $k_\phi(x) = \frac{m}{m'}$, where m' and m are the onesided derivatives of ϕ at the point x from the left and right respectively. We call $k_\phi(x)$ the slope ratio of ϕ at x . Thus, $x \in (0, 1)$ is a breakpoint for ϕ if and only if $k_\phi(x) \neq 1$.

Theorem 1.1. *The Chain Rule: Let γ and β be elements of $PL_0(I)$. Then,*

$$k_{\gamma\beta}(x) = k_\gamma(\beta(x))k_\beta(x)$$

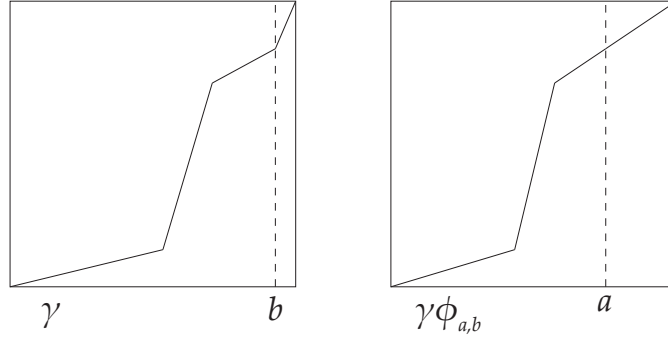


Figure 1.5: Grade Reduction by Right Multiplication

Proof. This result follows immediately from the following two equations, which are results of the chain rule for the ordinary derivative:

$$m_{\gamma\beta}(x) = m_{\gamma}(\beta(x))m_{\beta}(x)$$

$$m'_{\gamma\beta}(x) = m'_{\gamma}(\beta(x))m'_{\beta}(x)$$

□

Corollary 1.1. $k_{\gamma}(x) = \frac{1}{k_{\gamma^{-1}}(\gamma(x))}$.

Proof.

$$k_{\gamma^{-1}\gamma}(x) = k_{\gamma^{-1}}(\gamma(x))k_{\gamma}(x) = 1$$

□

Theorem 1.2. Let $k : (0, 1) \rightarrow \mathbb{R}^+$ be equal to 1 except at finitely many points of $(0, 1)$. Then, $k(x)$ is the slope ratio function of exactly one element of $PL_0(I)$.

Proof. We'll attempt to construct an element $\alpha \in PL_0(I)$ using $k(x)$ by assuming that α has initial slope m_0 . Let $x_1 < \dots < x_{n-1}$ be the breakpoints of α . Let $x_0 = 0$ and $x_n = 1$. Then,

$$\begin{aligned} \alpha(1) = & \\ & (x_1 - x_0)m_0 + (x_2 - x_1)m_0k(x_1) + (x_3 - x_2)m_0k(x_1)k(x_2) + \dots \\ & \dots + (x_n - x_{n-1})m_0k(x_1) \cdots k(x_{n-1}) \end{aligned}$$

$$= 1$$

It follows that m_0 is uniquely determined by $k(x)$. Thus, the function $k(x)$ uniquely determines α . □

Consider how slope ratios behave under left simple multiplication by $\phi = \phi_{a,b}$. For an element γ , the chain rule implies that

$$k_{\phi\gamma}(x) = k_\phi(\gamma(x))k_\gamma(x)$$

Thus, $k_{\phi\gamma}(x) = k_\gamma(x)$ almost everywhere. Left multiplication by $\phi_{a,b}$ changes the slope ratio of γ at exactly one x -coordinate in $(0, 1)$, namely at $x = \gamma^{-1}(a)$.

For right multiplication by $\phi = \phi_{a,b}$,

$$k_{\gamma\phi}(x) = k_\gamma(\phi(x))k_\phi(x)$$

so that

$$k_{\gamma\phi}(x) = k_\gamma(\phi(x)) \text{ for } x \neq a$$

and

$$k_{\gamma\phi}(a) = k_\gamma(\phi(a))k_\phi(a) = k_\gamma(b)k_\phi(a)$$

or, in terms of the correspondence map $x \mapsto \phi^{-1}(x) = x'$

$$k_{\gamma\phi}(x') = k_\gamma(x) \text{ for all } x' \neq a$$

$$k_{\gamma\phi}(a) = k_\gamma(b)k_\phi(a)$$

Theorem 1.3. (*Left Cancellation*) *Let P be the set of breakpoints of $\gamma \in PL_0(I)$. For each $p \in P$, there exists a unique simple element λ such that the set of breakpoints of $\lambda\gamma$ is $P \setminus \{p\}$. We call λ the simple element that left cancels the breakpoint p of γ .*

Proof. Let $\lambda = \phi_{a,b}$. Left multiplying γ by $\phi_{a,b}$ changes the slope ratio of γ only at $\gamma^{-1}(a)$. Thus, $a = \gamma(p)$. In addition, we require that

$$k_{\phi\gamma}(p) = 1.$$

Thus,

$$\begin{aligned} k_\phi(\gamma(p))k_\gamma(p) &= 1 \\ k_\phi(\gamma(p)) &= \frac{1}{k_\gamma(p)}. \end{aligned} \tag{1.2}$$

Now, observe that the slope ratio of $\phi_{a,b}$ at a is $k_\phi(a) = \frac{1/b - 1}{1/a - 1}$. Given $a = \gamma(p)$, we may solve for a unique b such that equation 1.2 is satisfied. \square

Theorem 1.4. (*Right Cancellation*) *Let P be the set of breakpoints of $\gamma \in PL_0(I)$. For each $p \in P$, there exists a unique simple element ρ such that the set of breakpoints of $\gamma\rho$ is*

$$\{\rho^{-1}(q) \mid q \in P, q \neq p\}$$

We call ρ the simple element that right cancels the breakpoint p of γ .

Proof. Saying that x is a breakpoint of α is equivalent to saying that $\alpha(x)$ is a breakpoint of α^{-1} . Then, given γ^{-1} with breakpoint set $\{\gamma(q) \mid q \in P\}$, we want to find a simple element ρ^{-1} such that $\rho^{-1}\gamma^{-1}$ has breakpoint set $\{\gamma(q) \mid q \in P, q \neq p\}$. By Theorem 1.3 ρ^{-1} , and thus ρ , is uniquely determined. \square

Theorem 1.5. *Given $\gamma \in PL_0(I)$, left or right multiplying γ by a simple element changes the grade of γ by 1, 0 or -1 .*

Proof. We'll take the left multiplication case first. Since left multiplication by a simple element changes the slope ratio of γ at only one point, grade changes of 1, 0 or -1 are the only possibilities. Left multiplying by a simple element that cancels one of the breakpoints of γ changes the grade of γ by -1 . Suppose that p is not a breakpoint of γ . Then, if $\gamma(p) \neq b$, the element $\phi_{\gamma(p),b}\gamma$ has one more breakpoint than γ . Suppose that p is a breakpoint of γ . In the proof of Theorem 1.3, we showed that $\phi_{\gamma(p),b}\gamma$ would have one less breakpoint than γ with an appropriate choice of b . If we choose some other b , $\phi_{\gamma(p),b}$ merely modifies the slope ratio of γ at p without making it 1. Thus, $\phi_{\gamma(p),b}\gamma$ has the same number of breakpoints as γ .

For right multiplication, observe that γ^{-1} has the same grade as γ and $\phi^{-1}\gamma^{-1}$ has the same grade as $\gamma\phi$. This case then follows from the proof for left multiplication. \square

Corollary 1.2. *The set S of simple elements generates R .*

Proof. Given a simple element γ of grade n , we may choose simple elements ϕ_1, \dots, ϕ_n such that $\gamma\phi_1 \dots \phi_n = I$ (the identity function). □

Chapter 2

Conjugacy Classes in the Piecewise Linear Group

We begin studying conjugacy classes in $PL_0(I)$ by looking at simple conjugation. That is, we consider conjugation of an element $\gamma \in PL_0(I)$ by a simple element $\phi_{a,b}$. Note that $\phi_{a,b}^{-1} = \phi_{b,a}$. Thus, simple conjugates of γ have the form $\phi_{b,a}\gamma\phi_{a,b}$. Geometrically, this is equivalent to drawing a horizontal line and a vertical line passing through the point (b, b) and sliding these lines to a pair passing through (a, a) while linearly rescaling the x and y coordinates of points in the graph of γ in each of the four regions created by the lines (Figure 2.1).

We will denote the conjugacy class of an element γ by $\mathfrak{C}(\gamma)$. Suppose that $\gamma' = \phi^{-1}\gamma\phi$. Consistent with our discussion of simple multiplication earlier, we view conjugation by ϕ as a map that sends $(x, \gamma(x))$ in the graph of γ to $(\phi^{-1}(x), \phi^{-1}\circ\gamma(x))$ in the graph of γ' . As in our geometric picture, x and y -coordinates are rescaled in the same way by ϕ^{-1} . In terms of labeling, a point labeled x in γ moves to a point labeled $x' = \phi^{-1}(x)$ in γ' .

We wish to describe what happens to the slope ratios of γ under conjugation by ϕ . By the chain rule,

$$k_{\gamma'}(x') = k_{\phi^{-1}}(\gamma\phi(x'))k_{\gamma}(\phi(x'))k_{\phi}(x').$$

But, observe that,

$$k_{\phi^{-1}}(\gamma\phi(x')) = \frac{1}{k_{\phi}(\phi^{-1}\gamma\phi(x'))} = \frac{1}{k_{\phi}(\gamma'(x'))}.$$

Then, the above expression becomes

$$k_{\gamma'}(x') = \frac{k_{\phi}(x')}{k_{\phi}(\gamma'(x'))}k_{\gamma}(x).$$

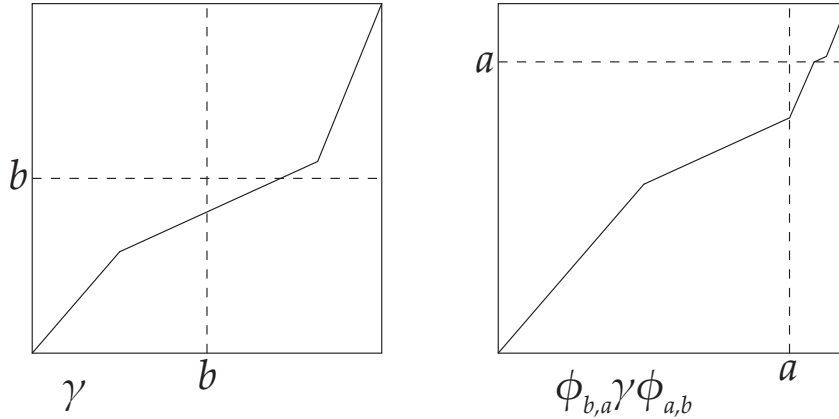


Figure 2.1: Conjugation by a Simple Element

Thus, the slope ratio of γ' at x' is the slope ratio of γ at x multiplied by the quotient $\frac{k_\phi(x')}{k_\phi(\gamma'(x'))}$. As expected from our geometric picture, simple conjugation generally changes the slope ratio at two points. We state this precisely in the following theorem:

Theorem 2.1. *Let $\gamma' = \phi_{b,a}\gamma\phi_{a,b}$. Let $x = \phi_{a,b}(x')$. If x' is not equal to a or $\gamma'^{-1}(a)$, then $k_{\gamma'}(x') = k_\gamma(x)$. In addition, $k_{\gamma'}(a) = k_{\phi_{a,b}}(a)k_\gamma(b)$ and $k_{\gamma'}(\gamma'^{-1}(a)) = \frac{k_\gamma(\gamma^{-1}(b))}{k_{\phi_{a,b}}(a)}$.*

We will formalize the relationship between pairs of points whose slope ratios are modified by simple conjugation.

Definition 2.1. We say that $x_1, x_2 \in [0, 1]$ are connected by γ if $\gamma(x_1) = x_2$ or $x_1 = \gamma(x_2)$. (Figure 2.2 gives an example of two points connected to a point P .)

Theorem 2.2. *Let $\gamma' = \phi^{-1}\gamma\phi$. Then, the correspondence map ϕ^{-1} sends connected point pairs of γ to connected point pairs of γ' .*

Proof. Let $x_2 = \gamma(x_1)$. Let $x'_1 = \phi^{-1}(x_1)$ and $x'_2 = \phi^{-1}(x_2)$. Then,

$$\gamma'(x'_1) = \phi^{-1}\gamma\phi\phi^{-1}(x_1) = \phi^{-1}\gamma(x_1) = x'_2$$

□

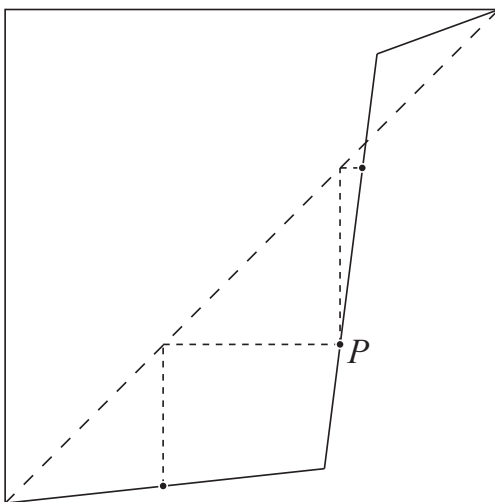


Figure 2.2: Connected Points

2.1 Orbits

We extend the concept of connected points by recalling the notion of an orbit.

Definition 2.2. We define $\mathcal{O}_\gamma(x)$ to be the set:

$$\{\gamma^n(x) | n \in \mathbb{Z}\}$$

We'll call this the orbit of γ containing x . An orbit of γ is any set that can be generated in this way by a single point.

Figure 2.3 shows the points in an orbit. We have the following corollary to Theorem 2.2.

Corollary 2.1. Let x and $x' = \phi^{-1}(x)$ be corresponding points of γ and γ' respectively, where $\gamma' = \phi^{-1}\gamma\phi$. Then,

$$\phi^{-1}(\mathcal{O}_\gamma(x)) = \mathcal{O}_{\gamma'}(x')$$

Proof.

$$\phi^{-1}(\gamma^n(x)) = \gamma'^n \phi^{-1}(x) = \gamma'^n(x')$$

□

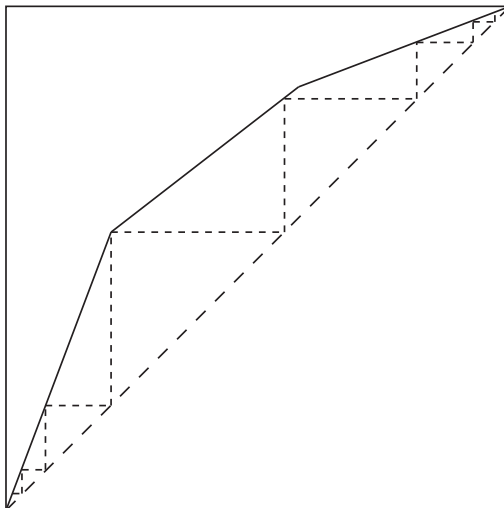


Figure 2.3: Points in an Orbit

Theorem 2.3. *Let \mathcal{O} be an orbit of γ . The orbit \mathcal{O} has exactly one of the following three properties:*

Increasing: $\gamma(x) > x \ \forall x \in \mathcal{O}$

Decreasing: $\gamma(x) < x \ \forall x \in \mathcal{O}$

Fixed: \mathcal{O} contains exactly one point which is a fixed point of γ .

Proof. Increasing: Suppose there is some $p \in \mathcal{O}$ such that $\gamma(p) > p$. Recall that γ is a strictly increasing function. It follows that γ^k is strictly increasing for each $k \in \mathbb{Z}$. For any $x \in \mathcal{O}$, there exists $k \in \mathbb{Z}$ such that $x = \gamma^k(p)$. Then,

$$\gamma^k(\gamma(p)) > \gamma^k(p)$$

$$\gamma^{k+1}(p) > x$$

$$\gamma(x) > x$$

Decreasing: Suppose that there is some $p \in \mathcal{O}$ such that $\gamma(p) < p$. A proof parallel to that for the increasing case shows that $\gamma(x) < x \ \forall x \in \mathcal{O}$.

Fixed: Suppose that no p exists in \mathcal{O} such that $\gamma(p) > p$ or $\gamma(p) < p$. Then, $\gamma(p) = p$ for all $p \in \mathcal{O}$. Then, p is a fixed point of γ and \mathcal{O} contains only the single point p . \square

Theorem 2.4. *Let \mathcal{O} be a non-fixed orbit of γ and let $p \in \mathcal{O}$. The sequences $\gamma^k(p)$ and $\gamma^{-k}(p)$ for $k \in \mathbb{N}$ converge to distinct fixed points x_1 and x_2 of γ . The function γ has no fixed points between x_1 and x_2 . We call x_1 and x_2 the boundary points of \mathcal{O} .*

Proof. Without loss of generality, assume that \mathcal{O} is an increasing orbit. As a bounded increasing sequence, $p_k = \gamma^k(p)$ must converge. Let $p_k \rightarrow x_1$. Then, $\gamma(p_k) = p_{k+1}$ and $\gamma(p_k) \rightarrow x_1$. By continuity, $\gamma(x_1) = x_1$. The sequence γ^{-k} is decreasing and thus must converge to $x_2 \neq x_1$. Now, suppose that γ has a fixed point x between x_1 and x_2 . There exists some $p' \in \mathcal{O}$ such that $p' < x$ and $\gamma(p') > x$. But, then $\gamma(p') > \gamma(x)$, contradicting the fact that γ is an increasing function. \square

2.2 A Conjugacy Invariant

Definition 2.3. Define the slope ratio of an orbit as

$$k(\mathcal{O}) = \prod_{p \in \mathcal{O}} k(p)$$

Theorem 2.5. *Let γ and γ' be conjugate and let \mathcal{O} and \mathcal{O}' be corresponding orbits of γ and γ' respectively. Then, $k_\gamma(\mathcal{O}) = k_{\gamma'}(\mathcal{O}')$.*

Proof. First, observe that $k_\gamma(p) = 1$ except at finitely many points p of γ due to the fact that γ is piecewise linear. Thus, $k_\gamma(\mathcal{O})$ is always defined. We first consider the case where γ' is a simple conjugate of γ . That is, $\gamma' = \phi^{-1}\gamma\phi$ and $\phi = \phi_{a,b}$. Then, $\mathcal{O}' = \phi^{-1}(\mathcal{O})$. Let $x' = \phi^{-1}(x)$. Recall from Theorem 2.1 that $k_{\gamma'}(x') \neq k_\gamma(x)$ only when $x' = a$ or $\gamma'^{-1}(a)$. These two points are either both in \mathcal{O}' or both not in \mathcal{O}' . If neither is in \mathcal{O}' , then $k_\gamma(\mathcal{O}) = k_{\gamma'}(\mathcal{O}')$ trivially.

Suppose that a and $\gamma'^{-1}(a)$ are in \mathcal{O}' . Using Theorem 2.1, we have

$$k_{\gamma'}(\mathcal{O}') = \prod_{x' \in \mathcal{O}'} k_{\gamma'}(x') = \left(\prod_{\substack{x' \in \mathcal{O}' \\ x' \neq a, \gamma'^{-1}(a)}} k_\gamma(\phi(x')) \right) k_{\gamma'}(a) k_{\gamma'}(\gamma'^{-1}(a))$$

$$\begin{aligned}
&= \left(\prod_{\substack{x' \in \mathcal{O}' \\ x' \neq a, \gamma'^{-1}(a)}} k_\gamma(\phi(x')) \right) k_\phi(a) k_\gamma(b) \frac{k_\gamma(\gamma^{-1}(b))}{k_\phi(a)} \\
&= \left(\prod_{\substack{x \in \mathcal{O} \\ x \neq b, \gamma^{-1}(b)}} k_\gamma(x) \right) k_\gamma(b) k_\gamma(\gamma^{-1}(b)) \\
&= \prod_{x \in \mathcal{O}} k_\gamma(x) \\
&= k_\gamma(\mathcal{O})
\end{aligned}$$

Now, suppose that $\gamma' = \beta^{-1}\gamma\beta$, where β is not simple. β may be factored into simple elements so that conjugation by β is equivalent to a series of simple conjugations. The theorem then holds in this case as well. \square

Definition 2.4. We call an orbit \mathcal{O} of γ *flat* if it contains no breakpoints and *broken* if it contains breakpoints. We say that \mathcal{O} is *k-flat* if $k(\mathcal{O}) = 1$ and *k-broken* if $k(\mathcal{O}) \neq 1$.

Theorem 2.6. *The breakpoints in the orbit \mathcal{O} of γ may be eliminated by conjugation if and only if \mathcal{O} is k-flat. If \mathcal{O} is k-broken, we may conjugate γ so that the orbit corresponding to \mathcal{O} has only 1 breakpoint.*

Proof. We will develop a general algorithm for reducing the number of breakpoints in \mathcal{O} . Choose $p \in \mathcal{O}$. Since \mathcal{O} has only finitely many breakpoints, there exist a minimum and maximum n such that $\gamma^n(p)$ is a breakpoint of \mathcal{O} . Let $d = n_{max} - n_{min}$. Suppose that \mathcal{O} contains more than 1 breakpoint. Then, $d > 0$. Choose a simple element ϕ that cancels the breakpoint $\gamma^{n_{max}}(p)$ of γ by right multiplication. (Theorem 1.4) Let $\gamma' = \phi^{-1}\gamma\phi$.

Suppose that $n < n_{min}$. We show that $k_{\gamma'}(\phi^{-1}\gamma^n(p)) = 1$ as follows:

$$k_{\gamma'}(\phi^{-1}\gamma^n(p)) = k_{\phi^{-1}}(\gamma^{n+1}(p))k_\gamma(\gamma^n(p))k_\phi(\phi^{-1}\gamma^n(p)) \quad (2.1)$$

Observe that ϕ^{-1} has its only breakpoint at $\gamma^{n_{max}}(p)$, so $k_{\phi^{-1}}(\gamma^{n+1}(p)) = 1$. Since $n < n_{min}$, we have that $k_\gamma(\gamma^n(p)) = 1$. The breakpoint of ϕ is at $\phi^{-1}\gamma^{n_{max}}$,

so $k_\phi(\phi^{-1}\gamma^n(p)) = 1$. Thus, $k_{\gamma'}(\phi^{-1}\gamma^n(p)) = 1$. Parallel arguments show that $k_{\gamma'}(\phi^{-1}\gamma^n(p)) = 1$ if $n > n_{max}$.

Now, suppose that $n = n_{max}$. We will use equation 2.1 again. Because ϕ right cancels the breakpoint $\gamma^{n_{max}}(p)$, it follows that $k_\gamma(\gamma^n(p))k_\phi(\phi^{-1}\gamma^n(p)) = 1$. In addition, $k_{\phi^{-1}}(\gamma^{n+1}(p)) = 1$, again using the fact that ϕ^{-1} has its breakpoint at $\gamma^{n_{max}}(p)$.

Consider the orbit $\mathcal{O}' = \phi^{-1}(\mathcal{O})$. If \mathcal{O}' contains a breakpoint, we define d' for \mathcal{O}' as we earlier defined d for \mathcal{O} . By the above analysis, \mathcal{O}' either contains no breakpoints or d' is less than d . Thus, by a finite number of simple conjugation steps, we may obtain an orbit \mathcal{O}'' with either no breakpoints or $d'' = 0$.

If $d'' = 0$, then $k(\mathcal{O}'') \neq 1$ and \mathcal{O} is k -broken by Theorem 2.5. Thus, \mathcal{O} being k -flat implies that \mathcal{O}'' is flat. Conversely, suppose that \mathcal{O}'' is flat. Then, \mathcal{O} is clearly k -flat, again by Theorem 2.5.

If \mathcal{O} is k -broken, then the only possibility left is that \mathcal{O}'' has exactly one breakpoint. □

2.3 Orbit-simple Elements

Definition 2.5. We say that an orbit \mathcal{O} is *reduced* if it has 1 or fewer breakpoints. Note that an orbit with exactly 1 breakpoint is k -broken. We say that γ is *orbit-simple* if each of its orbits is reduced. We say that γ is an *orbit-simple form* of β if γ is an orbit-simple conjugate of β .

Theorem 2.7. *Every element of $PL_0(I)$ has an orbit-simple form.*

Proof. From Theorem 2.1, we know that conjugation by a simple element affects slope ratios for connected point pairs. Thus, simple conjugation only effects slope ratios within a single orbit without changing ratios for points in any other orbit. It follows that the algorithm from Theorem 2.6 may be applied sequentially to orbits of an element γ . Since γ can have only finitely many breakpoints, we may create an orbit-simple form of γ in a finite number of steps. □

Theorem 2.8. *The only element of $PL_0(I)$ with no k -broken orbits is I .*

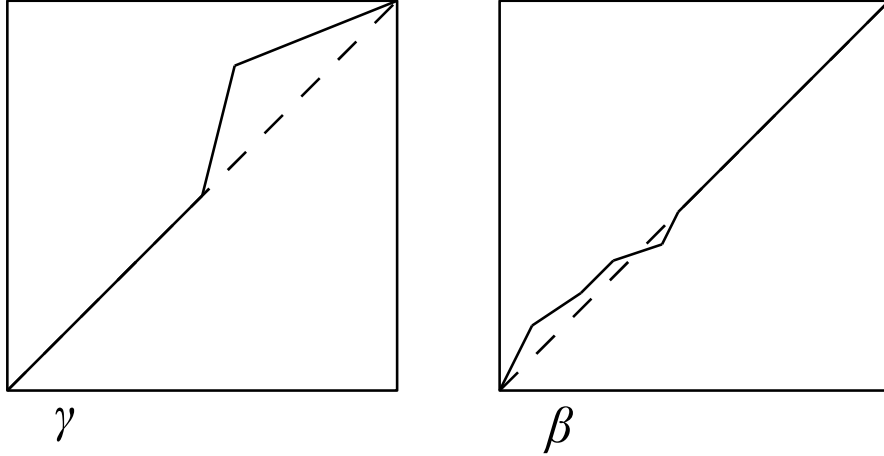


Figure 2.4: Commuting Elements

Proof. Let γ be an element with no k -broken orbits. By Theorem 2.7, there exists β such that $\beta^{-1}\gamma\beta$ is orbit-simple. The resulting element has no breakpoints. The only element of $PL_0(I)$ with no breakpoints is I , and its only conjugate is I , so $\gamma = I$. \square

Definition 2.6. Let γ be an orbit-simple element of $PL_0(I)$. We call β a *twist* of γ if $\beta^{-1}\gamma\beta$ is orbit-simple. If, in addition, β is a simple element, we say that β is a *simple twist* of γ . The element β is called a *trivial twist* of γ if $\beta^{-1}\gamma\beta = \gamma$ (β and γ commute).

It is, in fact, not difficult to find nontrivial elements γ and β which commute, with γ orbit-simple. See, for example, Figure 2.4.

Theorem 2.9. *Suppose that γ is an orbit-simple element of $PL_0(I)$. Suppose that γ has n breakpoints given by p_1, p_2, \dots, p_n . Let ρ be a simple element that right cancels one of these breakpoints p_j . Then, ρ is a simple twist of γ and $\gamma' = \rho^{-1}\gamma\rho$ has n breakpoints given by*

$$\rho^{-1}(p_1), \rho^{-1}(p_2), \dots, \rho^{-1}(p_{j-1}), \rho^{-1}\gamma^{-1}(p_j), \rho^{-1}(p_{j+1}), \dots, \rho(p_n)$$

Proof. First, observe that if $\rho^{-1}\gamma\rho$ indeed has the set of breakpoints given above, then ρ must be a simple twist of γ , for the points

$$\rho^{-1}(p_1), \rho^{-1}(p_2), \dots, \rho^{-1}(p_{j-1}), \rho^{-1}(p_j), \rho^{-1}(p_{j+1}), \dots, \rho^{-1}(p_n)$$

correspond to the breakpoints of γ and thus lie in distinct orbits of γ' . $\rho\gamma^{-1}(p_j) = \gamma'^{-1}(\rho^{-1}(p_j))$ and so lies in the γ' orbit of $\rho^{-1}(p_j)$ and thus does not lie in the same orbit as any of the other breakpoints of γ' .

Now, we show that these are indeed the breakpoints of γ' . Consider the element $\gamma\rho$. By Theorem 1.4, this element has all the breakpoints listed in the theorem for $\rho^{-1}\gamma\rho$ except $\rho^{-1}\gamma^{-1}(p_j)$. The breakpoint of ρ^{-1} is at p_j . Thus, left multiplying $\gamma\rho$ by ρ^{-1} modifies the slope ratio of $\gamma\rho$ only at $(\gamma\rho)^{-1}(p_j) = \rho^{-1}\gamma^{-1}(p_j)$. But, $\rho^{-1}\gamma^{-1}(p_j)$ is not a breakpoint of $\gamma\rho$ and so is a breakpoint of $\rho^{-1}\gamma\rho$. Thus, $\rho^{-1}\gamma\rho$ has the breakpoints given in the theorem. \square

Theorem 2.10. *Suppose that γ is an orbit-simple element of $PL_0(I)$. Suppose that γ has n breakpoints given by p_1, p_2, \dots, p_n . Let λ be a simple element that left cancels one of these breakpoints p_j . Then, λ^{-1} is a simple twist of γ and $\gamma' = \lambda\gamma\lambda^{-1}$ has n breakpoints given by*

$$\lambda(p_1), \lambda(p_2), \dots, \lambda(p_{j-1}), \lambda\gamma(p_j), \lambda(p_{j+1}), \dots, \lambda(p_n)$$

Proof. We'll prove this theorem by using Theorem 2.9. Observe that the breakpoints of γ^{-1} are $\gamma(p_1), \gamma(p_2), \dots, \gamma(p_n)$. If λ left cancels the breakpoint p_j of γ , then λ^{-1} right cancels the breakpoint $\gamma(p_j)$ of γ^{-1} . Thus, λ^{-1} is a simple twist of γ^{-1} . Then, $\lambda\gamma\lambda^{-1}$ is orbit-simple, so λ^{-1} is a simple twist of γ .

Now, $\lambda\gamma^{-1}\lambda^{-1}$ has breakpoints

$$\lambda\gamma(p_1), \lambda\gamma(p_2), \dots, \lambda\gamma(p_{j-1}), \lambda\gamma\gamma(p_j)\lambda\gamma(p_{j+1}), \dots, \lambda\gamma(p_n)$$

Then, $\lambda\gamma\lambda^{-1}$ has breakpoints

$$\lambda(p_1), \lambda(p_2), \dots, \lambda(p_{j-1}), \lambda\gamma(p_j)\lambda(p_{j+1}), \dots, \lambda(p_n)$$

\square

Theorem 2.11. *Every twist may be factored into simple twists. That is, if β is a twist of γ , we may choose simple elements $\phi_1, \phi_2 \dots \phi_n$ such that $\beta = \phi_1 \phi_2 \dots \phi_n$ and, for each natural number $m \leq n$*

$$\phi_m^{-1} \phi_{m-1}^{-1} \dots \phi_1^{-1} \gamma \phi_1 \dots \phi_{m-1} \phi_m$$

is orbit-simple.

Proof. Observe that the above theorem is equivalent to saying that given a twist β of γ , we may choose $\phi_1, \phi_2, \dots, \phi_n$ such that $\beta \phi_1 \phi_2 \dots \phi_n = I$ and

$$\phi_m^{-1} \phi_{m-1}^{-1} \dots \phi_1^{-1} \beta^{-1} \gamma \beta \phi_1 \dots \phi_{m-1} \phi_m$$

is orbit-simple for each natural number m less than or equal to n . We will prove this by looking at two cases.

Case I: Let $\gamma' = \beta^{-1} \gamma \beta$ and suppose that $k_{\gamma'}(\beta^{-1}(x)) = k_{\gamma}(x)$ for each $x \in (0, 1)$. Then,

$$k_{\gamma'}(\beta^{-1}(x)) = k_{\beta^{-1}}(\gamma(x)) k_{\gamma}(x) k_{\beta}(\beta^{-1}(x)) = k_{\gamma}(x)$$

$$k_{\beta^{-1}}(\gamma(x)) k_{\beta}(\beta^{-1}(x)) = 1$$

$$\frac{k_{\beta}(\beta^{-1}(x))}{k_{\beta}(\beta^{-1} \gamma(x))} = 1$$

$$\frac{k_{\beta}(\beta^{-1} \gamma^j(x))}{k_{\beta}(\beta^{-1} \gamma^{j+1}(x))} = 1$$

$$\frac{k_{\beta}(\gamma'^j \beta^{-1}(x))}{k_{\beta}(\gamma'^{j+1} \beta^{-1}(x))} = 1$$

Suppose that \mathcal{O} is a non-fixed orbit of γ' . Then, k_{β} is constant on this orbit. Since β can only have finitely many break points, $k_{\beta} = 1$ on the points of any non-fixed γ' orbit.

Let a be a breakpoint of β . The point a must be a fixed point of γ' . Choose a simple element ϕ such that ϕ right cancels the breakpoint of β at a . Let $\gamma'' = \phi^{-1} \gamma' \phi$. Then,

$$\begin{aligned} k_{\gamma''}(\phi^{-1}(a)) &= k_{\phi^{-1}}(\gamma' \phi \phi^{-1}(a)) k_{\gamma'}(\phi \phi^{-1}(a)) k_{\phi}(\phi^{-1}(a)) \\ &= k_{\phi^{-1}}(\gamma'(a)) k_{\gamma'}(a) k_{\phi}(\phi^{-1}(a)) \end{aligned}$$

$$\begin{aligned}
&= k_{\phi^{-1}(a)}k_{\gamma'}(a)k_{\phi}(\phi^{-1}(a)) \\
&= \frac{k_{\phi}(\phi^{-1}(a))}{k_{\phi}(\phi^{-1}(a))}k_{\gamma'}(a) \\
&= k_{\gamma'}(a)
\end{aligned}$$

Now, $k_{\gamma''}(\phi^{-1}(x)) = k_{\gamma'}(x)$ except when $x = a$ or $\gamma'^{-1}(a)$ by Theorem 2.1. But, $\gamma'^{-1}(a) = a$. It follows that $k_{\gamma''}(\phi^{-1}(x)) = k_{\gamma'}(x)$ for all $x \in (0, 1)$ and ϕ is a twist of γ' .

Now, we may prove the theorem for *Case I* by induction on the grade of β . Suppose that the theorem is true for *Case I* when $\text{grade}(\beta) = n$ and suppose that $\text{grade}(\beta) = n + 1$. Then, we may choose simple ϕ such that $\phi^{-1}\beta^{-1}\gamma\beta\phi$ is orbit-simple and $\text{grade}(\beta\phi) = n$. Thus, the theorem is true when $\text{grade}(\beta) = n + 1$. The theorem is clearly true when $\text{grade}(\beta) = 0$. Thus, the theorem is established by induction for *Case I*.

Before proceeding to *Case II*, we introduce a parameter N that describes how much breakpoints of γ move under conjugation by β . Specifically, let $\gamma' = \beta^{-1}\gamma\beta$ as before with γ and γ' orbit-simple. We may label the non-fixed broken orbits of γ as $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_l$. For each of these orbits, define the corresponding orbit $\mathcal{O}' = \beta^{-1}(\mathcal{O})$ so that we have $\mathcal{O}'_1, \mathcal{O}'_2, \dots, \mathcal{O}'_l$. Suppose that p is the breakpoint of \mathcal{O}_i . The corresponding orbit will have a breakpoint $\beta^{-1}\gamma^{n_i}(p)$, where n_i is an integer. Then, n_i measures how much the breakpoint of \mathcal{O}_i moves when γ is conjugated by β . We define $N = |n_1| + |n_2| + \dots + |n_l|$.

Suppose that $N = 0$. Then, $n_i = 0$ for each non-fixed broken orbit of γ and $k_{\gamma'}(\beta^{-1}(x)) = k_{\gamma}(x)$ for points in these orbits. If p is a fixed point of γ , then $k_{\gamma'}(\beta^{-1}(p)) = k_{\gamma}(p)$ by Theorem 2.5. Any points aside from these are not breakpoints. Thus, if $N = 0$, then $k_{\gamma'}(\beta^{-1}(x)) = k_{\gamma}(x)$ for all $x \in (0, 1)$ and the theorem is true by the proof for *Case I*.

Case II: Suppose that $N > 0$. Then, there exists some natural number j such that $n_j \neq 0$. We'll assume for the moment that $n_j > 0$. Let p be the breakpoint in the orbit \mathcal{O}_j . As before, the breakpoint of \mathcal{O}'_j is $\beta^{-1}\gamma^{n_j}(p)$ and $\gamma' = \beta^{-1}\gamma\beta$. Choose ρ to right cancel the breakpoint $\beta^{-1}\gamma^{n_j}$ and define $\gamma'' = \rho^{-1}\gamma\rho$. For each orbit \mathcal{O}_i , we

define n'_i to indicate the distance between the breakpoint of \mathcal{O}_i and the breakpoint of \mathcal{O}'_i just as we defined n_i to measure this distance for \mathcal{O}_i and \mathcal{O}'_i . Then, by Theorem 2.9, $n'_i = n_i$ for $i \neq j$ and $n'_j = n_j - 1$. Let $N' = |n'_1| + |n'_2| + \cdots + |n'_l|$. $N' = N - 1$

If $n_i < 0$, we construct a comparable argument using left cancellation to increase n_i by 1 and reduce N by 1.

The above work reduces the $N = m$ case to the $N = m - 1$ case. Since the theorem is true for $N = 0$, it is true for all N by induction. \square

2.4 One-bump Functions

Definition 2.7. We say that γ is *plus function* if $\gamma(x) > x$ for all $x \in (0, 1)$. We say that γ is *minus function* if $\gamma(x) < x$ for all $x \in (0, 1)$. Collectively, we call such elements *one-bump functions*

Theorem 2.12. *If γ is a plus or minus function, then every element of $\mathfrak{Cl}(\gamma)$ shares the same property.*

Proof. We'll show the plus function case:

$$\gamma(x) > x$$

$$\gamma\beta(x) > \beta(x)$$

$$\beta^{-1}\gamma\beta(x) > \beta^{-1}\beta(x)$$

$$\beta^{-1}\gamma\beta(x) > x$$

\square

Theorem 2.13. *The derivative of an element γ at $x = 0$ from the right is an invariant of conjugation.*

Proof.

$$\frac{d}{dx_+}(\beta^{-1}\gamma\beta)(0) = \frac{d\beta^{-1}}{dx_+}(0) \frac{d\gamma}{dx_+}(0) \frac{d\beta}{dx_+}(0) = \frac{d\gamma}{dx_+}(0)$$

\square

We will refer to the derivative of γ from the right at $x = 0$ as the initial slope of γ and represent it with the notation m_0 .

Theorem 2.14. *Suppose that γ is a one-bump orbit-simple element of $PL_0(I)$. Then, the simple twists of γ are elements of the form ρ and λ^{-1} , where ρ and λ respectively right and left cancel breakpoints of γ .*

Proof. Consider the simple twist $\phi_{a,b}$ of γ , where b is in a k -flat orbit of γ . Let $\gamma' = \phi_{b,a}\gamma\phi_{a,b}$. Then, by Theorem 2.1,

$$k_{\gamma'}(a) = k_{\phi_{a,b}}(a)k_{\gamma}(b) = k_{\phi_{a,b}}(a)$$

Thus, $\mathcal{O}_{\gamma'}(a)$ is not reduced, a contradiction.

Now, suppose that b lies in a broken orbit of γ . Then, once again by Theorem 2.1,

$$\begin{aligned} k_{\gamma'}(a) &= k_{\phi_{a,b}}(a)k_{\gamma}(b) \\ k_{\gamma'}(\gamma'^{-1}(a)) &= \frac{k_{\gamma}(\gamma^{-1}(b))}{k_{\phi_{a,b}}(a)} \end{aligned}$$

One of these two equations is 1, so we have either

$$k_{\gamma}(b) = \frac{1}{k_{\phi_{a,b}}(a)}, \quad k_{\gamma}(\gamma^{-1}(b)) = 1$$

or

$$k_{\gamma}(b) = 1, \quad k_{\gamma}(\gamma^{-1}(b)) = k_{\phi_{a,b}}(a)$$

In the first case, $\phi_{a,b}$ right cancels a breakpoint of γ at b , so that $\phi_{a,b} = \rho$, as described in the theorem. In the second case, note that $k_{\gamma}(\gamma^{-1}(b)) = \frac{1}{k_{\phi_{b,a}}(b)}$ so that $\phi_{b,a}$ left cancels a breakpoint of γ at $\gamma^{-1}(b)$ and $\phi_{b,a} = \lambda$. \square

Theorem 2.15. *An element with only one k -broken orbit has only one orbit-simple form.*

Proof. Suppose that γ has only one k -broken orbit. Then, γ has an orbit-simple form γ' , where γ' is simple. Suppose that ϕ is a simple twist of γ' . By Theorem 2.14, the breakpoint of γ' is either right cancelled by ϕ or left cancelled by ϕ^{-1} . But, γ'^{-1} is the

unique simple element that right cancels the breakpoint of γ' and the unique simple element that left cancels the breakpoint of γ' . (See Theorems 1.3 and 1.4.) It follows that $\phi = \gamma'^{-1}$ or γ' . Thus, every simple twist of γ' gives us back γ' . By Theorem 2.11, every twist of γ' gives back γ' . \square

Theorem 2.16. *Let γ be a one-bump function in $PL_0(I)$. Given $x \in (0, 1)$ and p between x and $\gamma(x)$, $\gamma^n(p)$ lies between $\gamma^n(x)$ and $\gamma^{n+1}(x)$ for each $n \in \mathbb{Z}$. Furthermore, the open interval between x and $\gamma(x)$ contains exactly one representative of every non-fixed orbit of γ except $\mathcal{O}(x)$.*

Proof. Let γ be a plus function. Because $x < p < \gamma(x)$, it follows that $\gamma^n(x) < \gamma^n(p) < \gamma^{n+1}(x)$ by the fact that γ is a strictly increasing function. A parallel argument works for the case where γ is a minus function.

Now, let p be a representative of some γ orbit $\mathcal{O} \neq \mathcal{O}(x)$. Then, for some n , p lies between $\gamma^n(x)$ and $\gamma^{n+1}(x)$. By the first part of the theorem, $\gamma^{-n}(p)$ is the only representative of \mathcal{O} between x and $\gamma(x)$. \square

2.5 Conjugacy Classes of One-bump Functions

We now wish to introduce a new way of describing the orbits of an element $\gamma \in PL_0(I)$ that will be useful later in constructing a complete set of invariants for conjugacy classes of $PL_0(I)$. Suppose that γ is a plus function. (This work is extended to the case where γ is a minus function by looking at the inverse of γ , which is a plus function.) Suppose that p is the smallest breakpoint of γ . Then, γ on the interval $[0, p]$ has the form m_0x , where m_0 is the initial slope of γ . The point p lies in some orbit of γ consisting of all points of the form $\gamma^n(p)$. For integers $n \leq 0$, $\gamma^n(p) = m_0^n p$. For real numbers $r \leq 0$, define $\gamma^r(p) = m_0^r p$. Define $\Gamma : (-1, 0] \rightarrow (p/m_0, p]$ by $\Gamma(r) = m_0^r p$. Observe that the interval $(p/m_0, p]$ contains exactly one representative of every nonfixed orbit of γ by Theorem 2.16. Then, we have the following definition:

Definition 2.8. We define ψ_γ to be a map from the orbits of γ to S^1 . Let x be the representative of \mathcal{O} in $(m_0/p, p]$. Then,

$$\psi_\gamma(\mathcal{O}) = e^{2\pi i \Gamma^{-1}(x)}$$

Define $\overline{\psi_\gamma}$ to be the equivalence class of all functions ψ' such that $\psi' = \rho \circ \psi_\gamma$, where ρ is a rigid rotation of S^1 .

Theorem 2.17. *Let $\gamma' = \beta^{-1}\gamma\beta$. Let \mathcal{O}_1 and \mathcal{O}_2 be orbits of γ . Suppose that $\psi_\gamma(\mathcal{O}_2) = e^{i\theta}\psi_\gamma(\mathcal{O}_1)$ for some real number θ . Let $\mathcal{O}'_1 = \beta^{-1}(\mathcal{O}_1)$ and $\mathcal{O}'_2 = \beta^{-1}(\mathcal{O}_2)$. Then, $\psi_{\gamma'}(\mathcal{O}'_2) = e^{i\theta}\psi_{\gamma'}(\mathcal{O}'_1)$.*

Proof. We will calculate the function $\psi_\gamma(\mathcal{O}(x))$ where $x \in (0, p]$. Observe that on the interval $(p/m_0, p]$, $\Gamma^{-1}(x) = \frac{\ln(x/p)}{\ln(m_0)}$. Let x be any point in $(0, p]$. Then, $x \in \mathcal{O}(x^*)$ for some $x^* \in (p/m_0, p]$. For some integer $n \leq 0$, $x = \gamma^n(x^*) = m_0^n x^*$. Then,

$$e^{2\pi i \frac{\ln(x/p)}{\ln(m_0)}} = e^{2\pi i \frac{\ln(m_0^n x^*/p)}{\ln(m_0)}} = e^{2\pi i \left(n + \frac{\ln(x^*/p)}{\ln(m_0)}\right)} = e^{2\pi i \frac{\ln(x^*/p)}{\ln(m_0)}}$$

Thus, $\psi_\gamma(\mathcal{O}(x)) = e^{2\pi i \frac{\ln(x/p)}{\ln(m_0)}}$ on $(0, p]$.

We may choose a representative x_1 of \mathcal{O}_1 such that x_1 is less than the first breakpoints of γ and β^{-1} and $\beta^{-1}(x_1)$ is less than the first breakpoint of γ' . We also choose a breakpoint x_2 of \mathcal{O}_2 using the same criteria. Let $m_{0,\gamma}$ be the initial slope of γ and γ' and let $m_{0,\beta}$ be the initial slope of β . Define $x'_1 = \beta^{-1}(x_1)$ and $x'_2 = \beta^{-1}(x_2)$. Then, $x'_1 = \frac{x_1}{m_{0,\beta}}$ and $x'_2 = \frac{x_2}{m_{0,\beta}}$. Observe that $\psi_\gamma(\mathcal{O}_1) = e^{2\pi i \frac{\ln(x_1/p)}{\ln(m_{0,\gamma})}}$ and $\psi_\gamma(\mathcal{O}_2) = e^{2\pi i \frac{\ln(x_2/p)}{\ln(m_{0,\gamma})}}$. Let p' be the first breakpoint of γ' . Let $\mathcal{O}'_1 = \beta^{-1}(\mathcal{O}_1)$ and $\mathcal{O}'_2 = \beta^{-1}(\mathcal{O}_2)$. Then,

$$\begin{aligned} \psi_{\gamma'}(\mathcal{O}'_1) &= e^{2\pi i \frac{\ln(x_1/(m_{0,\beta}p'))}{\ln(m_{0,\gamma})}} = e^{2\pi i \left(\frac{\ln(x_1)}{\ln(m_{0,\gamma})} - \frac{\ln(m_{0,\beta}p')}{\ln(m_{0,\gamma})}\right)} \\ \psi_{\gamma'}(\mathcal{O}'_2) &= e^{2\pi i \left(\frac{\ln(x_2)}{\ln(m_{0,\gamma})} - \frac{\ln(m_{0,\beta}p')}{\ln(m_{0,\gamma})}\right)} \\ \psi_\gamma(\mathcal{O}_1) &= e^{2\pi i \left(\frac{\ln(x_1)}{\ln(m_{0,\gamma})} - \frac{\ln(p)}{\ln(m_{0,\gamma})}\right)} \\ \psi_\gamma(\mathcal{O}_2) &= e^{2\pi i \left(\frac{\ln(x_2)}{\ln(m_{0,\gamma})} - \frac{\ln(p)}{\ln(m_{0,\gamma})}\right)} \end{aligned}$$

Comparison of these expressions establishes the theorem. \square

Definition 2.9. From the function class $\overline{\psi}$ defined for an element γ , choose some function ψ . Define a function $\sigma : S^1 \rightarrow \mathbb{R}^+$ by $\sigma(c) = k(\psi^{-1}(c))$, where $\psi^{-1}(c)$ is an orbit of γ . Let $\overline{\sigma}$ be the equivalence class of functions of the form $\sigma \circ \rho$, where ρ is

a rigid rotation of the circle. Note that this makes the definition of $\bar{\sigma}$ independent of the initial choice of $\psi \in \bar{\psi}$. We call $\bar{\sigma}$ defined for an element γ the *orbit spacing rotation class* of γ .

Theorem 2.18. *The function class $\bar{\sigma}$ is an invariant under conjugation.*

Proof. This follows from Theorems 2.17 and 2.5. □

Definition 2.10. Suppose that γ is a one bump function with a breakpoint p such that all the breakpoints of γ are contained in the interval $[p, \gamma(p))$. Then, we say that γ is *canonical*. If γ is a canonical conjugate of γ' , then we say that γ is a *canonical form* of γ' .

Theorem 2.19. *Canonical elements are orbit-simple.*

Proof. By Theorem 2.16, the interval $[p, \gamma(p))$ contains exactly one representative of each orbit of γ . Thus, each orbit of γ contains at most one breakpoint. □

Theorem 2.20. *Every one bump function $\gamma \in PL_0(I)$ has a canonical form.*

Proof. We may take γ to be orbit-simple and we will assume that γ is a plus function. A parallel proof will work for the minus function case. We will choose one of the breakpoints p_0 of γ as a basepoint. In each orbit-simple conjugate $\beta^{-1}\gamma\beta$ of γ we will use the breakpoint in $\mathcal{O}' = \beta^{-1}(\mathcal{O}_\gamma(p_0))$ as a basepoint.

We index the k -broken orbits of γ as $\mathcal{O}_0, \mathcal{O}_1, \dots, \mathcal{O}_l$, with $\mathcal{O}_0 = \mathcal{O}_\gamma(p_0)$. Each k -broken orbit \mathcal{O}_i contains one breakpoint p_i . This breakpoint must lie in the interval $(\gamma^{n_i}(p_0), \gamma^{n_i+1}(p_0))$ for some integer n_i . We define $N = |n_1| + |n_2| + \dots + |n_l|$.

Suppose that $N = 0$. Then, every breakpoint of γ lies in the interval $[p_0, \gamma(p_0))$ and γ is a canonical.

Suppose that $N > 0$. Then, $n_i \neq 0$ for some $i \in \mathbb{N}$. Suppose that $n_i > 0$. Then, $p_i \in (\gamma^{n_i}(p_0), \gamma^{n_i+1}(p_0))$. Choose a simple element ρ that right cancels with the breakpoint p_i of γ . Define $\gamma' = \rho^{-1}\gamma\rho$. For each p_i , define $p'_i = \rho^{-1}(p_i)$. By Theorem 2.9, the breakpoints of γ' are

$$p'_0, p'_1, \dots, p'_{i-1}, \gamma'^{-1}(p'_i), p'_{i+1}, \dots, p'_l$$

Define n'_j for γ' as we defined n_j for γ . For $j \neq i$, $n'_j = n_j$ and $n'_i = n_i - 1$. Then, $N' = |n'_1| + \dots + |n'_i| = N - 1$.

If $n_i < 0$, a comparable process allows us to increase n_i by 1 using left cancellation. As before, we will have $N' = N - 1$.

This proves the theorem by induction. □

Definition 2.11. We say that a map σ from S^1 to \mathbb{R} is finite if $\sigma(c) \neq 1$ for only finitely many points $c \in S^1$. An *abstract circle diagram* Σ is an ordered triple $(\Sigma_1, \Sigma_2, \Sigma_3)$ with $\Sigma_1 \in \{1, -1\}$, $\Sigma_2 \in \mathbb{R}^+$ and Σ_3 a class of finite functions from S^1 to \mathbb{R}^+ equivelant under rotation. For each one bump function $\gamma \in PL_0(I)$, we define an abstract circle diagram by letting Σ_1 be 1 if γ is a plus function and -1 if γ is a minus function; letting Σ_2 equal the initial slope m_0 of γ ; and letting Σ_3 equal the orbit spacing rotation class of γ .

Theorem 2.21. *Two one-bump functions γ_1 and γ_2 in $PL_0(I)$ are conjugate if and only if they have the same abstract circle diagram.*

Proof. Suppose that γ_1 and γ_2 have the same abstract circle diagram. We will show that γ_1 and γ_2 are conjugate to a common element.

Let γ_1 and γ_2 be plus functions in $PL_0(I)$ with common initial slope m_0 and orbit spacing rotation class $\bar{\sigma}$. (The case where γ is a minus function can be shown with a parallel proof.) We may also assume without loss of generality that γ_1 and γ_2 are orbit-simple.

Choose some breakpoint x_1 of γ_1 . Choose $\psi_1 \in \overline{\psi_{\gamma_1}}$ such that $\psi_1(\mathcal{O}_{\gamma_1}(x_1)) = 1$ in the complex plane. For $c \in S^1$, define $\sigma_1(c) = k(\psi_1^{-1}(c))$.

Because γ_2 has the same orbit function class as γ_1 , we may choose a breakpoint x_2 of γ_2 and $\psi_2 \in \overline{\psi_{\gamma_2}}$ with $\psi_2(\mathcal{O}_{\gamma_2}(x_2)) = 1$ such that $\sigma_2(c) = k(\psi_2^{-1}(c))$ is equal to $\sigma_1(c)$ for all $c \in S^1$. Conceptually, this means that $\mathcal{O}_{\gamma_1}(x_1)$ and $\mathcal{O}_{\gamma_2}(x_2)$ lie in comparable places in the sets of orbits of γ_1 and γ_2 respectively.

Now, by Theorem 2.20, γ_1 and γ_2 are conjugate to canonical forms. In the proof of Theorem 2.20, we may pick x_1 as a basepoint for the construction of the

canonical form of γ_1 and x_2 as a basepoint for the construction of the canonical form of γ_2 . If the canonical form of γ_1 is $\gamma'_1 = \beta^{-1}\gamma\beta$, the first breakpoint of γ'_1 will be $\beta^{-1}(x_1)$. Then, $\psi_{\gamma'_1}(\beta^{-1}(x_1)) = 1$ and by Theorems 2.17 and 2.5, $k\left(\psi_{\gamma'_1}^{-1}(c)\right) = \sigma_1(c)$ for all $c \in S^1$. Equivelant properties hold for the canonical form of γ_2 .

We now construct the canonical form $\gamma = \gamma'_1$. Let the first breakpoint of γ be p . Then, $\gamma(x) = m_0x$ on the interval $[0, p]$. From the proof of Theorem 2.17, we know that $\psi_\gamma(x) = e^{2\pi i \frac{\ln(x/p)}{\ln(m_0)}}$ on the interval $(0, p]$.

We can now use σ_1 to determine the breakpoints of γ with their respective slope ratios. Because γ is canonical, all of its breakpoints lie in the interval $[p, \gamma(p)) = [p, m_0p)$. For $x \in [p, m_0p)$,

$$\begin{aligned} k(x) &= k(\mathcal{O}_\gamma(x)) = k(\mathcal{O}_\gamma(\gamma^{-1}(x))) = \sigma_1(\psi_{\gamma'_1}(\mathcal{O}_\gamma(\gamma^{-1}(x)))) \\ &= \sigma_1(\psi_{\gamma'_1}(\mathcal{O}_\gamma(x/m_0))) = \sigma_1\left(e^{2\pi i \frac{\ln(x/(m_0p))}{\ln(m_0)}}\right) = \sigma_1\left(e^{2\pi i \frac{\ln(x/p) - \ln(m_0)}{\ln(m_0)}}\right) \\ &= \sigma_1\left(e^{2\pi i \frac{\ln(x/p)}{\ln(m_0)}}\right) \end{aligned}$$

Let $y = x/p$. If $x \in [p, m_0p)$, then $y \in [1, m_0)$. We then have

$$k(y) = \sigma_1\left(e^{2\pi i \frac{\ln(y)}{\ln(m_0)}}\right) \quad (2.2)$$

Let $1 = y_1 < y_2 < \dots < y_n$ be the set of points $y \in [1, m_0)$ such that $k(y) \neq 1$. Let k_j be the slope ratio at $y_j p$. Then, the breakpoints of γ are $x_1 = y_1 p, x_2 = y_2 p, \dots, x_n = y_n p$.

Now, we will apply the requirement of $\gamma(1) = 1$:

$$\begin{aligned} \gamma(1) &= \\ &= m_0x_1 + k_1m_0(x_2 - x_1) + \dots + k_{n-1} \dots k_1m_0(x_n - x_{n-1}) + k_n k_{n-1} \dots k_1m_0(1 - x_n) \\ &= m_0py_1 + k_1m_0(py_2 - py_1) + \dots + k_{n-1} \dots k_1m_0(py_n - py_{n-1}) + k_n k_{n-1} \dots k_1m_0(1 - py_n) \\ &= 1 \end{aligned}$$

$$m_0py_1 + k_1m_0(py_2 - py_1) + \dots + k_{n-1} \dots k_1m_0(py_n - py_{n-1}) - k_n k_{n-1} \dots k_1m_0py_n$$

$$= 1 - k_n k_{n-1} \cdots k_1 m_0$$

$$p = \frac{1 - k_n k_{n-1} \cdots k_1 m_0}{m_0 y_1 + k_1 m_0 (y_2 - y_1) + \cdots + k_{n-1} \cdots k_1 m_0 (y_n - y_{n-1}) - k_n k_{n-1} \cdots k_1 m_0 y_n}$$

This equation uniquely determines the value of p . The construction thus uniquely determines γ .

Now, let γ'_2 be the canonical form of γ_2 described above, where x_2 was used as a basepoint. Then, going through the construction above gives the same canonical form $\gamma = \gamma'_2$ that we obtained for γ'_1 . Thus, both γ_1 and γ_2 are conjugate to γ . \square

2.6 Conjugacy Classes of General Elements

For an element $\gamma \in PL_0(I)$, consider the set $x \in [0, 1]$ such that $\gamma(x) = x$. We call this *fixed set* of γ . Because γ is piecewise linear, this set consists of finitely many disjoint closed intervals, including single points. The set of points $x \in [0, 1]$ such that $\gamma(x) \neq x$ then must consist of finitely many open intervals. We call each of these intervals an *orbital*. Suppose that (a, b) is an orbital of γ . Let $f_{(a,b)}(x) = (b - a)x + a$. (This function is not generally an element of $PL_0(I)$.) Define a function $\hat{\gamma}_{(a,b)} = f_{(a,b)}^{-1} \circ \gamma \circ f_{(a,b)}$. Then, $\hat{\gamma}_{(a,b)}$ is piecewiselinear and increasing because each of its composition factors has these properties. In addition, $\gamma(a) = a$ and $\gamma(b) = b$, so $\hat{\gamma}_{(a,b)}(0) = 0$ and $\hat{\gamma}_{(a,b)}(1) = 1$. Because $\gamma(x) \neq x$ for $x \in (a, b)$, $\hat{\gamma}_{(a,b)}(x) \neq x$ for $x \in (0, 1)$. Thus, $\hat{\gamma}_{(a,b)}$ is a one-bump function in $PL_0(I)$ if we restrict it to the interval $[0, 1]$. For each orbital (a, b) of an element γ , we define an abstract circle diagram Σ equal to the abstract circle diagram of $\hat{\gamma}_{(a,b)}$ when restricted to $[0, 1]$.

Theorem 2.22. *Suppose that $\gamma' = \beta^{-1}\gamma\beta$. Then, x is in the fixed set of γ if and only if $\beta^{-1}(x)$ is in the fixed set of γ' .*

Proof. Suppose that x is in the fixed set of γ . Then,

$$\gamma'\beta^{-1}(x) = \beta^{-1}\gamma(x) = \beta^{-1}(x)$$

The converse follows from parallel reasoning. \square

The above theorem tells us that the orbitals of γ map to orbitals of the conjugate γ' under the correspondence map.

Theorem 2.23. *Suppose that $\gamma' = \beta^{-1}\gamma\beta$. Suppose that (a, b) is an orbital of γ and that $(a', b') = \beta^{-1}((a, b))$ is the corresponding orbital of γ' . Then, $\hat{\gamma}_{(a,b)}$ is conjugate to $\hat{\gamma}'_{(a',b')}$.*

Proof. For the purposes of this proof, we will treat γ , γ' and β as if their domains are all of \mathbb{R} . For $x \notin [0, 1]$, let $\gamma(x) = \gamma'(x) = \beta(x) = x$.

$$\gamma' = \beta^{-1}\gamma\beta$$

$$f_{(a',b')}^{-1}\gamma'f_{(a',b')} = f_{(a',b')}^{-1}\beta^{-1}\gamma\beta f_{(a',b')}$$

$$\hat{\gamma}'_{(a',b')} = f_{(a',b')}^{-1}\beta^{-1}f_{(a,b)}f_{(a,b)}^{-1}\gamma f_{(a,b)}f_{(a,b)}^{-1}\beta f_{(a',b')}$$

$$\hat{\gamma}'_{(a',b')} = f_{(a',b')}^{-1}\beta^{-1}f_{(a,b)}\hat{\gamma}_{(a,b)}f_{(a,b)}^{-1}\beta f_{(a',b')}$$

Define $\delta = f_{(a,b)}^{-1}\beta f_{(a',b')}$. Then,

$$\delta(0) = f_{(a,b)}^{-1}\beta(a') = f_{(a,b)}^{-1}(a) = 0$$

Likewise,

$$\delta(1) = 1$$

The function δ is increasing and piecewise linear because its composition factors have these properties. Thus, δ may be treated as an element of $PL_0(I)$ by restricting its domain to $[0, 1]$. Then, treating δ , $\hat{\gamma}_{(a,b)}$ and $\hat{\gamma}'_{(a',b')}$ as elements of $PL_0(I)$, $\hat{\gamma}_{(a,b)}$ and $\hat{\gamma}'_{(a',b')}$ are conjugate. \square

Corollary 2.2. *Suppose that $\gamma' = \beta^{-1}\gamma\beta$ and that (a, b) is an orbital of γ . Let $(a', b') = \beta^{-1}((a, b))$. Then, the orbital (a, b) of γ has the same abstract circle diagram as the orbital (a', b') of γ' .*

Theorem 2.24. *Let the set S consist of all the endpoints of orbitals of γ and the points 0 and 1. Define S' in the same way for γ' . The elements γ and γ' are conjugate if and only if the following conditions are met:*

1. S and S' have the same number of elements.
2. Let $0 = p_0 < p_1 < \cdots < p_{n-1} < p_n = 1$ be the points of S . Let $0 = p'_0 < p'_1 < \cdots < p'_{n-1} < p'_n = 1$ be the points of S' . For each $j \in \mathbb{Z}$ with $0 \leq j < n$, either the intervals $[p_j, p_{j+1}]$ and $[p'_j, p'_{j+1}]$ are in the fixed sets of γ and γ' respectively or the intervals (p_j, p_{j+1}) and (p'_j, p'_{j+1}) are orbitals of γ and γ' respectively and they have the same abstract circle diagram.

This theorem gives the general solution of the conjugacy problem in $PL_0(I)$ and represents the culmination of all our efforts in this paper.

Proof. Suppose that γ and γ' are conjugate. Then, the conditions follow from Theorems 2.22 and 2.23.

Conversely suppose that the conditions hold for elements γ and γ' of $PL_0(I)$. Then, we construct an element β of $PL_0(I)$ as follows:

On each interval $I = [p_j, p_{j+1}]$, where $j \in \mathbb{Z}$ and $0 \leq j < n$, define

$$\beta(x) = \frac{p'_{j+1} - p'_j}{p_{j+1} - p_j}(x - p_j) + p'_j$$

for $x \in I$.

Then, for each p_j , $\beta(p_j) = p'_j$. Let $\gamma'' = \beta^{-1}\gamma'\beta$. By Theorem 2.22, γ'' and γ have the same orbitals. In addition, if (a, b) is an orbital of both γ and γ'' , then this orbital will have the same abstract circle diagram for both γ and γ'' by Theorem 2.23.

Now, we may assume that γ and γ' have the same orbitals. Pick one of these orbitals (a, b) . We know that (a, b) has the same abstract circle diagram for γ and γ' . Then, by Theorem 2.21, $\hat{\gamma}_{(a,b)}$ and $\hat{\gamma}'_{(a,b)}$ are conjugate. For some β , $\hat{\gamma}_{(a,b)} = \beta^{-1}\hat{\gamma}'_{(a,b)}\beta$. Define $\bar{\beta} = f_{(a,b)}\beta f_{(a,b)}^{-1}$ for $x \in (a, b)$ and x for $x \notin (a, b)$. On (a, b) , $\bar{\beta}$ is a scaled down version of β . Following an analysis like that in the proof of 2.23, we find that $\bar{\beta}^{-1}\gamma'\bar{\beta}(x)$ is equal to $\gamma(x)$ on the interval (a, b) and equal to $\gamma'(x)$ otherwise. Since γ and γ' have only a finite number of orbitals, we can conjugate γ' to γ in a finite number of steps. \square

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