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Techniques for cubic algebraic surfaces I

Thomas W. Sederberg
tom@cs.byu.edu

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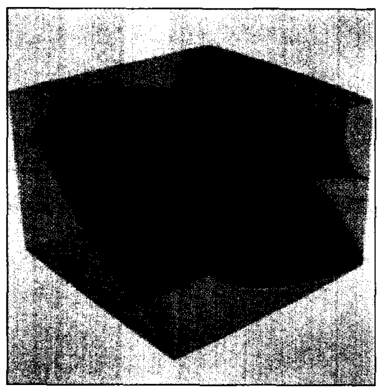
Surfaces

Tutorial
Part One

Techniques for Cubic Algebraic Surfaces

Thomas W. Sederberg
Brigham Young University



An algebraic surface can be defined by an implicit polynomial equation $f(x,y,z) = 0$. Degree 2 (and, of course, degree 1) algebraic surfaces are widely used in computer-aided geometric design and graphics. Free-form surface modeling, on the other hand, has traditionally been accomplished using parametric surface patches.

This two-part tutorial (part two will appear in *CG&A*'s September issue) discusses some techniques that may have potential for performing free-form modeling with algebraic surfaces. Specific attention is paid to cubic algebraic surfaces, although many of the ideas presented have application to algebraic surfaces of any degree. Topics addressed include piecewise constructions, interpolation to points and space curves, and parameterization.

Virtually any contemporary paper on free-form curves or surfaces deals with parametric equations. Bezier, B-spline, and Coons surface patches, for example, are based on parametric equations. In computer-aided geometric design and graphics the use of surfaces defined by implicit equations has largely been restricted to planes and quadrics. Some work has been done with polynomial implicit surfaces[1-4] and nonpolynomial implicit surfaces.[5-7]

This tutorial presents some tools for free-form modeling with algebraic surfaces, that is, surfaces that can be defined using an implicit polynomial equation $f(x,y,z) = 0$. Cubic algebraic surfaces (defined by an implicit equation of degree 3) are emphasized. While much of this material applies only to cubic surfaces, some of it applies to algebraic surfaces of any degree.

Part one of this tutorial introduces terminology, presents different methods for defining and modeling

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with cubic surfaces, and examines the power basis representation of algebraic surfaces. It also discusses methods of forcing an algebraic surface to interpolate a set of points or a space curve. Parametric definition of cubic surfaces by imposing base points is presented, along with the classical result that a cubic surface can be defined as the intersection locus of three two-parameter families of planes.

In part two, which will appear in *CG&A*'s September issue, we consider how to impose derivative continuity between two adjacent algebraic surfaces, and review piecewise algebraic surface patches. Then we discuss a method for combining them together into macro patches, and conclude with the important problem of parameterization.

At least three different algorithms can render algebraic surfaces. Such surfaces lend themselves readily to ray tracing.[8] Scan-line algorithms work well on algebraic surfaces of degree less than 6.[9] Alternately, algebraic surfaces can be polygonized and rendered using conventional polygon rendering programs.[10] This tutorial is illustrated with computer-generated images of algebraic surfaces created using a polygonization algorithm and Movie.BYU software.

Terminology

In this tutorial, an *implicit surface* means a surface whose initial definition is given in terms of a polynomial implicit equation $f(x,y,z) = 0$. A *parametric surface* is one whose initial definition is given by rational polynomial parametric equations

$$x = \frac{x(s,t)}{w(s,t)}$$

$$y = \frac{y(s,t)}{w(s,t)}$$

$$z = \frac{z(s,t)}{w(s,t)}$$

Given a parametric surface, it is possible to compute an implicit equation defining that same point locus. Determining the implicit equation of a parametric surface is known as *implicitization*. Likewise, for certain implicit surfaces (such as planes and quadric surfaces, and most cubic surfaces), it is possible to find a parametric representation. This process is referred to as *uniformization* or *parameterization*.

An *algebraic surface* is a surface that can be expressed using a polynomial implicit equation. A *rational surface* (in the classical sense) is a surface that can be expressed using parametric equations. Thus, all rational surfaces are algebraic (because they can be implicitized), and some algebraic surfaces (notably planes, quadrics, and most cubics) are rational. Since we need to distinguish between algebraic surfaces defined parametrically and those defined implicitly, we will use the term implicit surface to signify a surface defined by an implicit equation, and parametric surface to signify a surface defined by parametric equations.

Here we always take *cubic surface* to mean an algebraic surface whose implicit equation is degree 3. A *cubic triangular surface patch* is a parametric surface whose parametric equations are of degree 3.

Table 1

Surface Degree	Number of Terms
1	4
2	10
3	20
4	35
5	56
6	84

Power basis

The most familiar representation of an implicit surface is probably that given by a power basis implicit equation:

$$\sum_{i+j+k\le n} c_{ijk}x^iy^jz^k = 0$$

The number of terms grows quickly; the implicit equation of a degree n surface has $(n + 1)(n + 2)(n + 3)/6$ terms (see Table 1).

The power basis implicit equation provides little direct insight into the shape of the implicit surface. Consider a Bezier parametric surface patch: If a control point is moved, it is easy to predict how the surface will respond. By contrast, if the coefficient c_{ijk} is changed in an implicit equation, who knows how it will influence the surface?

Reducible surfaces

A possibility with implicit surfaces that does not exist with rational surfaces is that a single implicit

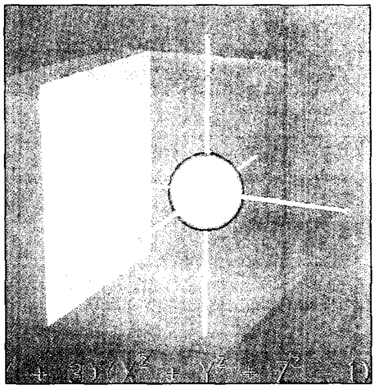


Figure 1. Reducible cubic surface.

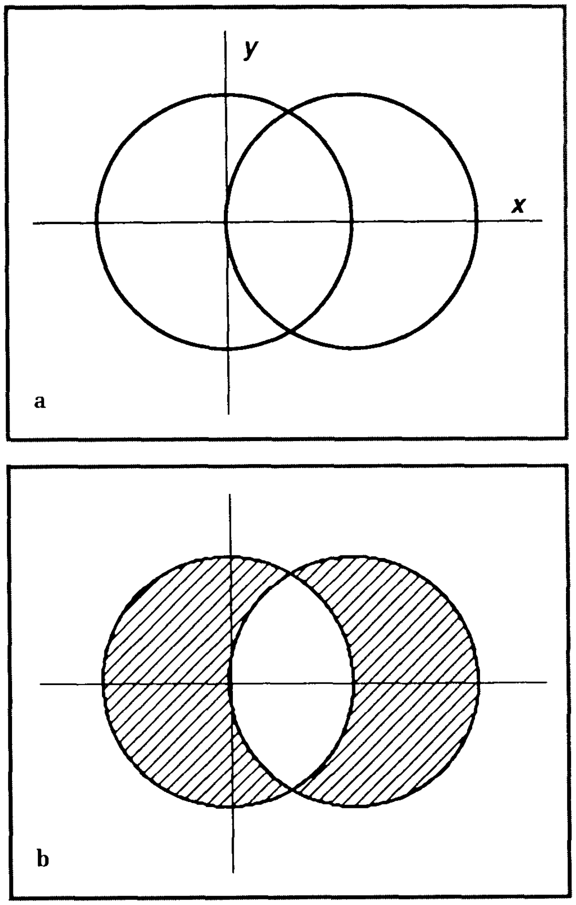


Figure 2. (a) Reducible quartic curve; (b) half-space of the quartic curve.

equation may simultaneously define two or more distinct implicit surfaces. For example, Figure 1 shows a cubic surface whose equation is

$$f(x,y,z) = x^3 + xy^2 + xz^2 + 3x^2 + 3y^2 + 3z^2 - x - 3 = 0$$

This equation can be factored into

$$f(x,y,z) = (x + 3)(x^2 + y^2 + z^2 - 1)=0$$

Any point that satisfies either the equation of the plane $x + 3 = 0$ or the equation of the sphere $x^2 + y^2 + z^2 - 1 = 0$ will satisfy the equation $f(x,y,z) = 0$. Any surface whose implicit equation can be factored is known as a *reducible* surface. Surfaces whose coefficients are chosen at random are generally *irreducible*.

This illustrates a curious property of implicit surfaces: The Boolean sum of two implicit surfaces (that is, the set of all points lying on either surface) can be expressed by multiplying their respective implicit equations. Unfortunately, this holds only for the surfaces themselves, not for the half-spaces bounded by the surfaces. A *half-space* is an important notion in solid geometric modeling, and refers to the (possibly infinite) region bounded by a surface. Each implicit surface $f(x,y,z) = 0$ defines two half-spaces, given respectively by the inequalities $f(x,y,z) < 0$ and $f(x,y,z) > 0$. If $f(x,y,z) > 0$ and $g(x,y,z) > 0$ are two half-spaces, then their *exclusive or* is given by $f(x,y,z) \times g(x,y,z) < 0$.

Figure 2a shows the degree 4 curve given by the implicit equation

$$(x^2 + y^2 - 4)((x - 2)^2 + y^2 - 4) = 0$$

This degree 4 curve is simply the two circles $x^2 + y^2 - 4 = 0$ and $(x - 2)^2 + y^2 - 4 = 0$. Figure 2b shows half-space

$$(x^2 + y^2 - 4)((x - 2)^2 + y^2 - 4) < 0$$

This half-space is the exclusive or of the two circles' interiors.

Consider again the reducible cubic surface in Figure 1, which consists of a sphere and a plane. If any coefficients of the implicit equation are perturbed, it seems reasonable that the surface will undergo only a slight alteration, although it is admittedly very difficult to predict what influence a given coefficient perturbation will have on the shape of the surface. Figures 3a and 3b show what happens to the surface in Figure 1 if its constant term is altered. The surface in Figure 3a is defined by the implicit equation

$$\begin{aligned} f(x,y,z) &= (x + 3)(x^2 + y^2 + z^2 - 1) - 3 \\ &= x^3 + xy^2 + xz^2 + 3x^2 + 3y^2 + 3z^2 - x - 6 = 0 \end{aligned}$$

This surface demonstrates another characteristic of some implicit surfaces that does not occur with parametric surfaces (at least not with a parametric surface

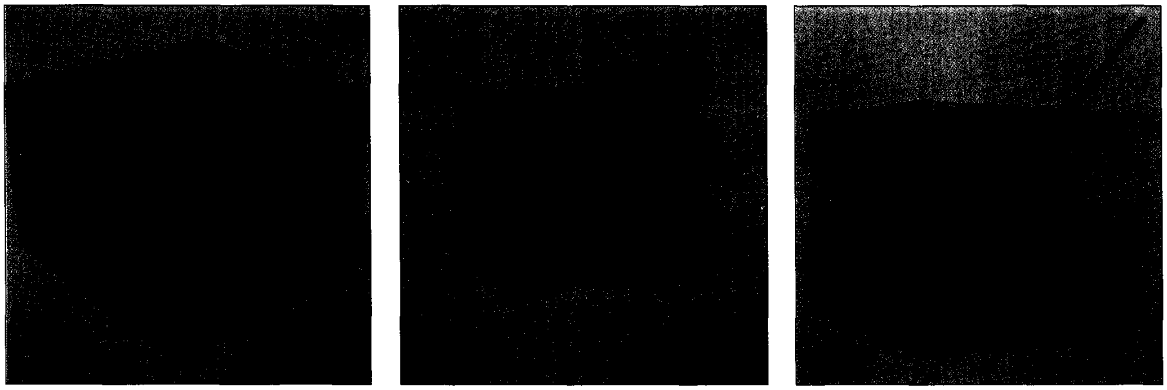


Figure 5. Cubic surface: (a) one line, (b) two lines, (c) three lines.

$$(2,2,3),\quad (1,1,3),\quad (1,2,2),\quad (1,3,3)$$

We evaluate the implicit Equation 1 at those four points. This creates four linear equations in (a,b,c,d):

$$a \cdot 2 + b \cdot 2 + c \cdot 3 + d = -(2^2 + 2^2 + 3^2)$$
$$a \cdot 1 + b \cdot 1 + c \cdot 3 + d = -(1^2 + 1^2 + 3^2)$$
$$a \cdot 1 + b \cdot 2 + c \cdot 2 + d = -(1^2 + 2^2 + 2^2)$$
$$a \cdot 1 + b \cdot 3 + c \cdot 3 + d = -(1^2 + 3^2 + 3^2)$$

from which

$$a = -2; \quad b = -4; \quad c = -6; \quad d = 13$$

and

$$x_c = 1; \quad y_c = 2; \quad z_c = 3; \quad r = 1$$

Curve interpolation

Bezout's theorem[11] states that a space curve of degree m either intersects a surface of degree n in exactly mn points (properly counting complex, infinite, and multiple intersections), or else it lies completely on the surface. This fact provides us with a method for forcing a surface to interpolate a space curve. In our case, the space curve is defined parametrically, and the surface is defined implicitly.

Consider the cubic surface in Figure 4. Recall that this surface was determined by specifying that it interpolate 19 points. If three of those points happen to be collinear as in Figure 4, then those three points constitute the complete intersection of the line with the surface. If we now specify that a fourth of our 19 points lies along that line, then the entire line must lie on the surface. This is shown in Figure 5a. Thus, 4 degrees of freedom are required to interpolate a line with a cubic surface.

This can also be shown algebraically. Suppose that the line is given parametrically by

$$x = x_0 + x_1 t; \quad y = y_0 + y_1 t; \quad z = z_0 + z_1 t$$

Then the intersection of the cubic surface $f(x,y,z) = 0$ with the line is

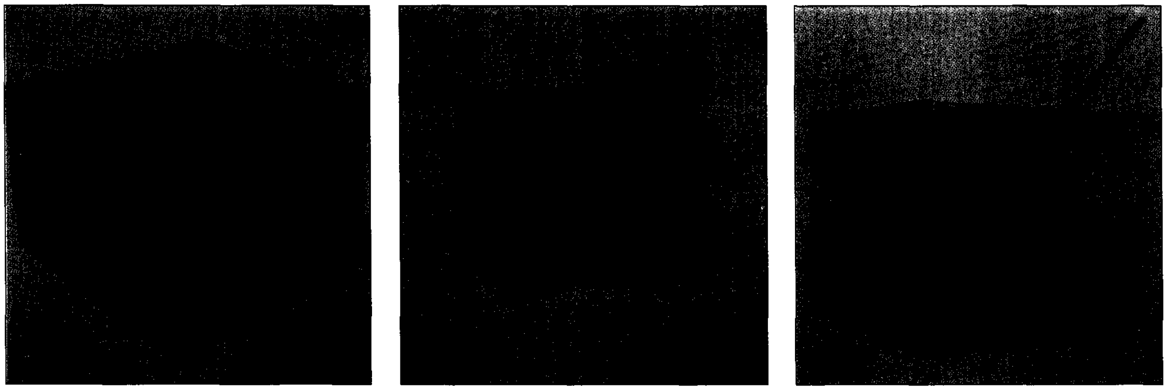
$$f(x_0 + x_1 t, y_0 + y_1 t, z_0 + z_1 t)$$
$$= a_3 t^3 + a_2 t^2 + a_1 t + a_0 = 0$$

which is a cubic polynomial in t whose coefficients a_i are linear in the c_{ijk} coefficients of the surface (for which we are solving). For the line to lie completely on the surface, we must have

$$a_3 = a_2 = a_1 = a_0 = 0$$

which gives four linear equations in c_{ijk}. Those four equations are equivalent to specifying that any four distinct points along the line lie on the surface.

Suppose we wish to force a second straight line, which intersects the first straight line, to lie on the surface, as shown in Figure 5b. In this case, it suffices to specify seven points, one of them being the point common to the two lines. Similarly, a third line, coplanar with the other two, can be specified with only two additional points, as shown in Figure 5c. Thus, 9 degrees of freedom are required to force a cubic surface to interpolate the three lines that meet pairwise in three distinct points.



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$$f(x_0 + x_1 t, y_0 + y_1 t, z_0 + z_1 t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0 = 0$$

which is a cubic polynomial in t whose coefficients a_i are linear in the c_{ijk} coefficients of the surface (for which we are solving). For the line to lie completely on the surface, we must have

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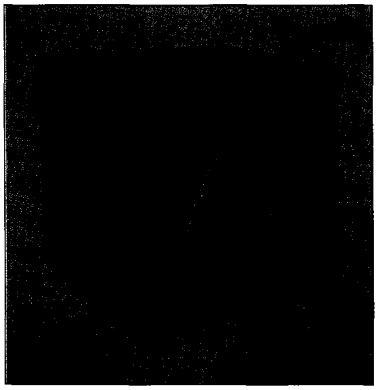

Figure 6. Cubic surface, three conic curves.

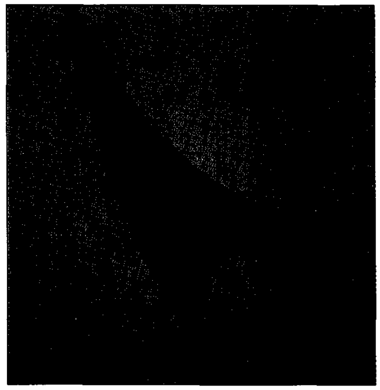

Figure 7. Steiner-like cubic surface.

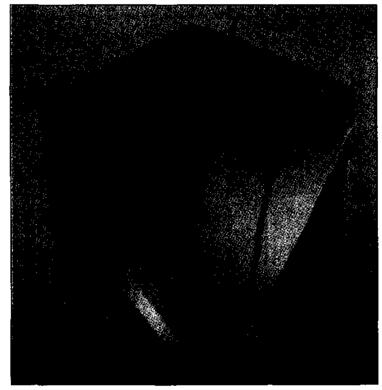

Figure 8. Cubic surface, two cubic curves.

Point and line interpolation is not an attractive mechanism for defining cubic surfaces, because the resulting surfaces behave unpredictably between the interpolated points or lines. We next look at forcing a cubic surface to interpolate three degree 2 curves, and we will see that this provides a potentially useful description.

A degree 2 curve intersects a cubic surface in six points, or else lies completely on it. Thus, if we select seven of our 19 points to lie on a degree 2 curve, the cubic surface is forced to interpolate that curve. Figure 6 shows interpolation of three parabolas. If the three degree 2 curves have three points in common pairwise, then only a total of 18 degrees of freedom is needed to cause interpolation. This leaves one extra point to interpolate, which can be viewed as sort of an interior control point.

Defining a cubic surface this way produces something strikingly similar to a Steiner patch.[12] A Steiner patch is a quadratic triangular parametric surface that can be implicitized to produce a degree 4 implicit surface. The three degree 2 boundary curves completely define the Steiner patch. By contrast, our cubic surface is defined by the three boundary curves, plus one additional interpolatory point. Figure 7 shows a cubic surface patch bounded by three conic boundary curves, and interpolating one interior point. So, even though the implicit degree of the Steiner surface is greater than the degree of the cubic surface, the cubic surface has one more degree of freedom.

Of course, a potential problem with the Steiner-like cubic is that the surface could possibly experience a flight to infinity between the three boundary curves. How to detect this, and how actually to model with cubic surfaces defined this way, remain open questions.

Finally, a twisted cubic curve (a cubic curve that does not lie in a plane) intersects a cubic surface in nine points. If it intersects it in 10 points, the surface interpolates the curve. This leaves 9 degrees of freedom, with which the surface can be forced to interpolate a second twisted cubic curve that intersects the other curve. Thus, two intersecting twisted cubic curves completely define a cubic surface, as illustrated in Figure 8. This is somewhat reminiscent of a cubic triangular surface patch, which is defined by three cubic curves plus an interior control point (and which has an implicit equation of degree 9).

One would expect (or at least hope) that a cubic surface defined this way would behave nicely in between the two boundary curves, something like a triangular cubic surface patch. Unfortunately, this has not been my experience. Figure 8 required several attempts to yield a reasonable example. Note here that a second component of the surface hovers nearby.

Base-point coercion

Our previous discussion has dealt with methods of defining algebraic surfaces using implicit equations. This section presents a method of defining low-degree algebraic surfaces using parametric equations.

Surface implicitization

In theory, it is possible to compute an implicit equation for any parametric surface. The implicit surface thus derived is equivalent to the parametric surface. As we saw earlier, the process of computing the im-

Table 2	
Surface Patch Type	**Degree**
Degree 1 triangle (plane)	1
Bilinear (hyperbolic paraboloid)	2
Quadratic Triangle (Steiner)	4
Biquadratic (tensor product)	8
Cubic Triangle	9
Bicubic	18

plicit equation of a parametric surface is called implicitization.[13-15]

Unfortunately, the degree of the implicit equation of a parametric surface is much higher than the degree of the parametric equations. A *tensor product* parametric surface can be defined by the power basis equation

$$\mathbf{X}(s,t) = \frac{\sum_{i=0}^{m}\sum_{j=0}^{n} \mathbf{c}_{ij}s^i t^j}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{ij}s^i t^j}$$

where $\mathbf{X}$ and $\mathbf{c}$ are 3D vectors. The implicit equation $f(x,y,z) = 0$ of such a surface is generally degree $2mn$. Thus, for example, a bicubic patch generally has an implicit equation of degree 18.

A *triangular* parametric surface can be defined by the power basis equation

$$\mathbf{X}(s,t) = \frac{\sum_{i+j\le n} \mathbf{c}_{ij}s^i t^j}{\sum_{i+j\le n} w_{ij}s^i t^j}$$

In this case, the degree of the implicit equation is generally n^2.

One motivation for modeling directly with implicit surfaces rather than parametric surfaces is that relatively low degree implicit surfaces exhibit design flexibility comparable to parametric surfaces. For example, a bicubic patch (with 64 scalar degrees of freedom) generally has an implicit equation of degree 18, whereas implicit surfaces of degree 6 have even more scalar degrees of freedom (83). Thus, implicitization does not appear feasible for creating low-degree implicit surfaces.

Table 2 lists the degree of the implicit equations of lowest degree parametric surfaces and raises a question: It is generally known that all quadric surfaces (degree 2 implicit equation) are rational. That is, they all can be expressed in parametric equations. Yet the only quadric surface in this table is the bilinear surface. Also, it is known that with one exception, any cubic surface can be expressed using parametric equations; yet cubic surfaces do not appear in this table. Since this table lists all of the lowest degree parametric surfaces, how are quadric and cubic surfaces expressed parametrically?

The answer to that question is centered in what are called base points.[11,15] Denoting the parametric equations of a surface as

$$x = \frac{f_1(s,t)}{f_4(s,t)}$$

$$y = \frac{f_2(s,t)}{f_4(s,t)}$$

$$z = \frac{f_3(s,t)}{f_4(s,t)}$$

a base point is any parameter pair (s,t) for which

$$f_1(s,t) = f_2(s,t) = f_3(s,t) = f_4(s,t) = 0$$

When we discussed the formulas for the degree of the implicit equation of a parametric surface ($2mn$ for tensor product surfaces and n^2 for triangular surface patches), we were careful to say that these applied to *general* parametric surfaces, that is, surfaces with no base points. If a parametric surface happens to have r base points, the implicit equation is degree $n^2 - r$ for triangular surface patches and degree $2mn - r$ for tensor product surface patches.

To explain this, we first examine why a triangular surface patch of degree n has an implicit equation of degree n^2. *The degree of the implicit equation of such a surface can be determined by counting the number of times that it is intersected by a generic straight line.* To determine the number of intersections between a line L and a rational surface S given by

$$x = \frac{f_1(s,t)}{f_4(s,t)}$$

$$y = \frac{f_2(s,t)}{f_4(s,t)}$$

$$z = \frac{f_3(s,t)}{f_4(s,t)}$$

where the f_i are all polynomials, we intersect the surface with two planes that contain L. Each plane intersects S in a curve that can be represented as an implicit equation in parameter space, and the intersections of those two curves identify the (s,t) values of the intersections of L and S.

Consider two distinct planes in general position $a_1x + a_2y + a_3z + a_4 = 0$ and $b_1x + b_2y + b_3z + b_4 = 0$. Recall that our goal is to determine how many times the line L at which those two planes meet intersects the parametric surface. These planes intersect the surface S in curves

$$\begin{aligned} a_1f_1(s,t) + a_2f_2(s,t) + a_3f_3(s,t) + a_4f_4(s,t) = 0 \\ b_1f_1(s,t) + b_2f_2(s,t) + b_3f_3(s,t) + b_4f_4(s,t) = 0 \end{aligned} \quad \textbf{(2)}$$

Each curve in Equation 2 is degree n in s and t, where n is the largest of the degrees of the f_i. By Bezout's theorem, these two curves intersect in n^2 points, each such point being the preimage of a point at which L intersects S. Therefore, n^2 is clearly the number of times that L intersects S. Thus, the degree of S, and by definition the degree of its implicit equation, is n^2.

Base points decrease the degree of a surface S as follows. If a base point exists, the two general plane section curves in Equation 2 each contain the base point: They intersect at the base point, and at $n^2 - 1$ other points. However, since the base point does not map to a unique point on S ($x = y = z = 0/0$ is undefined), this does not represent a point at which L intersects S. Therefore, the number of intersections between L and S is $n^2 - 1$, which must be the degree of the surface. Each additional simple base point diminishes the surface degree by 1. Base points at infinity occur when all plane section curves have a common asymptotic direction.

Base points can have a more complicated influence on the degree of the surface. For example, consider again the curves in Equation 2, generated by intersecting S with two planes in general position. If those two curves are tangent at a base point, they intersect twice at the base point, and the degree of the surface drops to $n^2 - 2$. Again, the reasoning is that L (a line in general position) intersects S in $n^2 - 2$ points. If the two curves have a double point at a base point, they intersect four times at that base point, and the degree becomes $n^2 - 4$. Thus, a general degree formula is $n^2 - r$, where r is the total number of times that the curves in Equation 2 (representing the intersection of S by two planes in general position) intersect at base points. This also assumes that the surface has a one-to-one parameterization.

A few examples will make these concepts more tangible.

Base point example 1

Let's begin by considering the quadric surface

$$x^2 + y^2 + xy + z^2 - x - y + z = 0$$

This surface can be parameterized by making the substitutions

$$x = sz, \quad y = tz$$

This yields

$$z^2(s^2 + t^2 + st - 1) + z(-s - t + 1) = 0$$

from which

$$z = \frac{s + t - 1}{s^2 + t^2 + st - 1}$$

$$x = s\frac{s + t - 1}{s^2 + t^2 + st - 1}$$

$$y = t\frac{s + t - 1}{s^2 + t^2 + st - 1}$$

Or, in homogeneous form,

$$\begin{aligned} f_1 &= s(s + t - 1) \\ f_2 &= t(s + t - 1) \\ f_3 &= (s + t - 1) \\ f_4 &= s^2 + t^2 + st - 1 \end{aligned} \quad \textbf{(3)}$$

We now have an example to illustrate the claims about base points. Equations 3 are degree 2 parametric equations derived by parameterizing a degree 2 algebraic surface. We therefore expect that these parametric equations have two base points. Through a little trial and error, we will see shortly that there are indeed two (and only two) base points: $(s,t) = (0,1), (1,0)$.

Base point example 2

Base points may be complex or infinite, and homogeneous parameters are required to make the identification. If we denote the homogeneous parameters by capital letters (S,T,U), the nonhomogeneous parameters s,t are related by $(s,t) = (S/U,T/U)$. The value of homogeneous parameters is that they permit a natural representation of parameter values at infinity by simply setting $U = 0$.

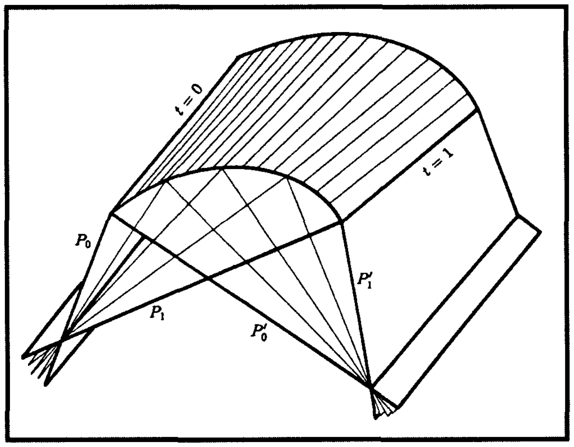

Figure 9. Intersecting pencils of planes.

The unit sphere $x^2 + y^2 + z^2 - 1 = 0$ can be parameterized (expressed in nonhomogeneous parametric equations):

$$x = 2s;\ \ y = 1 - s^2 - t^2;\ \ z = 2t;$$
$$w = s^2 + t^2 + 1$$

An example of homogeneous parametric equations for the sphere is

$$x = 2SU;\ \ y = U^2 - S^2 - T^2;\ \ z = 2TU;$$
$$w = S^2 + T^2 + U^2$$

As in example 1, since the parametric equations and the implicit equation are degree 2, we expect two base points in the parametric equations. Using the homogeneous form, we can identify those two base points as $(S,T,U) = (1,i,0)$ and $(i,1,0)$. These base points are both infinite and complex.

This discussion could continue in several fascinating directions. For example, we could show that a base point actually maps to an entire line on the surface, and that the line connecting two base points in parameter space (in the case of a quadric surface) actually maps to a single point on the surface. Furthermore, that single point is actually the center of projection of the quadric surface, or the point through which the parameterizing planes in Equation 3 pass.[12]

Defining cubic surfaces by imposing base points

Part two of this tutorial addresses the problem of parameterizing a cubic surface defined with an implicit equation. Here we look at the reverse problem of directly coming up with a parametric equation which, if it were implicitized, would define a cubic surface.

There are several ways to define a cubic surface using parametric equations (recall that here "cubic surface" means a surface whose implicit equation is degree 3): A Steiner patch with one base point, a biquadratic patch with five base points, or a cubic triangular patch with six base points all define a cubic surface. The Steiner patch with one base point cannot express the general cubic surface, but only ruled cubic surfaces.

The easiest way to force a parametric surface to contain base points is to define what those base points are, and then to use undetermined coefficients to solve for parametric equations that contain those base points. For example, suppose we want to define a ruled cubic surface by forcing a Steiner surface to contain the base point $(-1,0)$. If we work in power basis equations, the general expression for a Steiner surface is

$$\mathbf{X}(s,t) = \frac{\mathbf{x}_{20}s^2 + \mathbf{x}_{11}st + \mathbf{x}_{02}t^2 + \mathbf{x}_{10}s + \mathbf{x}_{01}t + \mathbf{x}_{00}}{d_{20}s^2 + d_{11}st + d_{02}t^2 + d_{10}s + d_{01}t + d_{00}}$$

To force a base point to occur at $(s,t) = (-1,0)$, the following linear conditions must be satisfied:

$$\mathbf{x}_{20}(-1)^2 + \mathbf{x}_{11}(-1)(0) + \mathbf{x}_{02}(0)^2 + \mathbf{x}_{10}(-1) + \mathbf{x}_{01}(0) + \mathbf{x}_{00} = 0$$

and

$$d_{20}(-1)^2 + d_{11}(-1)(0) + d_{02}(0)^2 + d_{10}(-1) + d_{01}(0) + d_{00} = 0$$

Other linear conditions can be imposed, such as specifying the Cartesian coordinates of the surface patch corners. From the appropriate number of linear conditions, the parametric equation coefficients can be computed.

This is not an efficient method of creating cubic surfaces. To impose six base points on a cubic triangular surface patch, six of the 10 Bezier control points must be constrained, which doesn't leave much control. Of course, the values of the base points are also variables that provide additional shape control, but there does not appear to be much intuitive relationship between the base points and the shape of the surface.

The next section discusses a method of defining a cubic surface in terms of a cubic triangular surface patch in which only three of the 10 control points are constrained.

Cubic surfaces from bundles of planes

A useful classical result from analytic geometry is that any cubic surface can be defined as the intersection locus of three two-parameter families of planes (where all three families of planes are controlled by the same two parameters).[16,17] We begin by showing how two pencils of planes intersect in a ruled quadric surface, and how three pencils of planes intersect in a cubic space curve.

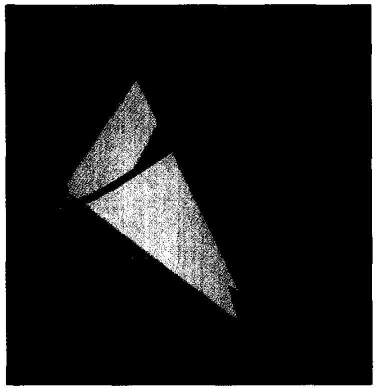

Figure 10. Three pencils of planes, cubic curve.

Pencils of planes

A pencil of planes is the family of all planes containing a given line, called the *pencil axis*. If the two planes are denoted

$$P_0 = a_0x + b_0y + c_0z + d_0 = 0$$

and

$$P_1 = a_1x + b_1y + c_1z + d_1 = 0$$

then the pencil of planes $P(t)$ can be expressed

$$P(t) = P_0(1 - t) + P_1t = 0$$

Consider two pencils of planes $P(t)$ and $P'(t)$. Each value of t defines the planes $P(t)$ and $P'(t)$, and for each value of t, those two planes intersect in a line. The intersection locus—that is, the set of all lines at which $P(t)$ and $P'(t)$ intersect for all values of t—turns out to be a ruled quadric surface. The implicit equation of the ruled quadric surface is $P_0P'_1 - P_1P'_0 = 0$. If the axes of $P(t)$ and $P'(t)$ are parallel, the resulting surface is a cylinder. If the axes are skew, the surface is a paraboloid, and if they intersect, the surface is a cone. Figure 9 shows two pencils of planes intersecting in a cylinder.

Next, consider three pencils of planes

$$P(t) = P_0(1 - t) + P_1t = 0$$
$$P'(t) = P'_0(1 - t) + P'_1t = 0$$
$$P''(t) = P''_0(1 - t) + P''_1t = 0$$

Each pair of pencils intersects in a ruled quadric surface, and those three ruled quadrics intersect in a twisted cubic curve. Figure 10 shows three pencils of planes intersecting in a twisted cubic curve. The planes P_0, P'_0, and P''_0 are white; P_1, P'_1, and P''_1 are red; and $P(.5)$, $P'(.5)$, and $P''(.5)$ are pink. The parametric equation of that curve is easily computed as follows:

$$x = \frac{\begin{vmatrix} b(t) & c(t) & d(t) \\ b'(t) & c'(t) & d'(t) \\ b''(t) & c''(t) & d''(t) \end{vmatrix}}{\begin{vmatrix} a(t) & b(t) & c(t) \\ a'(t) & b'(t) & c'(t) \\ a''(t) & b''(t) & c''(t) \end{vmatrix}}$$

$$y = -\frac{\begin{vmatrix} a(t) & c(t) & d(t) \\ a'(t) & c'(t) & d'(t) \\ a''(t) & c''(t) & d''(t) \end{vmatrix}}{\begin{vmatrix} a(t) & b(t) & c(t) \\ a'(t) & b'(t) & c'(t) \\ a''(t) & b''(t) & c''(t) \end{vmatrix}}$$

$$z = \frac{\begin{vmatrix} a(t) & b(t) & d(t) \\ a'(t) & b'(t) & d'(t) \\ a''(t) & b''(t) & d''(t) \end{vmatrix}}{\begin{vmatrix} a(t) & b(t) & c(t) \\ a'(t) & b'(t) & c'(t) \\ a''(t) & b''(t) & c''(t) \end{vmatrix}}$$

where $a(t)=a_0(1 - t) + a_1t$, etc. Figure 11 shows the twisted cubic curve lying on the ruled quadric surface defined by the intersection of pencils $P(t)$ and $P'(t)$.

Bundles of planes

A *bundle* of planes is the set of all planes containing a given point. If we denote the three planes

$$P_s = a_s x + b_s y + c_s z + d_s = 0$$
$$P_t = a_t x + b_t y + c_t z + d_t = 0$$
$$P_u = a_u x + b_u y + c_u z + d_u = 0$$

then the bundle can be expressed

$$P(s,t,u) = sP_s + tP_t + uP_u = 0$$

where we have $s + t + u = 1$.

Consider three bundles of planes, $P(s,t,u)$, $P'(s,t,u)$, and $P''(s,t,u)$. A given (s,t,u) identifies a plane from each bundle, and those three planes intersect in a point. As the parameters (s,t,u) vary, the point at which the three varying planes intersect sweeps out a cubic surface![16]

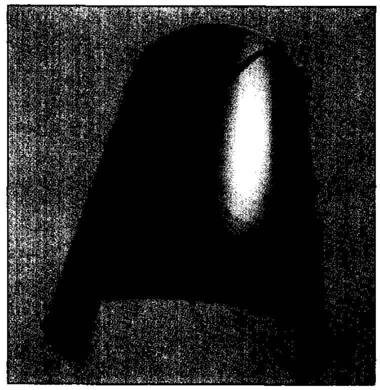

Figure 11. Cubic curves on a quadric surface.

The bundle representation of the cubic surface is a remarkable intermediate form from which the parametric and implicit equations of the cubic surface are readily derived. The implicit equation of the cubic surface is given by

$$f(x,y,z) = \begin{vmatrix} P_s & P_t & P_u \\ P'_s & P'_t & P'_u \\ P''_s & P''_t & P''_u \end{vmatrix} = 0$$

The parametric equations of the cubic surface are

$$x = \frac{\begin{vmatrix} b & c & d \\ b' & c' & d' \\ b'' & c'' & d'' \end{vmatrix}}{\Delta}$$

$$y = -\frac{\begin{vmatrix} a & c & d \\ a' & c' & d' \\ a'' & c'' & d'' \end{vmatrix}}{\Delta}$$

$$z = \frac{\begin{vmatrix} a & b & d \\ a' & b' & d' \\ a'' & b'' & d'' \end{vmatrix}}{\Delta}$$

where a, a', a'', b, etc., are all functions of s, t, and u; where $a(s,t,u) = a_s s + a_t t + a_u u$, etc.; and where

$$\Delta = \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}$$

These parametric equations are degree 3, so we conclude that there are six base points. This parametric representation has the form of a rational cubic triangular Bezier patch. The control points of that patch are easily computed from the nine defining planes, quite similar to how we computed the control points for the degree 2 rational Bezier curve in the section on pencils of planes. Unfortunately, all attempts at visualizing this process have so far proved too cluttered to be useful.

One way to use this technique is to select any seven of the Bezier control points and weights, from which the three bundles can be determined. These in turn fix the remaining three control points. However, if we try to adjoin a second patch with C^1 continuity to the first patch (which means that, at the patch boundary, all isoparameter curves on one patch have parametric first-order continuity with isoparameter curves on the other patch), this automatically constrains seven control points. The second patch is the same surface as the first patch. Thus, the bundle of planes approach does not appear to provide adequate flexibility to permit a C^1 mesh. Perhaps G^1 meshes would provide more flexibility; they require only that, at the patch boundary, normal vectors to identical points on adjacent patches are parallel. But that is an open question. Furthermore, the rational cubic Bezier surface obtained this way has the limitation that the cubic boundary curves are always nonplanar.

Conclusion

The concepts presented here have been gleaned primarily from classical literature and are quite basic. How much they can be applied to problems in computer graphics and geometric modeling remains to be seen. However, knowledge of these principles is useful in understanding current research in free-form modeling with implicit surfaces. ■

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Thomas W. Sederberg is an associate professor of civil engineering at Brigham Young University. His research interests are in computer-aided geometric design and computer graphics.

Sederberg received his PhD in mechanical engineering from Purdue University in 1983. He is a member of IEEE and ACM SIGGRAPH.

Sederberg can be reached at the Engineering Computer Graphics Laboratory, 368 Clyde Building, Brigham Young University, Provo, UT 84602.