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Techniques for cubic algebraic surfaces II

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Techniques for Cubic Algebraic Surfaces

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Research interest in algebraic surfaces is increasing. This survey of techniques for dealing with cubic algebraic surfaces includes some classical results as well as several recent innovations. We resume the survey begun in the July 1990 issue of *CG&A*[1] with the concept of derivative continuity.

Derivative continuity

The usefulness of implicit surfaces as free-form modeling primitives increases if they are amenable to being pieced together with derivative continuity. Parametric surface patches lend themselves to piecewise constructions more naturally than do implicit surfaces. Two reasons for this are that parametric patches are defined over a finite domain, and they have distinct boundary curves. In contrast, implicit surfaces can be of infinite extent. Much of the following discussion on continuity conditions for piecewise implicit surfaces was discussed in detail by Warren.[2]

The problem of derivative continuity between two implicit surfaces can be approached as follows: Given a surface $S_1(x, y, z) = 0$, find a second surface $S_2(x, y, z) = 0$ that is G^n continuous (continuous to n deriva-

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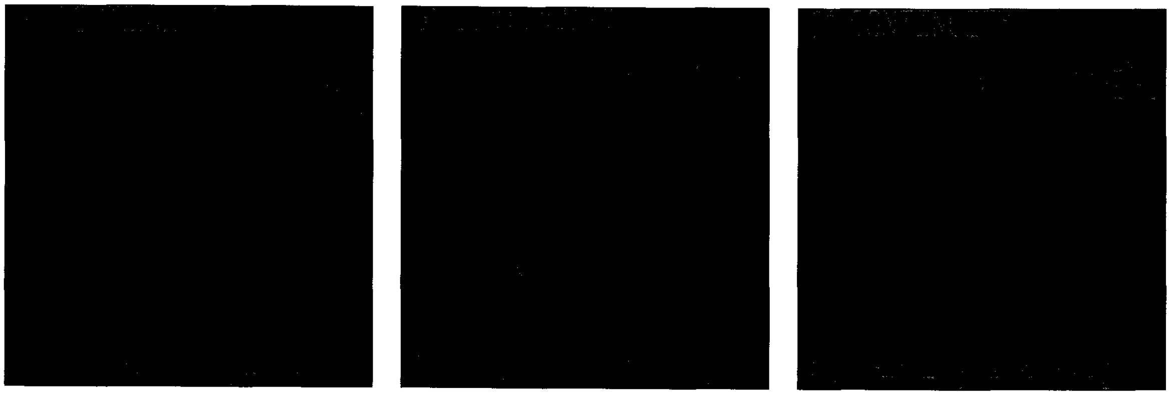


Figure 1. (a) G^0continuity, (b) G^1 continuity, (c) G^2 continuity.

tives) with S_1 along a boundary curve. The boundary curve is defined as the complete intersection of S_1 with an auxiliary surface $B(x, y, z) = 0$. Then $S_1(x, y, z) = 0$ and $S_2(x, y, z) = 0$ are G^n continuous along the curve of intersection of $S_1 \cap B$ if and only if

$$S_2(x, y, z) = F(x, y, z)S_1(x, y, z) + G(x, y, z)B^{n+1}(x, y, z)$$

where $F(x, y, z)$ and $G(x, y, z)$ are any polynomials.[2]

Let's discuss what this means for G^0 and G^1 continuity. G^0 (positional) continuity means that S_2 contains the curve $S_1 \cap B$. In other words, $S_2(x, y, z) = 0$ for any point (x, y, z) for which $S_1(x, y, z) = 0$ and $B(x, y, z) = 0$. We can easily verify this.

Here S_1 and S_2 are G^1 continuous if they are G^0 and also if they have the same normal direction at each point along $S_1 \cap B$. The (x, y, z) coordinates of the normal vector are given by the partial derivatives of the surface with respect to x, y, z. To verify that S_1 and S_2 are G^1, we must show that at any point along the curve $S_1 \cap B$, the normal vector for S_1—(S_{1x}, S_{1y}, S_{1z})—is parallel to the normal vector for S_2—(S_{2x}, S_{2y}, S_{2z}). Using the subscript x to indicate partial differentiation with respect to x,

$$S_{2x} = F_xS_1 + FS_{1x} + G_xB^2 + 2GBB_x$$

But along $S_1 \cap B$, $S_1 = B = 0$, so

$$S_{2x} = FS_{1x}$$

Therefore,

$$(S_{2x}, S_{2y}, S_{2z}) = F(S_{1x}, S_{1y}, S_{1z})$$

Figure 2. G^1 cubic elbow.

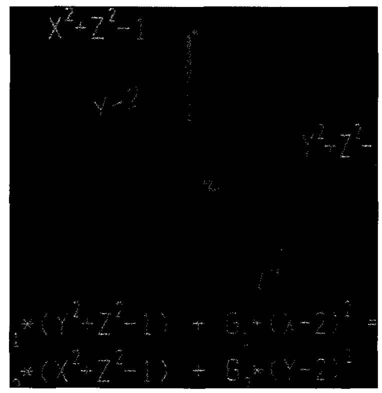


Figure 1 shows a cylinder $S_1(x, y, z) = y^2 + z^2 - 1$, a boundary plane $B(x, y, z) = x - 2$, and a cubic surface S_2. Figures 1a, 1b, and 1c show S_1 and S_2 with G^0, G^1, and G^2 continuity, respectively. Note that $F(x, y, z)$ is at most degree 1. Also, $G(x, y, z)$ is of degree 2 in Figure 1a, degree 1 in Figure 1b, and a constant in Figure 1c. Figure 2 shows an unexpected use for a cubic surface as an elbow that is G^1 with two circular cylinders.[2] An elbow surface is typically modeled using a quarter torus, which is a degree 4 surface.

Piecewise algebraic surfaces

In this section, we briefly review what has been termed *piecewise algebraic surface patches.*[3] The related scheme for defining planar algebraic curves has been discussed elsewhere.[4]

A piecewise algebraic surface patch has the following characteristics:

1. There is a meaningful relationship between the coefficient values and the shape of the surface.

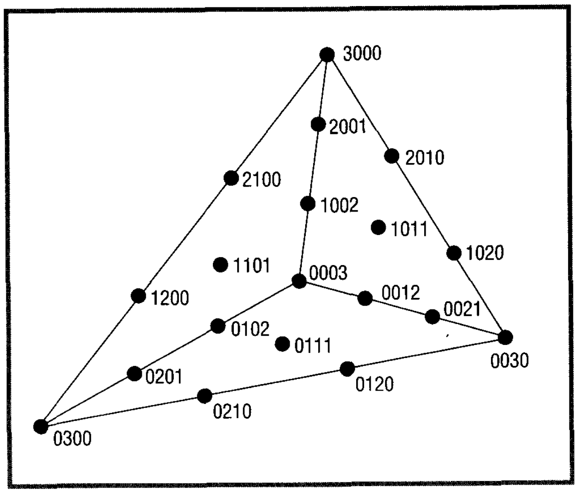


Figure 3. Cubic control points.

2. The patch has finite extent, being bounded by a tetrahedron.
3. A mesh of patches can be pieced together with derivative continuity.
4. It can be assured that the patch is single valued in a specified direction.

A piecewise algebraic surface patch is defined using a reference tetrahedron and a regular lattice of control points imposed on the tetrahedron. Coefficients assigned to the control points provide a meaningful way to control the shape of the surface patch. We define a continuous scalar function with respect to position within the tetrahedron. Thus, every point inside the tetrahedron has a scalar function value associated with it. Generally, some regions within the tetrahedron will have negative function values and some regions, positive function values. At the boundary between the positive and negative regions lies a surface whose function value is zero. This surface is the piecewise algebraic surface patch.

The scalar function is defined using barycentric coordinates s, t, u, v. These four coordinates are connected by the relation $s + t + u + v = 1$. For an arbitrary tetrahedron with vertices V_{n000}, V_{0n00}, V_{00n0}, and V_{000n}, the barycentric coordinates of a point P are the values s, t, u, v, as defined by

$$P = sV_{n000} + tV_{0n00} + uV_{00n0} + vV_{000n}$$
$$s + t + u + v = 1 \tag{1}$$

where P and the V's are points in 3D space.

The control points V_{ijkl} form a lattice on the tetrahedron such that

$$V_{ijkl} = \frac{i}{n}V_{n000} + \frac{j}{n}V_{0n00} + \frac{k}{n}V_{00n0} + \frac{l}{n}V_{000n}$$
$$i, j, k, l \geq 0; \quad i + j + k + l = n \tag{2}$$

A degree n surface patch requires $(n + 1)(n + 2)(n + 3)/6$ control points. Figure 3 shows the control points for the case $n = 3$. Note that V_{1110} is hidden in this view.

We denote by w_{ijkl} the control-point coefficients, and the scalar function is defined

$$f(s, t, u, v) = \sum_{i,j,k,l \geq 0} w_{ijkl} \frac{n!}{i!j!k!l!} s^i t^j u^k v^l$$
$$i + j + k + l = n; \quad s + t + u + v = 1 \tag{3}$$

The tetrahedral clipping is expressed by the inequality $s, t, u, v \geq 0$.

Let's take a closer look at the properties of piecewise algebraic surface patches.

- *Localized influence of control-point weights.* A control-point weight influences the function $f(s)$ most directly near the control point. In fact, the contribution of a particular control point's weight to the function $f(s)$ in Equation 3 can be shown to be maximum at the control point. Qualitatively, this means that if $f(V_{ijkl})$ is negative (positive), then decreasing (increasing) the value of w_{ijkl} will tend to push the surface $f(s) = 0$ away from V_{ijkl}, whereas increasing (decreasing) the value of w_{ijkl} will tend to attract the surface toward V_{ijkl}.

This type of control is illustrated in Figures 4a through 4c, which show a series of three cubic algebraic surface patches whose control-point weights are identical, except for the weight of the bottom right vertex. The value of that weight is –4 in Figure 4a, –2 in Figure 4b, and 0 in Figure 4c. As you can see, the effect of modifying one weight tends to be quite local, especially for corner control points. In these figures, white control points have a coefficient of zero, green means a positive coefficient, and red means negative.
- *Point interpolation.* The value of $f(s)$ at any of the four tetrahedral vertices is the value of the weight of that vertex. Equation 3 easily verifies this, showing that the algebraic surface $f(s)$ can be forced to interpolate a corner vertex by setting the weight of that vertex to zero. Note that this is not generally true for any

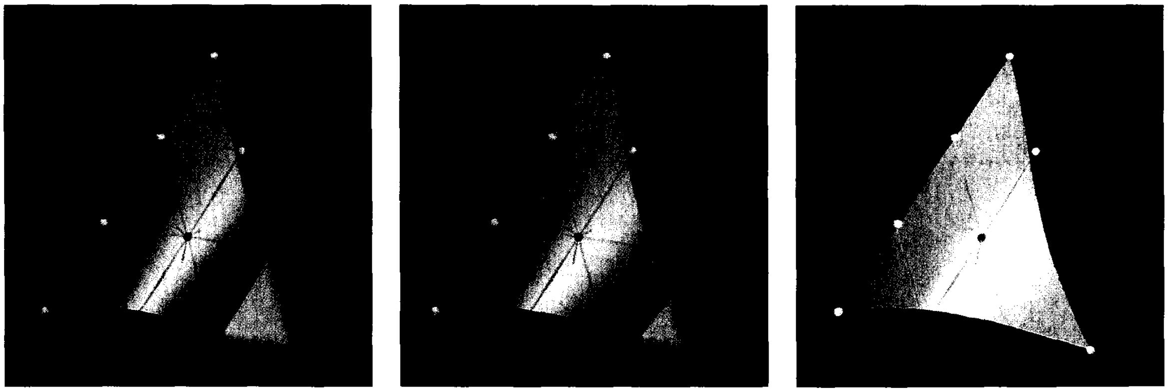


Figure 4. Cubic surface patch with weight of bottom right vertex (a) −4, (b) −2, (c) 0.

control point other than the corner vertices. Figure 4c illustrates point interpolation.

- *Line interpolation.* If all the weights of all $n + 1$ control points along an edge are zero, the entire edge interpolates the surface $f(s) = 0$. This is also easily verified from Equation 3. Figure 5 illustrates this with a hyperbolic paraboloid that interpolates four edges of the tetrahedron. Eight of the 10 control-point weights in Figure 5 are zero.
- *Gradient control.* If the weight of a corner control point is zero, along with the weight of two of its three nearest neighbors, the surface will be tangent to the plane defined by the three control points whose weights are zero.
- *Edge intersections.* The value of the function $f(s)$ along any of the six tetrahedral edges can be expressed as a univariate Bernstein polynomial whose coefficients are the $n + 1$ control-point weights along the edge. This means that if all the weights are positive (or negative), then the algebraic surface patch will not intersect that edge. It also means that the surface will intersect the edge exactly once if there is exactly one sign variation in the sequence of control-point weights along the edge.
- *Avoiding self-intersections.* A danger of algebraic surfaces is that a surface can intersect itself. We can avoid this within the region of the tetrahedron by imposing a monotonicity condition on the control points. Consider all lines defined by any two control points in the tetrahedron. Of that set of lines, consider all lines parallel to a given tetrahedral edge. If the weights of all control points on each of these lines increase (or decrease) monotonically in the same direction, then any line parallel to the edge will intersect the edge at most once. We can see this by examining the directional derivative of $f(s)$ in the direction of the edge. If the monotonicity condition is satisfied, then the directional derivative will be everywhere positive (or negative) within the tetrahedron.

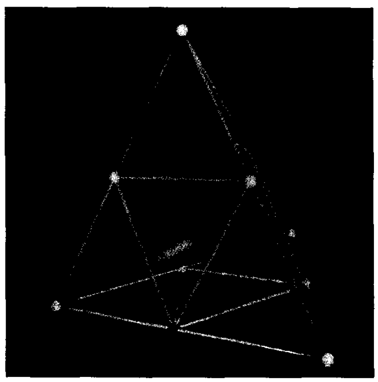


Figure 5. Hyperbolic paraboloid.

A crucial property of this algebraic surface patch formulation is that it inherits most of the tools of Bezier curves and surfaces. We can subdivide the surface by subdividing the tetrahedron; we can perform degree elevation and reduction; and we can impose cross-boundary derivative continuity. Derivative continuity is achieved simply by imposing derivative continuity on the $f(s)$ function of two adjacent tetrahedrons.[5] Figures 6a and 6b illustrate two C^1 cubic algebraic surface patches.

The continuity conditions can be easily stated. Figure 7 highlights the six subtetrahedrons on one face of a cubic piecewise algebraic surface patch, along with the six subtetrahedrons on the neighboring patch. Each subtetrahedron has four vertices with corresponding coefficients. Thus, we can view each subtetrahedron as a linear piecewise algebraic surface patch; the algebraic surface that each defines is a

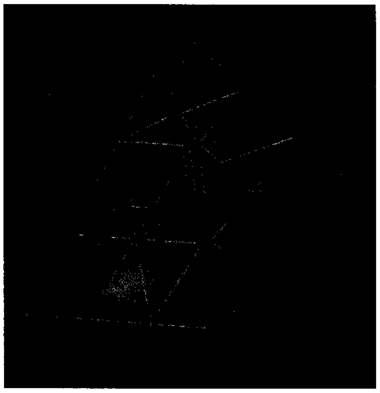
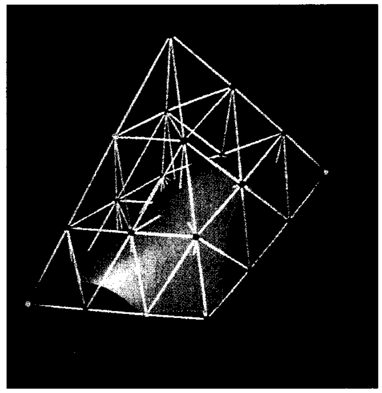


Figure 6. C^1 cubic surface patches.



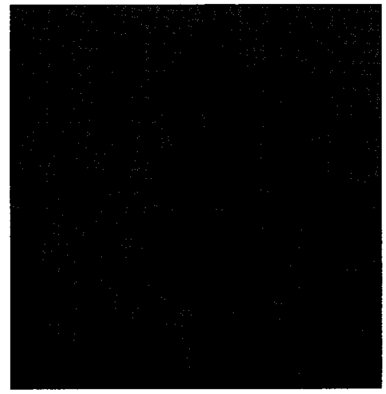


Figure 7. Subtetrahedrons.

plane. Two surfaces are C^1 if each of the six boundary subtetrahedrons defines the same plane as does its immediate counterpart on the neighboring patch.

An alternate approach is to define the patch within a parallelepiped[6,7] instead of within a tetrahedron, using tensor product blending functions. This has the disadvantage that the degree of the algebraic surface would be $3n$. The advantages are that parallelepipeds are much easier to work with than tetrahedrons and that we can define our functions with trivariate tensor product B-splines, solving the continuity problem.

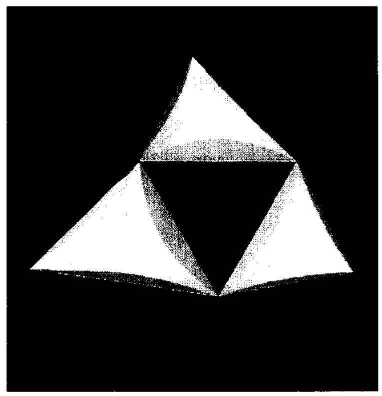


Figure 8. Three cubic surface patches.

Macro patches

It is challenging to work out all the continuity conditions for an extended mesh of piecewise algebraic surface patches. Indeed, that it is possible to fit an extended mesh of such surfaces with C^1 continuity is not obvious. This section presents one method of grouping together a number of piecewise algebraic surface patches into a *macro patch* that is always C^1 with three neighbors. There are several methods for forming a macro patch. Wolfgang Dahmen[8] and Andrew Worsey originated the idea of using macro patches. Dahmen determined that it is actually possible to obtain a mesh of C^1 quadric surfaces using macro patches. The macro patch presented here is a variation of his original idea.

Figure 8 illustrates the problem we wish to solve. It shows three cubic surface patches joined at their corners. At each corner, the two neighboring surfaces have the same tangent plane. The problem is to fill in the hole with a surface that is C^1 to the initial three surfaces. If we can solve this problem, we have solved the problem of how to construct an extended C^1 mesh of cubic surface patches.

It quickly becomes apparent that it is not possible to fill the hole using a single surface patch. The analogous problem for cubic triangular parametric surface patches is solved using the Clough-Tocher scheme of splitting a patch into three pieces, thereby generating additional degrees of freedom sufficient to satisfy the continuity conditions. Likewise, it turns out that if our piecewise algebraic surface patch is split into nine piecewise components, enough degrees of freedom are generated for C^1 continuity between all adjacent patches. Figure 9a shows the split, and Figure 9b shows the tetrahedrons that enclose the component surfaces.

Actually, this macro patch satisfies its continuity conditions with 10 degrees of freedom to spare. Thus we can see that the macro patch provides C^1 continuity with three neighbors and still has 10 coefficients left over to manipulate the shape of the macro patch. Figure 9c shows the patch with its 10 control points,

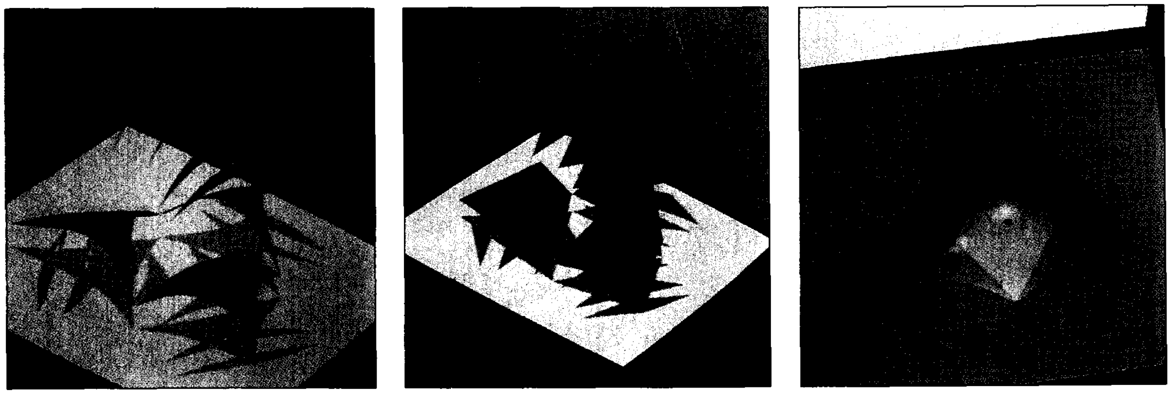


Figure 9. (a) Nine-component macro patch, (b) macro patch tetrahedrons, (c) macro patch with 10 control points.

which influence the shape of the surface just like the control points of an individual piecewise algebraic surface patch.

Parameterizing cubic algebraic surfaces

Cubic surfaces have the valuable property that they can be parameterized. Thus, models consisting of cubic piecewise algebraic surface patches (using, for example, macro patches) can be converted to a parametric representation. This section shows how to impose a parameterization on a cubic algebraic surface. More detailed discussions appear elsewhere.[9,10]

The 27 lines

One fascinating result from classical geometry is the fact that every nonsingular cubic algebraic surface contains exactly 27 straight lines. Nonsingular means that the surface does not contain any double points. These lines were discussed at length in the mathematical literature of 50 to 120 years ago, and even an entire book was devoted to the subject.[11] These lines hold the key to parameterization.

Existence of 27 lines

That the 27 lines exist can be seen most easily in the parametric equation of a cubic surface. Consider the parametric equation of a surface given by

$$x = \frac{P_x(s,t)}{P_w(s,t)}$$

$$y = \frac{P_y(s,t)}{P_w(s,t)}$$

$$z = \frac{P_z(s,t)}{P_w(s,t)}$$

where $P_x(s, t)$, $P_y(s, t)$, $P_z(s, t)$, and $P_w(s, t)$ are cubic polynomials in s and t. As discussed in the section on base-point coercion in part one of this tutorial,[1] a cubic implicit surface can be represented as a cubic parametric surface with six base points.

The existence of the 27 lines can be shown in terms of these six base points. To begin with, each base point maps to a line on the surface, because any plane intersects a base point exactly once. However, the only geometric object that intersects a plane exactly once is a straight line. This accounts for six of the lines.

A general line in parameter space maps to a cubic curve on the surface. However, if a line in parameter space contains a base point, its degree in three space is 2, and if a line passes through two base points in parameter space, its degree in three space is 1, making it a straight line on the surface. There are $\binom{6}{2} = 15$ ways that a straight line can pass through two base points, and they account for another 15 of the lines.

A general conic in parameter space maps to a curve of degree 6 in three space. However, if the conic contains five base points, it maps to a straight line. There are $\binom{6}{5} = 6$ ways that a conic can pass through five of the six base points, accounting for the final six straight lines.

If the parametric equations have real coefficients, then the base points must either be real or come in complex conjugate pairs. If all six base points are real, then all 27 lines are real. If two of the base points are complex (forming a conjugate pair), then 15 of the straight lines are real. If four of the base points are complex (forming two conjugate pairs), then seven lines are real. Finally, if all the base points are com-

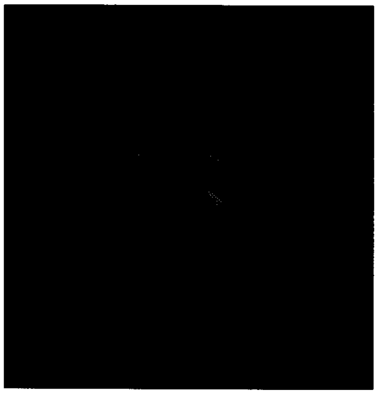


Figure 10. Cubic elbow surface.

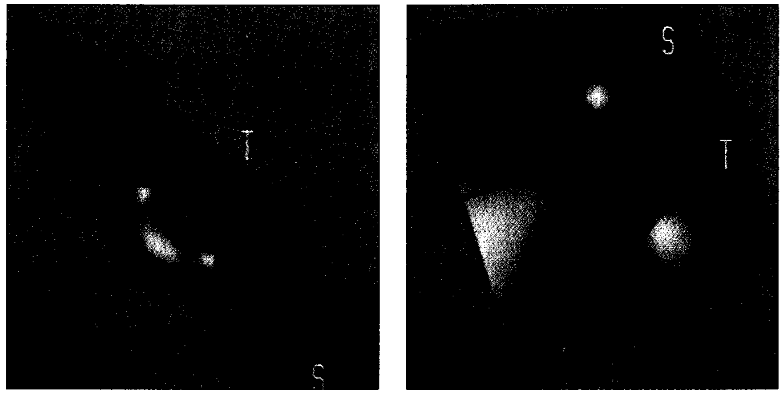


Figure 11. Two lines on a cubic surface viewed from different angles.

plex, then three lines are real. The three real lines are formed by straight lines in parameter space going through two base points that are a conjugate pair.

Real lines can be identified for the cubic surface in Figure 2. Figure 10 shows a more extended region of the surface, and Figures 11a and 11b show two of the real lines.

Computing the 27 lines

Locating the 27 lines is simple if we are given a parametric equation and six base points. However, we are forced to work from the implicit equation of the surface. Let's look at a robust method for computing all 27 lines. It is relatively expensive to compute one line, but having one line in hand, we can compute the remaining 26 lines systematically by solving a small number of quintic univariate polynomials.

There are exactly 27 lines on a cubic surface if that surface is nonsingular. If the surface happens to be singular, there are still straight lines on it. Of the several methods for parameterizing a cubic surface,[9] one method doesn't require any knowledge of the lines, but the parameterization involves a square root. A second method requires only one real line, but it is a two-to-one parameterization (every point on the surface is covered by two parameter pairs). The best parameterization (and the one we discuss here) requires two skew real lines. Thus, no parameterization method requires us to compute all 27 lines.

We can find the equation of one line as follows: If the implicit equation of the surface is given by

$$f(x,y,z) = c_{000} + c_{300}x^3 + c_{030}y^3 + c_{003}z^3 + c_{200}x^2 + c_{020}y^2 + c_{002}z^2 + c_{100}x + c_{010}y + c_{001}z + c_{210}x^2y + c_{120}xy^2 + c_{201}x^2z + c_{102}xz^2 + c_{021}y^2z + c_{012}yz^2 + c_{111}xyz + c_{110}xy + c_{101}xz + c_{011}yz$$

and the line is expressed parametrically as

$$x = t, \quad y = y_0 + y_1t, \quad z = z_0 + z_1t$$

then substituting the parametric equation of the line into the implicit equation of the surface yields an equation that is degree 3 in t. If a line lies entirely on the cubic surface, then this equation must be identically zero. The conditions for the equation to be identically zero are that the coefficients of t^3, t^2, t^1, and t^0 are all zero. Those coefficients are all cubic in y_0, y_1, z_0, and z_1:

- Coefficient of t^0:
$$c_{000} + c_{030}y_0^3 + c_{003}z_0^3 + c_{020}y_0^2 + c_{002}z_0^2 + c_{010}y_0 + c_{001}z_0 + c_{021}y_0^2z_0 + c_{012}y_0z_0^2 + c_{011}y_0z_0 = 0$$

- Coefficient of t^1:
$$3c_{030}y_1y_0^2 + 3c_{003}z_1z_0^2 + 2c_{020}y_1y_0 + 2c_{002}z_1z_0 + c_{100} + c_{010}y_1 + c_{001}z_1 + c_{120}y_0^2 + c_{102}z_0^2 + c_{021}y_0^2z_1 + 2c_{021}y_1y_0z_0 + 2c_{012}y_0z_1z_0 + c_{012}y_1z_0^2 + c_{111}y_0z_0 + c_{110}y_0 + c_{101}z_0 + c_{011}y_0z_1 + c_{011}y_1z_0 = 0$$

- Coefficient of t^2:
$$3c_{030}y_1^2y_0 + 3c_{003}z_1^2z_0 + c_{200} + c_{020}y_1^2 + c_{002}z_1^2 + c_{210}y_0 + 2c_{120}y_1y_0 + c_{201}z_0 + 2c_{102}z_1z_0 + 2c_{021}y_1y_0z_1 + c_{021}y_1^2z_0 + c_{012}y_0z_1^2 + 2c_{012}y_1z_1z_0 + c_{111}y_0z_1 + c_{111}y_1z_0 + c_{110}y_1 + c_{101}z_1 + c_{011}y_1z_1 = 0$$

Coefficient of t^3:
$c_{300} + c_{030}y_1^3 + c_{003}z_1^3 + c_{210}y_1 + c_{120}y_1^2 +$
$c_{201}z_1 + c_{102}z_1^2 + c_{021}y_1^2 z_1 + c_{012}y_1 z_1^2 +$
$c_{111}y_1 z_1 = 0$

We can find a solution to this set of four nonlinear equations in four unknowns using standard numerical methods, and we know that at least three real lines exist.

Now let's examine a procedure for finding the remaining 26 lines on the cubic surface, given one line. Recall that we do not need to compute all 26 of them, but we can stop as soon as we find a second real line skew to the first real line. After changing coordinates if necessary, the given line will be the z axis. The implicit equation must then vanish identically when x and y are both zero; hence all nonzero terms of F must have an x or a y. Thus F can be written

$$F(x,y,z,w) = xP(x,y,z,w) + yQ(y,z,w) \quad \textbf{(4)}$$

where P and Q are quadratic forms in the variables indicated. The pencil of planes through the z axis can be given as $x - \lambda y = 0$. Substituting λy for x in F gives the intersection curve of $F = 0$ with the plane $x = \lambda y$, which consists of the line $y = 0$ and a residual conic

$$Ay^2 + Byz + Cz^2 + Dyw + Ezw + Fw^2 = 0 \quad \textbf{(5)}$$

where $A, ..., F$ involve λ. The condition for the residual conic to degenerate to a pair of lines is

$$\frac{1}{2}\begin{vmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{vmatrix} = 0$$

and this turns out to be a polynomial of degree 5 in λ,

$$a\lambda^5 + b\lambda^4 + c\lambda^3 + d\lambda^2 + e\lambda + f = 0 \quad \textbf{(6)}$$

Once we find a root of Equation 6, we can factor the conic (Equation 5) to get two lines. In more detail, letting $F(x, y, z, w)$ be the homogeneous equation of the cubic surface, oriented with a line on the z axis

$$\begin{aligned}
F(x, y, z, w) &= a_{300}x^3 + a_{210}x^2y + a_{120}xy^2 + a_{030}y^3 + \\
&\quad a_{201}x^2z + a_{111}xyz + a_{021}y^2z + a_{102}xz^2 + a_{012}yz^2 + \\
&\quad a_{003}z^3 + a_{200}x^2w + a_{110}xyw + a_{020}y^2w + a_{101}xzw \\
&\quad + a_{011}yzw + a_{002}z^2w + a_{100}xw^2 + a_{010}yw^2 + \\
&\quad la_{010}zw^2 + a_{000}w^3 \\
&= x(a_{300}x^2 + a_{210}xy + a_{120}y^2 + a_{201}xz + a_{111}yz + \\
&\quad a_{102}z^2 + a_{200}xw + a_{110}yw + a_{101}zw + a_{100}w^2) \\
&\quad + y(a_{030}y^2 + a_{021}yz + a_{012}z^2 + a_{020}yw + a_{011}zw + \\
&\quad a_{010}w^2) + a_{003}z^3 + a_{002}z^2w + a_{001}zw^2 + a_{000}w^3
\end{aligned} \quad \textbf{(7)}$$

and by assumption these last four terms must be zero. The residual conic, obtained by substituting λy for x, is given by

$$\begin{aligned}
A &= a_{300}\lambda^3 + a_{210}\lambda^2 + a_{120}\lambda + a_{030} \\
B &= a_{201}\lambda^2 + a_{111}\lambda + a_{021} \\
C &= a_{102}\lambda + a_{012} \\
D &= a_{200}\lambda^2 + a_{110}\lambda + a_{020} \\
E &= a_{101}\lambda + a_{011} \\
F &= a_{100}\lambda + a_{010}
\end{aligned}$$

The coefficients of λ in the expanded determinant are then

$$\begin{aligned}
a &= 4a_{300}a_{102}a_{100} + a_{201}a_{101}a_{200} - a_{102}a_{200}^2 - \\
&\quad a_{300}a_{101}^2 - a_{100}a_{201}^2 \\
b &= 4(a_{300}a_{102}a_{010} + a_{300}a_{012}a_{100} + a_{210}a_{102}a_{100}) + \\
&\quad a_{201}a_{101}a_{110} + a_{201}a_{011}a_{200} + a_{111}a_{101}a_{200} - \\
&\quad 2a_{102}a_{110}a_{200} - a_{012}a_{200}^2 - 2a_{300}a_{011}a_{101} - \\
&\quad a_{210}a_{101}^2 - 2a_{100}a_{111}a_{201} - a_{010}a_{201}^2 \\
c &= 4(a_{300}a_{012}a_{010} + a_{210}a_{102}a_{010} + a_{210}a_{012}a_{100} + \\
&\quad a_{120}a_{102}a_{100}) + a_{201}a_{101}a_{020} + a_{201}a_{011}a_{110} \\
&\quad + a_{111}a_{101}a_{110} + a_{111}a_{011}a_{200} + a_{021}a_{101}a_{200} - \\
&\quad 2a_{102}a_{020}a_{200} - a_{102}a_{110}^2 - 2a_{012}a_{110}a_{200} \\
&\quad - a_{300}a_{011}^2 - 2a_{210}a_{011}a_{101} - a_{120}a_{101}^2 - \\
&\quad 2a_{100}a_{021}a_{201} - a_{100}a_{111}^2 - 2a_{010}a_{111}a_{201} \\
d &= 4(a_{210}a_{012}a_{010} + a_{120}a_{102}a_{010} + a_{120}a_{012}a_{100} + \\
&\quad a_{030}a_{102}a_{100}) + a_{201}a_{011}a_{020} + a_{111}a_{101}a_{020} \\
&\quad + a_{111}a_{011}a_{110} + a_{021}a_{101}a_{110} + a_{021}a_{011}a_{200} - \\
&\quad 2a_{102}a_{020}a_{110} - 2a_{012}a_{020}a_{200} - a_{012}a_{110}^2 \\
&\quad - a_{210}a_{011}^2 - 2a_{120}a_{011}a_{101} - a_{030}a_{101}^2 - \\
&\quad 2a_{100}a_{021}a_{111} - 2a_{010}a_{021}a_{201} - a_{010}a_{111}^2 \\
e &= 4(a_{120}a_{012}a_{010} + a_{030}a_{102}a_{010} + a_{030}a_{012}a_{100}) \\
&\quad + a_{111}a_{011}a_{020} + a_{021}a_{101}a_{020} + a_{021}a_{011}a_{110} \\
&\quad - a_{102}a_{020}^2 - 2a_{012}a_{020}a_{110} - a_{120}a_{011}^2 - \\
&\quad 2a_{030}a_{011}a_{101} - a_{100}a_{021}^2 - 2a_{010}a_{021}a_{111} \\
f &= 4a_{030}a_{012}a_{010} + a_{021}a_{011}a_{020} - \\
&\quad a_{012}a_{020}^2 - a_{030}a_{011}^2 - a_{010}a_{021}^2
\end{aligned}$$

Using these coefficients, we can compute the five roots of Equation 6.

Parameterizing a cubic surface

This section discusses how to parameterize cubic surfaces having two skew real lines. This parameterization requires two skew lines to be identified on the surface, which is accomplished using the development in the previous section.

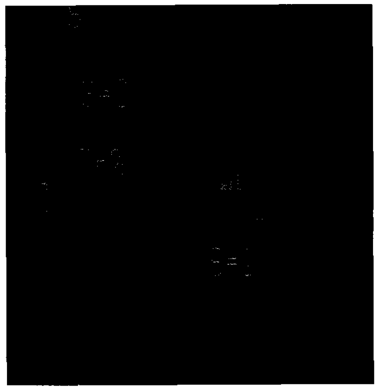


Figure 12. Cubic parameterization.

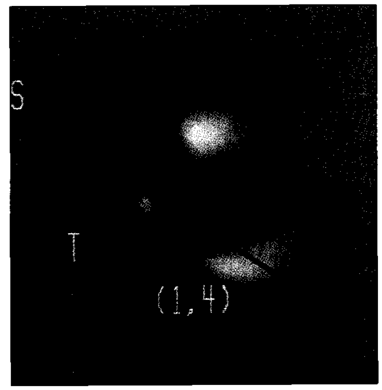


Figure 13. Biquadratic parameterization.

Begin by parameterizing each line. Call the parameter value on one line *s* and on the other line *t*, as illustrated in Figure 11b. The surface parameterization is created by imagining a variable line through an arbitrary point on the *s* line and a second arbitrary point on the *t* line. This variable line intersects the surface at three points: once along the *s* line, once along the *t* line, and at exactly one additional point. The additional point is assigned the *s* and *t* parameters that define the variable line. This is a one-to-one parameterization, because to each point on the surface there exists a unique variable line. Figure 12 shows a line that intersects the *s* line at $s = 1$ and the *t* line at $t = 4$. The third (and final) point at which that line intersects the surface is assigned parameter pair (1, 4).

The fact that to each point on the surface there exists a unique line connecting it with the *s* and *t* lines can be understood by viewing the two skew lines from any point on the surface. The skew lines will have one apparent intersection—the path of the unique line joining the point to the two skew lines.

We now see how to express the surface parametrically using the skew lines. This means that we derive equations for the *x*, *y*, *z* coordinates of points on the surface in terms of the parameters *s*, *t*. Let the first line be expressed $P_0(s) = (x(s), y(s), z(s), 1)$, where $x(s)$, $y(s)$, and $z(s)$ are all linear in *s*. Likewise, we denote the second line by $P_1(t)$. The variable line that contains a point on the *s* and *t* lines respectively is expressed

$$\lambda P_0(s) + \mu P_1(t), \quad \lambda + \mu = 1 \tag{8}$$

Then the points at which the variable line hits the surface can be found from the equation

$$f(\lambda P_0(s) + \mu P_1(t)) = 0 \tag{9}$$

where $f(P)$ is the homogeneous equation of the cubic surface as in Equation 7. We expand Equation 9 by Taylor's series to get

$$f(P_0)\lambda^3 + (\nabla f(P_0) \cdot P_1)\lambda^2\mu + (\nabla f(P_1) \cdot P_0)\lambda\mu^2 + f(P_1)\mu^3 \tag{10}$$

There are three roots (λ, μ) to the homogeneous cubic Equation 10 which, when substituted into Equation 8, produce the three points at which the variable line hits the surface. However, since $P_0(s)$ and $P_1(t)$ always lie on the surface for any value of *s* and *t*, the third point at which the variable line hits the surface can be found from Equation 10 to be

$$(\nabla f(P_0) \cdot P_1\lambda + (\nabla f(P_1)) \cdot P_0)\mu = 0 \tag{11}$$

With the constraint $\lambda + \mu = 1$, we can solve Equation 11 to get

$$\lambda = \frac{\nabla f(P_1) \cdot P_0}{\nabla f(P_1) \cdot P_0 - \nabla f(P_0) \cdot P_1}$$

$$\mu = \frac{\nabla f(P_0) \cdot P_1}{\nabla f(P_0) \cdot P_1 - \nabla f(P_1) \cdot P_0}$$

Combining Equations 8 and 11, we obtain the parameterization of the surface as

$$P(s,t) = P_0(s)\frac{\nabla f(P_1(t)) \cdot P_0(s)}{\nabla f(P_1(t)) \cdot P_0(s) - \nabla f(P_0(s)) \cdot P_1(t)} + P_1(t)\frac{\nabla f(P_0(s)) \cdot P_1(t)}{\nabla f(P_0(s)) \cdot P_1(t) - \nabla f(P_1(t)) \cdot P_0(s)} \tag{12}$$

This turns out to be biquadratic in *s* and *t*. Thus, the skew-line parameterization results in a rational biquadratic parametric surface. Figure 13 shows an example. We can be sure that such a parametric surface equation has five base points.

Conclusions

This tutorial has presented several tools for defining cubic algebraic surfaces. Cubic surfaces have the appeal of being low degree, while possessing adequate flexibility to provide tangent continuity. We have also seen that cubic surfaces defined using implicit equations can be parameterized.

Designing by interpolating points and curves makes controlling the surface between interpolated entities difficult. The bundle of planes idea also shows little practical design merit. Warren[12] recently made the valuable observation that base points of arbitrary multiplicity can nicely be imposed on a rational surface by assigning zero weights to a corner control point and possibly some of its neighbors. In this way, the curve that the base point maps to is easily identified in terms of Bezier control points and in fact can be treated as a boundary curve of the patch. This insight has led to a clever approach to designing n-sided patches, and can also be used to define low-degree algebraic surfaces using parametric equations. However, for cubic surfaces, the boundary curves of the patch turn out to include straight lines.

The methods we have just seen in part two of this tutorial seem somewhat promising, but further work is needed to examine the behavior of macro patches and to determine whether they can be used for serious geometric modeling. ■

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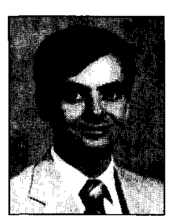

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