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William V. Smith
smithw@mathematics.byu.edu

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THE ANALYSIS OF A MODEL FOR WAVE MOTION IN A LIQUID SEMICONDUCTOR: BOUNDARY INTERACTION AND VARIABLE CONDUCTIVITY*

WILLIAM V. SMITH†

Abstract. The theory of conducting fluids in relative motion with small conductivity is studied with a model including the Maxwell displacement current. The model is linearized, and the interaction of waves with a plane boundary in three space is studied for two orientations of the external magnetic field. It is found that two *families* of boundary conditions preserve energy in one orientation (external field orthogonal to the boundary), while in the other (external field parallel to the boundary) only one condition exists which preserves energy. It is shown that generalized Fourier transforms exist, generated from the generalized eigenfunction expansions. Further, it is shown that surface waves are not supported by this model, indicating that their presence is unstable when relative motion of the fluid is allowed (surface waves exist in the still fluid case). Finally, the problem of variable conductivity (decaying to zero at infinity) is studied and steady-state and time dependent solutions are shown to exist for certain force terms.

Key words. eigenfunction expansions, energy preserving boundaries, variable conductivity, Maxwell's equations, magnetofluiddynamics, liquid semiconductor

AMS(MOS) subject classifications. 35L50, 76W05

1. Introduction. The theoretical modeling of problems in "magnetofluiddynamics" is a rich source of interesting and unusual systems of partial differential equations and corresponding wave motions [LL]. The problem we consider here involves waves of *finite energy* in a fluid-like conducting medium which we assume to be a relatively dense poor conductor, and we treat the *Maxwell displacement current as significant.* The model (like most in this area) can be said to be "physical" only in a certain range of the parameters. For example, in the model studied here, frequency must be relatively high but not high enough to require a particle treatment. (It may also be assumed that permitivity is high relative to free space.) The problem we study here is also of physical interest in a true gas where the constitutive equations (see (1.2)) are much simpler, but we want to examine the fluid case first as, perhaps, a kind of transition state (the theory of liquid semiconductors is still in a rather primitive state with few settled issues [C]). The conductivity appears in only one of the model equations explicitly (see (1.17)). Our model, at apparent zero conductivity, does not reduce (formally) to the uncoupled Maxwell equatoins and fluid motion equations. This is because the (finite) conductivity is implicitly present in the equations containing E'. Mixed frame equations of this type are useful in studying dissipative nonlinear processes since they remove terms which are second order in time. It is this fact that makes the model useful to consider for the undamped behavior of small amplitude waves in a rather dense poor conductor, as this effect makes it possible to study the essentially dissipative problem (1.17) as the bounded perturbation of a symmetric problem. Hence the solutions of (1.17) will be like those of its associated symmetric problem modulo an exponentially decreasing (with time) factor. Elsewhere [S1], [S2], we have already studied the MHD fluid case (perfect conductor), and we refer the reader there for a more detailed

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† Mathematics Department, Brigham Young University, Provo, Utah 84602.

exposition at various points of our treatment here. We note that while this introductory section is mathematically informal (but rather typical of the theoretical treatments of the subject) the following sections are completely rigorous in nature and are founded securely on functional analysis and in particular the Hilbert space theory of differential equations. However, at the end of this section we shall give a brief summary of the results contained in this paper, a comparison to related problems and some comments about the computational difficulties in discussing the differential equations.

The derivation of the problem considered here is founded on Maxwell's equations. The properties of the medium are assumed to be enough like those of a fluid that the continuum approach is reasonably close to reality. We will assume that all fluid *velocities are nonrelativistic*, and that acceleration is small in magnitude (compared to the velocity of light). In order to illustrate the differences between our model and the classical case of a perfect conductor, we will indicate the contrasting assumptions that lead to these two models in their respective derivations.

The Maxwell equations in RMKS units are

$$\nabla \circ D = \rho_e, \qquad \nabla \circ B = 0,$$
$$\nabla \times E = -\frac{\partial B}{\partial t}, \qquad \nabla \times H = J + \frac{\partial D}{\partial t}, \tag{1.1}$$

which hold in any frame of reference, either the rest frame (with respect to the fluid) or the laboratory frame. The constitutive equations must be used and these are (in the laboratory frame):

$$D = \varepsilon_0 E + \varepsilon_0 V \times B - \frac{1}{c^2} V \times H,$$
$$B = \mu_0 H - \mu_0 V \times D + \frac{1}{c^2} V \times E. \tag{1.2}$$

Here as usual, ε_0 is the electric permitivity (for free space), μ_0 the magnetic permeability, E the electric field, H the magnetic field, B the magnetic flux, and D the electric flux. We assume a linear isotropic medium so that μ_0 and ε_0 are scalar constants (this can be modified somewhat—see §§ 4 and 5). c is the speed of light. V is the fluid velocity. We now assume the electric fields to be of order $V \times B$, that is, of the order of magnitude of the induced effects. In other words, the induced magnetic field is much smaller than the externally applied magnetic field. From this it is easily shown that the magnetic induction is the same in all reference frames. Of course, because B is the same in all frames of reference does not mean the same is true for H, but we will see that this is the case under our assumptions on V and the electric field. Let us write H' for the rest frame field and H the laboratory frame. By our Newtonian assumption and the Lorentz transformation,

$$H' = H - V \times (\varepsilon(E + V \times B) - 1/c^2(V \times H)), \tag{1.3}$$

so that $H = H'$ is valid if the magnitude of $V\varepsilon\mu$ is $\ll 1$ ($\varepsilon\mu = 1/c^2$), or in other words, E is approximated by $V \times B$ (here we have used $B = \mu H$). For E' we have

$$E' = E + V \times B. \tag{1.4}$$

E' must always be considered, since to get $H' = H$ as noted, E and $V \times B$ must have the same order of magnitude. We now assume that the period of variation of the fields is large compared to the mean free time of the conduction electrons and that the

Larmor frequency is small compared with the mean free time of the conduction electrons. (In rarefied gases this may break down). This allows the assumption of a constant conductivity σ [LL] (see § 5 below). (We will also assume it is a scalar quantity to begin with—see the remarks on ρ_e below.)

In the MHD approximation, the displacement current $\partial D/\partial t$ would be neglected compared to J, at least when σ is significant. (In a dielectric J is virtually zero.) Here we assume that the displacement current is *not* trivial (in a true metal, for example, the displacement current is essentially meaningless, except at frequencies where the other hypotheses we use begin to break down). In Ohm's law, ρ_e (the space charge) may usually be neglected in a liquid (it must be retained in some gases—we ignore this); hence we have

$$J=\sigma E'+\rho_e V=\sigma(E+V\times B)+(V\circ J/c^2)V. \tag{1.5}$$

The second term is small compared to the first (the coefficient of V in the second term being ρ_e). Thus we take

$$J=\sigma(E+V\times B).$$

Now the Maxwell equations become (note that ρ_e is not present now)

$$\begin{aligned} -\nabla\times E'+\nabla\times(V\times B)&=\frac{\partial B}{\partial t},\\ \nabla\times H&=\sigma E'+\frac{\partial D}{\partial t},\\ \nabla\circ J&=0,\\ \nabla\circ B&=0, \end{aligned} \tag{1.6}$$

Ohm's law:

$$J=\sigma(E+V\times B).$$

The fluid equations are

$$\frac{\partial p}{\partial t}+\nabla\circ(\rho V)=0 \qquad \text{(continuity)}, \tag{1.7}$$

(1.8)

$$\rho\left(\frac{\partial V}{\partial t}+\nabla\left(\frac{V\circ V}{2}\right)-V\times(\nabla\times V)\right)=-\nabla P-\rho\nabla\psi+\nabla\circ\tau'+J\times B+\Sigma \qquad \text{(motion)},$$

with the other terms on the right of the motion equation (Σ) depending on the displacement current if it is considered in a fluid—in a gas Σ is zero (see below). Here ψ is the gravitational potential and τ' is the shear part of the mechanical stress tensor τ. From the Maxwell equations,

$$J\times B=(\sigma E'\times B)=-\sigma B\times E' \tag{1.9}$$

(*in the perfect conductor case, we would use the Maxwell equation to get J here—see below* (1.12)) and so the motion equation becomes

$$\rho\left(\frac{\partial V}{\partial t}+(V\circ\nabla)V\right)=-\nabla P-\rho\nabla\psi+\nabla\circ\tau'-\sigma B\times E', \tag{1.10}$$

$$\rho\left(\frac{\partial V}{\partial t}+(V\circ\nabla)V\right)=-\nabla P-\rho\nabla\psi+\nabla\circ\tau'-\mu H\times(\nabla\times H) \tag{1.11}$$

and replacing τ' by its value in terms of velocity and viscosity,

(1.12)
$$\rho\left(\frac{\partial V}{\partial t}+(V\circ\nabla)V\right)=-\nabla P-\rho\nabla\psi+\rho\nu\nabla^2 H+\left(\zeta+\frac{1}{3}\rho\nu\right)\nabla(\nabla\circ V)+\mu(\nabla\times H)\times H,$$

where the second and third equations are the perfect conductor case ($\partial D/\partial t=0$ and so $J=\nabla\times H$). We write this down because it turns out that (1.10) is correct for the case of a gas only, while (1.12) is correct for a fluid because it includes the complete body force (see (1.16) below). Here we have used P for pressure; ν and ζ are, respectively, the first and second coefficients of viscosity.

In many problems it is useful to make the assumption of infinite conductivity in order to obtain qualitative information about physical situations, since this assumption generally allows a much simpler mathematical formulation. An important application of the concept of infinite conductivity is in high temperature plasma studies, such as those associated with fusion devices. In interstellar matter, the decay of the magnetic fields is so slow that infinite conductivity gives a good approximation. In such cases (where infinite conductivity is assumed) the results differ from physical reality by a damping term. When the displacement current is neglected, we can combine the two curl Maxwell equations using Ohm's law and the divergence equation for B to obtain

$$\frac{\partial B}{\partial t}=\eta\nabla^2 B+\nabla\times(V\times B), \tag{1.13}$$

the so-called magnetic transport equation. Here $\eta=(1/\sigma\mu)$ and is called the magnetic diffusivity by obvious analogy. *For $\sigma=\infty$ we have, formally, that B becomes "frozen into" the fluid (the transport term vanishes).* The neglect of gravitational force and viscosity together with the appropriate equation of state yields the well-known equations of magnetohydrodynamics:

$$\begin{aligned}&\frac{\partial B}{\partial t}=\nabla\times(V\times B),\\&\rho\frac{DV}{Dt}=-a^2\nabla\rho+\mu(\nabla\times H)\times H,\\&\frac{D\rho}{Dt}=0,\\&\nabla\circ B=0,\end{aligned} \tag{1.14}$$

where we have used the convective derivative notation in the second and third equations. a is the sound speed from the equation of state.

Equations (1.14) are mentioned only for comparison's sake since the case of interest here is when σ is relatively small. So, supposing that the displacement current is significant compared to the conduction current, and if we note the equation for D given above, that is,

$$D=\varepsilon_0 E'-\frac{1}{c^2}V\times H,$$

then we have

$$\varepsilon_0\frac{\partial E'}{\partial t}=\nabla\times H-\sigma E'+\frac{1}{c^2}\frac{\partial V\times H}{\partial t}. \tag{1.15}$$

The essential assumptions made so far are that V is relatively small, external forces (gravity) may be neglected, and dissipation from viscous effects is small. Now, using (1.15), and the constitutive expression for D (there is some question as to the proper form for the body force here but we take the one implied by the Abraham tensor [LL]) and considering the motion equation with the body force term, assuming no steady D field, and neglecting space charge effects, we have for the body force density (last terms in the motion equation)

$$J\times B+\frac{\partial}{\partial t}(D\times B)-\frac{1}{c^2}\frac{\partial}{\partial t}(E\times H)=J\times B+\Sigma. \tag{1.16}$$

Now if the permitivity ε of the medium is comparable to frequency over conductivity (in sea water for example, at frequencies above about 200 megacycles), (or else $\varepsilon \gg \varepsilon_0$), we may neglect the last term on the left side of (1.16) compared to the others. If the magnitude of acceleration is small compared to c we can neglect the last term in (1.15) and so finally obtain

$$\begin{aligned}
&\varepsilon_0\frac{\partial E'}{\partial t}=\nabla\times H-\sigma E',\\
&\frac{\partial B}{\partial t}=\nabla\times(V\times B)-\nabla\times E',\\
&\rho\frac{DV}{Dt}=-a^2\nabla\rho+(\nabla\times B)\times\mu^{-1}B,\\
&\frac{D\rho}{Dt}=0,\\
&\nabla\circ B=0,\qquad \nabla\circ J=0.
\end{aligned} \tag{1.17}$$

The reader will notice that formally, (1.17) reduces to (1.1) when $V=0$ or to (1.14) when $\sigma=\infty$.

Introducing small disturbances about a steady-state condition and neglecting the second-order terms, we finally arrive at a form to observe in terms of wave motion (see (2.1)). It is well known that in the MHD approximation (1.14) ($\eta=0$, or $\sigma=\infty$) there exist essentially three modes of propagation, namely, Alfven waves, and the slow and fast magnetosonic waves [LL], [A], [K], etc. The Alfven waves do not involve acoustic effects but are simply disturbances in the velocity and magnetic fields. As we will see below, the "Alfven waves" in (1.17) degenerate in the sense that they appear as disturbances which are like sound waves (the external field is not "frozen into" the medium) but move more rapidly in the direction of the external field.

It has often been stated that surface wave phenomena are important in the physics of conducting fluids. But as noted in [A], for example, and shown rigorously in [S2], (1.14) does not support surface waves. The displacement current term is needed to generate surface wave phenomena but as we will see, the *presence of surface waves is unstable*: *Whether conductivity is high* (MHD *case*) *or low* (*the case studied here*) *surface waves do not exist when the fluid is in motion. We may say that such surface disturbances are convected away by the fluid. At zero velocity however, a type* I *boundary for orthogonal external fields or the boundary for horizontal external field both support surface waves.* These matters are fully explained below.

As to boundary conditions that are appropriate for the system (1.17), these may be derived from the boundary conditions for Maxwell fields plus the appropriate conditions on the fluid equations. We remark here that the boundary conditions discovered in [Sc1] are related to those derived here in the case where the external magnetic field is parallel to the boundary. This might be expected since the derivation of (1.17) is based on the Maxwell equations. In fact, the boundary conditions in this case are (taking account of the larger number of variables) the so-called "strange" boundary conditions of [Sc1]. It is perhaps then of some surprise that no surface waves exist in this configuration. This is the instability just mentioned.

In terms of the confinement of fusion plasmas, a number of simple conditions have been studied. No matter the shape of the confinement device for a conducting fluid the boundary problem may frequently be reduced to the consideration of a half-space [J]. That is, the problem may be studied as though the medium occupies a volume with a plane boundary, at least locally.

The boundary conditions for (1.17) with $\sigma = 0$ in a half-space, which are energy conserving, are particularly useful in the study of the dissipative problem $\sigma > 0$. These are derived in the next section.

We have said that one of the main (negative?) results we prove is the absence of surface waves for this model. Another is the absence of steady-state motion for low frequencies. This is proved even for the anisotropic case (see § 5). The reader may consult [S3] and [S4] for a treatment of general systems of the type considered here.

The results given below require very complex computations involving large symbolic matrices and polynomials in several variables. Nearly all of these were carried out using a combination of certain observations about the structure of the matrices involved and certain computer-based symbolic algebra routines constructed by the author as well as those standard packages available from commercial vendors, most computations being done in MAPLE and MACSYMA and a few in MATHEMATICA. A frontal attack on the problems leads nowhere, however, and considerable pattern recognition/reduction is required on the human side. Such techniques are nearly always very specific to the problem and are of an ad hoc nature. Once required objects were derived, checking was done by essentially the same methods, i.e., a combination of human observation and machine interaction. There are several methods of computing the large eigenprojector matrices used here. But they are based on the following facts.

Suppose $A(p)$ is a real symmetric matrix depending on the parameter(s) $p \neq 0$. The spectrum of $A(p)$ is real for all p. $A(p)$ is assumed to have the property that all its entries are linear combinations of the parameter(s) p. The positive and negative eigenvalues of $A(p)$ are equal in number and as continuous functions of p may be enumerated as an ordered list (counting possible multiplicities) as

$$\lambda_k(p) \geqq \lambda_{k-1}(p) \geqq \cdots \geqq 0 (= \lambda_0(p)) \geqq \lambda_{-1}(p) \geqq \cdots \geqq \lambda_{-k}(p).$$

The $\lambda_j(p)$ have the two properties

(1) $\lambda_j(\alpha p) = \alpha \lambda_j(p)$ for all $\alpha > 0$,

(2) $\lambda_j(-p) = -\lambda_{-j}(p)$.

The $\lambda_j(p)$ are roots of the minimal polynomial for $A(p)$ which has the form

$$S = \lambda^{\pi(r(p))}\{\lambda^{2\alpha(p)} + S_1(p)\lambda^{2\alpha(p)-1} + \cdots + S_{2\alpha(p)}(p)\}.$$

In case $A(p)$ has constant rank, $r(p)$ (=dimension of A minus the rank of A) and $\alpha(p)$ are constant. ($\pi(p) = 0$ or 1 depending on whether A is of full rank or not.) We need only deal with the constant rank case in our problem. $D(p)$, the discriminate of S in λ is a homogeneous polynomial and hence the set $\beta = \{p \mid D = 0\}$ is an algebraic cone (in n space for some n). β is the locus of points p where one or more of the functions $\lambda_j(p)$ coincide (and is a set of Lebesgue measure zero). $\lambda_j(p)$ is an analytic function of p on $R^n - \beta$. The orthogonal projection of C^m (A is $m \times m$ and m is related to k in the obvious way) onto the eigenspace for $\lambda_j(p)$ is given by

$$P_j(p) = -\frac{1}{2}\pi i \int_{\gamma_j(p)} (A(p) - z)^{-1}\, dz, \qquad p \in R^n - \beta,$$

where $\gamma_j(p) = \{z \,\|\, z - \lambda_j(p)| = \rho_j(p)\}$ and the $\rho_j(p)$ are chosen so small that the $\gamma_j(p)$

do not intersect. The $P_j(p)$ have the properties ($p \neq 0$)

(1) $P_j(p)$ is analytic on $R^n - \beta$,
(2) $P_j(p) = P_j(\alpha p)$ for all $\alpha > 0$ and $p \in R^n - \beta$,
(3) $P_j(-p) = P_{-j}(p)$ for $p \in R^n - \beta$,
(4) $\sum_j P_j(p) = I$ (the identity matrix),
(5) $A(p)P_j(p) = \lambda_j(p)P_j(p)$.

Each of these facts plays a role in the actual computation of the matrices $P_j(p)$ the results of which are given in § 3 below for a certain $A(p)$ defined by the system of partial differential equations studied here. The path integral for $P_j(p)$ may be computed in a number of ways in a given example; the Cauchy integral theorem is an obvious method of attack. Many of the wave propagation problems of classical physics present with symbols ($A(p)$) of a particularly simple and useful form (the nonzero entries are contained in two nonintersecting submatrices each being the transpose of the other) [Sc1] but the problem we consider here is one of the interesting exceptions to that rule. Hence the computations are more difficult and resolution of the problem requires more basic methods, particularly since we need to extend one of the real parameters p into the complex plane.

2. Boundary conditions. The energy-preserving boundary conditions for the case of a perfect conductor (1.14), $\sigma = \infty$, were characterized in [S1] (see also [S2] and [S5]). The computations are somewhat more complex for the case of (1.17), and since they are carried out in essentially the same manner as in [S1], we will not give the complete details. We will nevertheless construct a complete set of boundary conditions. The divergence equations in (1.17) are contained in the other equations and so will not be needed here. It may be expected that E is divergence free as well. In fact, by the Lorentz transformation of E, the divergence of E' will be related to the divergence of V (the reader will recall that the space charge was neglected in the derivation of (1.17)). This requires that $\nabla \circ (E' - V \times B) = 0$. We will discuss this further below (see (4.18), (4.19)).

Since there is a boundary to consider, the direction of the external magnetic field may not be trivialized by the choice of a convenient coordinate system. The complications arising by treatment of general magnitude and direction of the external field require a great deal of space; we will treat two special cases, namely, external fields which are either parallel or orthogonal to the boundary plane. Oblique fields may be considered at a later time, provided a way can be discovered to sufficiently compress the expressions in a meaningful way.

The linearized version of (1.17) is

$$(2.1)\qquad \begin{aligned} \frac{\partial B}{\partial t} &= \nabla \times (V \times B_0) - \nabla \times E', \\ \varepsilon_0 \frac{\partial E'}{\partial t} &= \nabla \times H - \sigma E', \\ \rho_0 \frac{\partial V}{\partial t} &= -a^2 \nabla \rho + (\nabla \times B) \times \mu^{-1} B_0, \\ \frac{\partial \rho}{\partial t} &= -\rho_0 \nabla \circ V. \end{aligned}$$

Here, ρ_0 is the equilibrium density, $B_0 = (h_1, h_2, h_3)$ is the external magnetic field, a is the equilibrium speed of sound, μ is the magnetic permeability, B is the internal magnetic field, V is the velocity field, and ρ is the density. If we choose units in which $\varsigma\mu^2 \approx \sqrt{\varepsilon_0}/a^3$ and $h_i \approx \sqrt{\varepsilon_0}/\sqrt{\mu\rho_0}$ numerically, (these may be nonstandard units for these quantities) then a change of variables in (2.1) allows us to take ρ_0, ε_0, a, and μ as unity, and h_3 or $h_2 = 1$ depending on which external field we consider. *We will assume this to be the case from now on* except where it is necessary to record the location of the external field components. σ has a somewhat different expression, but this is unimportant for our purposes and we still refer to it with the same notation.

We may then write (2.1) in matrix form (we have rearranged the order of the equations in (2.1) as indicated by the definition of u below) as

$$-i\frac{\partial u}{\partial t} = -i\sum_{j=1}^{3} A_j \frac{\partial u}{\partial x_j} + iKu, \quad K = \operatorname{diag}(0,0,0,0,0,0,0,\sigma,\sigma,\sigma), \quad u = (V, B, \rho, E)^t. \tag{2.2}$$

The superscipt t in (2.2) means transpose and $i = \sqrt{-1}$ (this is added for later convenience), "diag" means the square diagonal matrix with entries as shown and the A_j are given by

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & -h_2 & -h_3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & h_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -h_2 & h_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -h_3 & 0 & h_1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.3}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & h_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -h_1 & 0 & -h_3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & h_2 & 0 & 0 & 0 & 0 \\ h_2 & -h_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -h_3 & h_2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.4}$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & h_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -h_1 & -h_2 & 0 & -1 & 0 & 0 & 0 \\ h_3 & 0 & -h_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & h_3 & -h_2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.5}$$

We write D_j for $-i\partial/\partial x_j$ and $D=(D_1, D_2, D_3)$. For the right-hand side of (2.2) (taking $K=0$), we write $A(D)$. As we noted above, we will consider the case of a half-space where the vector B_0 is given by either $(0, 0, h_3)$ or $(0, h_2, 0)$. When it is necessary to distinguish between these two cases, we will do so by using a superscript as $A^3(D)$ and $A^2(D)$. By $\mathscr{n}$, we mean the inward unit normal vector to ∂G (=boundary of G), where G is some domain in $\mathbb{R}^3$. From here on, $G=\mathbb{R}^3_+=\{x \mid x=(x_1, x_2, x_3),\ x_3>0\}$.

Definition 2.1 [LP]. A subspace $\mathscr{S}(\mathscr{n})$ of $\mathbb{R}^{10}$ is a maximal conservative boundary space for $A(D)$ in G if and only if $\zeta \circ A(\mathscr{n})\zeta=0$ for all ζ in $\mathscr{S}(\mathscr{n})$ and $\mathscr{S}(\mathscr{n})$ is maximal with respect to this property.

To proceed further, it is necessary to consider the eigenvalues of the symbol of $A(D)$. These are the solutions to the equation $\det(A(p)-\lambda I)=0$, where $p=(p_1, p_2, p_3)\in\mathbb{R}^3\setminus\{0\}$ (the plane wave speeds). They are given by (for A^3):

$$
\begin{aligned}
&{}_3\lambda_0(p)=0 \qquad \text{(multiplicity 4)},\\
&{}_3\lambda_{\pm1}(p)=\pm(2p_3^2+|n|^2)^{1/2} \qquad \text{(each with multiplicity 1)},\\
&{}_3\lambda_{\pm2}(p)=\frac{\pm(-(p_3^2(p_3^2+6|n|^2)+5|n|^4)^{1/2}+3(p_3^2+|n|^2))^{1/2}}{\sqrt{2}},\\
&{}_3\lambda_{\pm3}(p)=\frac{\pm((p_3^2(p_3^2+6|n|^2)+5|n|^4)^{1/2}+3(p_3^2+|n|^2))^{1/2}}{\sqrt{2}}
\end{aligned}
\tag{2.6}
$$

(each with multiplicity 1 for almost all p). Here we have used the notation $|n|=(p_1^2+p_2^2)^{1/2}$ and in (2.7), $|n_1|=(p_1^2+p_3^2)^{1/2}$. For A^2,

$$
\begin{aligned}
&{}_2\lambda_0(p)=0 \qquad \text{(multiplicity 4)},\\
&{}_2\lambda_{\pm1}(p)=\pm(|n_1|^2+2p_2^2)^{1/2},
\end{aligned}
\tag{2.7}
$$

and similarly for the rest, exchanging p_2 and p_3, n and n_1 in (2.6). For future reference, we record the following: (cf. (3.24) and also (3.33))

$$
\begin{aligned}
&\frac{\partial_3\lambda_{\pm1}}{\partial\tau}=\frac{2\tau}{{}_3\lambda_{\pm1}},\\
&\frac{\partial_3\lambda_{\pm2}}{\partial\tau}=\frac{-1}{2\,{}_3\lambda_{\pm2}}\left(\frac{2\tau(\tau^2+6|n|^2)+2\tau^3)}{\tau^2(\tau^2+6|n|^2)+5|n|^4)^{1/2}}+3\tau\right),\\
&\frac{\partial_3\lambda_{\pm3}}{\partial\tau}=\frac{1}{2\,{}_3\lambda_{\pm3}}\left(\frac{2\tau(\tau^2+6|n|^2)+2\tau^3)}{\tau^2(\tau^2+6|n|^2)+5|n|^4)^{1/2}}+3\tau\right)
\end{aligned}
\tag{2.6a}
$$

with the expressions for ${}_2\lambda_{\pm j}$ obtained in a similar fashion. The multiplicity of the second and third eigenvalues in (2.6) may change for p of certain direction and magnitude ($p=(0, 0, \pm1)$). This is important for the application of Lemma 2.2 below. We will refer to ${}_i\lambda_{\pm1}$ (2.6), (2.7) as the quasi-Alfven wave speeds since the constant speed surfaces of these waves have the same relation to the electromagnetosonic constant speed surfaces as do Alfven waves for the MHD slow and fast magnetosonic waves (i.e., roughly speaking, first the fast wave arrives, then the Alfven wave, and finally the slow wave; in the direction of the external field, the Alfven wave may arrive at the same time as either the slow or fast wave depending upon certain relationships of the parameters (see [CH]).

It is evident from (2.6), (2.7) that $A(D)$ is strongly propagative [Wi]. It is instructive to compare this with the MHD case [S2]. For MHD there are (almost everywhere) three nonzero plane wave speeds, and they are given (using the notation above) by

$$
\begin{aligned}
&{}_3\lambda_0 \equiv 0,\\
&{}_3\lambda_{\pm 1} \equiv \pm p_3,\\
&{}_3\lambda_{\pm 2} \equiv \pm(|p|^2 - |n||p|)^{1/2},\\
&{}_3\lambda_{\pm 3} \equiv \pm(|p|^2 + |n||p|)^{1/2},
\end{aligned}
\tag{2.8}
$$

and

$$
\begin{aligned}
&{}_2\lambda_0 \equiv 0,\\
&{}_2\lambda_{\pm 1} \equiv \pm p_2,\\
&{}_2\lambda_{\pm 2} \equiv \pm(|p|^2 - |n_1||p|)^{1/2},\\
&{}_2\lambda_{\pm 3} \equiv \pm(|p|^2 + |n_1||p|)^{1/2}.
\end{aligned}
\tag{2.9}
$$

Here, if p is orthogonal to B_0 $(=(0, 0, 1)$ or $(0, 1, 0)$ in (2.8) or (2.9), respectively), then $\lambda_{\pm 1,2}$ vanish. $\lambda_{\pm 1,2}$ are the Alfven and slow magnetosonic wave speeds, respectively [CH]. It is instructive to consider the slow magnetosonic speed profile (see Fig. 1) (the normal surface or "slowness surface" [CH], [Wi]) compared to the electromagnetosonic profiles of (2.1) (see Figs. 2, 3, and 4—the grids in these figures are the same relative size). These are just the unit level surfaces of the functions ${}_i\lambda_j(p)$ in p space—in Figs. 2-4, the plane is the $p_1 p_2$ plane, $i = 3$. From Fig. 1 we see the constant speed (normal) surface for the slow magnetosonic wave. It is unbounded (it tends to the direction of the Alfven wave surface (a plane!) to which it is parallel at ∞), while that for the slow electromagnetosonic wave (Fig. 2) is roughly inverse to that of Fig. 1; it is bounded. The quasi-Alfven surface is caught between the slow and fast surfaces just as for MHD (see the illustrations on page 615 of [CH] for a two-dimensional cross

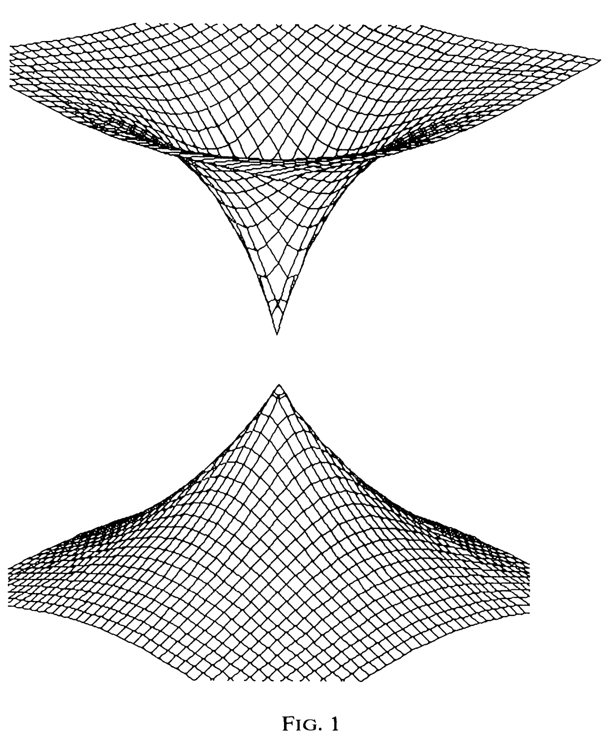


FIG. 1

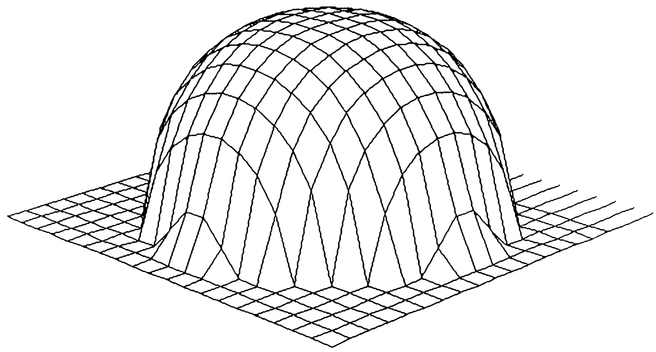


FIG. 2

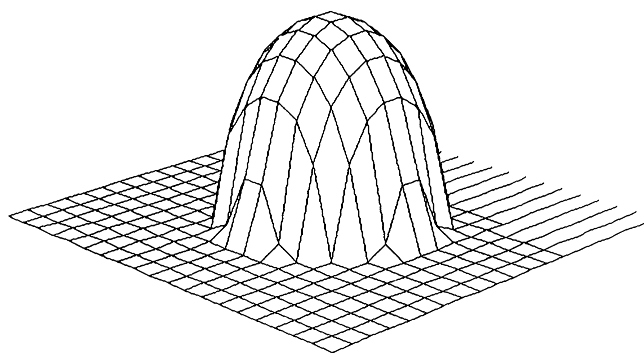


FIG. 3

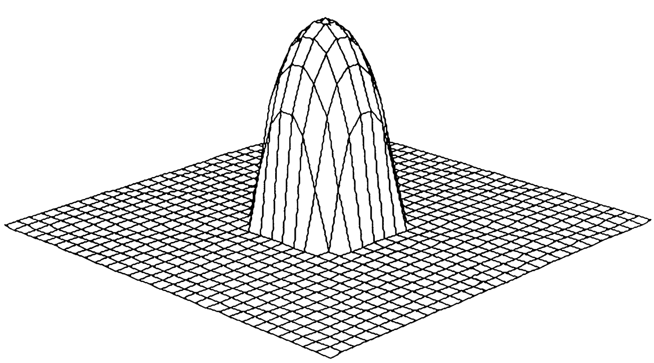


FIG. 4

section for MHD). In our choice of units and external field intensity, the quasi-Alfven surface meets the fast wave surface at the external magnetic axis (vertical in all figures) and is disjoint from the slow wave surface.

By the positive and negative eigenvectors we mean those corresponding to the positive and negative eigenvalues, respectively. Their number depends on B_0.

LEMMA 2.2 [Sc1]. *Let $\mathcal{N}(A(\varkappa))$, $\mathcal{X}(\varkappa)$, $\mathcal{Y}(\varkappa)$ denote, respectively, the null space of $A(\varkappa)$, the subspace spanned by the positive eigenvectors of $A(\varkappa)$, and the subspace spanned by the negative eigenvectors of $A(\varkappa)$. Let ζ_j be any orthonormal base of $\mathcal{N}(A(\varkappa))$. Let ξ_j be any base of $\mathcal{X}(\varkappa)$ that is orthonormal with respect to $A(\varkappa)$, i.e., $\xi_i \circ A(\varkappa)\xi_j = \delta_{ij}$, and*

η_j be any base of $\mathcal{Y}(n)$ orthonormal with respect to $-A(n)$ (j as for $\mathcal{X}(n)$). Suppose $\mathcal{S}^{3,2}(n)$ is the subspace of $\mathbb{R}^{10}$ spanned by $\{\zeta_j, \xi_j+\eta_j \text{ (all } j)\}$. Then $\mathcal{S}^{3,2}(n)$ is a maximal conservative boundary space for $A(D)$ and any such boundary space may be constructed in this way.

The lemma is obvious when the eigenvalues of $A(n)$ ($=A_3$, see (2.5)) are computed. (Recall that $n=(0,0,1)$.)

To classify such spaces, we proceed as in [S1] and [S5]. Consider any basis of $\mathcal{X}\oplus\mathcal{Y}$, say for A^3. We have ξ_1, ξ_2, ξ_3 and η_1, η_2, η_3 with $\lambda_1\leftrightarrow\xi_1$, $\lambda_2\leftrightarrow\xi_2$, $\lambda_3\leftrightarrow\xi_3$, etc. Let e_2^1, e_2^2, e_1^3, e_{-2}^1, e_{-2}^2, e_{-1}^3 be any such fixed basis. Then we have

$$(2.10)\qquad \begin{aligned} &\eta_i = d_{i1}e_{-2}^1 + d_{i2}e_{-2}^2 \qquad (i=1,2),\\ &\eta_3 = d_3 e_{-1}^3,\\ &\xi_i = c_{i1}e_2^1 + c_{i2}e_2^2 \qquad (i=1,2),\\ &\xi_3 = c_3 e_1^3, \end{aligned}$$

In order that the orthonormality conditions be satisfied, it must be that $c_{i1}c_{j1}+c_{i2}c_{j2}=\delta_{ij}$, and thus the matrix $[c_{ij}]$ must be orthogonal and the same is true of $[d_{ij}]$. The constants d_3 and c_3 must have the value 1. Thus by letting $C=[c_{ij}]$ and $D=[d_{ij}]$ run through all possible such matrices, we obtain all possible orientations of the boundary spaces associated with A^3. This allows us to compute operators whose kernels identify the boundary spaces for A^3 (and by a similar process, for A^2). For the details, we refer the reader to [S1] and [Sc1]. In any case, the boundary operators obtained by this process for A^3 consist of two one-parameter families which can be written (here we include the effect of the external field intensity) as

$$(2.11)\qquad \begin{aligned} G_{1,\lambda} &= \begin{bmatrix} h_3\lambda & -h_3 & 0 & 0 & 0 & 0 & 0 & 1 & \lambda & 0\\ 0 & 0 & 0 & 1 & \lambda & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},\\ G_{2,\lambda} &= \begin{bmatrix} h_3\lambda & 0 & 0 & 0 & -1 & 0 & 0 & 0 & \lambda & 0\\ 0 & h_3\lambda & 0 & 1 & 0 & 0 & 0 & -\lambda & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

One thing which is immediately apparent from (2.11) is that for orthogonal external magnetic fields, the fluid density must vanish at the boundary, if the energy is confined to a half space. This has been proposed in the physical literature, see [A] for example. This fact is in contrast to the orthogonal field case in MHD, where the component of velocity orthogonal to the boundary must vanish ([S1] or [S2]) (nothing is required of the density) and the boundary conditions do not depend on the field intensity. $G_{1,\lambda}$ couples the velocity and electric fields at the boundary while $G_{2,\lambda}$ couples all three fields. Neither condition requires anything from the induced field components orthogonal to the boundary.

For the case of the parallel external field, there is but a single boundary condition for which energy is preserved, and it does not depend on the external field intensity. The boundary condition is

$$(2.12)\qquad G_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Once again we see that the density must vanish at the boundary. The reader should compare (1.7) with the "strange" boundary condition of [Sc2] to see that they are the same, modulo the density term. Again there is no condition on the velocity field, while the magnetic and electric field components in the direction of the external field must vanish at the boundary. The parallel field case in MHD also gives a single energy-preserving boundary condition but there the boundary condition depends on the external field intensity. However (see [S2]), in MHD the density and induced field component in the direction of the external field are coupled at the boundary ($h_2H_2+r=0$ at $x_3=0$). Thus in the MHD case, if the density does vanish at the boundary, so also must H_2, which is reminiscent of G_2. One other comparison between MHD and the present system should be noted: the modes are uncoupled in the parallel case for MHD (an incident slow wave generates only a slow wave, etc. [S2, Thm. 3.6], and they are here, too.

DEFINITION 2.3. The operators $A^{3,2}$ in $L_2(\mathbb{R}^2\times\mathbb{R}_+,\mathbb{C}^{10})=\mathscr{H}$ with their associated boundary spaces $\mathscr{S}^{3,2}$ are defined with domains:

$$D(A^{3,2})=\{u\mid u,\ A^{3,2}u \text{ are in } \mathscr{H} \text{ and } G_{i,\lambda}u \text{ or } G_2u=0 \text{ if } x_3=0\ (i=1 \text{ or } 2)\}.$$

The proof of self-adjointness is essentially the same as in Theorem 3.1 of [Sc2] and will not be repeated here. We note the following, which may be proved in a manner similar to that of [Sc1].

THEOREM 2.4. *If u is in $\mathscr{D}(A)\cap\mathscr{N}(A)^{\perp}$, then the D_3 derivative of u lies in $L_2(\mathbb{R}_+,\mathscr{H}^{-1})$ where $\mathscr{H}^{-1}$ is the usual Sobolev space, $u(\cdot,0)$ is in $\mathscr{S}$ in the $\mathscr{H}^{-1/2}$ sense and there exists a sequence*

$$\{u_k\}\subset\mathscr{H}^1(\mathbb{R}^2\times\mathbb{R}_+,\mathbb{C}^{10})\cap C(\mathbb{R}^2\times(\mathbb{R}_+\backslash\{0\}),\mathbb{C}^{10})\cap\mathscr{D}(A)\cap\mathscr{N}(A)^{\perp}$$

such that $u_k(x_1,x_2,0)$ is in $\mathscr{S}$ (with either orientation), $u_k(\cdot,0)\to u(\cdot,0)$ in $\mathscr{H}^{-1/2}$, and $u_k\to u$ in graph norm.

3. Resolvent kernels. The analysis here is based on Stone's theorem for the construction of the spectral family of a self-adjoint operator A in a Hilbert space $\mathscr{H}$ with inner product $(\,,)$. Let $R(\lambda)=(A-\lambda I)^{-1}$ (the resolvent of A) and let $E(\lambda)$ be the (right continuous) spectral family of A. Then for (a,b) a finite interval and for f, g in $\mathscr{H}$,

$$\begin{aligned}&\left(\frac{(E(b)+E(b-))f}{2}-\frac{(E(a)+E(a-))f}{2},g\right)\\ &\qquad=\lim_{\varepsilon\to0}\int_a^b((R(k+i\varepsilon)-R(k-i\varepsilon))f,g)\frac{dk}{2\pi i}.\end{aligned}\tag{3.1}$$

Using the well-known relations (* signifies adjoint operator),

$$\begin{aligned}&R^*(\lambda)=R(\bar\lambda),\\ &R(\lambda_1)-R(\lambda_2)=(\lambda_1-\lambda_2)R(\lambda_1)R(\lambda_2).\end{aligned}\tag{3.2}$$

Using the second equation of (3.2), the integral of the right-hand side of (3.1) may be rewritten as

$$\lim_{\varepsilon\to0^+}\int_a^b(R(k-i\varepsilon)f,R(k-i\varepsilon)g)\,dk(\varepsilon/\pi).\tag{3.3}$$

Taking $f=g$ and using the first equation of (3.2) we have

$$\left(\frac{(E(b)+E(b-))}{2}f-\frac{(E(a)+E(a-))}{2}f,f\right)=\lim_{\varepsilon\to0^+}\frac{\varepsilon}{\pi}\int_a^b|R(k-i\varepsilon)f|^2\,dk\tag{3.4}$$

with $|\cdot|$ representing the norm in $\mathscr{H}$. Equation (3.4) gives (2.1) upon polarization. Therefore, we seek to compute (3.4) for A^3 and A^2. When it is not necessary to distinguish these operators we simply write A.

We will need the Fourier transform. On $\mathscr{S}(\mathbb{R}^n, \mathbb{C}^m)$, the space of smooth, rapidly decreasing $\mathbb{C}^m$-valued functions on $\mathbb{R}^n$, the Fourier transform is defined ($x \circ y = \sum x_i y_i$) as:

$$\Phi_n f(p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\circ p} f(x)\, dx \tag{3.5}$$

with $\Phi_n^{-1} = \Phi_n^*$ defined by

$$(\Phi_n^{-1} f)(p) = (\Phi_n f)(-p). \tag{3.6}$$

Φ_n is an isomorphism on $\mathscr{S}$ which extends by duality to $\mathscr{S}'$ the continuous dual of $\mathscr{S}$ and by continuity to $L^2(\mathbb{R}^n, \mathbb{C}^m)$ (see [R], for example). We will employ the notation $\mathscr{H}$ for $L^2(\mathbb{R}^3_+, \mathbb{C}^7)$. Now, using Parseval's formula in the case of Φ_3, (3.4) may be written (here and below, χ_c is the characteristic function of the set c) as

$$\lim_{\varepsilon\to 0^+} \frac{\varepsilon}{\pi} \int_a^b |\Phi_3(\chi_{R^3_+} R(k-i\varepsilon) f)|^2\, dk \tag{3.7}$$

$$= \lim_{\varepsilon\to 0^+} \int_{R^3} \frac{\varepsilon}{\pi} \int_a^b |(\Phi_3 \chi_{R^3_+} R(k-i\varepsilon) f)(p)|^2\, dk\, dp. \tag{3.8}$$

We first wish to obtain

$$\frac{\varepsilon}{\pi} (\Phi_3 \chi_{\mathbb{R}^3} R(k-i\varepsilon) f)(p) \tag{3.9}$$

in a form which can be studied as $\varepsilon \to 0+$. To this end, we need to compute the "resolvent kernel" of $R(\lambda)$. This is a function $R(x, y; z)$ such that for f in $\mathscr{H}$,

$$R(z) f(x) = \int_{\mathbb{R}^3_+} R(x, y; z) f(y)\, dy. \tag{3.10}$$

The idea is to seek $R(x, y; z)$ in the form

$$\mathscr{E}(x-y; z) - F(x, y; z), \tag{3.11}$$

where $\mathscr{E}(x-y; z)$ is a solution in $\mathscr{S}'$ of

$$(A(D) - zI)\mathscr{E}(x; z) = \delta(x) I_{10\times 10}, \tag{3.12}$$

and F satisfies the three conditions:

$$(A(D) - zI) F(x, y; z) = 0, \qquad x, y \in \mathbb{R}^3_+ \text{ (differentiation on } x\text{)}, \tag{3.13}$$

$$G^i_{\lambda,j} F(x_1, x_2, 0, y; z) = G^i_{\lambda,j} \mathscr{E}(x-y; z)|_{x_3=0}, \qquad y \in \mathbb{R}^3_+, \tag{3.14}$$

$$\int_{\mathbb{R}^3_+} F(x, y; z) f(y)\, dy \text{ is in } \mathscr{H} \text{ for } f \text{ in } \mathscr{H}. \tag{3.15}$$

Let us define $A(p)$ to be $\sum A_j p_j$ for all nonzero p in $\mathbb{R}^3$. Then it is clear from our definition of Φ_3 that in $\mathscr{S}'$,

$$\mathscr{E}(\cdot\,; z) = (2\pi)^{-3/2} \Phi_3^* (A(p) - zI)^{-1} \Phi_3. \tag{3.16}$$

Taking the Fourier transform Φ_2 (on (x_1, x_2)) of (3.13) and (3.14) results in a first-order initial value problem in x_3. To solve this, it is necessary to compute $\Phi_2 G^i_{\lambda,j}\mathscr{E}(x-y; z)|_{x_3=0}$. It is evident that we will need $\Phi_2\mathscr{E}(x-y; z)|_{x_3=0}$ explicitly. In $\mathscr{S}'$ this means the evaluation of the integral $(\operatorname{im}(z)\neq 0)$ (which may be regarded as a member of $\mathscr{S}'$ or it may be computed in the usual way, by insertion of an appropriate exponential factor $e^{-p_3\varepsilon}\chi_{(0,\infty)}(p_3)$, for example, then letting $\varepsilon\to 0$)

$$(3.17)\qquad (2\pi)^{-2}\, e^{-iy'\circ n}\int_{-\infty}^{\infty} e^{-iy_3p_3}[A(n, p_3)-z]^{-1}\, dp_3,$$

where we have used the notation $n=(p_1, p_2)$, $y'=(y_1, y_2)$. This will be done by means of the residue theorem through deforming the integration into the lower half plane. It is therefore necessary to consider the integrand as being extended as a function of p_3 into $\mathbb{C}$; n, z are not zero.

We write $\tau=p_3+i\alpha$. We must consider the zeros of

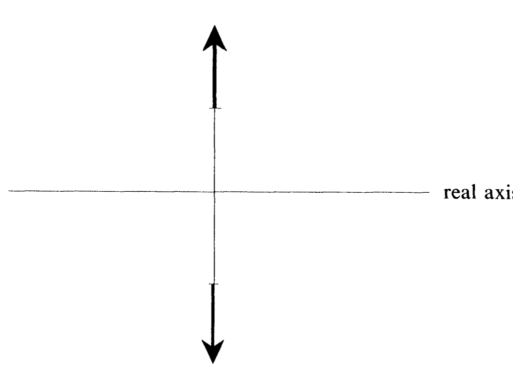
$$(3.18)\qquad \det([A(n, \tau)-z])$$

in τ. These occur in the upper and lower half plane at values $\tau_\pm$, respectively. We consider the cases $A=A^3$ and $A=A^2$ separately now. The roots of $\det(A(p)-\lambda I)$ are given by (2.6), (2.7) above.

For $i=2,3$ and $j=0,1,2,3$ let ${}_iP_{\pm j}(p)$ be the associated eigenprojectors on $\mathbb{C}^{10}$ of $A^i(p)$. By the spectral theorem,

$$(3.19)\qquad [A^i(p)-z]^{-1}=\sum_{j=-3}^{3}({}_i\lambda_j(p)-z)^{-1}{}_iP_j(p).$$

We wish to extend (in single-valued fashion) ${}_i\lambda_j(n, p_3)$ to ${}_i\lambda_j(n, \tau)$ and likewise ${}_iP_j(n, p_3)$ to ${}_iP_j(n, \tau)$ so that (3.19) remains valid, with all poles determined by the coefficients $({}_i\lambda_j(n, \tau)-z)^{-1}$. For ${}_i\lambda_{\pm 1}$, ${}_i\lambda_{\pm 2}$ and ${}_i\lambda_{\pm 3}$ we will make branchcuts in the τ plane (see Fig. 5) along the intervals $[(-i\infty, -i\sqrt{(2p_2^2+p_1^2)}), (i\sqrt{(2p_2^2+p_1^2)}, i\infty)]$, $[(-i\infty, -i\sqrt{(2p_2^2+p_1^2)}), (i\sqrt{(2p_2^2+p_1^2)}, i\infty)]$, $[(-i\infty, -in), (in, i\infty)]$, respectively, for A^2 and $[(-i\infty, -in/\sqrt{2}), (in/\sqrt{2}, i\infty)]$, $[(-i\infty, -in/\sqrt{2}), (in/\sqrt{2}, i\infty)]$, $[(-i\infty, -in), (in, i\infty)]$, respectively, for A^3.



τ-plane branch cuts

FIG. 5. *τ-plane branchcuts.*

The (τ) zeros of ${}_i\lambda_j(n,\tau)-z$ in the plane are given by

$$
\begin{aligned}
{}_2\tau_{\pm1} &= \pm(z^2-(2p_2^2+p_1^2))^{1/2},\\
{}_2\tau_{\pm2} &= \pm\frac{((5z^4-6p_2^2z^2+p_2^4)^{1/2}+3z^2-3p_2^2-2p_1^2)^{1/2}}{\sqrt{2}},\\
{}_2\tau_{\pm3} &= \pm\frac{(-(5z^4-6p_2^2z^2+p_2^4)^{1/2}+3z^2-3p_2^2-2p_1^2)^{1/2}}{\sqrt{2}},
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
{}_3\tau_{\pm1} &= \frac{\pm(z^2-n^2)^{1/2}}{\sqrt{2}},\\
{}_3\tau_{\pm2} &= \frac{((z^4+6n^2z^2+n^4)^{1/2}+3z^2-3n^2)^{1/2}}{2},\\
{}_3\tau_{\pm3} &= \frac{(-(z^4+6n^2z^2+n^4)^{1/2}+3z^2-3n^2)^{1/2}}{2}.
\end{aligned}
\tag{3.21}
$$

Here branchcuts are made for (3.20) $(A2)$ on the intervals

$$
\begin{gathered}
[(-\infty,-\sqrt{(2p_2^2+p_1^2)}),(\sqrt{(2p_2^2+p_1^2)},\infty)],\\
\left[\left(-\infty,\frac{-3n^2-(9n^4-4p_2^2n^2)^{1/2})^{1/2}}{\sqrt{2}}\right),\left(\frac{(3n^2-(9n^4-4p_2^2n^2)^{1/2})^{1/2}}{\sqrt{2}},\infty\right)\right],\\
\left[\left(-\infty,\frac{(3n^2+(9n^4-4p_2^2n^2)^{1/2})^{1/2}}{\sqrt{2}}\right),\right.\\
\left(\frac{(3n^2-(9n^4-4p_2^2n^2)^{1/2})^{1/2}}{\sqrt{2}},\frac{(3n^2-(9n^4-4p_2^2n^2)^{1/2})^{1/2}}{\sqrt{2}}\right)\\
\left.\left(\frac{(3n^2+(9n^4-4p_2^2n^2)^{1/2})^{1/2}}{\sqrt{2}},\infty\right)\right],
\end{gathered}
\tag{3.22}
$$

respectively, and for (3.21) (A^3) we make the branchcuts (see Fig. 6)

$$
\begin{gathered}
[(-\infty,-n),(n,\infty)]\\
\left[\left(-\infty,-\left(\frac{3-\sqrt{5}}{2}\right)^{1/2}n\right),\left(\left(\frac{3-\sqrt{5}}{2}\right)^{1/2}n,\infty\right)\right]\\
\left[\left(-\infty,-\left(\frac{3+\sqrt{5}}{2}\right)^{1/2}n\right),\left(-\left(\frac{3-\sqrt{5}}{2}\right)^{1/2}n,\left(\frac{3-\sqrt{5}}{2}\right)^{1/2}n\right),\left(\left(\frac{3+\sqrt{5}}{2}\right)^{1/2}n,\infty\right)\right].
\end{gathered}
\tag{3.23}
$$

It is easily verified that $\pm\text{im}\,({}_i\tau_{\pm j})>0$. Using the residue theorem, we obtain for (3.17) the expression (see Fig. 7):

$$
-(2\pi i)^{-1}\,e^{-y'\circ n}\sum_{j=1}^{3} e^{iy_i\tau_j}\,{}_ic_j\,{}_iP_j(n,-{}_i\tau_j,z),
\tag{3.24}
$$

where the expression ${}_ic_j$ is determined by l'Hopital's rule as the reciprocal of

$$
\left.\frac{\partial_i\lambda_j}{\partial\tau}\right|_{\tau={}_i\tau_j}.
$$

The matrices ${}_iP_j(n,-{}_i\tau_j,z)$ are obtained from (3.19) by substitution of ${}_i\tau_j$ for p_3. They are given here (note that in ${}_2P_2$ and ${}_2P_3$, $a_1=s_{12}^2-z^2$, $a_2=s_{12}^2-2z^2$, $a_3=s_{12}^2-3z^2$,

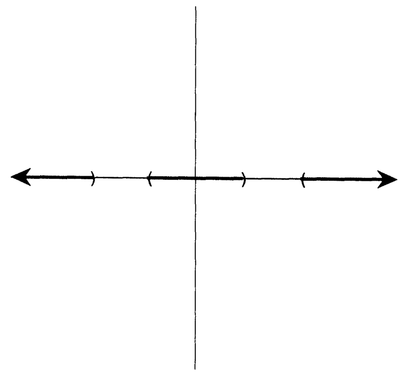


FIG. 6. *z-plane branchcuts (center cut used only for λ_3).*

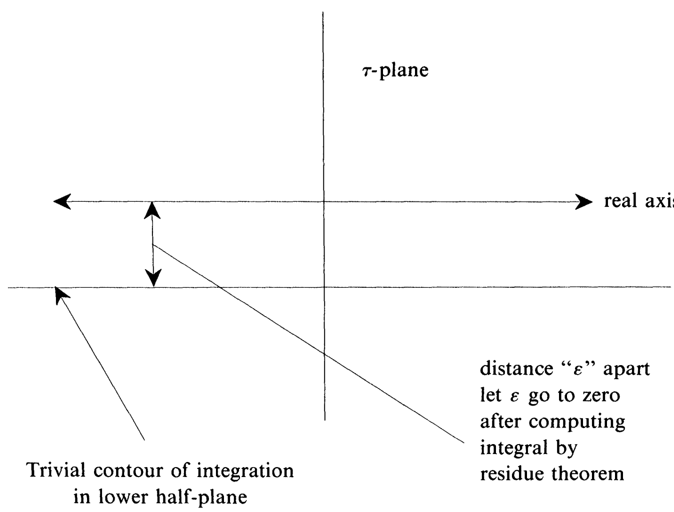


FIG. 7

$b_1 = s_{13}^2 - z^2$, $b_2 = s_{13}^2 - 2z^2$, $b_3 = s_{13}^2 - 3z^2$, $s_{ij} = {}_2\lambda_1({}_i\tau_j)$ and in ${}_3P_2$, ${}_3P_3$, $a_1 = n^2 + {}_3\tau_2^2 - z^2$, $a_2 = 2_3\tau_2^2 + n^2 - 2z^2$, $b_1 = z^2 - {}_3\tau_3^2 - n^2$, $b_2 = 2z^2 - 2_2\tau_3^2 - n^2$, the functions f_{ij} are normalization factors)(see Figs. 8–10).

We are able to write down the resolvent kernel now. First, we note that in the solution of (3.13), (3.14) we have

$$\Phi_{x'} F(n, x_3, y, z) = \frac{-1}{2\pi i} e^{-iy' \circ n} \sum_{j=1}^{3} e^{ix_3 {}_i\tau_j} {}_ic_j \, {}_iP_j \, {}_iM_j,$$

where the matrices ${}_iM_j$ are selected so that (3.14) is satisfied. Generally there are many possible choices for the ${}_iM_j$. The idea is to select the simplest among these for each of the boundary conditions. The ${}_iM_j$ are functions of z, t, n and are bounded except near points z where the so-called Lopatinski determinant vanishes. These (real) points yield the speeds of any surface waves. We discuss this further in the next section. We note that for $h_3 \neq 1$ or $h_2 \neq 1$, the development above is completely parallel except for the explicit formulas of the ${}_ic_j$ and ${}_iP_j$.

DEFINITION. For $k \neq j \neq 0$, let β be the set of points in p space where any ${}_i\lambda_j$ coincides with another ${}_i\lambda_k$. It is easy to see that this is a set of measure zero in p space.

$$ {}_2P_1 = \frac{1}{2(\tau_1^2+p_1^2)z^2} = \begin{bmatrix} p_2^2\tau_1 & 0 & -p_1p_2^2\tau_1 & p_2\tau_1^2 z & 0 & -p_1p_2^2\tau_1 & 0 & -p_1p_2^2\tau_1 & n_1^2p_2\tau_1 & -p_2^2\tau_1^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -p_1p_2^2\tau_1 & 0 & p_1^2p_2^2 & -p_1p_2\tau_1 z & 0 & p_1^2p_2 z & 0 & p_1^2p_2^2 & -n_1^2p_1p_2 & p_1p_2^2\tau_1 \\ p_2\tau_1^2 z & 0 & -p_1p_2\tau_1 z & \tau_1^2z^2 & 0 & -p_1\tau_1 z^2 & 0 & -p_1p_2\tau_1 z & n_1^2\tau_1 z & -p_2\tau_1^2 z \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -p_1p_2\tau_1 z & 0 & p_1^2p_2 z & -p_1\tau_1 z^2 & 0 & p_1^2z^2 & 0 & p_1^2p_2 z & -n_1^2p_2 z & (p_1p_2\tau_1 z) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -p_1p_2^2\tau_1 & 0 & p_1^2p_2^2 & -p_1p_2\tau_1 z & 0 & p_1^2p_2 z & 0 & p_1^2p_2^2 & -n_1^2p_1p_2 & p_1p_2^2\tau_1 \\ n_1^2p_2\tau_1 & 0 & -n_1^2p_1p_2 & n_1^2\tau_1 z & 0 & -n_1^2p_1 z & 0 & -n_1^2p_1p_2 & n_1^4 & -n_1^2p_2\tau_1 \\ -p_2^2\tau_1^2 & 0 & p_1p_2^2\tau_1 & -p_2\tau_1^2 z & 0 & p_1p_2\tau_1 z & 0 & p_1p_2^2\tau_1 & -n_1^2p_2\tau_1 & p_2^2\tau_1^2 \end{bmatrix}. \tag{3.25} $$

$$ {}_2p_2 = \frac{1}{{}_2f_2(z,n)} = \begin{bmatrix} a_2^2p_1^2z^2 & a_1a_2p_1p_2z^2 & a_2^2p_1\tau_2z^2 & a_2a_3p_1^2p_2z & a_1a_2^2p_1z & a_2a_3p_1p_2\tau_2z & a_1a_2p_1z^3 & -a_2p_1\tau_2z^4 & 0 & a_2p_1^2z^4 \\ a_1a_2p_1p_2z^2 & a_1^2p_2^2z^2 & a_1a_2p_2\tau_2z^2 & a_1a_3p_1p_2^2z & a_1^2a_2p_2z & a_1a_3p_2^2\tau_2z & a_1^2p_2z^3 & -a_1p_2\tau_2z^4 & 0 & a_1p_1p_2z^4 \\ a_2^2p_1z_2a^2 & a_1a_2p_2\tau_2z^2 & a_2^2\tau_2^2z^2 & a_2a_3p_1p_2\tau_2z & a_1a_2^2\tau_2z & a_2a_3p_2\tau_2^2z & a_1a_2\tau_2z^3 & -a_2\tau_2^2z^4 & 0 & a_2p_1\tau_2z^4 \\ a_2a_3p_1^2p_2z & a_1a_3p_1p_2^2z & a_2a_3p_1p_2\tau_2z & a_3^2p_1^2p_2^2 & a_1a_2a_3p_1p_2 & a_3^2p_1p_2^2\tau_2 & a_1a_3p_1p_2z^2 & -a_3p_1p_2\tau_2z^3 & 0 & a_3p_1^2p_2z^3 \\ a_1a_2^2p_1z & a_1^2a_2p_2z & a_1a_2^2\tau_2z & a_1a_2a_3p_1p_2 & a_1^2a_3^2 & a_1a_2a_3p_3\tau & a_1^2a_2z^2 & -a_1a_2\tau_2z^3 & 0 & a_1a_2p_1z^3 \\ a_2a_3p_1p_2\tau_2z & a_1a_3p_2^2\tau_2z & a_2a_3p_2\tau_2^2z & a_3^2p_1p_2^2\tau_2 & a_1a_2a_3p_2\tau_2 & a_3^2p_3^2\tau_2^2 & a_1a_3p_2\tau_2z^2 & -a_3p_2\tau_2^2z^3 & 0 & a_3p_1p_2\tau_2z^3 \\ a_1a_2p_1z^3 & a_1^2p_2z^3 & a_1a_2\tau_2z^3 & a_1a_3p_1p_2z^2 & a_1^2a_2z^2 & a_1a_3p_2\tau_2z & a_1\tau_2z^5 & -a_1\tau_2z^5 & 0 & a_1p_1z^5 \\ -a_2p_1\tau_2z^4 & -a_1p_2\tau_2z^4 & -a_2\tau_2^2z^4 & -a_3p_1p_2\tau_2z^3 & -a_1a_2\tau_2z^3 & -a_3p_2\tau_2^2z^3 & -a_1\tau_2z^5 & \tau_2^2z^6 & 0 & -p_1\tau_2z^6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_2p_1^2z^4 & a_1p_1p_2z^4 & a_1p_1\tau_2z^4 & a_3p_1^2p_2z^3 & a_3p_1\tau_2z^3 & a_3p_1p_2\tau_2z^3 & a_1p_1z^5 & -p_1\tau_2z^6 & 0 & p_1^2z^6 \end{bmatrix} \tag{3.26} $$

FIG. 8

$$(3.27)\qquad {}_2P_3=\frac{1}{({}_2f_3(z))}=\begin{bmatrix}
b_2^2p_1^2z^2 & b_1b_2p_1p_2z^2 & b_2^2p_1\tau_3z^2 & b_2b_3p_1^2p_2z & b_1b_2^2p_1z & {}_2b_3p_1p_2\tau_3z & -b_1b_2p_1z^3 & -b_2p_1\tau_3z^4 & 0 & b_2p_1^2z^4\\
b_1b_2p_1p_2z^2 & b_1^2p_2^2z^2 & b_1b_2p_2\tau_3z^2 & b_1b_3p_1p_2^2z & b_1b_2p_2z & b_1b_3p_2^2\tau_3z & -b_1^2p_2z^3 & -b_1p_2\tau_3z^4 & 0 & b_1p_1p_2z^4\\
b_2^2p_1\tau_3z^2 & b_1b_2p_2\tau_3z^2 & b_2^2\tau_3^2z^2 & b_2b_3p_1p_2\tau_3z & b_1b_2^2\tau_3z & b_2b_3p_2\tau_3^2z & -b_1b_2\tau_3z^3 & -b_2\tau_3^2z^4 & 0 & b_2p_1\tau_3\\
b_2b_3p_1^2p_2z & b_1b_3p_1p_2^2z & b_2b_3p_1p_2\tau_3z & b_3^2p_1^2p_2^2 & b_1b_2b_3p_1p_2 & b_3^2p_1p_2^2\tau_3 & -b_1b_3p_1p_2z^2 & -b_3p_1p_2\tau_3z^3 & 0 & b_3p_1^2z^3\\
b_1b_2^2p_1z & b_1^2b_2p_2z & b_1b_2^2\tau_3z & b_1b_2b_3p_1p_2 & b_1^2b_2^2 & b_1b_2b_3p_2\tau_3 & -b_1^2b_2z^2 & -b_1b_2\tau_3z^3 & 0 & b_1b_2p_1z^3\\
b_2b_3p_1p_2\tau_3z & b_1b_3p_2^2\tau_3z & b_2b_3p_2\tau_3^2z & b_3^2p_1p_2^2\tau_3 & b_1b_2b_3p_2\tau_3 & b_3^2p_2^2\tau_3^2 & -b_1b_3p_2\tau_3z^2 & -b_1p_2\tau_3^2z^3 & 0 & b_3p_1p_2\tau_3z^3\\
-b_1b_2p_1z^3 & -b_1^2p_2z^3 & -b_1b_2\tau_3z^3 & -b_1b_3p_1p_2z^2 & -b_1^2b_2z^2 & -b_1b_3p_2\tau_3z^2 & b_1^2z^4 & b_1\tau_3z^5 & 0 & -b_1p_1z^5\\
-b_2p_1\tau_3z^4 & -b_1p_2\tau_3z^4 & -b_2\tau_3^2z^4 & -b_3p_1p_2\tau_3z^3 & -b_1b_2\tau_3z^3 & -b_3p_2\tau_3^2z^3 & b_1\tau_3z^5 & \tau_3^2z^6 & 0 & -p_1\tau_3z^6\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
b_2p_1^2z^4 & b_1p_2p_2z^4 & b_2p_1\tau_3z^4 & b_1b_2p_1z^3 & b_3p_1p_2z^3 & b_3p_1p_2\tau_3z^3 & -b_1p_1z^5 & -p_1\tau_3z^6 & 0 & p_1^2z^6
\end{bmatrix}$$

$$(3.28)\qquad {}_3f_1(z)\,{}_3P_1=\begin{bmatrix}
a_2^2\tau_1^2 & -p_1p_2\tau_1^2 & 0 & p_2^2\tau_1z & -p_1p_2\tau_1z & 0 & 0 & p_1p_2\tau_1^2 & p_2^2\tau_1^2 & -n^2p_2\tau_1\\
-p_1p_2\tau_1^2 & p_1^2\tau_1^2 & 0 & -p_1p_2\tau_1z & p_1^2\tau_1z & 0 & 0 & -p_1^2\tau_1^2 & -p_1p_2\tau_1^2 & n^2p_2\tau_1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
p_2^2\tau_1z & -p_1p_2\tau_1z & 0 & p_2^2z^2 & -p_1p_2z^2 & 0 & 0 & p_1p_2\tau_1z & p_2^2\tau_1z & -n^2p_2z\\
-p_1p_2\tau_1z & p_1^2\tau_1z & 0 & -p_1p_2z^2 & p_1^2z^2 & 0 & 0 & -p_1^2\tau_1z & -p_1p_2\tau_1z & n^2p_1z\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
p_1p_2\tau_1^2 & -p_1^2\tau_1^2 & 0 & p_1p_2\tau_1z & -p_1^2\tau_1z & 0 & 0 & p_1^2\tau_1^2 & p_1p_2\tau_1^2 & -n^2p_1\tau_1\\
p_2^2\tau_1^2 & -p_1p_2\tau_1^2 & 0 & p_2^2\tau_1z & -p_1p_2\tau_1z & 0 & 0 & p_1p_2\tau_1^2 & p_2^2\tau_1^2 & -n^2p_2\tau_1\\
-n^2p_2\tau_1 & n^2p_1\tau_1 & 0 & -n^2p_2 & n^2p_1z & 0 & 0 & -n^2p_2\tau_1 & -n^2p_2\tau_1 & n^4
\end{bmatrix}$$

FIG. 9

$$(3.29)\quad \frac{1}{{}_3f_2(z)}\,{}_3P_2=\begin{bmatrix}
a_1^2a_2^2p_1^2 & a_1^2a_2^2p_1p_2 & a_1a_2n^2p_1\tau_2z^2 & -a_1a_2^2p_1^2\tau_2z & -a_1a_2^2p_1p_2\tau_2z & a_1a_2^2n^2p_1z & -a_1a_2n^2p_1z^3 & a_1^2a_2p_1p_2z^2 & -a_1a_2^2p_1^2(\tau_2^2+n^2) & 0\\
a_1^2a_2^2p_1p_2 & a_1^2a_2^2p_2^2 & a_1a_2n^2p_1\tau_2z^2 & -a_1a_2^2p_1p_2\tau_2z & -a_1a_2^2p_2^2\tau_2z & a_1a_2^2n^2p_2z & -a_1a_2n^2p_2z^3 & a_1^2a_2p_2^2z^2 & -a_1a_2^2p_1p_2(\tau_2^2+n^2) & 0\\
a_1a_2n^2p_1\tau_2z^2 & a_1a_2n^2p_2\tau_2z^2 & n^4\tau_2^2z^4 & -a_2n^2p_1p_2\tau_2^2z^3 & -a_2n^2p_2\tau_2^2z^3 & a_2n^4\tau_2z^3 & -n^4\tau_2z^5 & a_1n^2p_2\tau_2z^4 & -a_2n^2p_1\tau_2(\tau_2^2+n^2)z^2 & 0\\
-a_1a_2^2p_1^2\tau_2z & -a_1a_2^2p_1p_2\tau_2z & -a_2n^2p_1\tau_2^2z^3 & a_2^2p_1^2\tau_2^2z^2 & a_2^2p_1p_2\tau_2^2z^2 & -a_2^2n^2p_1\tau_2z^2 & a_1n^2p_1\tau_2z^4 & -a_1a_2p_1p_2\tau_2z^3 & a_2^2p_1^2\tau_2(\tau_2^2+n^2)z & 0\\
-a_1a_2^2p_1p_2\tau_2z & -a_1a_2^2p_2^2\tau_2z & -a_2n^2p_2\tau_2^2z^3 & a_2^2p_1p_2\tau_2^2z^2 & a_2^2p_2^2\tau_2^2z^2 & -a_2^2n^2p_2\tau_2z^2 & a_2n^2p_2\tau_2z^4 & -a_1a_2p_2^2\tau_2z^3 & a_2^2p_1p_2\tau_2(\tau_2^2+n^2)z & 0\\
a_1a_2^2n^2p_1z & a_1a_2^2n^2p_2z & a_2n^4\tau_2z^3 & -a_2^2n^2p_1\tau_2z^2 & -a_2^2n^2p_2\tau_2z^2 & a_2^2n^4z^2 & -a_2n^4z^4 & a_1a_2n^2p_2z^3 & -a_2^2n^2p_1(\tau_2^2+n^2)z & 0\\
-a_1a_2n^2p_1z^3 & -1_1a_2n^2p_2z^3 & -n^4\tau_2z^5 & a_2n^2p_1\tau_2z^4 & a_2n^2p_2\tau_2z^4 & -a_2n^4z^4 & n^4z^6 & -a_1n^2p_2z^5 & a_2n^2p_1(\tau_2^2+n^2)z^3 & 0\\
a_1^2a_2p_1p_2z^2 & a_1a_2p_2^2z^2 & a_1n^2p_2\tau_2z^4 & -a_1a_2p_1p_2\tau_2z^3 & -a_1a_2p_2^2\tau_2z^3 & a_1a_2n^2p_2z^3 & -a_1n^2p_2z^5 & a_1^2p_2^2z^4 & -a_1a_2p_1p_2(\tau_2^2+n^2)z^2 & 0\\
-a_1a_2^2p_1^2(\tau_2^2+n^2) & -a_1a_2^2p_1p_2(\tau_2^2+n^2) & -a_2n^2p_1\tau_2(\tau_2+n^2)z^2 & a_2^2p_1^2\tau_2(\tau_2^2+n^2)z & a_2^2p_1p_2\tau_2(\tau_2^2+n^2)z & -a_2^2n^2p_1(\tau_2^2+n^2)z & a_2n^2p_1(\tau_2^2+n^2)z^3 & -a_1a_2p_1p_2(\tau_2^2+n^2)z^2 & a_2^2p_1^2(\tau_2^2+n^2)^2 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$(3.30)\quad {}_3f_3(z)\,{}_3P_3=\begin{bmatrix}
b_1^2b_2^2p_1^2 & b_1^2b_2^2p_1p_2 & b_1b_2n^2p_1\tau_3z^2 & b_1b_2^2p_1^2\tau_3z & b_1b_2^2p_1p_2z & -b_1b_2^2n^2p_1z & -b_1b_2n^2p_1z^3 & -b_1^2b_2p_1p_2z^2 & b_1b_2^2p_1^2(\tau_3^2+n^2) & 0\\
b_1^2b_2^2p_1p_2 & b_1^2b_2^2p_2^2 & b_1b_2n^2p_2\tau z^2 & b_1b_2^2p_1p_2\tau_3z & b_1b_2^2p_2^2\tau_3z & -b_1b_2^2n^2p_2z & -b_1b_2n^2p_2z^3 & -b_1^2b_2p_2^2z^2 & b_1b_2^2p_1p_2(\tau_3^2+n^2) & 0\\
b_1b_2n^2p_1\tau_3z^2 & b_1b_2n^2p_2\tau_3z^2 & n^4t_3^2z^4 & b_2n^2p_1\tau_3^2z^3 & b_2n^2p_2\tau_3^2z^3 & -b_2n^4\tau_3z^3 & -n^4\tau_3z^5 & -b_1n^2p_2\tau_3z^4 & b_2n^2p_1\tau_3(\tau_3^2+n^2)z^2 & 0\\
b_1b_2^2p_1^2\tau z & b_1b_2^2p_1p_2\tau_3z & b_2n^2p_1\tau_3^2z^3 & b_2^2p_1^2\tau_3^2z^2 & b_2^2p_1p_2\tau_3^2z^2 & -b_2^2n^2p_1\tau_3z^2 & -b_2n^2p_1\tau_3z^4 & -b_1b_2p_1p_2\tau_3z^3 & b_2^2p_1^2\tau_3(\tau_3^2+n^2)z & 0\\
b_1b_2^2p_1p_2\tau_3z & b_1b_2^2p_2^2\tau_3z & b_2n^2p_2\tau_3^2z^3 & b_2^2p_1p_2\tau_3^2z^2 & b_2^2p_2^2\tau_3^2z^2 & -b_2^2n^2p_2\tau_3z^2 & -b_2n^2p_2\tau_3z^4 & -b_1b_2p_2^2\tau_3z^3 & b_2^2p_1p_2\tau_3(\tau_3^2+n^2)z & 0\\
-b_1b_2^2n^2p_1z & -b_1b_2^2n^2p_2z & -b_2n^4\tau_3z^3 & -b_2^2n^2p_1\tau_3z^2 & -b_2^2n^2p_2\tau_3z^2 & -b_2^2n^4z^2 & b_2n^4z^4 & b_1b_2n^2p_2z^3 & -b_2^2n^2p_1(\tau_3^2+n^2)z & 0\\
-b_1b_2n^2p_1z^3 & -b_1b_2n^2z_2z^3 & -n^4\tau_3z^5 & -b_2n^2p_1\tau_3z^4 & -b_2n^2p_2\tau_3z^4 & b_2n^4z^4 & n^4z^6 & b_1n^2p_2z^5 & -b_2n^2p_1(\tau_3^2+n^2)z^3 & 0\\
-b_1^2b_2p_1p_2z^2 & -b_1^2b_2p_2^2z^2 & -b_1n^2p_2\tau_3z^4 & -b_1b_2p_1p_2\tau_3z^3 & -b_1b_2p_2^2\tau_3z^3 & b_1b_2n^2p_2z^3 & b_1n^2p_2z^5 & b_1^2p_2^2z^4 & -b_1b_2p_1p_2(\tau_3^2+n^2)z^2 & 0\\
b_1b_2^2p_1^2(\tau_3+n^2) & b_1b_2^2p_1p_2(\tau_3^2+n^2) & b_2n^2p_1\tau_3(\tau_3^2+n^2)z^2 & b_2^2p_1^2\tau_3(\tau_3^2+n^2)z & b_2^2p_1p_2\tau_3(\tau_3^2+n^2)z & -b_2^2p_1(\tau_3^2+n^2)z & -b_2n^2p_1(\tau_3^2+n^2)z^3 & -b_1b_2p_1p_2(\tau_3^2+n^2)z^2 & b_2^2p_1(\tau_3^2+n^2)^2 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

FIG. 10

We may now write down $\Phi_3 \chi_{\mathbb{R}^3_+} R(p, y, z)$ by applying Φ_1 in x_3 to (3.31), using (3.11) to get

$$(3.32)\quad \begin{aligned} &(2\pi)^{-3/2} e^{-iy'\circ n}\left(e^{-iy_3 p_3}[A^3(p)-zI]^{-1} + \sum_{j=1}^{3} \frac{e^{iy_3\,{}_3\tau_j}}{({}_3\tau_j + p_3)}\, {}_3c_j\, {}_3P_j(n, -{}_3\tau_j, z)\right) \\ &+ 2\pi\, e^{-iy'\circ n} \sum_{j=1}^{3} \left(\frac{{}_3c_j}{{}_3\tau_j - p_3}\right) {}_3P_j(n, {}_3\tau_j, z)\, {}_3M_j \end{aligned}$$

with a completely analogous expression for A^2. Here it is helpful to note that the functions ${}_ic_j$ are not singular. For later reference we also note the facts:

$$(3.33)\quad \begin{aligned} &\text{(i)} \qquad \lim_{z \to {}_i\lambda_{kj}(p) \pm i0} {}_i\tau_j(p', z) = \pm k|p_3|, \qquad k = \pm 1, \\ &\text{(ii)} \qquad \lim_{z \to {}_i\lambda_{kl}(p) \pm i0} {}_i\tau_j(p', z) \neq \pm k|p_3|, \qquad l \neq j. \end{aligned}$$

4. Eigenfunction expansions. In the computation of the spectral families of the various operators A^i_λ arising from the different combinations of external fields and boundary conditions, the (first-order) singularities of the resolvent kernel give rise to the terms of the spectral family. These singularities include the eigenvalues ${}_i\lambda_j$ but may also include singularities of the matrices ${}_iM_j$. The singularities of the ${}_iM_j$ are the surface wave speeds and may be computed directly from (3.14). This reduces to the search for real zeros (in z) of the Lopatinski determinant [Wa]. This is defined as follows.

DEFINITION 4.1 (A^3). The Lopatinski determinant is the family of determinants $\det[G_3P^j_1, G_3P^k_2, G_3P^l_3]$. A number $s(n, z)$ is a zero of the Lopatinski determinant if it is a zero for each member of the above family. Here, G is fixed to be one of the A^3 boundary conditions (2.11) and $G_3P^j_k$ is the jth column of G_3P_k. The definition for A^2 is entirely similar.

THEOREM 4.2. *The Lopatinski determinant has no real zeros for either A^3 or A^2, and hence neither of these supports surface waves.*

The proof is rather tedious but is just a matter of finding one of each family of determinants that has no real zeros. We give the computation for A^2 as an example. The matrices G_2P_j, $j = 1, 2, 3$, respectively, are given as in Fig. 11. The Lopatinski determinant is seen to be essentially $a_2 + b_2$. This has no real zeros. In fact, we may give (up to a nice scalar factor determined by (3.14) and (3.31)) the matrices ${}_2M_j$, $j = 1, 2, 3$ (respectively) as

$$(4.1)\quad \begin{gathered} \operatorname{diag}(-1, a, 1, -1, a, 1, a, 1, 1, -1), \\ \operatorname{diag}(1, 1, -1, 1, 1, -1, 1, -1, a, 1), \\ \operatorname{diag}(1, 1, -1, 1, 1, -1, 1, -1, a, 1), \end{gathered}$$

where "a" means the entry is arbitrary. Combining this with the fact that (1.17) reduces to the Maxwell equations when $V = 0$, and results of [Sc2] together with the remarks above (cf. (2.11)) we have the following corollary.

COROLLARY 4.3. *The presence of surface waves is unstable in a liquid semiconductor modeled by* (1.17) *for either parallel or orthogonal external fields.*

DEFINITION 4.4.

$$(4.2)\quad {}_i\psi^*_j = ({}_i\lambda_j(p) - z^*)\chi_{\mathbb{R}^3 \setminus \beta}(p)\Phi_3 \chi_{\mathbb{R}^3_+} R(p, y, z^*),$$

$$(4.3)\quad {}_i\hat{f}_j(p, z) = \int_{\mathbb{R}^3_+} {}_i\psi^*_j(p, y, z) f(y)\, dy,$$

$$G_2P_1=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_1^2p_2\tau_1 & 0 & -n_1^2p_1p_2 & n_1^2\tau_1 z & 0 & -n_1^2p_1z & 0 & 0 & -n_1^2\tau_1p_2 & n_1^4 & -n_1^2p_2\tau_1 \end{bmatrix},$$

$$G_2P_2=\begin{bmatrix} a_1a_2^2p_1z & a_1^2a_2p_2z & a_1a_2^2\tau_2z & a_1a_2a_3p_1p_2 & a_1^2a_2^2 & a_1a_2a_3p_2\tau_2 & a_1^2a_2z^2 & -a_1a_2\tau_2z^3 & 0 & a_1a_2p_1z^3 \\ a_1a_2p_1z & a_1^2p_2z^3 & a_1a_2\tau_2z^3 & a_1a_3p_1p_2z^2 & a_1^2a_2z^2 & a_1a_3p_2\tau_2z^2 & a_1^2z^4 & -a_1\tau_2z^5 & 0 & a_1p_1z^5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G_2P_3=\begin{bmatrix} b_1b_2^2p_1z & b_1^2b_2p_2z & b_1b_2^2\tau_3z & b_1b_2b_3p_2 & b_1^2b_2^2 & b_1b_2b_3p_2\tau_3 & -b_1^2b_2z^2 & -b_1b_2\tau_3z^3 & 0 & b_1b_2p_1z^3 \\ -b_1b_2p_1z^3 & -b_1^2p_2z^3 & -b_1b_2\tau_3z^3 & -b_1b_3p_1p_2z^2 & -b_1^2b_2z^2 & -b_1b_3p_2\tau_3z^2 & b_1^2z^4 & b_1\tau_3z^5 & 0 & -b_1p_1z^5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

FIG. 11

where f is smooth and has bounded support.

Set $k=\pm 1$ and

$$(4.4)\qquad {}_i\psi^*_{kj}(p, y)=\lim_{z\to {}_i\lambda_{kj}(p)+i0} {}_i\psi^*_j(p, y, z)$$

$$(4.5)\qquad \begin{aligned}&=(2\pi)^{-3/2}\chi_{\mathbb{R}^3\setminus\beta}(p)\chi_{\mathbb{R}_{-k}}(p_3)\,{}_iP_j(p)\\ &\quad\times\{e^{iy\circ p}I-{}_iM_j(n, {}_i\lambda_{kj}(p)-i0, y_3)\,e^{-iy'\circ n}\}.\end{aligned}$$

LEMMA 4.5. *If f is smooth and has compact support, then*

$$(4.6)\qquad {}_i\hat{f}_{\pm j}(p)=\lim_{z\to {}_i\lambda_{\pm j}(p)+i0} {}_j\hat{f}_j(p, z)=\int_{\mathbb{R}^3_+} {}_i\psi^*_{\pm j}(p, y)f(y)\,dy$$

defines a function which is smooth and rapidly decreasing almost everywhere.

Proof. Equations (4.2) and (4.5) show that the function on the left-hand side of (4.3) in this case converges by the definition of ${}_iM_j$ and the dominated convergence theorem as indicated. The fact that the Fourier transform of f is smooth and rapidly decreasing together with (4.5) gives the result.

Most of the results in this section are essentially independent of external field direction, at least in their statements. So that the notation does not become unwieldy, we will omit the front subscript from most expressions. The generalized Fourier transforms are defined by (4.6). These will also be denoted by expression Φ. Whether the ordinary or generalized transform is meant should be clear from the context.

LEMMA 4.6.

$$(4.7)\qquad \begin{aligned}&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}\frac{\varepsilon}{\pi}\int_a^b|\Phi_3\chi_{\mathbb{R}^3_+}R(k-i\varepsilon)f(p)|^2\,dk\,dp\\ &=\int_{\mathbb{R}^3_+}\lim_{\varepsilon\to 0^+}\frac{\varepsilon}{\pi}\int_a^b|\Phi_3\chi_{\mathbb{R}^3_+}R(k-i\varepsilon)f(p)|^2\,dk\,dp.\end{aligned}$$

There is no problem in switching the order of integration for positive e, since the integrand is continuous in k and measurable in p and nonnegative. The proof of this lemma is tedious but straightforward.

THEOREM 4.7.

$$(4.8)\qquad \lim_{\varepsilon\to 0^+}\frac{\varepsilon}{\pi}\int_a^b|\Phi_3\chi_{\mathbb{R}^3_+}R(k-i\varepsilon)f(p)|^2\,dk=\sum_{j\neq 0}\chi_{\lambda_j(p)\in(a,b)}(p)|\hat{f}_j(p)|^2$$

for all f in the orthogonal complement of the null space of A^i.

THEOREM 4.8. *The modes of propagation are uncoupled for A^2 and for A^3 with type* II *boundary at $\lambda=\infty$. For A^3 with type* II *boundary at $\lambda=0$, the quasi-Alfven mode is uncoupled.*

Proof of Theorem 4.7. We apply the classical elementary fact:

$$(4.9)\qquad \lim_{\varepsilon\to 0^+}\frac{\varepsilon}{\pi}\int_a^b\frac{f(x)}{(k-x)^2+\varepsilon^2}\,dk=\chi_{(a,b)}(x)f(x)$$

for any continuous f.

For $p\notin\beta$, and δ small, the sets

$$(4.10)\qquad \Delta_j=(a, b)\cap(\lambda_j(p)-\delta, \lambda_j(p)+\delta)$$

are pairwise disjoint. Making the appropriate substitution for f above, we have ($z^*=k-i\varepsilon$)

$$(4.11)\qquad \sum_j \lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_{\Delta_j}\frac{\varepsilon}{(\lambda_j(p)-k)^2+\varepsilon^2}|(\lambda_j(p)-z^*)\Phi_3\chi_{\mathbb{R}^3_+}R(k-i\varepsilon)f(p)|^2\,dk$$

from which the result follows.

Proof of Theorem 4.8. Here we must use (3.14) to obtain the equations for M_j as follows:

$$(4.12)\qquad \sum_{j=1}^{3} G(e^{iy_3\tau_j}c_j(-\tau_j)P_j(n,-\tau_j,z)-c_jP_j(n,\tau_j,z)M_j)=0.$$

Now from (4.5) it follows that one mode is uncoupled from the others when M_j can be found for that j so that

$$(4.13)\qquad G(e^{iy_3\tau_j}c_j(-\tau_j)P_j(n,-\tau_j,z)-c_jP_j(n,\tau_j,z)M_j)=0.$$

The result for A^2 now follows from (4.1) and for A^3, $\lambda=\infty$,

$$(4.14)\qquad \begin{gathered}\operatorname{diag}(1,1,a,-1,-1,a,a,1,1,-1),\\ \operatorname{diag}(1,1,-1,-1,-1,1,1,1,1,a),\\ \operatorname{diag}(1,1,-1,-1,-1,1,1,1,1,a),\end{gathered}$$

and for $\lambda=0$ for

$$(4.15)\qquad \operatorname{diag}(-1,-1,1,1,a,a,-1,-1,1).$$

To check that coupling occurs for the other boundary conditions is a straightforward computation and is omitted.

In order to continue, we must define the null spaces associated with the operators A^2 and A^3. These null spaces are determined, respectively, by the sums of two collections of orthoprojectors given by the pseudodifferential operator kernels

$$(4.16)\qquad {}_2P_{01}=\frac{1}{p^2}(0,0,0,p_1,p_2,p_3,0,0,0,0)\otimes(0,0,0,p_1,p_2,p_3,0,0,0,0),$$

$$(4.17)\qquad {}_2P_{02}=\frac{1}{p^2}(0,0,0,0,0,0,0,p_1,p_2,p_3)\otimes(0,0,0,0,0,0,0,p_1p_2,p_3),$$

$$(4.18)\qquad {}_2P_{03}={}_2m_3\otimes{}_2m_3,$$

$$(4.19)\qquad {}_2P_{04}={}_2m_4\otimes{}_2m_4,$$

where

$$(4.20)\qquad {}_2m_3=(-p^2p_2p_3,p^2p_1p_3,0,0,0,0,0,p_1p_2p_3^2,p_2^2p_3^2,-n^2p_2p_3)/({}_2\lambda_1p|p_3|n),$$

$$(4.21)\qquad {}_2m_4=(-p_1^2p_3,-p_1p_2p_3,n^2p_1,0,0,0,0,-p_1p_2^2,p_1^2p_2,0)/({}_2\lambda_1n|p_1|),$$

and for A^3

$$(4.22)\qquad {}_3P_{01}={}_2P_{01},$$

$$(4.23)\qquad {}_3P_{02}={}_2P_{02},$$

$$(4.24)\qquad {}_3m_3=(-p^2p_2,p^2p_1,0,0,0,0,0,p_1p_3^2,p_2p_3^2,-n^2p_3)/({}_3\lambda_1np),$$

$$(4.25)\qquad {}_4m_4=(-p_1^2p_3,-p_1p_2p_3,n^2p_1,0,0,0,0,-p_1p_2p_3,p_1^2p_3,0)N({}_3\lambda_1n|p_1|),$$

From this we see that the usual Maxwell divergence equations continue to hold for B and E' for both A^2 and A^3. The other two auxiliary conditions are more complex, relating E' and V.

Defining ${}_2P_0 = {}_2P_{01} + {}_2P_{02} + {}_2P_{03} + {}_2P_{04}$ and similarly for ${}_3P_0$, we obtain the following result.

THEOREM 4.9. *If* $g, h \in (I - P_0)\mathcal{H}$ *then*

$$(E(I)g, h) = \sum_{j \neq 0} \int_{\mathbb{R}^3} \chi_{(\lambda_j(p) \in I)}(p) \hat{g}_j(p)^* \hat{h}_j(p)\, dp, \tag{4.26}$$

where E *is the spectral family for* A *and* I *is any subinterval of* $\mathbb{R}$.

Proof. This follows from (4.9) and the polarization identity. (See Lemma 4.5 for the notation ^.) We can define the generalized transforms now.

DEFINITION 4.10. For $g \in \mathcal{D}(\mathbb{R}^3_+)$ define

$$\Phi_j g(p) = \hat{g}_j(p) \tag{4.27}$$

and by Theorem 4.9 extend to all of $\mathcal{H}$. The adjoints of the maps Φ_j are given by

$$\Phi_j^* f(x) = \int_{\mathbb{R}^3} \psi_j^{**}(p, x) f(p)\, dp. \tag{4.28}$$

This follows easily for functions in $\mathcal{D}$ by definition and the general case follows by extension. The maps Φ_j yield the reduction of the unitary groups e^{-itA^i}. To check that they are orthogonal in the sense that the range of Φ_j^* is in the null space of Φ_j, $k \neq j$, suppose f is smooth and rapidly decreasing. Then the expression

$$\begin{aligned} \Phi_j f(r) &= \int_{\mathbb{R}^3_+} \psi_j^*(x, r) f(x)\, dx \\ &= (2\pi)^{-3/2} \chi_{\mathbb{R}^3 \setminus \beta} \chi_{\mathbb{R}}(r_3) P_j(r) \int_{\mathbb{R}^3_+} \{e^{-ix \circ r} I - M_j(r)\, e^{-ix' \circ r'}\} f(x)\, dx \end{aligned} \tag{4.29}$$

makes sense pointwise and further (here we have assumed $j, k > 0$),

$$\begin{aligned} \Phi_k^* g(x) &= \int_{\mathbb{R}^3} \psi_k^{**}(x, s) g(s)\, ds \\ &= (2\pi)^{-3/2} \int_{\mathbb{R}^3} \chi_{\mathbb{R}_-}(s_3) \chi_{\mathbb{R}^3 \setminus \beta}(s) \{e^{ixs} I - e^{ix' \circ s'} M_k^*(s) P_k(s) g(s)\, ds \end{aligned} \tag{4.30}$$

is a smooth rapidly decreasing function if g is, and if g vanishes in a neighborhood of β for a fixed p on a neighborhood of the set of s such that $\lambda_k(s) = \lambda_j(p)$. If F satisfies this condition, then $g(s) = F(s)/(\lambda_j(p) - \lambda_k(s))$ also satisfies the same condition.

Let $\mathcal{D}_1 = \{f \in \mathcal{D}(\mathbb{R}^3, \mathbb{C}^7) \mid \beta \cap \operatorname{supp}(f) = \varnothing\}$. Fix $F \in \mathcal{D}_1$ and $p \in \mathbb{R}^3$, and set $g(s)$ as above; $g \in \mathcal{D}_1$. Then $\Phi_k^* g$ is smooth, rapidly decreasing, and satisfies the boundary conditions and so is in the domain of A, and $A\Phi_k^* g = \Phi_k^* \lambda_k(\cdot) g$. Hence $\Phi_j A \Phi_k^* g = \Phi_j \Phi_k^* \lambda_k(\cdot) g$. But also $\Phi_j A \Phi_k^* g = \lambda_j(p) \Phi_j \Phi_k^* g(p)$. Subtracting, we obtain $\Phi_j \Phi_k^* F(p) = 0$. Since p is arbitrary and $\mathcal{D}_1$ is dense, this proves the required relation.

It follows from the preceding that the maps $[{}_i\Phi_j^*\, {}_i\Phi_j]$ are projections on $\mathcal{H}$.

In a similar way, we may show the spectral representation

$$e^{-itA^i} f = \sum_{j \neq 0} {}_i\Phi_j^*\, e^{-i|p|t}\, {}_i\Phi_j f. \tag{4.31}$$

The theory of potential scattering in a half-space may be studied for the operators A^i. We will not do this here. The interested reader may consult Theorems 3.8–3.10 of [S2], where this was done in the perfect conductor case. The method is entirely similar for the present problem. Instead, we take up the problem of when $\sigma = \sigma(x)$, or $\sigma = \sigma(t, x)$ is nonzero but decays at infinity in an appropriate sense. The case of σ that do not decay will be studied elsewhere.

5. Variable conductivity. We now wish to consider the problem of nonvanishing conductivity which may vary in space and/or time. First we consider the spatial variation only. We allow for possible anisotropy of the medium.

Assumption. Let $\sigma(x)$ be any two-tensor of dimension 3 whose components σ_{ij} $(i, j = 1, 2, 3)$ are almost everywhere uniformly bounded and satisfy the condition

$$|\sigma_{ij}(x)| < 0(|x|^{-1-\varepsilon}) \quad \text{as } |x| \to \infty \tag{5.1}$$

for some $\varepsilon > 0$.

We wish to study the operator determined by the right-hand side of (2.2) but where $B = i[\sigma_{ij}]$. We write $\Lambda(D, x)u = A(D)u + B(x)u$, where $A(D)$ is given by the first terms on the right side of (2.2). It is easily established that $\Lambda(D, x)$ is maximal dissipative in $L^2(\mathbb{R}^3, \mathbb{C}^{10})$. We will show that steady-state solutions of

$$-i\frac{\partial u}{\partial t} = \Lambda(D, x)u \tag{5.2}$$

exist in certain weighted spaces when the initial disturbance lies in the dual of the given weighted space. The interesting concept of "spectral barrier" arises here (see the appendix of [S3]).

We define the weighted spaces $L_{2,\alpha}(\mathbb{R}^3, \mathbb{C}^{10})$ as

$$L_{2,\alpha}(\mathbb{R}^3, \mathbb{C}^{10}) = \left\{ f \,\middle|\, \int_{\mathbb{R}^3} (1+|x|^2)^\alpha |f(x)|^2 \, dx < \infty, f : \mathbb{R}^3 \to \mathbb{C}^{10} \right\}.$$

It is noted here that the λ_j satisfy the "strongly propagative" hypothesis (λ_j is either bounded away from zero or is identically zero—see [Wi]).

The steady-state form of (5.2) at frequency λ is given by

$$A(D)u + B(x)u - \lambda u = f, \tag{5.3}$$

where f is assumed to belong to $L_{2,\alpha}$ with $\alpha > \frac{1}{2}$ and u is sought in a space $L_{2,-\beta}$, $\beta > \frac{1}{2}$. Note that $L_{2,-\beta} \supseteq L^2 \supseteq L_{2,\alpha}$. Note that B, considered as a multiplication operator, maps $L_{2,-\beta}$ to $L_{2,\alpha}$ if α and β are sufficiently close to $\frac{1}{2}$ (we will assume they are from now on). We will use the notation $\mathbb{C}^{\pm} = \{z \in \mathbb{C} \mid \pm\text{imaginary part of } z > 0\}$. Without loss of generality, we may assume $A(D) = A^3(D)$ by choice of coordinates since we are working in all of $\mathbb{R}^3$. Let $P_1 = I - {}_3P_0$, $P_0 = {}_3P_0$ in L^2. Assume $\lambda \in \mathbb{C}^{\pm}$ and operate on both sides of (5.2) with $(A(D) - \lambda I)^{-1}$ in the sense of L^2. This makes sense because $A(D)$ is self-adjoint. From (2.9) of [We] we may conclude that $P_1(A(D) - \lambda I)^{-1}$, thought of as mapping $L_{2,\alpha}$ to $L_{2,-\beta}$ is continuous in $\mathbb{C}^{\pm}$ and has continuous extensions $P_1(A(D) - \lambda I)^{-1}_{\pm}$ to the closure of $\mathbb{C}^+$ or $\mathbb{C}^-$ (i.e., down to or up to the real axis), that assume compact values as operators from $L_{2,\alpha}$ to $L_{2,-\beta}$. P_0 has a bounded extension to $L_{2,-\beta}$. We may "solve" for u now when λ is real as

$$u_{\pm}(x, \lambda) = (I - P_0(B/\lambda) + \lambda P_1(A(D) - \lambda I)^{-1}_{\pm}(B/\lambda))^{-1}(A(D) - \lambda I)^{-1}_{\pm} f.$$

The Fredholm theory (see [S4], for example) now allows us to say $u_{\pm}$ exists (in $L_{2,-\beta}$) when $|\lambda|$ is sufficiently large. (There may be some other exceptional values of λ besides the "small" values for which a solution fails to exist—these form a countable nowhere dense set of linear measure zero [S4].) The difficulty for small λ is that the operator $(I - P_0(B/\lambda))^{-1}$ may not exist. In fact, using the explicit formula for P_0 given above, it is possible to construct examples exhibiting this difficulty. $u_\pm$ exists provided λ does not belong to the set of exceptional values or to the spectrum of P_0B (the spectrum of P_0B is the "spectral barrier."

For $\sigma = \sigma(t, x)$, a similar technique can be employed. We quote the following result from [S3, Thm. 4.2], adapted to the present situation.

THEOREM 5.1. *Suppose $B(t, x)$ is measurable in (t, x) and $t \to B(t, x)$ is a continuous map from $\mathbb{R}$ to the set of bounded operators on L^2. If $|B(t, x)| \leqq C(1+|t|^{-1-\varepsilon}$ $(\varepsilon > 0)$ then for any $f(t, x)$ in the space $L_{2,\alpha}(\mathbb{R}, L^2(\mathbb{R}^3, \mathbb{C}^{10}))(\alpha > \frac{1}{2})$ there is a solution $u(t, x)$ of*

$$-i\frac{\partial u}{\partial t} = \Lambda(D, x)u + f(t, x)$$

in the space $L_{2,-\beta}(\mathbb{R}, L^2(\mathbb{R}^3, \mathbb{C}^{10}))$.

In fact, since the medium is a semiconductor, we may assume that C in the statement of Theorem 5.1 is small. In that case, the continuity hypothesis on B may be discarded (see Theorem 4.1 of [S3]).

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