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Classifcation of Conics in the Tropical Projective Plane

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CLASSIFICATION OF CONICS IN THE TROPICAL PROJECTIVE PLANE

by

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This thesis has been read by each member of the following graduate committee and by majority vote has been found to be satisfactory.

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As chair of the candidate's graduate committee, I have read the thesis of Amanda Ellis in its final form and have found that (1) its format, citations, and bibliographical style are consistent and acceptable and fulfill university and department style requirements; (2) its illustrative materials including figures, tables, and charts are in place; and (3) the final manuscript is satisfactory to the graduate committee and is ready for submission to the university library.

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ABSTRACT

CLASSIFICATION OF CONICS IN THE TROPICAL PROJECTIVE PLANE

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This paper defines tropical projective space, $\mathbb{TP}^n$, and the tropical projective general linear group $\mathbb{TPGL}(n)$. After discussing some simple examples of tropical polynomials and their hypersurfaces, a strategy is given for finding all conics in the tropical projective plane. The classification of conics and an analysis of the coefficient space corresponding to such conics is given.

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Table of Contents

1 Introduction to Tropical Mathematics

In this paper we will be considering the tropical semi-ring, $\mathcal{R} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$, as discussed by Richter-Gebert, Sturmfels, and Theobald in [1]. Here tropical addition, $\oplus$, and tropical multiplication, $\odot$, are defined to be:

$$a \odot b = a + b$$

$$a \oplus b = min\{a, b\}.$$

Note that both $\oplus$ and $\odot$ are commutative, and that some other nice properties hold: To prove that the distributive property holds, consider $a \odot (b \oplus c) = a + min\{b, c\} = min\{a + b, a + c\} = a \odot b \oplus a \odot c$. For associativity of $\oplus$, examine $a \oplus (b \oplus c) = min\{a, min\{b, c\}\} = min\{a, b, c\} = min\{min\{a, b\}, c\}$. Note that $\odot$ is associative, since it is regular addition. We can extend this definition to $\{\mathbb{R} \cup \infty\}^n$, giving a module over $\mathcal{R}$, in the following way:

Coordinatewise addition:

$$(a_1, a_2, \ldots, a_n) \oplus (b_1, b_2, \ldots, b_n) = (a_1 \oplus b_1, a_2 \oplus b_2, \ldots, a_n \oplus b_n)$$

Scalar multiplication:

$$r \odot (a_1, a_2, ..., a_n) = (r \odot a_1, r \odot a_2, \ldots, r \odot a_n)$$

We consider tropical algebraic varieties in tropical projective space $\mathbb{TP}^{n-1} = \frac{\mathbb{R}^n}{(1,1,\ldots,1)}$. This means for $r \neq \infty$, $(a_1, a_2, \ldots, a_n) \sim (r + a_1, r + a_2, \ldots, r + a_n)$, where not all n entries can be equal to ∞ (the additive identity).

Define the open set $U_i = \left\{(a_1, \ldots, a_n) \in \mathbb{TP}^{n-1} \middle| a_i \neq \infty\right\} \subset \mathbb{TP}^{n-1}$. Each point, $(a_1, a_2, \ldots, a_n) \sim (a_1 - a_i, a_2 - a_i, \ldots, a_{i-1} - a_i, 0, a_{i+1} - a_i, \ldots, a_n - a_i)$ in U_i, corresponds to the point $(a_1 - a_i, a_2 - a_i, \ldots, a_{i-1} - a_i, a_{i+1} - a_i, \ldots, a_n - a_i) \in \mathbb{A}^{n-1}$. Thus $\mathbb{TP}^{n-1} \ncong \{\mathbb{R} \cup \infty\}^{n-1}$, but $\{\mathbb{R} \cup \infty\}^{n-1} \cong U_i \subsetneq \mathbb{TP}^{n-1}$. Whenever $a_i = \infty$, it is possible to find a similar open set U_j for some coordinate $a_j \neq \infty$. Hence,

$$\mathbb{TP}^{n-1} \cong U_1 \cup U_2 \cup \ldots \cup U_n,$$

a series of affine open sets. In general, we will use the affine representation U_n in our analysis of objects in $\mathbb{TP}^{n-1}$. In particular, polynomials in variables x, y, and z have hypersurfaces in $\mathbb{TP}^2 \cong \{\mathbb{R} \cup \infty\}^2 \cup \{\mathbb{R} \cup \infty\}^2 \cup \{\mathbb{R} \cup \infty\}^2$, where each copy of $\{\mathbb{R} \cup \infty\}^2$ is the affine open set where a fixed one of the three coordinates is finite. Unless otherwise stated, the affine representation used in this paper is $U_3 = \left\{(x, y, z) \in \mathbb{TP}^2 \middle| z \neq \infty\right\}$. Note that one of the limitations of choosing an affine representation like U_3, is that it does not include any points where $a_3 = \infty$. If it is necessary to let a_3 range through ∞, we can use $\mathbb{TP}^2$ itself, in terms of ordered triples, or choose a different affine representation.

Definition 1.1. A **tropical polynomial of degree** m is an expression of the form:

$$f(x_1, \ldots, x_n) = \bigoplus_{i_1+\ldots+i_n \leq m} (a_{i_1 i_2 \ldots i_n} \odot x_1^{i_1} \ldots \odot x_n^{i_n})$$

where $a_{i_1 \ldots i_n} \in \mathbb{R} \cup \{\infty\}$ and $m \in \mathbb{Z}_+$.

Definition 1.2. For a given tropical polynomial $f(x_1, \ldots, x_n)$, the **tropical hypersurface** associated with f is defined to be the set

$$\begin{aligned} Z(f) = \{(x_1, \ldots, x_n) \big| f(x_1, \ldots, x_n) \\ = min_{i_1,\ldots,i_n}\{a_{i_1 i_2 \ldots i_n} \odot x_1^{i_1} \ldots \odot x_n^{i_n}\} \text{ is achieved by two monomials of f}\} \end{aligned}$$

To see a demonstration of finding the hypersurface of a tropical polynomial, skip to Section 3.

The following proposition is immediate.

Proposition 1.3. *For any two tropical polynomials*

$$g(x_1, \dots, x_n) = \bigoplus_{i_1+\ldots+i_n \le m} (a_{i_1 i_2 \ldots i_n} \odot x_1^{i_1} \ldots \odot x_n^{i_n}), \text{ and}$$

$$\tilde{g}(x_1, \dots, x_n) = \bigoplus_{i_1+\ldots+i_n \le m} (a_{i_1 i_2 \ldots i_n} \odot c \odot x_1^{i_1} \ldots \odot x_n^{i_n}), \text{ and fixed } c \in \mathbb{R}$$

we have $Z(g) = Z(\tilde{g})$

Since the coefficients of tropical polynomials are in the semiring, $(\mathbb{R} \cup \infty)$, consider the hypersurfaces that arise when one or more of the coefficients is equal to ∞. In the case where only one coefficient is finite, since hypersurfaces are generated by places where the minimum is achieved twice, the monomial whose coefficient is finite will always be the only minimum, unless at least one of the variables in the monomial is infinite. The zero locus of such a polynomial will be be sets wherever one or more of the variables in the monomial with finite coefficient is infinite. Now, if all but two of the coefficients are ∞, whenever the two monomials with finite coefficients are equal, that will be the minimum, and also the only place where the minimum is achieved twice. Thus all polynomials with only two finite coefficients define hypersurfaces which consist of a classical line, infinite in both directions and values where both monomials contain an infinite variable. For three or more finite coefficients, more complicated calculations are necessary.

Proposition 1.4. $Z(f \odot g) = Z(f) \cup Z(g)$

Proof. (Due to Aaron Hill)

Define two polynomials f and g with r and s monomials, respectively in the following way:

$$f(x_1, x_2, \dots, x_n) = a_1 x_1^{i_{11}} \ldots x_n^{i_{1n}} \oplus a_2 x_1^{i_{21}} \ldots x_n^{i_{2n}} \oplus \ldots \oplus a_r x_1^{i_{r1}} \ldots x_n^{i_{rn}}, \text{ and}$$

$$g(x_1, x_2, \dots, x_n) = b_1 x_1^{j_{11}} \ldots x_n^{j_{1n}} \oplus b_2 x_1^{j_{21}} \ldots x_n^{j_{2n}} \oplus \ldots \oplus b_s x_1^{j_{s1}} \ldots x_n^{j_{sn}}$$

First we will show that $Z(f \odot g) \subseteq Z(f) \cup Z(g)$. Let $p = (p_1, p_2, \dots, p_n) \in Z(f \odot g)$. Then there are two terms of $f \odot g$ which achieve the minimum and are equal to each other, that is

$$a_k p_1^{i_{k1}} \dots p_n^{i_{kn}} \odot b_l p_1^{j_{l1}} \dots p_n^{j_{ln}} = a_m p_1^{i_{m1}} \dots p_n^{i_{mn}} \odot b_q p_1^{j_{q1}} \dots p_n^{j_{qn}} \tag{1.1}$$

Suppose that $a_k p_1^{i_{k1}} \dots p_n^{i_{kr}} \neq a_m p_1^{i_{m1}} \dots p_n^{i_{mr}}$. Then without loss of generality $a_k p_1^{i_{k1}} \dots p_n^{i_{kr}} < a_m p_1^{i_{m1}} \dots p_n^{i_{mr}}$. This implies that $b_l p_1^{j_{l1}} \dots p_s^{j_{ls}} > b_q p_1^{j_{q1}} \dots p_s^{j_{qs}}$. But consider the term $a_k p_1^{i_{k1}} \dots p_n^{i_{kr}} \odot b_q p_1^{j_{q1}} \dots p_s^{j_{qs}}$. This term would then be smaller than the terms listed in Equation (1.1). Hence, $a_k p_1^{i_{k1}} \dots p_n^{i_{kr}} = a_m p_1^{i_{m1}} \dots p_n^{i_{mr}}$, and $b_l p_1^{j_{l1}} \dots p_s^{j_{ls}} = b_q p_1^{j_{q1}} \dots p_s^{j_{qs}}$. Note that if $k = m$, and $l = q$, then the two terms listed in Equation (1.1) are really the same term, and hence the minimum is only achieved once. Thus, either $k \neq m$ in which case $p \in Z(f)$, or $l \neq q$ in which case $p \in Z(g)$. We have shown that $Z(f \odot g) \subseteq Z(f) \cup Z(g)$.

For the other direction, suppose without loss of generality that $p \in Z(f)$. Then for some k and l, $a_k p_1^{i_{k1}} \dots p_n^{i_{kn}} = a_l p_1^{i_{l1}} \dots p_n^{i_{ln}}$, and this quantity must be less than all the other monomials of f evaluated at p. Note that for some m, $b_m p_1^{j_{m1}} \dots p_n^{j_{mn}} = g(p)$. Then $b_m p_1^{j_{m1}} \dots p_n^{j_{mn}} \odot a_k p_1^{i_{k1}} \dots p_n^{i_{kn}} = b_m p_1^{j_{m1}} \dots p_n^{j_{mn}} \odot a_l p_1^{i_{l1}} \dots p_n^{i_{ln}}$ and this product is less than the other monomials of $f \odot g$ evaluated at p. Thus $p \in Z(f \odot g)$ implies $Z(f) \cup Z(g) \subseteq Z(f \odot g)$, in fact $Z(f \odot g) = Z(f) \cup Z(g)$.

□

2 Linear Morphisms

Definition 2.1. Let $X \subseteq \mathbb{TP}^n$ and $Y \subseteq \mathbb{TP}^m$. A set theoretical function $f : X \to Y$ is a tropical linear morphism if for

$$f(x_0, x_1, \cdots, x_n) = (y_0, y_1, \cdots, y_m) \in Y$$

each coordinate y_j can be represented as a tropical linear polynomial in $x_0, x_1, \dots x_n$.

Note that, as in the regular case, if a function is a tropical linear morphism from

$\mathbb{TP}^n$ to $\mathbb{TP}^m$, it can be represented by an $(m+1) \times (n+1)$ matrix with coefficients in $\mathcal{R}$.

Definition 2.2. $\mathbb{TPGL}(n)$ is the set of invertible tropical linear morphisms in $\mathbb{TP}^{n-1}$.

Note here that $\mathbb{TPGL}(n)$ corresponds to those tropical linear morphisms whose matrix representations are invertible.

Define the Identity matrix to be the matrix that has the multiplicative ($\odot$) identity, 0, along the diagonal, and the additiive ($\oplus$) identity, ∞, everywhere else.

$$I : \begin{bmatrix} 0 & \infty & \infty & \dots & \infty \\ \infty & 0 & \infty & \dots & \infty \\ \infty & \infty & 0 & \dots & \infty \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \infty & \infty & \dots & \infty & 0 \end{bmatrix}$$

It is easy to verify that this matrix does indeed act as the identity.

One type of morphism in $\mathbb{TPGL}(n)$ consists of translations. They take the algebraic form:

$$\begin{aligned} f : (a_1, a_2, \cdots, a_n) &\mapsto (c_1 \odot a_1, c_2 \odot a_2, \cdots, c_n \odot a_n) \\ &= (c_1 + a_1, c_2 + a_2, \cdots, c_n + a_n) \end{aligned} \tag{2.1}$$

Let us call this group of morphisms T, and the translation described in Equation (2.1) we will call $f_{(c_1,\dots,c_n)}$. It is easy to see that these morphisms are invertible by taking the translation $f_{(-c_1,-c_2,\dots,-c_n)}$. This gives

Proposition 2.3. $T \subseteq \mathbb{TPGL}(n)$.

Note that a translation as in Equation (2.1) has a unique matrix representation of the form

$$A_f : \begin{bmatrix} c_1 & \infty & \infty & \dots & \infty \\ \infty & c_2 & \infty & \dots & \infty \\ \infty & \infty & c_3 & \dots & \infty \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \infty & \infty & \dots & \infty & c_n \end{bmatrix}$$

Another type of morphism in $\mathbb{TPGL}(n)$ is the set of permutations of the coordinates $(a_1, a_2, \cdots, a_n) \in \mathbb{TP}^{n-1}$. That is,

$$\sigma : (a_1, \cdots, a_n) \mapsto (a_{\sigma(1)}, a_{\sigma(2)}, \cdots, a_{\sigma(n)}) \tag{2.2}$$

This defines a subgroup of $\mathbb{TPGL}(n)$, isomorphic to the symmetric group, which we denote by S_n. Again, it is easy to see that these morphisms are invertible by taking the permuation morphism defined by the permutation σ^{-1}. This gives

Proposition 2.4. $S_n \subseteq \mathbb{TPGL}(n)$

Note that these also have a unique matrix representation $A_\sigma = (a_{ij})$ defined by $a_{ij} = 0$ whenever $j = \sigma(i)$ and $a_{ij} = \infty$ everywhere else.

Proposition 2.5. $\mathbb{TPGL}(n) = S_n T$. *That is, any element of* $\mathbb{TPGL}(n)$ *can be written* st*, where* $s \in S_n$ *and* $t \in T$.

Proof. If σ is a linear morphism taking $\mathbb{TP}^n$ to $\mathbb{TP}^m$, then it is equivalent to an $m \times n$ matrix in $(\mathbb{R}, \oplus, \odot)$. We claim that the set of invertible $n \times n$ matrices consist of exactly those which have exactly one finite entry in every row and column. If this claim is true, then the set of invertible matrices are all combinations of permutation matrices and diagonal matrices, which gives the result we want.

Proof of Claim: Let $A = [a_{ij}]$ be an invertible $n \times n$ matrix, and let $A' = [a'_{ij}]$ be its inverse. First let's suppose that for some row i_0 in A there is no finite entry, then we know that

$$A \odot A' = I, \text{ so } I_{ij} = \bigoplus_{k=1}^{n} (a_{ik} \odot a'_{kj}), \tag{2.3}$$

thus $I_{i_0,i_0} = 0 = min_{\{k=1,\ldots,n\}}(a_{i_0k} + a_{ki_0}) = \infty$ gives us a contradiction. Thus at least one entry in each row must be finite. A similar argument with $A' \times A = I$ gives us the same result for columns. Now Suppose that for some column j_0 in A, there are two finite entries, a_{i_1,j_0}, and a_{i_2,j_0}. Using Equation (2.1) we know that for all $i \neq j, min_k\{a_{ik} + a'_{kj}\} = \infty$ implies $a_{ik} + a_{kj} = \infty$. Thus for all $j \neq i_1$ we have $a_{i_1j_0} + a'_{j_0j} = \infty$ implies $a'_{j_0j} = \infty$ and for all $j \neq i_2$ we have $a'j_0j = \infty$. Since, by hypothesis, $i_1 \neq i_2$. We have an entire row, j_0, in A', which is infinite. A' is invertible gives us a contradiction, thus each column must have exactly one finite entry. Suppose that some row, i_0 in A has two finite entries, a_{i_0,j_1}, and a_{i_0,j_2}. Note that each of the other $n-1$ rows there is at least one finite entry. These $n-1$ or more finite entries can fall in the $n-2$ empty columns, or in columns j_1, or j_2. By the Pigeonhole Principle, there will be some column which will have more than one finite entry. But we know that an invertible matrix has only one finite entry in each column, so we have our result. □

Consider what happens to objects in $\mathbb{TP}^2$ when their coordinates are permuted.

$$\sigma : (a_1, a_2, a_3) \mapsto (a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$$

Whenever $a_3 \neq \infty$ this corresponds to an action on $\mathbb{R}^2$, which we shall also denote σ:

$$\sigma : (a_1 - a_3, a_2 - a_3) \mapsto (a_{\sigma(1)} - a_{\sigma(3)}, a_{\sigma(2)} - a_{\sigma(3)}).$$

If $a_3 = \infty$, it is just as easy to pick one of the other coordinates (recall that not all can be infinite). Suppose, for example, that $a_2 \neq \infty$, then the affine representation would be

$$\sigma : (a_1 - a_2, a_3 - a_2) \mapsto (a_{\sigma(1)} - a_{\sigma(2)}, a_{\sigma(3)} - a_{\sigma(2)}).$$

If $a_2 = a_3 = \infty$ then $a_1 \neq \infty$. Throughout the remainder of this paper, let σ be the permutation (12), and let τ be (13). Then in the case where $a_3 \neq \infty$,

$$\sigma(a_1 - a_3, a_2 - a_3) = (a_2 - a_3, a_1 - a_3)$$

$$\tau(a_1 - a_3, a_2 - a_3) = (a_3 - a_1, a_2 - a_1)$$

There is a unique matrix representation for the permutation group S_3 in terms of these two homomorphisms, since they generate it. Below are the elements of the permutation group S_3 as it acts on $\mathbb{R}^2 \cong U_3$ and their corresponding matrices.

$$\sigma : \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma\tau : \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \qquad 1 : \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\tau : \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \qquad \tau\sigma : \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \qquad \tau\sigma\tau : \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

Note that $(\sigma)^2 = (\tau)^2 = (\tau\sigma\tau)^2 = 1$ while $(\tau\sigma)^3 = (\sigma\tau)^3 = 1$.

Proposition 2.6. *The group of translations is a normal subgroup in the tropical linear group* $\mathbb{TPGL}(n)$*.*

Proof. Consider a translation $f_{(c_1,c_2,\ldots,c_n)}$. By Proposition 2.5, any element $\phi \in \mathbb{TPGL}(n) = \rho g$ where $\rho \in S_n$, and $g \in T$. Thus we need to show that $\rho g f g^{-1} \rho^{-1} \in T$. Note that $g f g^{-1} \in T$ since T is a subgroup of $\mathbb{TPGL}(n)$. It will suffice to show that $\rho f \rho^{-1}$ is in the group of translations. Note that

$$\begin{aligned}
&\rho f_{(c_1,c_2,\ldots,c_n)} \rho^{-1}(a_1, a_2, \ldots, a_n) \\
=&\rho f_{(c_1,c_2,\ldots,c_n)}(a_{\rho^{-1}(1)}, a_{\rho^{-1}(2)}, \ldots, a_{\rho^{-1}(n)}) \\
=&\rho(a_{\rho^{-1}(1)} + c_1, a_{\rho^{-1}(2)} + c_2, \ldots, a_{\rho^{-1}(n)} + c_n) \\
=&(a_1 + c_{\rho(1)}, a_2 + c_{\rho(2)}, \ldots, a_n + c_{\rho(n)}) \\
=&f_{(c_{\rho(1)},c_{\rho(2)},\ldots,c_{\rho(n)})}(a_1, a_2, \ldots, a_n) \in T.
\end{aligned}$$

□

Corollary 2.7. $\mathbb{TPGL}(n) = T \rtimes S_n$.

At this point it will be interesting to consider what the permutation group does to simple lines and segments in $\mathbb{R}^2$, as they will become important in the later discussion of tropical lines and conics.

Proposition 2.8. *σ and τ take segments in the first set of columns to segments in the second set of columns, as summarized by Table 1 on the following page.*

Table 1: Images of segments under permutation generators

SLOPE	ENDPOINTS		SLOPE	ENDPOINTS	
σ takes:			to:		
∞	(a,b)	(a,c)	0	(b,a)	(c,a)
0	(a,b)	(c,b)	∞	(b,a)	(b,c)
1	(a,b)	(a+c,b+c)	1	(b,a)	(b+c,a+c).
2	(a,b)	(a+c, b+2c)	$\frac{1}{2}$	(b,a)	(b+2c,a+c)
$\frac{1}{2}$	(a,b)	(a+2c,b+c)	2	(b,a)	(b+c, a+2c)
-1	(a,b)	(a+c,b-c)	-1	(b,a)	(b-c,a+c).
τ takes			to:		
∞	(a,b)	(a,c)	∞	(-a,b-a)	(-a,c-a)
0	(a,b)	(c,b)	1	(-a,b-a)	(-c,b-c)
1	(a,b)	(a+c,b+c)	0	(-a,b-a)	(-a-c,b-a)
2	(a,b)	(a+c,b+2c)	-1	(-a,b-a)	(-a-c,b+c-a)
$\frac{1}{2}$	(a,b)	(a+2c,b+c)	$\frac{1}{2}$	(-a,b-a)	(-a-2c,b-a-c)
-1	(a,b)	(a+c,b-c)	2	(-a,b-a)	(-a-c,b-a-2c)

It is easy an easy exercise to verify the above. Note that this table also gives the actions of each generator on lines and half rays as well, by letting the endpoints range through infinity.

3 Lines in $\mathbb{TP}^2$

Consider tropical polynomials of degree one:

$$f(x, y, z) = a \odot x \oplus b \odot y \oplus c \odot z. \tag{3.1}$$

Unlike in the affine case, in $\mathbb{TP}^n$ tropical polynomials must be homogeneous in order for the definition of their hypersurface to make sense. In fact, Proposition 1.3 shows that while homogeneous tropical polynomials do not have a well-defined value, they do have well-defined hypersurfaces. Recall that the zero locus $Z(f)$ of this linear expression consists of those ordered triples (x, y, z), where the value of the polynomial is achieved by more than one of the monomials of f. This could be written:

$$\begin{aligned} Z(f) = &\{(x,y,z) \big| a + x = b + y \le c + z, \\ \text{or} \quad & \quad b + y = c + z \le a + x, \\ \text{or} \quad & \quad a + x = c + z \le b + y\}. \end{aligned} \tag{3.2}$$

Recall from Section 1 that if two of the coefficients are infinite (only one is finite), then $Z(f)$ will consist of all points where the variable in the monomial with finite coefficient is infinite, that is

$$\text{Case 1a: if } a = b = \infty, \text{ then } Z(f) = \{(x, y, \infty)\}$$

$$\text{Case 1b: if } a = c = \infty, \text{ then } Z(f) = \{(x, \infty, z)\}$$

$$\text{Case 1c: if } b = c = \infty, \text{ then } Z(f) = \{(\infty, y, z)\}$$

If one coefficient is infinite (two are finite), then $Z(f)$ is the line formed by equality of the two monomials with finite coefficients, that is,

$$\text{Case 2a: if } a = \infty \text{ then } Z(f) = \{(x, y, z) \big| b + y = c + z\}$$

$$\text{Case 2b: if } b = \infty \text{ then } Z(f) = \{(x, y, z) \big| a + x = c + z\}$$

$$\text{Case 2c: if } c = \infty \text{ then } Z(f) = \{(x, y, z) \big| a + x = b + y\}$$

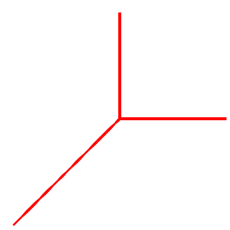
Otherwise all three coefficients are finite. Note that

$$\{z = \infty\} \cap Z(f) = \{(x, y, \infty) \big| a + x = b + y\}.$$

Otherwise $z \neq \infty$ in which case we may tropically divide (which is the same as classical subtraction) all equations by z. With the substitutions $x' = x - z$ and $y' = y - z$, set (3.2) can now be written:

$$U_3 \cap Z(f) = \{z \neq \infty\} \cap Z(f) = \{(x', y') | a + x' = b + y' \leq c,$$
$$\text{or} \qquad b + y' = c \leq a + x',$$
$$\text{or} \qquad a + x' = c \leq b + y'\}$$

Then $Z(f) \cap U_1 \cap U_2 \cap U_3$ is easily plotted as the following "tropical line" with vertex $(c-a, c-b)$ in $\mathbb{R}^2$

Case 3: 

3.1 $\mathbb{TP}^2$ as the Coefficient Space of Lines

In the case of such linear polynomials, since we don't permit the case where all three coefficients are infinite, and by Proposition 1.3 the polynomials with coefficients (a, b, c) and $(a + d, b + d, c + d)$ define the same hypersurface, the space of coefficients (a, b, c) is $\mathbb{TP}^2$. The hypersurfaces associated with these linear polynomials will be called **tropical lines**. Since $\mathbb{TPGL}(3)$ acts on $\mathbb{TP}^2$, it acts on both the coefficient space and the set of lines in $\mathbb{TP}^2$. Consider the diagram

$$\begin{array}{ccc} \mathbb{TP}^2 & \rightarrow & \{Lines\} \\ \downarrow & & \downarrow \\ \mathbb{TP}^2/T & \cong & \{Lines\}/T \\ \downarrow & & \downarrow \\ \mathbb{TP}^2/\mathbb{TPGL}(3) & \cong & \{Lines\}/\mathbb{TPGL}(3) \end{array}$$

Each piece of the above diagram will be discussed presently.

Consider first

$$\mathbb{TP}^2/T \cong \{\{(a,b,c)\} \setminus \{(\infty,\infty,\infty)\}/\{\odot(d_1,d_2,d_3)\},$$

where d_1, d_2, and $d_3 \in \mathbb{R}$. This gives

$$(a,b,c) \sim (a+d_1, b+d_2, c+d_3)$$

The coefficient space $\mathbb{TP}^2$ consists of points of the form (a,b,c), (∞,b,c), (a,∞,c), (a,b,∞), (∞,∞,c), (∞,b,∞), and (a,∞,∞), where a, b, $c \in \mathbb{R}$. Under the substitutions $d_1 = -a$, $d_2 = -b$, and $d_3 = -c$, the preceding triples are equivalent to the seven points $(0,0,0)$, $(\infty,0,0)$, $(0,\infty,0)$, $(0,0,\infty)$, $(\infty,\infty,0)$, $(\infty,0,\infty)$, and $(0,\infty,\infty)$. Thus the quotient $\mathbb{TP}^2/T$ is the set of points

$$\{(0,0,0),(\infty,0,0),(0,\infty,0),(0,0,\infty),(\infty,\infty,0),(\infty,0,\infty),(0,\infty,\infty)\}.$$

Now consider $\mathbb{TP}^2/\mathbb{TPGL}(3) \cong \mathbb{TP}^2/\{T \rtimes S_3\}$. Under S_3, $(\infty,0,0) \sim (0,\infty,0) \sim (0,0,\infty)$, and $(\infty,\infty,0) \sim (\infty,0,\infty) \sim (0,\infty,\infty)$. So the quotient $\mathbb{TP}^2/\mathbb{TPGL}(3)$ consists of exactly three distinct points,

$$\{(0,0,0),(\infty,0,0),(\infty,\infty,0)\}$$

Next, note that the set of tropical lines, $\{Lines\}$, consists of seven basic types, listed as Cases 1a, 1b, 1c, 2a, 2b, 2c and 3. Consider the group $\{Lines\}/T$. For any Case 3 conic there is some element in T that will move the conic so that its vertex is on the origin. Thus all Case 3 conics are in the same orbit under T. Next, consider one of the Case 2 conics. Again, some element of T will shift such a conic so that it passes through the origin. This means Case 2a, 2b, and 2c each correspond to one coset in $\{Lines\}/T$. Finally note that the conics in Case 1 are fixed under elements of T. So, each of those three lines are in their own orbit. Thus,

$\{Lines\}/T \cong$

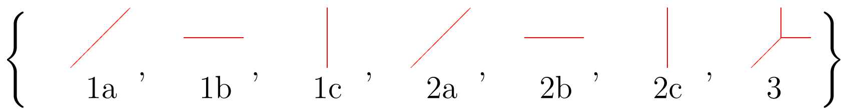
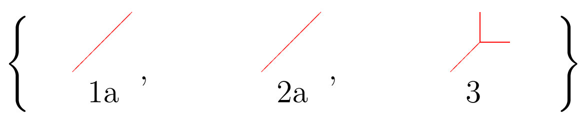


Table 1 tells us that under S_3,

$$\text{Case 1a} \sim \text{Case 1b} \sim \text{Case 1c}\ , \text{ and } \text{Case 2a} \sim \text{Case 2b} \sim \text{Case 2c}\ .$$

Thus

$\{Lines\}/\mathbb{TPGL}(3) \cong$



4 Algorithm for Computing Dual Graphs of Hypersurfaces

In this section we will outline a method for finding the combinatorial structure the hypersurface of a tropical polynomial. We will use this method to classify combinatorial types of hypersurfaces. In order to proceed we first must define polytopes. There are two standard ways of doing this.

Definition 4.1. A polyhedron in $\mathbb{R}^n$ is any set obtained as the intersection of finitely many halfspaces in $\mathbb{R}^n$. Mathematically, it has the form $P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$ where $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$.

Definition 4.2. A bounded polyhedron (as described above) is called a **polytope**.

Alternatively we could define polytopes in the following way:

Definition 4.3. A **convex combination** of $\mathbf{p}_1, \mathbf{p}_2, \ldots \mathbf{p}_n \in \mathbb{R}^n$ is any point of the form $\sum_{i=1}^{t} \lambda_i \mathbf{p}_i$ where $\lambda_i \geq 0$, for all $i = 1, \ldots, t$ and $\sum_{i=1}^{t} \lambda_i = 1$. The set of all convex combinations of $\mathbf{p}_1, \ldots, \mathbf{p}_n$ is called their **convex hull**. We denote it as $conv(\{\mathbf{p}_1, \ldots, \mathbf{p}_n\})$.

Proposition 4.4. *A polytope in $\mathbb{R}^n$ is the convex hull of a finite number of points in $\mathbb{R}^n$. Mathematically, it is a set of the form $P = conv(\{\mathbf{p}_1, \ldots, \mathbf{p}_n\})$.*

Definition 4.5. For a polytope P in $\mathbb{R}^n$, a point $p = (x_1, \ldots, x_n)$ of P, is in the **lower envelope** of P if the line between $p = (x_1, x_2, \ldots, x_n)$ and $p_0 = (0, x_2, x_3, \ldots, x_n)$ intersects P only at p.

Here is a method given in [1] for computing the combinatorial structure of the finite representations of hypersurfaces associated with the polynomial of degree m

$$f(x, y) = \bigoplus_{i+j \leq m} a_{i,j} \odot x^i \odot y^j,$$

where $a_{i,j} \in \mathbb{R} \cup \{\infty\}$. For each monomial, $a_{i,j} \odot x^i \odot y^j$ in $f(x, y)$ create the ordered triple $(i, j, a_{i,j})$. Then take the lower envelope of the convex hull of the set of such triples, and project this lower envelope onto the xy-plane. This will be a regular subdivision of the convex hull of the set of ordered pairs (i, j) in $\mathbb{R}^2$. This subdivision Δ is a dual graph to the hypersurface of the original expression. That is, the hypersurface will contain one ray or segment for every segment in Δ, each with slope perpendicular to the segment in Δ.

Consider what happens when we use this dual graph algorithm in the case of a polynomial with one or more infinite coefficient. If a coefficient is infinite, neither the vertex corresponding to that coefficient, nor any lines touching this vertex will appear in the dual graph.

Recall that the vertex, which gives us the location of a line, is given in terms of the coefficients a_i. The dual graph method outlined above, however, "forgets" these coefficients when the lower envelope of the convex hull is projected onto the x-y plane.

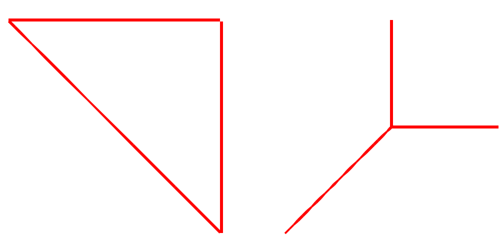
These coefficients not only determine location, but leg lengths as well in the cases of hypersurfaces with finite segments. The dual graph algorithm, then, gives us away to find or determine only the combinatorial types of hypersurfaces.

The proof for this algorithm is given for *max*-plus algebra in [2]. However, we want our result in tropical, or *min*-plus algebra, we must consider the fact that $min(a, b) = -max(-a, -b)$. Thus, to find the dual graph to our hypersurface, we actually want to consider the same subdivision except on the indices $(-i, -j)$, or in other words, taking x to $-x$ and y to $-y$.

Because Δ is defined as the lower envelope of a convex hull of points, which is a regular subdivision of a convex polytope, we have the following proposition.

Proposition 4.6. *The line segments composing the dual graph Δ of a tropical polynomial can cross only at the vertices of the graph, and any non-convex polytope in Δ must be the union of convex polytopes.*

Let us compute the dual graph for the the affine tropical line associated with the linear expression $a_1 \odot x \oplus a_2 \odot y \oplus a_3$, where each $a_i < \infty$. We start with the ordered triples $(a_1, 1, 0), (a_2, 0, 1),$ and $(a_3, 0, 0)$ yield the subdivision:



It is easy to verify that this is the dual graph to the tropical line with all finite coefficients. Note that the dual graphs generated in this algorithm are preserved under the translation group T, since they give the only the combinatorial structure of the hypersurface and not its location. The elements of S_3 act on Δ in a way similar to how we think of the dihedral group acting on an equilateral triangle. Our underlying set in Δ is not regular, but is equivalent to an equilateral triangle. σ exchanges the x and y coordinates in the graph, τ exchanges the sides with slopes 0 and -1, $\tau\sigma\tau$ exchanges sides with slopes ∞ and -1. $\sigma\tau$ moves all segments one rotation clockwise, and $\tau\sigma$ moves all segments one rotation counter clockwise. Thus, once we have one dual graph giving us a particular class of hypersurfaces, we can easily find the orbit of these classes under $\mathbb{TPGL}(3)$.

5 Conics

We now want to study tropical polynomials of degree two:

$$f(x_1, x_2, x_3) = ax_1^2 \oplus bx_1x_2 \oplus cx_2^2 \oplus dx_1x_3 \oplus ex_2x_3 \oplus fx_3^2 \tag{5.1}$$

and their hypersurfaces in $\mathbb{TP}^2$, which are called **conics**. The space of conic coefficients (a, b, c, d, e, f) naturally corresponds to the space $\mathbb{TP}^5$ by Proposition 1.3.

5.1 $\mathbb{TP}^5$ as the Coefficient Space of Conics

The group $\mathbb{TPGL}(3)$ acts on the coefficient space $\mathbb{TP}^5$, as well as on the conics themselves. As in the case of lines, the group of translations simply translates the picture of the conic in $\mathbb{R}^2$ while preserving its combinatorial structure.

Proposition 5.1. *The translation* $g_{(r_1,r_2,r_3)} \in T \subset \mathbb{TPGL}(3)$ *takes an element* $(a, b, c, d, e, f) \in \mathbb{TP}^5$ *to the element*

$$(a + 2r_1, b + r_1 + r_2, c + 2r_2, d + r_1 + r_3, e + r_2 + r_3, f + 2r_3).$$

Proof. Note that $g_{(r_1,r_2,r_3)}$ takes (x_1, x_2, x_3) to $(x_1 + r_1, x_2 + r_2, x_3 + r_3)$. This means that it takes the tropical polynomial $f(x_1, x_2, x_3)$ to

$$f(x_1 + r_1, x_2 + r_2, x_3 + r_3)$$

$$= \min\{a + 2x_1 + 2r_1, b + x_1 + x_2 + r_1 + r_2, c + 2x_2 + 2r_2,$$

$$d + x_1 + x_3 + r_1 + r_3, e + x_2 + x_3 + r_2 + r_3, f + 2x_3 + 2r_3\}$$

$$= (a + 2d_1) \odot x_1^2 \oplus (b + r_1 + r_2) \odot x_1 \odot x_2 \oplus (c + 2r_2) \odot x_2^2$$

$$\oplus (d + r_1 + r_3) \odot x_1 \odot x_3 \oplus (e + r_2 + r_3) \odot x_2 \odot x_3 \oplus (f + 2r_3) \odot x_3^2$$

Thus

$$g_{(r_1,r_2,r_3)}(a, b, c, d, e, f) = (a + 2r_1, b + r_1 + r_2, c + 2r_2, d + r_1 + r_3, e + r_2 + r_3, f + 2r_3).$$

□

What about for S_3? The way in which the elements of S_3 act on the dual graphs of tropical polynomials has already been discussed, so combinatorially they will act the same way on the conics themselves. Table 1 tells how exactly the legs of a particular conic will be permuted and shifted under a permutation. It remains to consider how S_3 acts on the coefficients of a conic. Using a similar argument as in the proof of the last proposition, we have the following proposition:

Proposition 5.2. *The action of S_3 on $\mathbb{TP}^5$ is given as follows,*

$$\sigma(a, b, c, d, e, f) = (c, b, a, e, d, f)$$

$$\tau(a, b, c, d, e, f) = (f, e, c, d, b, a)$$

$$\tau\sigma\tau(a, b, c, d, e, f) = (a, d, f, b, e, c)$$

$$\tau\sigma(a, b, c, d, e, f) = (f, d, a, e, b, c)$$

$$\sigma\tau(a, b, c, d, e, f) = (c, e, f, b, d, a).$$

Propositions 5.2 and 5.1 completely describe the orbits of points in $\mathbb{TP}^5$ under the action of T and S_3 and hence $\mathbb{TPGL}(3)$.

As already described, there is a map which takes a point in $\mathbb{TP}^5$ to a point in $\mathcal{C}$, the space of all conics. This section will discuss each of these spaces as well as their quotient spaces when identified by either the group of translations, T, or the full tropical linear group $\mathbb{TPGL}(3)$. A diagram for this relationship is given on the following page.

Table 3:

$$\begin{array}{ccc} \mathbb{TP}^5 & \to & \mathcal{C} \\ \downarrow & & \downarrow \\ \mathbb{TP}^5/T & \to & \mathcal{C}/T \\ \downarrow & & \downarrow \\ \mathbb{TP}^5/\mathbb{TPGL}(3) & \to & \mathcal{C}/\mathbb{TPGL}(3) \end{array}$$

Consider first

$$\mathbb{TP}^5/T = \{(a,b,c,d,e,f)\}/\{\odot(2r_1, r_1+r_2, 2r_2, r_1+r_3, r_2+r_3, 2r_3)\}, \qquad (5.2)$$

where $r_1, r_2, r_3 \in \mathbb{R}$. Equation (5.2) gives us the equivalence:

$$(a,b,c,d,e,f) \sim (a+2r_1, b+r_1+r_2, c+2r_2, d+r_1+r_3, e+r_2+r_3, f+2r_3)$$

Recall the stipulation made in Section 1 that not all coordinates $a,b,c,d,e,$ and f can be infinite. Let U_b be the set where $b \neq \infty$, let $r_1 = -b - r_2$. Then

$$(a,b,c,d,e,f) \sim (a-2b-2r_2, 0, c+2r_2, d-b-r_2+r_3, e+r_2+r_3, f+2r_3)$$

Next, if $a=c=d=e=f=\infty$, then this is the point $(\infty,0,\infty,\infty,\infty,\infty)$. Otherwise, define the set $U_{bd} \subset U_b$ to be the subset where $d \neq \infty$, then let $r_2 = d - b + r_3$. Then

$$(a,b,c,d,e,f) \sim (a-2d-2r_3, 0, c+2d-2b+2r_3, 0, e+d-b+2r_3, f+2r_3)$$

Now, if $a=c=e=f=\infty$ then this is the point $(\infty,0,\infty,0,\infty,\infty)$. Otherwise, define the set $U_{bde} \subset U_{bd}$ to be the subset where $e \neq \infty$, and let $2r_3 = b-d-e$. Then

$$(a,b,c,d,e,f) \sim (a+e-d-b, 0, c+d-b-e, 0, 0, f-d-e+b)$$

Then

$$\begin{aligned} \mathbb{TP}^5/T \supsetneq U_b &\supsetneq \{(\infty,0,\infty,\infty,\infty,\infty)\} \cup U_{bd} \\ &\supsetneq \{(\infty,0,\infty,\infty,\infty,\infty)\} \cup \{(\infty,0,\infty,0,\infty,\infty)\} \cup U_{bde} \\ &\supsetneq \{(\infty,0,\infty,\infty,\infty,\infty)\} \cup \{(\infty,0,\infty,0,\infty,\infty)\} \cup \{\mathbb{R} \cup \infty\}^3. \end{aligned}$$

It is possible to carry out this calculation for all of the 120 possible permutations of three letters with six choices. This gives $\mathbb{TP}^5/T$ as the union of open sets and problematic point sets where three or more of the coordinates are infinite. As it turns out, depending on the choice of variables to eliminate, there may be entire lines that are problematic. One such example is given here:

This time start with the first coordinate: Let $U_a \subset \mathbb{TP}^5/T$ be the subset where $a \neq \infty$ then let $2r_1 = -a$, then

$$(a,b,c,d,e,f) \sim (0, b - \frac{1}{2}a + r_2, c + 2r_2, d - \frac{1}{2}a + r_3, e + r_2 + r_3, f + 2r_3)$$

Now if $b = c = d = e = f = \infty$ then this is the point $(0,\infty,\infty,\infty,\infty,\infty)$. Otherwise, Let $U_{ab} \subset U_a$ where $b \neq \infty$, then let $2r_2 = a - 2b$. Then,

$$(a,b,c,d,e,f) \sim (0, 0, c + a - 2b, d - \frac{1}{2}a + r_3, e + \frac{1}{2}a - b + r_3, f + 2r_3)$$

Notice that if $d = e = f = \infty$ then this is the entire line $\{(0,0,c,\infty,\infty,\infty)\big|c \in \mathbb{R}\cup\infty\}$. Otherwise, Let $U_{abd} \subset U_{ab}$ be the subset where $d \neq \infty$, and let $2r_3 = a - 2d$. Then,

$$(a,b,c,d,e,f) \sim (0,0,c+a-2b,0,e+a-b-d,f+a-2d)$$

This gives

$$\begin{aligned}\mathbb{TP}^5/T \supsetneq U_a &\supsetneq \{(0,\infty,\infty,\infty,\infty,\infty)\} \cup U_{ab}\\ &\supsetneq \{(0,\infty,\infty,\infty,\infty,\infty)\} \cup \{(0,0,c,\infty,\infty,\infty)\big|c \in \mathbb{R}\cup\infty\} \cup U_{abd}\\ &\supsetneq \{(0,\infty,\infty,\infty,\infty,\infty)\} \cup \{(0,0,c,\infty,\infty,\infty)\big|c \in \mathbb{R}\cup\infty\} \cup \{\mathbb{R}\cup\infty\}^3.\end{aligned}$$

Similar computations for all 120 combinations of these three-dimensional open sets, give the following proposition:

Proposition 5.3. $\mathbb{TP}^5/T$ *can be described as the union of seventeen copies of* $\{\mathbb{R}\cup\infty\}^3$*, twelve points, and nine lines. Specifically:*

$$U_{abc} \cup U_{abe} \cup U_{abf} \cup U_{acd} \cup U_{ace} \cup U_{acf} \cup U_{ade} \cup U_{adf} \cup U_{aef} \cup$$
$$U_{bcd} \cup U_{bcf} \cup U_{bde} \cup U_{bdf} \cup U_{bef} \cup U_{cde} \cup U_{cdf} \cup U_{cef} \cup$$
$$\{(0,\infty,\infty,\infty,\infty,\infty)\} \cup \{(\infty,0,\infty,\infty,\infty,\infty)\} \cup \{(\infty,\infty,0,\infty,\infty,\infty)\} \cup$$
$$\{(\infty,\infty,\infty,0,\infty,\infty)\} \cup \{(\infty,\infty,\infty,\infty,0,\infty)\} \cup \{(\infty,\infty,\infty,\infty,\infty,0)\} \cup$$
$$\{(0,\infty,\infty,\infty,0,\infty)\} \cup \{(\infty,0,\infty,0,\infty,\infty)\} \cup \{(\infty,0,\infty,\infty,0,\infty)\} \cup$$
$$\{(\infty,0,\infty,\infty,\infty,0)\} \cup \{(\infty,\infty,0,0,\infty,\infty)\} \cup \{(\infty,\infty,\infty,0,0,\infty)\} \cup$$
$$\{(0,0,c,\infty,\infty,\infty)\big|c \in \mathbb{R} \cup \infty\} \cup \{(0,c,0,\infty,\infty,\infty)\big|c \in \mathbb{R} \cup \infty\} \cup$$
$$\{(c,0,0,\infty,\infty,\infty)\big|c \in \mathbb{R} \cup \infty\} \cup \{(0,\infty,\infty,0,\infty,c)\big|c \in \mathbb{R} \cup \infty\} \cup$$
$$\{(0,\infty,\infty,c,\infty,0)\big|c \in \mathbb{R} \cup \infty\} \cup \{(c,\infty,\infty,0,\infty,0)\big|c \in \mathbb{R} \cup \infty\} \cup$$
$$\{(\infty,\infty,0,\infty,0,c)\big|c \in \mathbb{R} \cup \infty\} \cup \{(\infty,\infty,0,\infty,c,0)\big|c \in \mathbb{R} \cup \infty\} \cup$$
$$\{(\infty,\infty,c,\infty,0,0)\big|c \in \mathbb{R} \cup \infty\}.$$

Proposition 5.2 gives the orbits of each of the sets in the above proposition under S_3. Using that information, the following proposition follows from straightforward arguments.

Proposition 5.4. $\mathbb{TP}^5/\mathbb{TPGL}(3)$ *can be described as the union of five copies of* $\{\mathbb{R}\cup\infty\}^3$, *four points, and two lines. Specifically:*

$$U_{abc} \cup U_{abe} \cup U_{abf} \cup U_{acf} \cup U_{bde}$$
$$\{(0,\infty,\infty,\infty,\infty,\infty)\} \cup \{(\infty,0,\infty,\infty,\infty,\infty)\}$$
$$\{(0,\infty,\infty,\infty,0,\infty)\} \cup \{(\infty,0,\infty,0,\infty,\infty)\}$$
$$\{(0,0,c,\infty,\infty,\infty)\big|c \in \mathbb{R} \cup \infty\} \cup \{(0,c,0,\infty,\infty,\infty)\big|c \in \mathbb{R} \cup \infty\}.$$

5.2 Classification of Conics Modulo the Action of T

To find all possible combinatorial types of conics we will use our knowledge of dual graphs as presented in Section 4. Only after finding the combinatorial type of a conic, will we look at the conic itself in order to add location and leg length information which is not included in the dual graph.

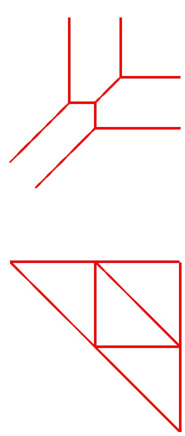
First, we will find the most "general" types of conics, called the **nondegenerate** conics. These correspond to the dual graphs which cannot be subdivided further, without violating the conditions described in Proposition 4.6. Then, systematically removing lines following these same conditions will allow us to find all other possible types.

Recall that using dual graphs as discussed in Section 4 gives the finite representation of the conic in the affine set which we called U_3, that is, when x_1, x_2, and x_3 are finite. Whenever this is the case $(x_1, x_2, x_3) \sim (x_1 - x_3, x_2 - x_3, 0)$ in $\mathbb{TP}^2$. So we can consider the affine subset $(x, y) \in \{\mathbb{R} \cup \infty\}^2$, where $x = x_1 - x_3$ and $y = x_2 - x_3$. Thus Equation (5.1) corresponds to

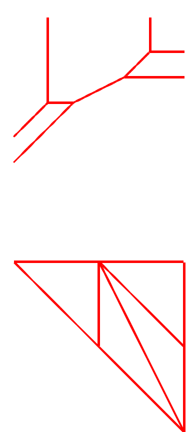
$$f(x, y) = ax^2 \oplus bxy \oplus cy^2 \oplus dx \oplus ey \oplus f.$$

We label combinatorial types with the number of unbounded rays in their affine representations. The nondegenerate conics have one of essentially two types of dual graph, which we will label A or B. Note that the name of each conic begins with the number of unbounded rays that appear in that conic.

$Case\ 6A$

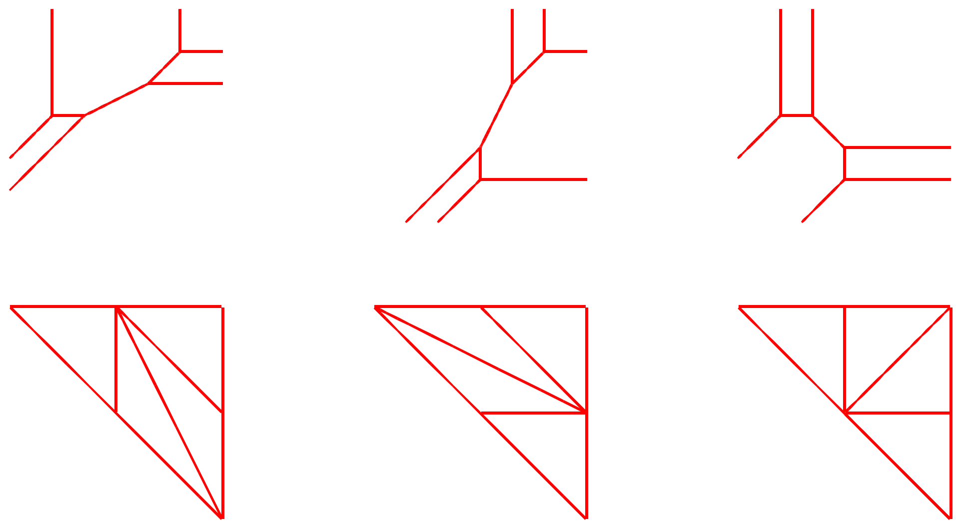


$Case\ 6B$



There are three subtypes of Type B conics, and these can be described in terms of the slope of their central line segment: $\frac{1}{2}$, -1, or 2.

$Case\ 6B\{\frac{1}{2}\}$ $Case\ 6B\{2\}$ $Case\ 6B\{-1\}$



All the cases named 6B are in the same orbit under S_3, while Case 6A is in an orbit by itself, so it will only be necessary to gather any information about one of the three 6B Conics, and then use the group action on the dual graph and coefficient sets of this conic, to find the equivalent information about the other two 6B cases. We will use $6B\{\frac{1}{2}\}$.

It will be useful to determine the set of all coefficients which define a combinatorial type of conic, in order to determine how the different classes of conics relate to each other. To start, obtain the conditions on x and y that define the conic, as was done for the case of lines. Recall that to do this we require that at least two of the monomials are equal and that they are less than or equal to the rest of the monomials. These monomials, in classical notation, are $a+2x, b+x+y, c+2y, d+x, e+y,$ and f. Not only must the two monomials be equal to each other, they must also be equal to

$$\min\{a + 2x, b + x + y, c + 2y, d + x, e + y, f\}.$$

Or in other words, it must be equal to the value of the tropical polynomial

$$f(x, y) = a \odot x^2 \oplus b \odot x \odot y \oplus c \odot y^2 \oplus d \odot x \oplus e \odot y \oplus f.$$

Setting the monomials equal to each other two at a time gives a set of fifteen, or $\binom{6}{2}$ possible equations, one for each pairing of tropical monomials, which will only show up in the hypersurface if and when it is less than the other four monomials. The lines in the conic come in six types, classified by their slopes $1, -1, \frac{1}{2}, 2, 0,$ and ∞ (where a slope of infinity corresponds to a vertical line). As discussed above, whether a particular line, which comes from equality of two monomials, shows up in the conic is completely determined by whether or not it is less than or equal to the other four monomials. The resulting inequalities also give the interval over which the line appears. If a particular line appears in the conic can be determined by checking to see whether the coefficients satisfy some given inequalities. All lines, inner and outer, and their conditions for appearing are described below. We have numbered each inequality, to be able to reference specific conditions later in the paper.

Proposition 5.5. *The appearance of any of the 15 possible lines, rays, and segments is determined by the inequalities given in Table 4.*

Table 4: Lines, Rays, and Segments in Conics

LINES, RAYS, AND SEGMENTS	*CONDITIONS*	*INEQUALITY #*
$y = x + a - b$	$2b < a + c$	*1*
		continued on next page

continued from previous page

LINES, RAYS, AND SEGMENTS	*CONDITIONS*	*INEQUALITY #*
$y = x + b - c$	$2b < a + c$	*1*
$y = x + \frac{a-c}{2}$	$2b \geq a + c$	*not 1*
$x = d - a$	$2d < a + f$	*2*
$x = f - d$	$2d < a + f$	*2*
$x = \frac{f-a}{2}$	$2d \geq a + f$	*not 2*
$y = e - c$	$2e < c + f$	*3*
$y = f - e$	$2e < c + f$	*3*
$y = \frac{f-c}{2}$	$2e \geq c + f$	*not 3*
$y = d - b$	$2b < a + c$	*1*
	$2d < f + a$	*2*
	$d - a < e - b (= d - e < a - b)$	*4*
$x = e - b$	$2b < a + c$	*1*
	$2e < f + c$	*3*
	$e - c < d - b (= b - c < d - e)$	*5*
$y = x + d - e$	$2d < f + a$	*2*
	$2e < f + c$	*3*
	$d - b < f - e (= e - b < f - d)$	*6*
$y = 2x + a - e$	$2e < f + c$	*3*
	$a - b < d - e$	*4 reversed*
	$2e + a < 2b + f$	*7*
	$2e + a < 2d + c$	*8*
$y = -x + f - b$	$2b < a + c$	*1*

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LINES, RAYS, AND SEGMENTS	*CONDITIONS*	*INEQUALITY #*
	$f - e < d - b$	*6 reversed*
	$2b + f < 2e + a$	*7 reversed*
	$2b + f < 2d + c$	*9*
$y = \frac{x}{2} + \frac{d-c}{2}$	$2d < f + a$	*2*
	$d - b < e - c$	*5 reversed*
	$2d + c < 2e + a$	*8 reversed*
	$2d + c < 2b + f$	*9 reversed*

Note that Proposition 5.5 does not give the intervals over which the lines appear, it only determines if they appear at all. It does, however, make it possible to compute which coefficients define a given type of conic.

Case 6A: For Type 6A conics we define the **vertex** to be the point where the inner line segments meet. For all other conics derived from 6A, their vertices will be defined strictly to be consistent with this. (This is also true about the vertex for the 6B conics which we will define shortly). Modulo the action of T, it is necessary only to consider conics whose vertex is at the origin, hence we may assume the origin lies on all three of the inner lines in 6A. Their equations imply $b = d = e \neq \infty$ for the 6A conics. Now let γ_0 be the length of the horizontal inner segment of the conic, γ_1 the length of the diagonal inner segment, and γ_∞ the length of the vertical inner segment. That is, let γ_i be the length of the inner segment of the conic which has slope i. Each of these can be expressed in terms of the coefficients corresponding to the conic using Table 4.

$$\gamma_0 = e - b - (d - a), \gamma_1 = f - e - (d - b), \text{ and } \gamma_\infty = d - b - (e - c)$$

Fix $\gamma_0, \gamma_1,$ and γ_∞. Then the coefficients corresponding to conic 6A must be of the form $(\gamma_0 + b, b, \gamma_\infty + b, b, b, \gamma_1 + b)$, which in $\mathbb{TP}^5$ is equal to $(\gamma_0, 0, \gamma_\infty, 0, 0, \gamma_1)$. To check that this particular coefficient set does correspond to 6A, examine the inequalities listed in Table 4. For the lines in a type 6A conic, Inequalities 1, 2, 3, 4, 5, and 6 must all be satisfied. This happens if and only if $\gamma_0, \gamma_1, \gamma_\infty \in \mathbb{R}_{>0}$.

Case 6B $\{\frac{1}{2}\}$: For a similar analysis of the Type $6B\{\frac{1}{2}\}$ note that the point that is fixed under all permutations is the point where the inner lines with slopes 0 and 1 would intersect if they were extended. We call this point the **vertex**. Then, for Type $6B\{\frac{1}{2}\}$ conics $b = d = e \neq \infty$. Now, as in the previous case, assign $\gamma_0, \gamma_{\frac{1}{2}},$ and γ_1 to be the lengths of the inner segments of the conic which have corresponding slopes. These can be evaluated as above to be:

$$\gamma_0 = a - b - (b - c), \gamma_{\frac{1}{2}} = e - c - (d - b), \gamma_1 = f - e - (e - c).$$

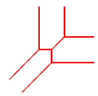
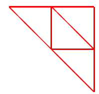
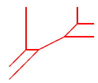
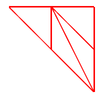
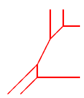
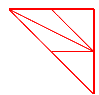
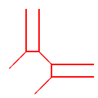
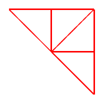
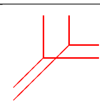
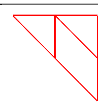
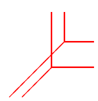
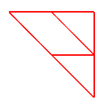
This implies that the coefficients that correspond to a nondegenerate conic of type $6B\{\frac{1}{2}\}$ must be of the form $(\gamma_0 + \gamma_{\frac{1}{2}} + b, b, -\gamma_{\frac{1}{2}} + b, b, b, \gamma_1 + \gamma_{\frac{1}{2}} + b)$, which in $\mathbb{TP}^5$ is $(\gamma_0 + \gamma_{\frac{1}{2}}, 0, -\gamma_{\frac{1}{2}}, 0, 0, \gamma_1 + \gamma_{\frac{1}{2}})$. Once again, to verify that this does correspond to the right conic, check that it satisfies Inequalities 1, 2, 3, 4, 5R, 6, 8R, and 9R. This happens if and only if $\gamma_0, \gamma_{\frac{1}{2}},$ and $\gamma_1 \in \mathbb{R}_{>0}$. Similarly, using Proposition 5.2 we see the coefficients $6B\{2\}$ are $(-\gamma_2, 0, \gamma_\infty + \gamma_2, 0, 0, \gamma_1 + \gamma_2)$, with $\gamma_0, \gamma_2, \gamma_\infty \in \mathbb{R}_{>0}$,and for $6B\{-1\}$ they are of the form $(\gamma_0 + \gamma_{-1}, 0, \gamma_\infty + \gamma_{-1}, 0, 0, -\gamma_{-1})$, where $\gamma_0, \gamma_{-1}, \gamma_\infty \in \mathbb{R}_{>0}$.

Similar computations can be done for each type of conic, once the dual graph has been found. The following table summarizes the conics, their orbits under S_3, and corresponding subsets of $\mathbb{TP}^5/T$.

The first column in the table gives the name of the conic, which consists of the number of unbounded rays in the conic, followed by a letter and other identifiers of the particular conic. The letters A and B refer to the two types of nondegenerate conics.

The letter R implies that as a geometric object the conic is reducible, meaning that it is the union of two lines. Note that such a conic is not necessarily algebraically reducible. The letter I implies that as a geometric object the conic is irreducible. Finally, the part of the name any conic in braces following these letters refers to the slopes of the inner segment(s) of the conic, or, if there are no inner segments, it corresponds to the slopes of the slopes of the lines in the conic. The second column shows a picture of such a conic, while a picture of the dual graph as referred to in Section 4 is in the third column. The fourth through twelfth columns reference the inequalities given in Table 4. That is, in order to have the specific lines and segments of any given conic, it must satisfy the inequalities required for that line or segment. Note that if the letter Y is in the column for an inequality, that implies that that inequality must be satisfied. In there is an N, then it is not satisfied. If there is an R, the reverse inequality must be satisfied. If there is a y, n, or r that implies that the conic automatically satisfies the inequaltiy (or does not satisfy or satisfies the reverse), because of an infinite coefficient. The last column shows the points in $\mathbb{TP}^5/T$ that correspond to the particular type of conic. Horizontal lines separate the different orbits of conics in $\mathcal{C}/T$.

Table 5: Classes of Conics

TYPE	CONIC	DUAL	1	2	3	4	5	6	7	8	9	(a, b, c, d, e, f)
$6A\{0, 1, \infty\}$			Y	Y	Y	Y	Y	Y				$(\alpha, 0, \beta, 0, 0, \gamma)$ where $\alpha, \beta, \gamma \in \mathbb{R}_{>0}$
$6B\{\frac{1}{2}\}$			Y	Y	Y		R			R	R	$(\alpha + \beta, 0, -\beta, 0, 0, \gamma + \beta)$
$6B\{2\}$			Y	Y	Y	R			Y	Y		$(-\beta, 0, \alpha + \beta, 0, 0, \gamma + \beta)$
$6B\{-1\}$			Y	Y	Y			R	R		Y	$(\alpha + \beta, 0, \gamma + \beta, 0, 0, -\beta)$ where $\alpha, \beta, \gamma \in \mathbb{R}_{>0}$
$6R\{0, 1\}$			Y	Y	Y	Y	=	Y				$(\alpha, 0, 0, 0, 0, \beta)$
$6R\{1, \infty\}$			Y	Y	Y	=	Y	Y				$(0, 0, \beta, 0, 0, \alpha)$

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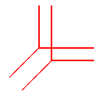
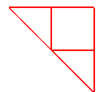
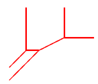
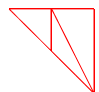
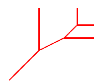
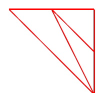
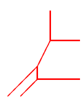
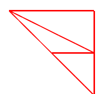
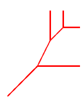
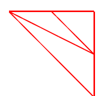
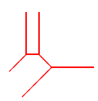
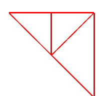
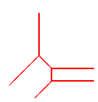
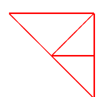
TYPE	CONIC	DUAL	1	2	3	4	5	6	7	8	9	(a, b, c, d, e, f)
$6R\{0, \infty\}$			Y	Y	Y	Y	Y	=				$(\alpha, 0, \beta, 0, 0, 0)$ where $\alpha, \beta \in \mathbb{R}_{>0}$
$5I\{0, \frac{1}{2}\}a$			Y	Y	N	Y	R			R	R	$(\alpha + \beta, 0, -\beta, 0, e, \beta)$
$5I\{\frac{1}{2}, 1\}a$			N	Y	Y		R	Y		R	R	$(\beta, e, -\beta, 0, 0, \alpha + \beta)$
$5I\{\infty, 2\}a$			Y	N	Y	R	Y		Y	Y		$(-\beta, 0, \alpha + \beta, e, 0, \beta)$
$5I\{2, 1\}a$			N	Y	Y	R		Y	Y	Y		$(-\beta, e, \beta, 0, 0, \alpha + \beta)$
$5I\{0, -1\}a$			Y	Y	N	Y		R	R		Y	$(\alpha + \beta, 0, \beta, 0, e, -\beta)$
$5I\{-1, \infty\}a$			Y	N	Y		Y	R	R		Y	$(\beta, 0, \alpha + \beta, e, 0, -\beta)$

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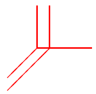
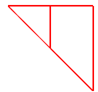
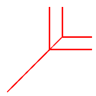
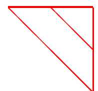
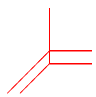
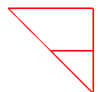
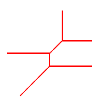
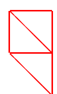
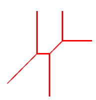
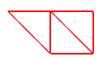
TYPE	CONIC	DUAL	1	2	3	4	5	6	7	8	9	(a, b, c, d, e, f)
												where $\alpha, \beta \in \mathbb{R}_{>0}$, $e \in \mathbb{R}_{\geq 0} \cup \{\infty\}$
$5R\{0\}a$			Y	Y	N	Y					=	$(\alpha, 0, 0, 0, e, 0)$
$5R\{1\}a$			N	Y	Y			Y		=		$(0, e, 0, 0, 0, \alpha)$
$5R\{\infty\}a$			Y	N	Y		Y		=			$(0, 0, \alpha, e, 0, 0)$ where $\alpha, \in \mathbb{R}_{>0}$ and $e, \in \mathbb{R}_{\geq 0} \cup \{\infty\}$
$5I\{\infty, 1\}$			y	y	Y	y	Y	Y				$(\infty, 0, \alpha, 0, 0, \beta)$
$5I\{0, 1\}$			y	Y	y	Y	y	Y				$(\alpha, 0, \infty, 0, 0, \beta)$

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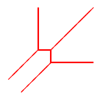
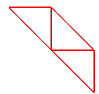
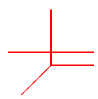
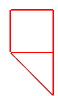
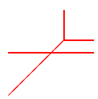
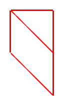
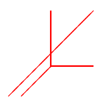
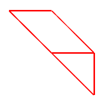
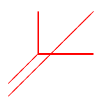
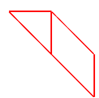
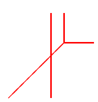
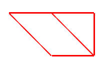
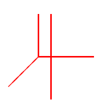
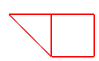
TYPE	CONIC	DUAL	1	2	3	4	5	6	7	8	9	(a, b, c, d, e, f)
$5I\{0, \infty\}$			Y	y	y	Y	Y	y				$(\alpha, 0, \beta, 0, 0, \infty)$ where $\alpha, \beta \in \mathbb{R}_{>0}$
$5R\{0\}ai$			y	y	Y	y	Y	=				$(\infty, 0, \alpha, 0, 0, 0)$
$5R\{0\}aii$			y	y	Y	y	=	Y				$(\infty, 0, 0, 0, 0, \alpha)$
$5R\{1\}ai$			Y	y	y	=	Y	y				$(0, 0, \alpha, 0, 0, \infty)$
$5R\{1\}aii$			Y	y	y	Y	=	y				$(\alpha, 0, 0, 0, 0, \infty)$
$5R\{\infty\}ai$			y	Y	y	=	y	Y				$(0, 0, \infty, 0, 0, \alpha)$
$5R\{\infty\}aii$			y	Y	y	Y	y	=				$(\alpha, 0, \infty, 0, 0, 0)$

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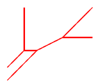
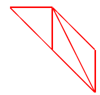
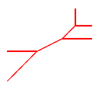
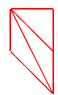
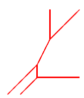
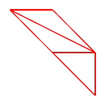
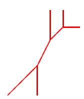
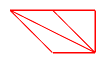
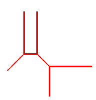
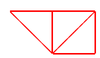
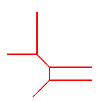
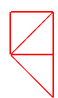
TYPE	CONIC	DUAL	1	2	3	4	5	6	7	8	9	(a,b,c,d,e,f)
												where $\alpha \in \mathbb{R}_{>0}$
$5I\{0,\frac{1}{2}\}b$			Y	y	y	Y	R	y		R	R	$(\alpha+\beta,0,-\beta,0,0,\infty)$
$5I\{\frac{1}{2},1\}b$			y	y	Y	y	R	Y		R	R	$(\infty,0,-\beta,0,0,\alpha+\beta)$
$5I\{\infty,2\}b$			Y	y	y	R	Y	y	Y	Y		$(-\beta,0,\alpha+\beta,0,0,\infty)$
$5I\{2,1\}b$			y	Y	y	R	y	Y	Y	Y		$(-\beta,0,\infty,0,0,\alpha+\beta)$
$5I\{0,-1\}b$			y	Y	y	Y	y	R	R		Y	$(\alpha+\beta,0,\infty,0,0,-\beta)$
$5I\{-1,\infty\}b$			y	y	Y	y	Y	R	R		Y	$(\infty,0,\alpha+\beta,0,0,-\beta)$ where $\alpha,\beta \in \mathbb{R}_{>0}$

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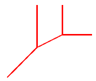
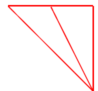
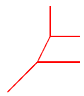
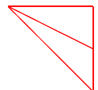
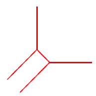
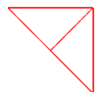
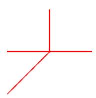
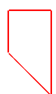
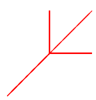
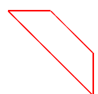
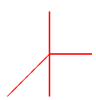
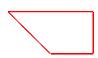
TYPE	CONIC	DUAL	1	2	3	4	5	6	7	8	9	(a, b, c, d, e, f)
$4I\{\frac{1}{2}\}a$			N	Y	N		R			R	R	$(\alpha, b, -\alpha, 0, e, \alpha)$
$4I\{2\}a$			N	N	Y	R			Y	Y		$(-\alpha, b, \alpha, e, 0, \alpha)$
$4I\{-1\}a$			Y	N	N			R	R		Y	$(\alpha, 0, \alpha, b, e, -\alpha)$
												where $\alpha \in \mathbb{R}_{>0}$,
												$b, e \in \mathbb{R}_{\geq 0} \cup \{\infty\}$
$4R\{0\}$			y	y	N	y					=	$(\infty, 0, 0, 0, e, 0)$
$4R\{1\}$			N	y	y			y		=		$(0, e, 0, 0, 0, \infty)$
$4R\{\infty\}$			y	N	y		y		=			$(0, 0, \infty, e, 0, 0)$
												where $e \in \mathbb{R}_{\geq 0} \cup \{\infty\}$

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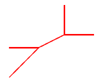
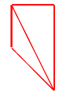
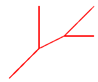
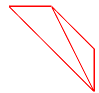
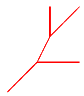
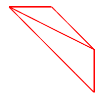
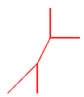
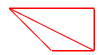
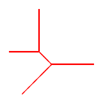
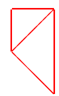
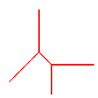
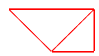
TYPE	CONIC	DUAL	1	2	3	4	5	6	7	8	9	(a, b, c, d, e, f)
$4I\{\frac{1}{2}\}bi$			y	y	N	y	R			R	R	$(\infty, 0, -\alpha, 0, e, \alpha)$
$4I\{\frac{1}{2},\}bii$			N	y	y		R	y		R	R	$(\alpha, e, -\alpha, 0, 0, \infty)$
$4I\{2\}bi$			N	y	y	R		y	Y	Y		$(-\alpha, e, \alpha, 0, 0, \infty)$
$4I\{2\}bii$			y	R	y	R	y		Y	Y		$(-\alpha, 0, \infty, e, 0, \alpha)$
$4I\{-1\}bi$			y	y	N	y		R	R		Y	$(\infty, 0, \alpha, 0, e, -\alpha)$
$4I\{-1\}bii$			y	N	y		y	R	R		Y	$(\alpha, 0, \infty, e, 0, -\alpha)$
												where $\alpha \in \mathbb{R}_{>0}$
												where $e \in \mathbb{R}_{\geq 0} \cup \{\infty\}$

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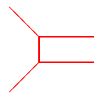
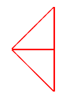
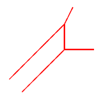
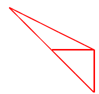
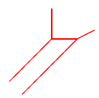
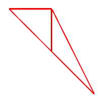
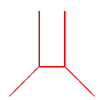
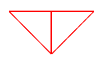
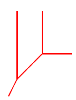
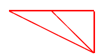
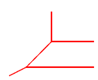
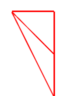
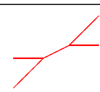
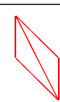
TYPE	CONIC	DUAL	1	2	3	4	5	6	7	8	9	(a, b, c, d, e, f)
$4I\{\infty\}ai$			y		Y		y	r	r		y	$(\infty, 0, 0, \infty, \alpha, 0)$
$4I\{\infty\}aii$			Y		y	r	y		y	y		$(0, \alpha, 0, \infty, 0, \infty)$
$4I\{0\}ai$			Y	y		y	r			r	r	$(0, \alpha, 0, 0, \infty, \infty)$
$4I\{0\}aii$			y	Y		y		r	r		y	$(0, 0, \infty, \alpha, \infty, 0)$
$4I\{1\}ai$				Y	y	r		y	y	y		$(0, \infty, \infty, \alpha, 0, 0)$
$4I\{1\}aii$				y	Y		r	y		r	r	$(\infty, \infty, 0, 0, 0, \alpha, 0)$ where $\alpha \in \mathbb{R}_{>0}$
$4I\{\frac{1}{2}\}b$			y	y	y	y	R	y		r	r	$(\infty, 0, -\alpha, 0, 0, \infty)$

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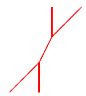
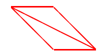
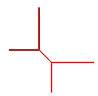
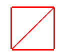
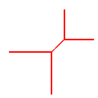
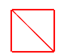
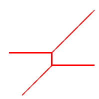
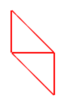
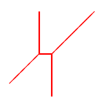
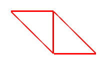
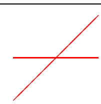
TYPE	CONIC	DUAL	1	2	3	4	5	6	7	8	9	(a,b,c,d,e,f)
$4I\{2\}b$			y	y	y	R	y	y	y	y		$(-\alpha,0,\infty,0,0,\infty)$
$4I\{-1\}b$			y	y	y	y	y	R	r		y	$(\infty,0,\infty,0,0,-\alpha)$
												where $\alpha\in\mathbb{R}_{>0}$
$4I\{1\}b$			y	y	y	y	y	Y				$(\infty,0,\infty,0,0,\alpha)$
$4I\{\infty\}b$			y	y	y	y	Y	y				$(\infty,0,\alpha,0,0,\infty)$
$4I\{0\}b$			y	y	y	Y	y	y				$(\alpha,0,\infty,0,0,\infty)$
												where $\alpha\in\mathbb{R}_{>0}$
$4R\{0,1\}$			y	y	y	y	=	y				$(\infty,0,0,0,0,\infty)$

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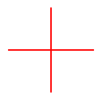
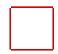
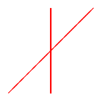
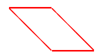
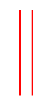
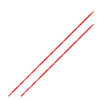
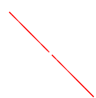
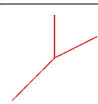
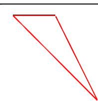
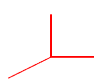
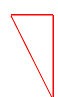
TYPE	CONIC	DUAL	1	2	3	4	5	6	7	8	9	(a, b, c, d, e, f)
$4R\{0, \infty\}$			y	y	y	y	y	=				$(\infty, 0, \infty, 0, 0, 0)$
$4R\{\infty, 1\}$			y	y	y	=	y	y				$(0, 0, \infty, 0, 0, \infty)$
$4R\{0, 0\}$					Y							$(\infty, \infty, 0, \infty, e, 0)$
$4R\{\infty, \infty\}$				Y								$(0, e, 0, \infty, \infty, \infty)$
$4R\{1, 1\}$			Y									$(0, \infty, \infty, e, \infty, 0)$ where $e \in \mathbb{R}_{<0}$
$3\{\frac{1}{2}, 1, \infty\}$			N	y			r			r	r	$(0, e, 0, 0, \infty, \infty)$
$3\{\frac{1}{2}, 0, \infty\}$				y	N		r			r	r	$(\infty, \infty, 0, 0, e, 0)$

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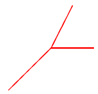
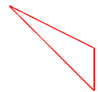
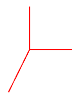
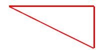
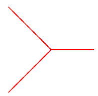
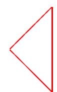
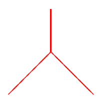
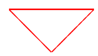
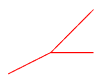
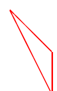
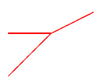
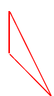
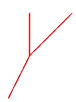
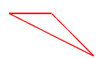
TYPE	CONIC	DUAL	1	2	3	4	5	6	7	8	9	(a, b, c, d, e, f)
$3\{2, 1, 0\}$			N		y	r			y	y		$(0, e, 0, \infty, 0, \infty)$
$3\{2, 0, \infty\}$				N	y	r			y	y		$(0, \infty, \infty, e, 0, 0)$
$3\{-1, 1, 0\}$			y		N			r	r		y	$(\infty, 0, 0, \infty, e, 0)$
$3\{-1, 1, \infty\}$			y	N				r	r		y	$(0, 0, \infty, e, \infty, 0)$
												where $e \in \mathbb{R}_{\geq 0} \cup \{\infty\}$
$3\{\frac{1}{2}, 0, 1\}i$				y	y		r	y		r	r	$(\infty, \infty, 0, 0, 0, \infty)$
$3\{\frac{1}{2}, 0, 1\}ii$			y	y		y	r			r	r	$(\infty, 0, 0, 0, \infty, \infty)$
$3\{2, 1, \infty\}i$				y	y	r		y	y	y		$(0, \infty, \infty, 0, 0, \infty)$

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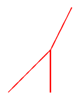
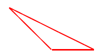
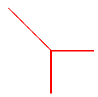
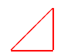
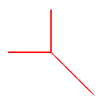
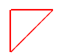
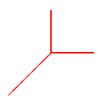
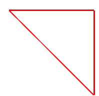
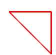
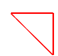
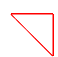
TYPE	CONIC	DUAL	1	2	3	4	5	6	7	8	9	(a,b,c,d,e,f)
$3\{2,1,\infty\}ii$			y		y	r	y		y	y		$(0,0,\infty,\infty,0,\infty)$
$3\{-1,o,\infty\}i$			y		y		y	r	r		y	$(\infty,0,\infty,\infty,0,0)$
$3\{-1,0,\infty\}ii$			y	y		y		r	r		y	$(\infty,0,\infty,0,\infty,0)$
$3R$			N	N	N							$(0,b,0,d,e,0)$
			y	y		y						$(0,0,\infty,0,\infty,\infty)$
			y		y		y					$(\infty,0,0,\infty,0,\infty)$
				y	y			y				$(\infty,\infty,\infty,0,0,0)$ where $b,d,e \in \mathbb{R}_{\geq 0} \cup \{\infty\}$

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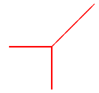
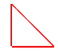
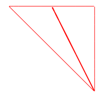
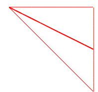
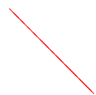
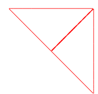
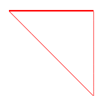
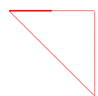
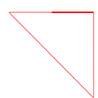
TYPE	CONIC	DUAL	1	2	3	4	5	6	7	8	9	(a, b, c, d, e, f)
$3I$			y	y	y	y	y	y				$(\infty, 0, \infty, 0, 0, \infty)$
$2R\{\frac{1}{2}\}$				y			r			r	r	$(\infty, \infty, 0, 0, \infty, \infty)$
$2R\{2\}$					y	r			y	y		$(0, \infty, \infty, \infty, 0, \infty)$
$2R\{-1\}$			y					r	r		y	$(\infty, 0, \infty, \infty, \infty, 0)$
$2R\{\infty\}$				N								$(0, \infty, \infty, d, \infty, 0)$
				y								$(0, \infty, \infty, 0, \infty, \infty)$
				y								$(\infty, \infty, \infty, 0, \infty, 0)$

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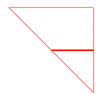
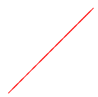
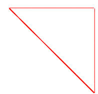
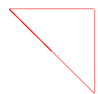
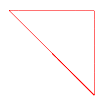
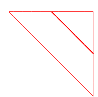
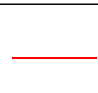
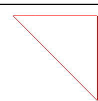
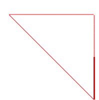
TYPE	CONIC	DUAL	1	2	3	4	5	6	7	8	9	(a, b, c, d, e, f)
			y		y		y					$(\infty, 0, \infty, \infty, 0, \infty)$
$2R\{1\}$			N									$(0, b, 0, \infty, \infty, \infty)$
			y									$(0, 0, \infty, \infty, \infty, \infty)$
			y									$(\infty, 0, 0, \infty, \infty, \infty)$
				y	y			y				$(\infty, \infty, \infty, 0, 0, \infty)$
$2R\{0\}$					N							$(\infty, \infty, 0, \infty, e, 0)$
					y							$(\infty, \infty, 0, \infty, 0, \infty)$

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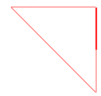
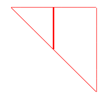
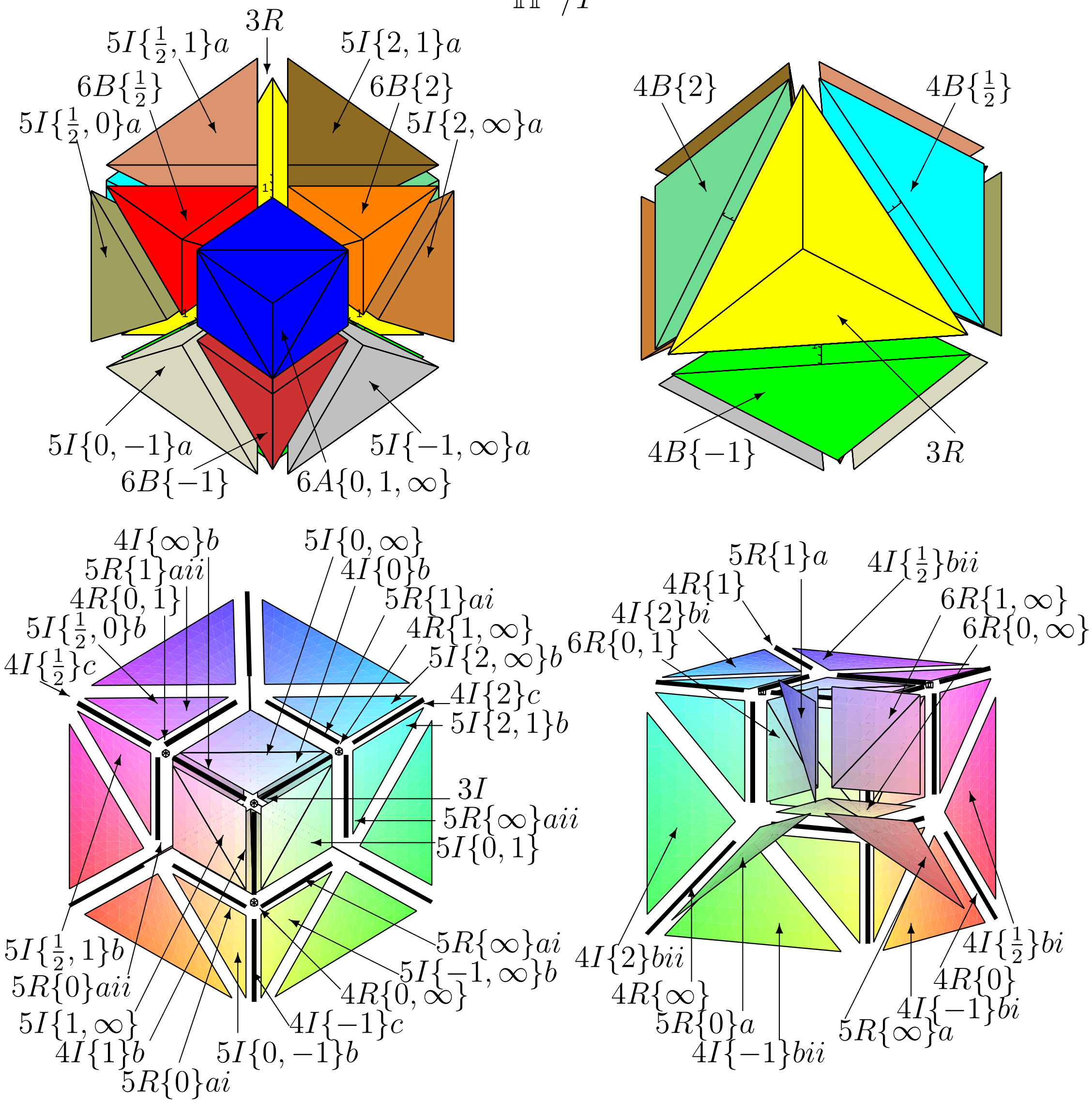
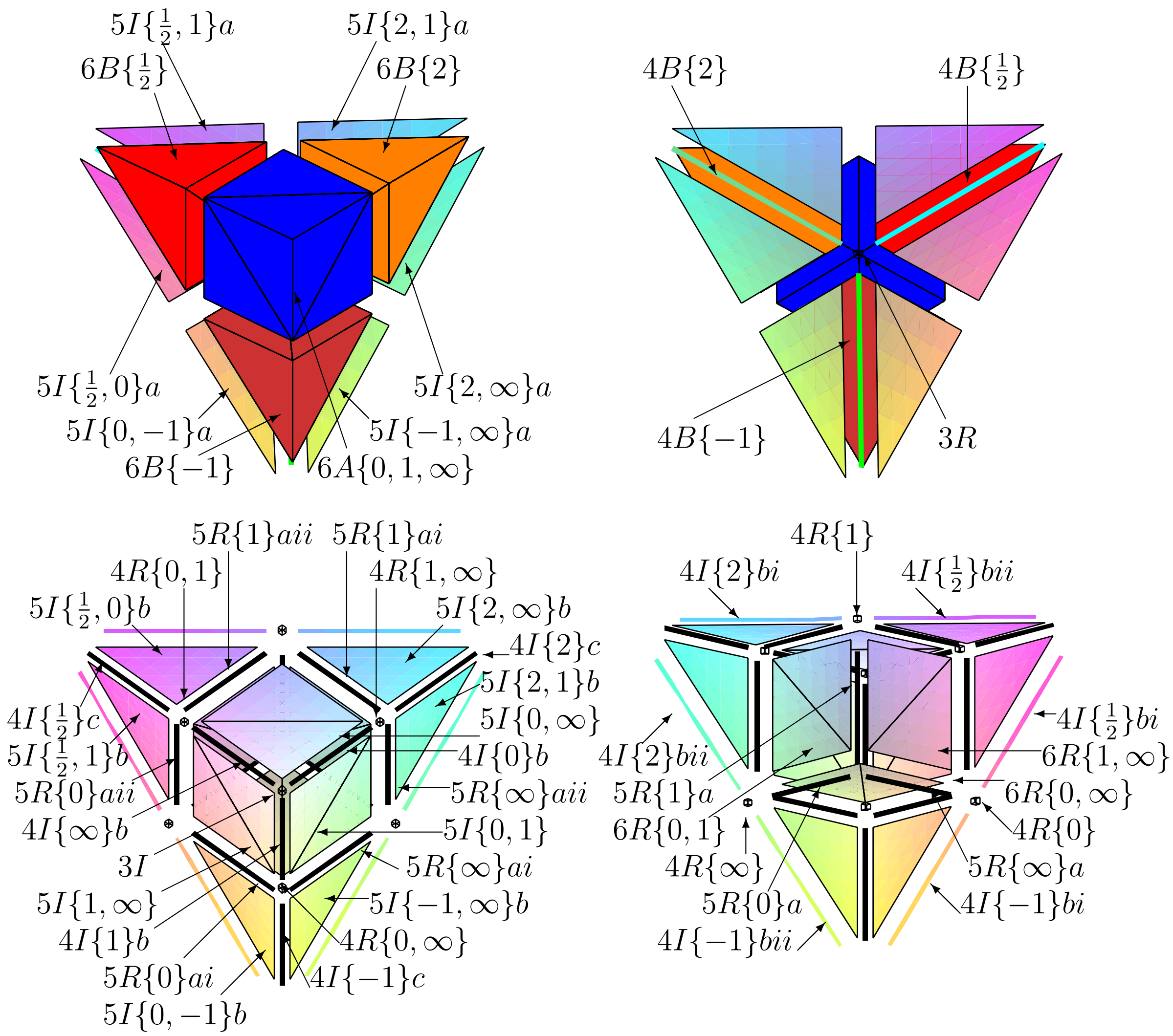
TYPE	CONIC	DUAL	1	2	3	4	5	6	7	8	9	(a, b, c, d, e, f)
					y							$(\infty, \infty, \infty, \infty, 0, 0)$
			y	y		y						$(\infty, 0, \infty, 0, \infty, \infty)$ where $e \in \mathbb{R}_{\geq 0} \cup \{\infty\}$

Recall from Section 5.1 that the coefficient space $\mathbb{TP}^5/T$, as well as the space $\mathbb{TP}^5/\mathbb{TPGL}(3)$ are each isomorphic to some number of copies of $\{\mathbb{R} \cup \infty\}^3$ and some points and lines. Consider the set $U_{bde} \cong \{\mathbb{R} \cup \infty\}^3$ as defined in Section 5.1. It is easy to determine which parts of $U_{bde} \cong \{\mathbb{R} \cup \infty\}^3$ as coefficient cosets get taken to which types of conics. Then under the quotient map which takes coefficients to conics, this gives a representation of the space of conics. Finally, quotient by S_3 to get the last part of the Diagram 3. The results are given pictorally below.

$\mathbb{TP}^5/T$

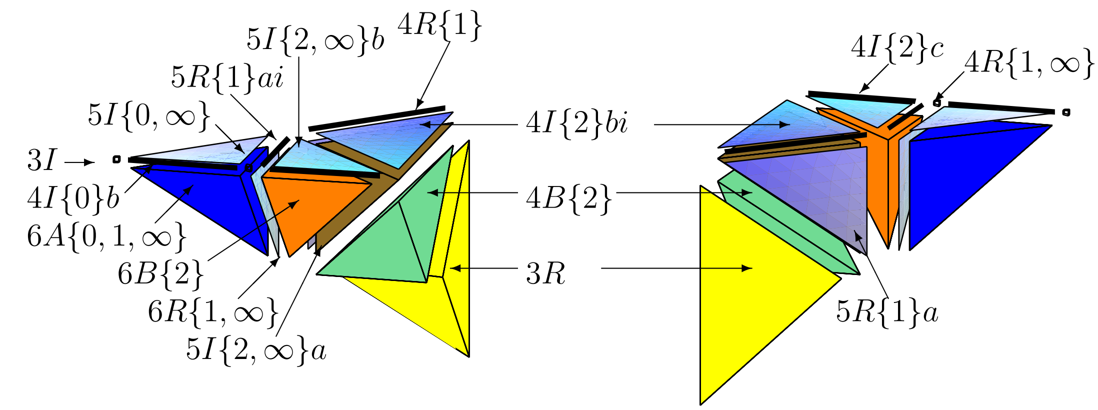


$\mathcal{C}/T$

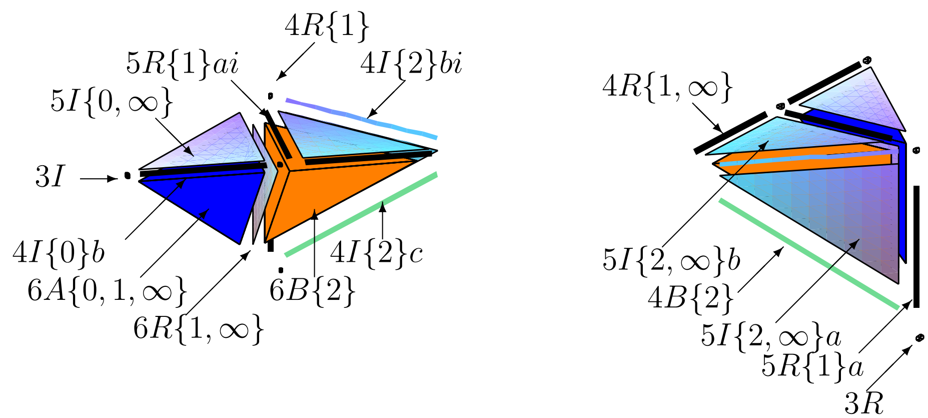


$5I\{\frac{1}{2},1\}a$
$5I\{2,1\}a$
$6B\{\frac{1}{2}\}$
$6B\{2\}$
$5I\{\frac{1}{2},0\}a$
$5I\{2,\infty\}a$
$5I\{0,-1\}a$
$5I\{-1,\infty\}a$
$6B\{-1\}$
$6A\{0,1,\infty\}$
$4B\{2\}$
$4B\{\frac{1}{2}\}$
$4B\{-1\}$
$3R$
$5R\{1\}aii$
$5R\{1\}ai$
$4R\{0,1\}$
$4R\{1,\infty\}$
$5I\{\frac{1}{2},0\}b$
$5I\{2,\infty\}b$
$4I\{2\}c$
$5I\{2,1\}b$
$4I\{\frac{1}{2}\}c$
$5I\{0,\infty\}$
$5I\{\frac{1}{2},1\}b$
$4I\{0\}b$
$5R\{0\}aii$
$5R\{\infty\}aii$
$4I\{\infty\}b$
$5I\{0,1\}$
$3I$
$5R\{\infty\}ai$
$5I\{1,\infty\}$
$5I\{-1,\infty\}b$
$4I\{1\}b$
$4R\{0,\infty\}$
$5R\{0\}ai$
$4I\{-1\}c$
$5I\{0,-1\}b$
$4R\{1\}$
$4I\{2\}bi$
$4I\{\frac{1}{2}\}bii$
$4I\{\frac{1}{2}\}bi$
$4I\{2\}bii$
$6R\{1,\infty\}$
$5R\{1\}a$
$6R\{0,\infty\}$
$6R\{0,1\}$
$4R\{0\}$
$4R\{\infty\}$
$5R\{\infty\}a$
$5R\{0\}a$
$4I\{-1\}bi$
$4I\{-1\}bii$

$\mathbb{TP}^5/\mathbb{TPGL}(3)$

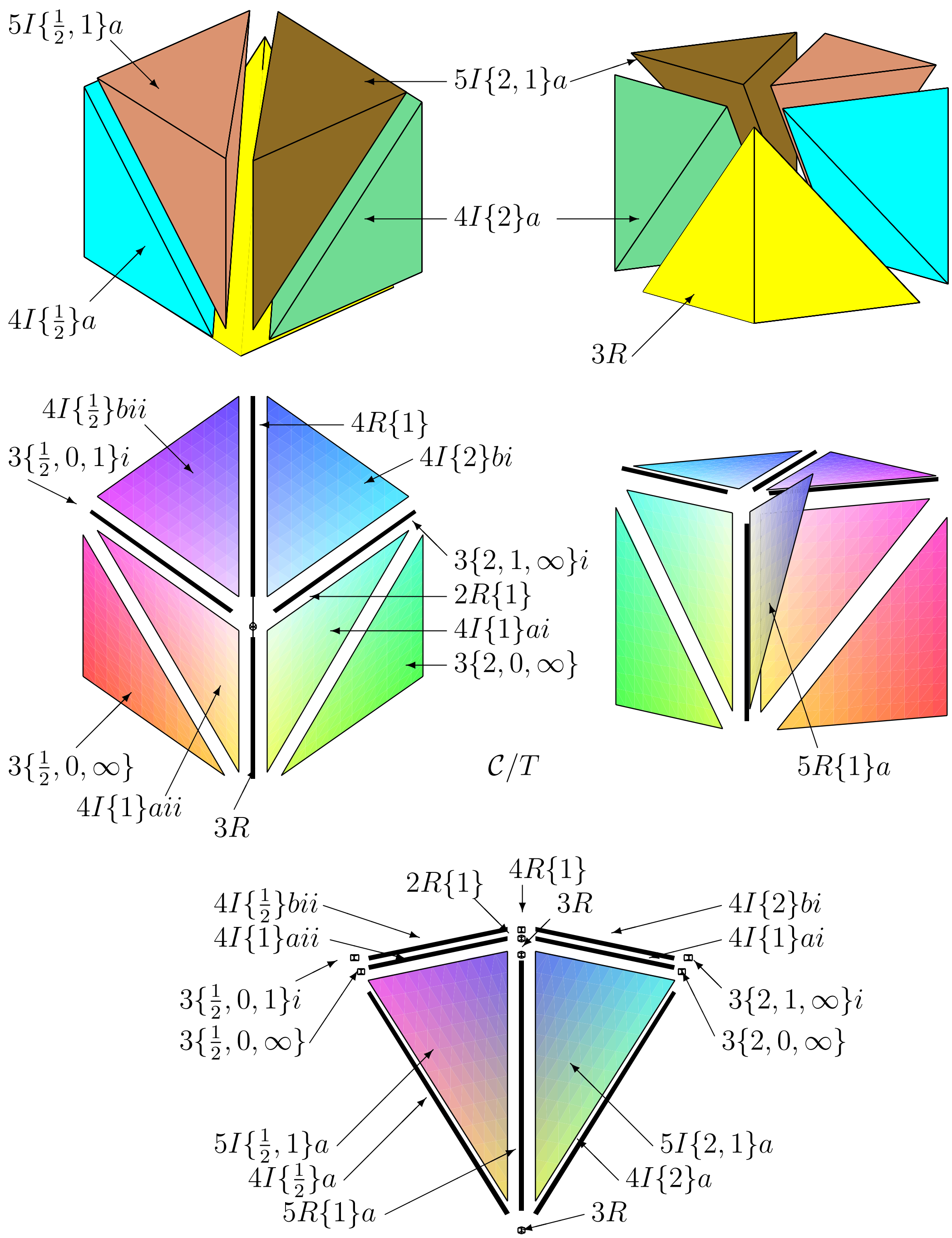


$\mathcal{C}/\mathbb{TPGL}(3)$

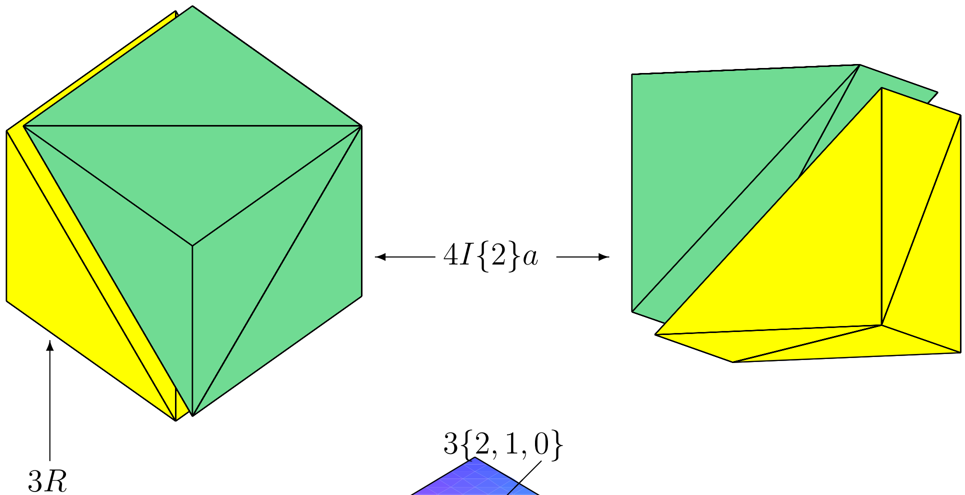


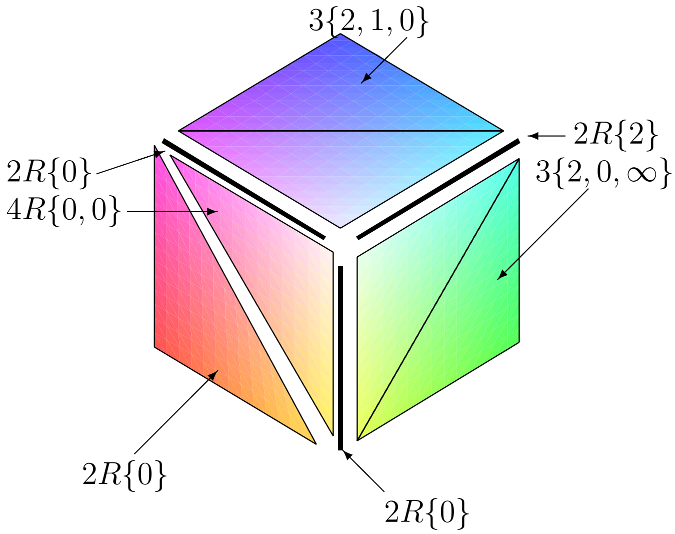
As mentioned earlier, the limitation of using only the open set U_{bde}, is that it misses some points and lines mentioned in Proposition 5.3, as well as points which are in any one of the other sixteen open sets, but not in U_{bde}. To resolve this, consider first the case where $b = \infty$. We could go through all of the analysis of the coefficient space in the same three coordinates a, c, and f, this time with $b = \infty$. This gives another coefficient space and a corresponding conic space. We could also do this for the cases where $d = \infty$, $e = \infty$, $b = d = \infty$, $b = e = \infty$, $d = e = \infty$, and $b = d = e = \infty$. Altogether we have eight coefficient spaces and eight corresponding conic spaces, which, together provide all of the information about conics. Modulo the action of S_3, there are four sets of spaces, namely those where $b, d, e < \infty$, $b = \infty$, $b = d = \infty$, and $b = d = e = \infty$. When $b = d = e = \infty$ this reduces to the cases described in Section 3 for tropical lines. The other two spaces are shown on the following pages. Note that a line in the coefficient space could be collapsed into a single point in the space of conics. Similarly, a plane could be collapsed to a line or a point, and a three dimensional set could be collapsed to a plane, a line, or a point. This means that in the conic space there may be points or lines adjacent to each other.

$b = \infty \subseteq \mathbb{TP}^5/T$

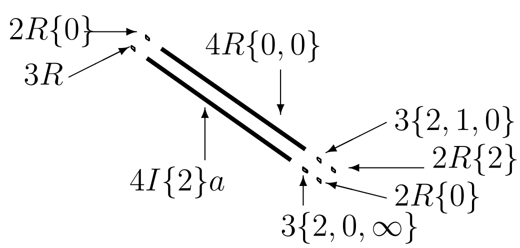


$b = d = \infty \subseteq \mathbb{TP}^5/T$





$\mathcal{C}/T$



References

[1] Richter-Gebert, Jürgen; Sturmfels, Bernd; Theobald, Thorsten. *First steps in tropical geometry.* In Idempotent mathematics and mathematical physics, 289–317, Contemp. Math., 377, Amer. Math. Soc., Providence, RI, 2005.

[2] Zur Ishakian, *Duality of Tropical Curves*, arXiv:math.AG/0503691 v2, Preprint, July 2005.