Ortho-Planar Mechanisms for Microelectromechanical Systems

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ORTHO-PLANAR MECHANISMS FOR
MICROELECTROMECHANICAL SYSTEMS

by

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Doctor of Philosophy

Department of Mechanical Engineering
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This dissertation has been read by each member of the following graduate committee and by majority vote has been found to be satisfactory.

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<td>Larry L. Howell, Chair</td>
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ABSTRACT

ORTHO-PLANAR MECHANISMS FOR
MICROELECTROMECHANICAL
SYSTEMS

Craig P. Lusk
Mechanical Engineering
Doctor of Philosophy

A method for representing the design space of ortho-planar mechanisms has been developed. The method is based on the Theorem of Equality of Orientation Set Measures (TEOSM) which allows mechanisms to be represented by points in an abstract space. The method is first developed for single loop planar folded mechanisms with revolute joints, and later extended to mechanisms with prismatic joints and to spherical folded mechanisms. Functions which assign a value to each point in design space can be used to describe classes of mechanisms and evaluate their utility for MEMS design. Additionally, this work introduces the use of spherical mechanisms in MEMS design. Spherical mechanisms have characteristics that may be useful in MEMS, including the capability of spatial positioning of a link and the ability to convert rotation about an axis perpendicular to the substrate to rotation about an axis parallel to the substrate.
Spherical kinematics has been used to develop three novel mechanisms, the Micro Helico-Kinematic Platform (MHKP), the Spherical Bistable Mechanism (SBM), and the Three-degree-of-freedom Platform (3DOFP). Mathematical models of these devices have been developed and MEMS prototypes have been designed and fabricated.
ACKNOWLEDGMENTS

There are many individuals whose help has made a difference in my life during the course of writing this dissertation. First and foremost, I am grateful to my wife, Mary, for the many sacrifices she has endured in order to further my education. She has been a wonderful friend and support to me over the years. Not only has she helped to create a wonderful home and family environment, she has often been the first to edit my drafts, a truly unenviable task. I love and appreciate her and sincerely hope that her sacrifices will be well rewarded. I appreciate my sons, Kyle, Joshua and Alec. Their smiles and enthusiasm make life a joy. I thank my parents who supported my education in so many ways.

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I have received financial support from Brigham Young University, from the Utah Center of Excellence, and from the National Science Foundation. I appreciate their faith in me, and hope that their investment will return dividends, in honor and in my contributions to science.

Finally, I am grateful for the abundant blessings and tender mercies of a loving Heavenly Father.
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Chapter 1

INTRODUCTION

In the last twenty years, researchers have spun off a new technology from the electronic circuit industry. Using similar techniques as those used to produce the integrated circuits found in computers, scientists have created devices that not only compute but also can sense and move and interact with the environment on a miniature scale. These small machines are called microelectromechanical systems (MEMS), and some have already been integrated with consumer products. Two of the most successful are the AD-XL50 integrated accelerometer [1] used to trigger car airbags, and the Texas Instrument Digital Light Processor, which is the core technology of business video projectors in which thousands of minuscule mirrors are individually controlled to produce extraordinary picture quality [2]. Industry visionaries predict applications for MEMS ranging from inertial navigation units on a chip which could help cars prevent rollovers [3], optical communications network interconnects which could dramatically increase the speed of the internet [4], extraordinary control of the surfaces of aircraft, leading to reduced drag and improved aerodynamic control [5], and medical applications in which micro-robots operate on individual living cells [6].

The strength of MEMS technology lies in what are referred to as the three M’s: miniaturization, multiplicity, and micro-electronics [3]. Miniaturization refers to the ability of MEMS to interact with the environment at scales previously inaccessible to humans. Multiplicity refers to the ability technologists have of fabricating large numbers of MEMS devices very cheaply. Micro-electronics refers to the fact that MEMS devices can be very compatible with micro-electronics, meaning that MEMS devices can be
integrated with sophisticated computational and control elements giving them the ability to react swiftly and precisely to their environment.

In particular, a form of MEMS fabrication called surface micromachining that uses many of the techniques developed for microelectronics has attracted the interest of many researchers. Surface micromachining is less expensive and more versatile than alternative forms of MEMS fabrication. For these reasons much of current MEMS research is devoted to this technique. On the other hand, the major disadvantage of the surface micromachining technology is that it is inherently a two-dimensional process [7]. This is because surface micromachined devices are typically hundreds of microns (10^{-6} meters) in length and width, but less than ten microns thick, meaning that they are essentially flat. Most MEMS designs are thus limited to moving back-and-forth and side-to-side (two-dimensional motion). Considerable design ingenuity is required to make mechanisms that also move up-and-down (three-dimensional motion). For many of the applications mentioned above, it may be necessary for MEMS devices to execute three-dimensional motions. Kota et al. [8] acknowledged that the flat (two-dimensional) nature of MEMS is a difficulty and stated that it could either be surmounted through breakthroughs in microfabrication technology or circumvented through novel designs. Parise [9] coined the term “ortho-planar mechanism” to describe mechanisms that are fabricated in a plane and then have motion out of that plane.

In his thesis [10], Parise describe several different types of ortho-planar (OP) mechanisms, including rigid-body OP mechanisms, compliant OP mechanisms, metamorphic OP mechanisms, and OP linear motion springs. His work on metamorphic and compliant mechanisms was advanced by Carroll [11]. Parise’s work on rigid-body OP mechanisms was focused on planar mechanisms, and included discussion of the planar crank-slider, the planar four- and five-bars, the Watt and Stephenson chains, and pantographs. In his work, he developed a set of postulates that described the conditions under which planar four-bars and planar five-bars may be OP mechanisms. The postulates are the necessary conditions for four- and five-bar mechanisms to be mobile and ortho-planar. The effort to make those posulates more rigorous and more general led to the theoretical developments, including the development of a theorem, described in Chapter 2 of this work.
MEMS devices satisfying the definition of ortho-planar mechanisms have been demonstrated by a number of researchers. A primary application for OP MEMS devices is integrated optical devices such as variable optical attenuators, micro Fresnel lenses, micro gratings, multiplexers, and micro-mirrors (see for example [12, 13, 14, 15, 16, 17, 18, 19, 20, 21]). Other applications include micro-gimbals [22], micro-gyros [23, 24], flow control [25], and shear sensing [25, 26]. Researchers have been able to move the mechanisms up from the plane by the use of special surface treatments [27, 28, 29], the use of electrical or magnetic attraction [30, 31, 32], and geometric constraints on a hinge or flexure [33, 34, 35]. Electrostatic actuation is the method of choice of many of the researchers [22, 23, 24, 30, 37], although scratch drive actuators [12, 17], pneumatic actuators [25], electromagnetic actuators [26] and thermal actuators [15] are also used. Most of the devices are fairly simple, consisting of a single, compliant, monolithic device (such as in [37, 13, 38, 39]), or a hinged link attached to the substrate (see [18] for a number of examples). Some are as complicated as a planar fourbar [19] or a crank slider [12]. Important work has also been done in developing systems for measuring the out-of-plane motion of MEMS devices [38, 40, 41, 42, 43, 44, 45].

One particular application that may be important is in MEMS-based adaptive optics [46]. Adaptive optics include deformable mirrors in which individual pixels are controlled and moved in ways that can compensate for distortions in optical wavefronts as they pass through turbulence or density gradients. In other words, adaptive optics can be used to create high power telescopes that can compensate for atmospheric effects, or cameras that can be used to detect abnormalities on the retina of the human eye. In order to achieve these results, designs are needed that allow for rapid and accurate spatial positioning of arrays of micro-mirrors.

Fukushige et al. [37] state “If a long stroke in the out-of-plane direction, a large output force, and high integration can be simultaneously realized, several new applications, including tactile displays, active braille, variable inductors, and micro-optical systems such as actuation of micromirrors become possible”. Notwithstanding all the previous work done developing MEMS with motion up from the plane, there is a need for
deeper and better understanding of the fundamentals and constraints associated with the design of ortho-planar MEMS devices.

1.1 Objective

The purpose of the research is to discover geometric fundamentals of ortho-planar mechanisms and to demonstrate those fundamentals by designing and modeling devices. The reason for studying the fundamentals of anything is to move from intuitive, heuristic approaches to more refined, logic-based and mathematical approaches. The benefits from studying OP mechanism fundamentals are that problematic or excessively time-consuming approaches are identified and avoided and promising approaches are identified.

Because ortho-planar mechanism research is particularly relevant for MEMS, novel MEMS devices were designed, fabricated, and tested. Information about these mechanisms has been distilled in the form of validated mathematical models that characterize motion and force-deflection relationships.

In addition to the design of specific devices, it is desirable to create broad descriptions of different classes of mechanisms utilizing planar and spherical kinematics and to provide information that makes it easy for other researchers to replicate and, if need be, improve and specialize the devices. To this end, mathematical models were created that allow manipulation of the size and shape of the device. By modifying the mathematical model, the designer can see how changing the size and shape of the device would affect its motion.

The results that were obtained from this research include the further development of the fundamentals of ortho-planar planar and spherical mechanisms, and the introduction of a number of novel mechanisms to the MEMS research community, including several mechanisms that display encouraging qualitative behaviors.
1.2 Research Approach

As part of the development of novel MEMS designs, a fundamental examination of the geometry of planar and spherical ortho-planar mechanisms is conducted. Here, geometry is taken in the very broad sense suggested by Felix Klein [47], who defined geometry as the study of the invariants of a set, or in other words, the properties of a system that are always true. Thus, geometry includes not just the physical geometry of the tangible parts of the mechanism, but also conceptual geometries such as those that describe the mechanism’s motion and even more generally, the geometry that describes the set of all mechanisms that can be constructed by varying certain standard parameters such as the lengths of individual links. By describing the geometry of ortho-planar mechanisms in precise mathematical terms, very broad conclusions can be made about the capabilities of such mechanisms.

The first step taken is to investigate the geometry of ortho-planar mechanisms using planar kinematics and revolute joints. In planar kinematics, all joint axes are parallel and all mechanism geometry moves in a plane. After establishing the geometry of planar ortho-planar mechanisms with revolute joints, the geometry of planar ortho-planar mechanisms with prismatic joints is also considered. Prismatic joints are a practical necessity for the integration of linear planar actuators, which generally provide the motive force for planar ortho-planar mechanisms. Thus, a discussion of the use of such actuators in powering devices is a practical consideration.

The focus of the research then shifts to an investigation of the geometry of ortho-planar mechanisms employing spherical kinematics. In spherical kinematics, the joint axes of a mechanism intersect at a point. Here, as in planar mechanisms, the mathematical consequences of spherical mechanism geometry is developed. Drawing on the work done on planar mechanisms, this additional research shows the distinctions and similarities between spherical and planar ortho-planar mechanisms and the ensuing consequences for design strategies for the arrangement of links, joints and actuation.

These geometrical insights are then utilized for a number of ortho-planar mechanisms that are designed using spherical and planar kinematic techniques. The first
mechanism is the Micro Helico-Kinematic Platform (MHKP), which uses spherical crank-sliders to cause a platform to translate vertically and rotate. The second mechanism described combines the in-plane motion of a bistable mechanism (the Young mechanism), with a spherical crank-slider to achieve a bistable ortho-planar mechanism. The third mechanism is a three-degree-of-freedom platform, which uses three independently actuated spherical crank-sliders to achieve controllable tilting and vertical translation of a platform.

Chapter 2 describes work that has been done in describing the geometry of planar ortho-planar mechanisms with revolute joints. The work completed on this research to date has been published in connection with the 2004 Design Engineering Technical Conference (DETC) and is in review for publication in the *Journal of Mechanical Design*. Chapter 3 gives some illustration of the significance of the methods developed in Chapter 2, by demonstrating that broad classes of mechanisms that might be considered for fabrication do not function properly due to interference problems. Chapter 3 also contains other illustrations of analyses that can be performed on broad classes of planar OP mechanisms. Chapter 4 extends the methods developed in Chapter 2 for application to planar OP mechanisms with prismatic joints. Chapter 5 gives background on spherical kinematics and describes the extension of the techniques of Chapter 2 to spherical ortho-planar mechanisms. Chapter 6 describes the use of spherical mechanisms in MEMS and has been accepted for publication in the 2005 DETC conference. Chapter 7 describes ortho-planar mechanisms designed using spherical and planar design techniques and gives possible MEMS applications for the devices. In the final chapter, the conclusions that can be drawn for the work are given, the contributions of the work are summarized and recommendations for future work are made.
Chapter 2

THE FUNDAMENTALS OF PLANAR OP MECHANISMS

2.1 Introduction

This chapter develops a framework for the description and analysis of single-loop planar folded mechanisms. This framework is applicable to mechanisms with an arbitrary number of links and with arbitrary link lengths. A method is developed for assigning a coordinate in an abstract design space to the kinematic skeleton of possible single-loop folded mechanisms. Functions of those coordinates can describe geometric and kinematic limitations of the mechanisms. The motivation for the work is to explore the limitations of folded mechanisms used in microelectromechanical systems (MEMS) applications. Although extending the capability of MEMS is the primary justification for this work, the results are general and can be applied to other domains, such as deployable structures [48], and orthoplanar mechanisms [9].

The term folded mechanism has been applied in the past to planar 4R, slider-crank, and spherical 4R mechanisms [49]. Folded planar 4R and slider-crank mechanisms can have all their joints along a line. In the case of the spherical 4R mechanism, all four joints are aligned in a plane. Here, we wish to broaden the definition of folded mechanisms to
planar mechanisms with more than four links. Thus, a folded planar $NR$ mechanism has a configuration in which all joints lie along a line.

Planar folded mechanisms are used in MEMS devices when it is desired to create a mechanism with kinematically simple, i.e. planar, motion up from the plane of fabrication. Folded mechanisms make this motion possible because the joints lie along the line of intersection between the (vertical) plane of motion and the (horizontal) plane of fabrication. Surface micromachined MEMS devices that use planar kinematics and move up from the fabrication plane have been demonstrated by several researchers [50, 34, 35, 18]. An example folded mechanism is shown in its folded configuration in Figure 2.1 and in an out-of-plane configuration in Figure 2.2 [50].

Folded mechanisms present numerous design challenges. First, they typically have transmission angles that are near 0° or 180°. These transmission angles can make it difficult for actuators to cause the out-of-plane motion. Second, many of these mechanisms are \textit{change-point mechanisms} which means that their motion out of the folded configuration may be unpredictable in that it may arbitrarily move into one of two (or more) configurations. Finally, for micro folded mechanisms, the joints available in many MEMS processes are far from ideal. Problems with joint clearances and rotation limits may further limit design possibilities.

\section*{2.2 Basic Elements}

In this work we focus on the folded position of the links, and so the links of the $N$-link loop can be modeled with real-valued vectors. The ground link, denoted $r_1$, is defined to have positive or rightward orientation. Proceeding in a counter-clockwise
Figure 2.1: A typical folded micromechanism in its folded configuration [50]
fashion around the loop, the links are denoted as $r_2$, $r_3$, ..., $r_N$, as shown in Figure 2.3. In the folded position, in which all the links are parallel with the ground link, the orientation of the links can be rightward (positive) or leftward (negative).

Links are connected by joints, and in a single loop mechanism there are an equal number of links and joints. In this work, joints are numbered such that the joint has the same number as the link that is counter-clockwise of it as one proceeds around the loop (see Figure 2.3). Joints between links of the same orientation are called open joints (Figure 2.4(a)). Joints between links of different orientations are called closed joints. Closed joints can further be distinguished as right- and left-closed joints, as shown in Figure 2.4(b) and 2.4(c), respectively. One of the left closed joints and one of the right-closed joints represent the furthest extents of the mechanism, and as such are denoted extreme joints. Under specific geometric conditions, more closed joints may be extreme,
Figure 2.3: Illustration of a folded N-bar, including a) a sketch of an example N-bar mechanism, and b) a vector loop representation of the same mechanism

Figure 2.4: Schematic depiction of joints: a) open joint, b) right-closed joint and c) left-closed joint

i.e. two left- (or right-) closed joints with the same axis of rotation. The extreme joints for the mechanism shown in Figure 2.3 are joints 2 and 8.

2.3 Groups of Links and Joints

The three groups of links and joints basic to our discussion of the design space of folded linkages are orientation sets, adjacency sets, and oriented paths. Some terminology is helpful in discussing these groups or sets of links. The number of links in a set $M$ is designated the cardinality of that set and is denoted $\text{card}(M)$. To facilitate discussion, we define the measure of a set $M$ of links as the sum of the link lengths in the set and denote it symbolically as $\text{measure}(M)$. 

An orientation set is a collection of links with the same orientation in the folded position. Thus all right-laying links belong to the right-orientation set. Similarly, the left-laying links belong to the left-orientation set. For example, the right-orientation set for the mechanism in Figure 2.3 includes $r_1, r_3, r_8,$ and $r_9$.

An adjacency set consists of all the adjacent links between two sequential closed joints. Thus there are as many adjacency sets as there are closed joints. All the links between closed joints are joined by open joints and belong to the same orientation set. For example, the mechanism shown in Figure 2.3 has four closed joints and thus it has four adjacency sets: \{r_8, r_9, r_1\}, \{r_2\}, \{r_3\}, and \{r_4, r_5, r_6, r_7\}. In a closed loop, there is an even number of closed joints.

The number of closed joints (i.e. the number of adjacency sets) in a mechanism is called the plication of the mechanism and is represented by the variable $Q$. Thus, the mechanism in Figure 2.3 has a plication of $Q = 4$. The term plication comes from the verb plicate, which means to arrange in folds, like those of a fan. Plication is significant in MEMS because it gives a measure of how many links are co-linear, or overlapping in a planar projection. This has implications for either the number of stacked links and/or the number of side by side links in the mechanisms. Because the number of mechanically releasable layers is usually fixed for a specific fabrication process, there is typically an upper bound on the number of links that can be fabricated one on top of another. Thus, mechanisms with large plication values usually imply many links side by side. This typically represents an increase in the width of the mechanism and hence an increase in its overall size.
An oriented path is defined as the sequential collection of links as one proceeds in a CCW direction from one extreme joint to the other. Proceeding from the left extreme joint to the right extreme joint is the right-oriented path and from the right extreme joint to the left extreme joint is the left-oriented path. For example, the mechanism in Figure 2.3 has a right-oriented path \( \{r_8, r_9, r_1\} \).

These three groups are complete and exclusive, meaning that each link belongs to one and only one orientation set, one and only one adjacency set, and one and only one oriented path for a given folded position.

Figure 2.5 shows an additional three example mechanisms which are analyzed in terms of orientation sets, adjacency sets, and oriented paths. Table 2.1 lists the number of links, the right-laying links, the left-laying links, the initial links of each adjacency set, the right-closed joints, the left-closed joints, the extreme joints, and the plication of each example linkage. In Figure 2.5 the left-laying links are represented by leftward pointing arrows and the right-laying links by right pointing arrows. The initial link of each adjacency set is the link which is immediately CCW from a closed joint. The extreme joints depend on the relative lengths of the links in the different adjacency sets. The plication of each linkage is illustrated by a vertical shift between adjacency sets.

2.4 Construction of the Design Space

The design space of single-loop folded mechanisms is the catalogue of possible single-loop folded mechanisms. There is considerable value in developing a scheme for identifying each folded mechanism with a unique point in a continuous, bounded finite-dimensional space, \( U_N \). In this kind of scheme, the link lengths and link orientations of
Figure 2.5: Folded mechanisms described in Table 2.1

Table 2.1: Oriented path, orientation and adjacency set terminology for the mechanisms in Figures 2.3 and 2.5

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<td>Number of Links</td>
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<td>4</td>
<td>9</td>
<td>6</td>
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<td>Right-Laying Links</td>
<td>$r_1, r_3$, $r_8, r_9$</td>
<td>$r_1, r_3$</td>
<td>$r_1, r_3, r_5$, $r_6, r_8, r_9$</td>
<td>$r_1, r_2,$</td>
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<td>Left-Laying Links</td>
<td>$r_2, r_4, r_5$, $r_6, r_7$</td>
<td>$r_2, r_4$</td>
<td>$r_2, r_4$, $r_5$, $r_7$</td>
<td>$r_5,$</td>
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<tr>
<td>Initial Links of</td>
<td>$r_8, r_2$, $r_3, r_4$</td>
<td>$r_1, r_2$, $r_3, r_4$</td>
<td>$r_8, r_2, r_3$, $r_4, r_5, r_7$</td>
<td>$r_1,$</td>
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<td>2, 4, 7</td>
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<td>2, 3</td>
<td>5, 2</td>
<td>1, 5</td>
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<td>Links in Right</td>
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<td>$r_3, r_4, r_1$</td>
<td>$r_5$ to $r_9, r_1$</td>
<td>$r_1$ to $r_4$</td>
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<td>Oriented Path</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Links in Left</td>
<td>$r_2$ to $r_7$</td>
<td>$r_2$</td>
<td>$r_2$ to $r_4$</td>
<td>$r_5, r_6$</td>
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<tr>
<td>Oriented Path</td>
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<tr>
<td>Plication, $Q$</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>2</td>
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a folded linkage are used to designate a specific point in $U_N$. In order to ensure that a mechanism is uniquely represented in $U_N$ space, the orientation of the ground link and the sense (CCW) in which the loop is traversed is specified to ensure that only the front view (and not the back view) of a mechanism is included. On the other hand, because the motion of the folded mechanisms is not analyzed, if the mechanism can be moved to a different folded position, then the mechanism will be represented by a point in $U_N$ for each folded position. In the next section, the design space $U_N$ is characterized by the constraints of loop closure and dimensional similarity. These constraints lead to a theorem about the orientation sets which are true regardless of the number of links in each orientation set. The design space is then examined from a viewpoint that accounts for the number of links in each orientation set and develops distinctions based on the orientations of individual links.

2.4.1 Fundamental Constraints on Folded Mechanisms

We can write an expression for the total length of all the links in the linkage, i.e. the loop perimeter, $p$, as

\[ p = \text{measure}(M_R) + \text{measure}(M_L) \]  \hspace{1cm} (2.1)

where $M_R$ is the set of right-laying links and $M_L$ is the set of left-laying links. We may also write the constraint that guarantees loop closure as

\[ \text{measure}(M_R) - \text{measure}(M_L) = 0 \]  \hspace{1cm} (2.2)
Equations (2.1) and (2.2) can be solved simultaneously to express the measures of the orientation sets, \( \text{measure}(M_R) \) and \( \text{measure}(M_L) \), in terms of the perimeter. This result is fundamental to other developments, and is called the *Theorem of Equality of Orientation Set Measures (TEOSM)*, or

\[
\text{measure}(M_R) = \text{measure}(M_L) = \frac{p}{2} \quad (2.3)
\]

The TEOSM requires that, regardless of their placement in the linkage or the cardinality of either set, the measure of the links with a left orientation equal the measure of the links with a right orientation, and are each equal to half the perimeter of the linkage. This expresses the fundamental constraint on N-bar single-loop folded mechanisms.

There are two distinct features of the TEOSM. The first is that the links in the mechanism must be partitioned into right-laying links and left-laying links. The second is that once the links have been partitioned, the constraints on the link lengths of the left- and right-laying links are independent. The first aspect is discrete and combinatorial in nature, requiring a description of the number of different ways that the links may be partitioned into right- and left-laying links. The second aspect permits a continuous variation among links with the same orientation. The geometrical implications of the two aspects of the TEOSM for an N-bar with a specified number of links is best appreciated at this point by analogy. The regions of the design space having the same partition of links into left- and right orientations are like the faces of a polyhedron. Different points on the same ‘face’ represent mechanisms in which similarly partitioned links have the same orientation but different link lengths. Different ‘faces’ represent different partitions
of the links into left and right orientations. As in a polyhedron, the different ‘faces’ join together to form a closed convex hull—the design space, $U_N$. As an example, the design space of folded four-bar mechanisms, $U_4$, is shown in Figure 2.6 which shows the surface of an Archimedean solid called the cuboctahedron [52]. Antipodal points on the cuboctahedron represent the same mechanism where one point represents the front side of the mechanism and the other point represents the back side. A non-redundant version (i.e. only the front side of the mechanisms) of $U_4$ is shown in Figure 2.7. In order for the design space to be regarded as closed, the remaining antipodal points in Figure 2.7 should be thought of as being sewn onto each other\(^1\). In order to further justify this geometric characterization, it is necessary to describe mathematically the mapping of the catalogue of mechanisms into specific points in the design space $U_N$.

2.4.2 Mapping the Design Space

We continue our discussion of using the TEOSM to describe the design space by noting that the general folded $N$-bar can be described by an $N$-dimensional vector, $R$, whose components are the real numbers, $r_i$, that describe the length and orientation of each link. A positive value for $r_i$ represents a right-laying link, and a negative value is a left-laying link. The design space can be simplified by non-dimensionalizing the components of each mechanism. Thus, mechanisms that differ only by scale are mapped

\(^1\)The design space $U_4$ is topologically equivalent to the Real Projective Plane [53].
Figure 2.6: Illustration of the design space, $U_4$, (a cuboctahedron) of four-bar folded mechanisms. Points on the same face represent mechanisms whose links are partitioned in the same way into orientation sets. Antipodal points on the cuboctahedron represent the front and back sides of the same mechanism.

Figure 2.7: A second representation of the design space of four-bar folded mechanisms with redundant (antipodal) points removed. The dashed lines in this figure indicate an open set of points.
to the same point in the design space and are regarded as the same mechanism. This
non-dimensionalization is achieved by dividing the vector by half of the perimeter, $p$, or

$$
\rho_i = \frac{2r_i}{p}
$$

where $\rho_i$ is the normalized length and $r_i$ is the dimensional length of the $i^{th}$ link. The
perimeter, $p$, is given by

$$
p = \sum_i \abs(r_i)
$$

where $\abs(r_i)$ denotes the absolute value of $r_i$.

This scaling ensures that the nondimensional perimeter is 2 and the measure of
each orientation set is 1. This choice of normalization is preferred since it preserves
the symmetry of design space with respect to each link and highlights the dimensional
requirement that the TEOSM puts on each orientation set. The remainder of the chapter
assumes the use of normalized components.

Thus, equation (2.2), which expresses the requirement that the vector loop be
closed, can be written in normalized component form as

$$
\sum_{i=1}^{N} \rho_i = 0
$$

Equation (2.1), which expresses the perimeter in terms of the linkage components,
can be written

$$
\sum_{i=1}^{N} \abs(\rho_i) = 2
$$

The geometric significance of the absolute value in equation (2.7) is that the design
space has corners at locations where a component $\rho_i$ changes orientation. The fact that
the absolute value function produces corners is seen in the function $y = \text{abs}(x)$, which is plotted in Figure 2.8. When $x > 0$, $y = \text{abs}(x) = x$ and when $x < 0$, $y = \text{abs}(x) = -x$. When $x = 0$, both forms, $y = -x$ and $y = x$ are satisfied. Thus, at the corner when $x = 0$, the equation $y = \text{abs}(x)$ changes from one form to another and satisfies both forms.

The same principles apply in a multi-dimensional sense to the edges (such as the edges of the cuboctahedron shown in Figure 2.6) that results from the constraint expressed in equation (2.7).

Suppose that a four-bar design, $R_1$, is selected from the design space (see Figure 2.7). $R_1$ has components $\rho_1 > 0$, $\rho_2 < 0$, $\rho_3 > 0$, and $\rho_4 < 0$. Equation (2.7) could then be written

$$\sum_{i=1}^{4} \text{abs}(\rho_i) = \rho_1 - \rho_2 + \rho_3 - \rho_4 = 2$$

(2.8)
where \( \text{abs}(\rho_1) = \rho_1, \text{abs}(\rho_2) = -\rho_2, \text{abs}(\rho_3) = \rho_3, \) and \( \text{abs}(\rho_4) = -\rho_4. \) A different four-bar design, \( R_2, \) is also selected (see Figure 2.7) that has components \( \rho_1 > 0, \rho_2 < 0, \rho_3 > 0, \) and \( \rho_4 > 0. \) For this design, equation (2.7) can be written

\[
\text{for } R_2 : \sum_{i=1}^{4} \text{abs}(\rho_i) = \rho_1 - \rho_2 + \rho_3 + \rho_4 = 2 \quad (2.9)
\]

where \( \text{abs}(\rho_1) = \rho_1, \text{abs}(\rho_2) = -\rho_2, \text{abs}(\rho_3) = \rho_3, \) and \( \text{abs}(\rho_4) = \rho_4. \) Because the equation (2.7) has a different form for mechanism \( R_1 \) than for \( R_2 \) \( (\rho_4 \) changes sign), we can conclude that there is an edge in design space in between the two mechanisms, and that the edge occurs when \( \rho_4 = 0. \)

There are edges in design space whenever a component changes sign and equation (2.7) takes on a different form. Recall that the orientation of the ground link, \( \rho_1, \) was defined to be positive. The other \( N-1 \) links in the folded mechanism may be either right-laying or left-laying (positive or negative), yielding \( 2^{(N-1)} \) different possible ways of partitioning the the links of an \( N \)-bar into orientation sets. For a four-bar, the different forms of equation (2.7) can be expressed as a matrix:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 \\
1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & -1
\end{bmatrix}
\begin{bmatrix}
\rho_1 \\
\rho_2 \\
\rho_3 \\
\rho_4
\end{bmatrix}
= 
\begin{bmatrix}
2 \\
2 \\
2 \\
2 \\
2 \\
2 \\
2 \\
2
\end{bmatrix}
\quad (2.10)
\]
Note that the equation contained in the first row of the matrix,

\[ \rho_1 + \rho_2 + \rho_3 + \rho_4 = 2 \] (2.11)

indicates that all the links are right-laying (positive), which contradicts the four-bar form of equation (2.6), which is

\[ \rho_1 + \rho_2 + \rho_3 + \rho_4 = 0 \] (2.12)

and so there are no mechanisms represented by the first row of the matrix (equation (2.10)), that satisfy both equation (2.7) and equation (2.6). Thus, equation (2.7) takes on \(2^{(N-1)} - 1\) applicable forms, each of which when paired with equation (2.6) describes a region of design space known as an adjacency set patch. The term adjacency set patch serves to emphasize that the adjacency sets are the same for all the mechanisms described by the same equation forms. The union of the \(2^{(N-1)} - 1\) adjacency set patches is the design space, \(U_N\).

Adjacency set patches are named according to a convention which assigns the same letter (A, B, etc.) to patches that have similar arrangements of links into adjacency sets. For example, all four-bar patches with a plication of two that allocate three links to one adjacency set and one to the other are assigned the same letter. The patches are then assigned a pattern of numbers. Superscripts indicate links belonging to the right orientation set and subscripts indicate links belonging to the left orientation set. The numbers indicate the first link in each adjacency set beginning with the adjacency set that includes the ground link and proceeding CCW around the loop. For example, \(B_2^1\) represents an adjacency set patch in which \(\rho_1\) is right-laying and \(\rho_2\) to \(\rho_N\) are left-laying. The plication (number of adjacency sets) of the mechanisms represented in an adjacency
set patch is indicated by the total number of subscripts and superscripts in the patch’s name.

The TEOSM equations for the four-bar adjacency set patches are listed in Table 2.2. The patches $B_2^1$, $B_3^2$, $B_4^3$, and $B_4^1$ are equilateral triangles (shown in Figure 2.9). The ‘$B$’ patches describe mechanisms in which the longest link is the only link in its orientation set. The mechanisms in these patches are immobile, and have plication, $Q$, equal to two. Patches $C_3^1$, $C_2^4$, and $D_{13}^{24}$ are squares (shown in Figure 2.10). Patches $C_3^1$ and $C_2^4$ both consist of mechanisms with plication equal to two. Patch $D_{13}^{24}$ describes mechanisms with plication equal to 4, and can be subdivided into four triangular regions each of which has different extreme joints. Note that the $B$ patches are the triangular faces and the $C$ patches and the $D$ are the square faces of the cuboctahedron shown in Figure 2.6.

In an adjacency set patch, every possible mixture of lengths from the right orientation set is paired with every possible mixture of lengths from the left orientation set. Thus, every point in an adjacency set patch represents a mechanism and every possible mechanism with links having the specified orientation is represented in an adjacency set patch. The mechanisms represented by the points $\alpha^i_j$ in Figures 2.6 2.7 2.9 and 2.10 are points in which $\rho_i = 1$, and $\rho_j = -1$ and all other components equal to zero. For example, for a fivebar, $\alpha_2^1 = [1, -1, 0, 0, 0]^T$. These $\alpha^i_j$ points are the vertexes of the patches and are useful in constructing the design space but do not represent practical mechanisms.
Figure 2.9: Four-bar adjacency set patches representing immobile linkages: a) Patch $B_1^1$: Linkages satisfying $\rho_1 = -(\rho_2 + \rho_3 + \rho_4)$ b) Patch $B_2^2$: Linkages satisfying $\rho_2 = -(\rho_3 + \rho_4 + \rho_1)$ c) Patch $B_3^3$: Linkages satisfying $\rho_3 = -(\rho_4 + \rho_1 + \rho_2)$ d) Patch $B_4^1$: Linkages satisfying $\rho_4 = -(\rho_1 + \rho_2 + \rho_3)$

Table 2.2: Atlas of adjacency set patches in the design space of four-bar folded mechanisms

<table>
<thead>
<tr>
<th>Patch</th>
<th>Eq. 2.6 Eq. 2.7</th>
<th>TEOSM Equations</th>
<th>Mobility</th>
<th>Plication, Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$\rho_1 + \rho_2 + \rho_3 + \rho_4 = 0$ $\rho_1 + \rho_2 + \rho_3 + \rho_4 = 2$</td>
<td>Contradiction</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B_1^1$</td>
<td>$\rho_1 + \rho_2 + \rho_3 + \rho_4 = 0$ $\rho_1 - \rho_2 - \rho_3 - \rho_4 = 2$</td>
<td>$\rho_1 = 1$ $\rho_2 + \rho_3 + \rho_4 = -1$</td>
<td>Immobile</td>
<td>2</td>
</tr>
<tr>
<td>$B_2^2$</td>
<td>$\rho_1 + \rho_2 + \rho_3 + \rho_4 = 0$ $\rho_1 - \rho_2 + \rho_3 + \rho_4 = 2$</td>
<td>$\rho_3 + \rho_4 + \rho_1 = 1$ $\rho_2 = -1$</td>
<td>Immobile</td>
<td>2</td>
</tr>
<tr>
<td>$B_3^3$</td>
<td>$\rho_1 + \rho_2 + \rho_3 + \rho_4 = 0$ $\rho_1 + \rho_2 - \rho_3 + \rho_4 = 2$</td>
<td>$\rho_4 + \rho_1 + \rho_2 = 1$ $\rho_3 = -1$</td>
<td>Immobile</td>
<td>2</td>
</tr>
<tr>
<td>$B_4^1$</td>
<td>$\rho_1 + \rho_2 + \rho_3 + \rho_4 = 0$ $\rho_1 + \rho_2 + \rho_3 - \rho_4 = 2$</td>
<td>$\rho_1 + \rho_2 + \rho_3 + \rho_4 = 1$ $\rho_4 = -1$</td>
<td>Immobile</td>
<td>2</td>
</tr>
<tr>
<td>$C_1^1$</td>
<td>$\rho_1 + \rho_2 + \rho_3 + \rho_4 = 0$ $\rho_1 + \rho_2 - \rho_3 - \rho_4 = 2$</td>
<td>$\rho_1 + \rho_2 = 1$ $\rho_3 + \rho_4 = -1$</td>
<td>Mobile</td>
<td>2</td>
</tr>
<tr>
<td>$C_2^2$</td>
<td>$\rho_1 + \rho_2 + \rho_3 + \rho_4 = 0$ $\rho_1 - \rho_2 - \rho_3 + \rho_4 = 2$</td>
<td>$\rho_1 + \rho_1 = 1$ $\rho_2 + \rho_3 = -1$</td>
<td>Mobile</td>
<td>2</td>
</tr>
<tr>
<td>$D_1^3$</td>
<td>$\rho_1 + \rho_2 + \rho_3 + \rho_4 = 0$ $\rho_1 - \rho_2 + \rho_3 - \rho_4 = 2$</td>
<td>$\rho_1 + \rho_3 = 1$ $\rho_2 + \rho_4 = -1$</td>
<td>Mobile</td>
<td>4</td>
</tr>
</tbody>
</table>
Figure 2.10: Four-bar adjacency set patches representing mobile mechanisms: a) Patch $C_{3}^{1}$: Linkages $\rho_{1} + \rho_{2} = -(\rho_{3} + \rho_{4})$, b) Patch $C_{2}^{4}$: Linkages satisfying $\rho_{4} + \rho_{1} = -(\rho_{2} + \rho_{3})$, c) Patch $D_{24}^{13}$: Linkages satisfying $\rho_{1} + \rho_{3} = -(\rho_{2} + \rho_{4})$
2.4.3 Special Change-Point Mechanisms

A previous look at the design space of four-bar mechanisms by Barker [54] divides four-bar change-point (folded) mechanisms into six different categories; change-point crank-crank crank (CPCCC), change-point crank rocker rocker (CPCRR), change-point rocker crank rocker (CPRCR), change-point rocker rocker crank (CPRRC), double change-point (CP2X), and triple change-point (CP3X). Barker’s nomenclature is a shorthand where the letters CP indicate that the mechanisms are change-point mechanisms, and the other three letters indicate something about the mechanism’s motion. The three letters following CP have place value, i.e. the first letter refers to $\rho_2$, the second letter refers to $\rho_3$, and the third letter refers to $\rho_4$. If a link can rotate with respect to the ground link, it is designated with a ‘C’ for crank, if not, it is designated with an ‘R’ for rocker. The CP2X change-point has two pairs of links with equal lengths and all the links are of equal length for the CP3X change-point mechanisms. Barker’s mechanism categories are useful for comparison and correlate to distinct locations in the square patches of $U_4$ as shown in Figure 2.10. Table 2.3 gives the distinguishing criterion (based on Grashof’s Law [55]) for each category, and also gives the shape of each region in patches $C_3^1$, $C_2^4$ and $D_{24}^{13}$. The dashed lines in Figure 2.10 represent CP2X mechanisms, which have two pairs of links of equal length. The dashed lines connecting $\alpha_j^i$ points in which $i$ and $j$ are sequential represent deltoid (or kite) mechanisms, in which the link pairs of equal length are next to each other in the loop. The dashed lines (CP2X mechanisms) connecting $\alpha_j^i$ points in which $i$ and $j$ are not sequential represent parallelogram/antiparallelogram mechanisms in which the link pairs are opposite to each other in the loop.
Table 2.3: Barker’s [54] distinctions on the patch

<table>
<thead>
<tr>
<th>Change-Point Category</th>
<th>Dimensional Criteria</th>
<th>Shape of Region of the Patch in Figure 2.10</th>
<th>Vertex Points of Region on each Patch</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPCCC</td>
<td>$\rho_1$ is shortest link</td>
<td>Triangles</td>
<td>$C_{13}^1: \alpha_1^2, \alpha_3^2, \beta_{24}^3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$C_{24}^2: \alpha_3^1, \alpha_4^3, \beta_{23}^4$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_{24}^{13}: \alpha_2^4, \alpha_2^3, \beta_{34}^1$</td>
</tr>
<tr>
<td>CPCRR</td>
<td>$\rho_2$ is shortest link</td>
<td>Triangles</td>
<td>$C_{13}^1: \alpha_1^4, \alpha_3^4, \beta_{24}^3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$C_{24}^2: \alpha_4^1, \alpha_4^3, \beta_{23}^4$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_{24}^{13}: \alpha_1^4, \alpha_3^4, \beta_{34}^1$</td>
</tr>
<tr>
<td>CPRCR</td>
<td>$\rho_3$ is shortest link</td>
<td>Triangles</td>
<td>$C_{13}^1: \alpha_1^3, \alpha_3^3, \beta_{24}^3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$C_{24}^2: \alpha_3^1, \alpha_4^3, \beta_{23}^4$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_{24}^{13}: \alpha_1^3, \alpha_4^3, \beta_{34}^1$</td>
</tr>
<tr>
<td>CRRR</td>
<td>$\rho_4$ is shortest link</td>
<td>Triangles</td>
<td>$C_{13}^1: \alpha_1^2, \alpha_4^2, \beta_{34}^3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$C_{24}^2: \alpha_2^1, \alpha_4^3, \beta_{23}^4$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_{24}^{13}: \alpha_2^1, \alpha_4^3, \beta_{34}^1$</td>
</tr>
<tr>
<td>CP2X</td>
<td>Two pairs of equal length lines</td>
<td>Pairs of intersecting lines</td>
<td>$C_{13}^1: \alpha_1^4$ to $\alpha_4^4$, $\alpha_3^2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$C_{24}^2: \alpha_1^3$ to $\alpha_4^3$, $\alpha_3^2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_{24}^{13}: \alpha_2^3$ to $\alpha_4^3$, $\alpha_1^2$</td>
</tr>
<tr>
<td>CP3X</td>
<td>All links equal length</td>
<td>Points</td>
<td>$C_{3}^{13}: \beta_{24}^3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$C_{24}^2: \beta_{23}^4$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_{24}^{13}: \beta_{24}^1$</td>
</tr>
</tbody>
</table>

In Figure 2.10 we also include the CP3X mechanisms, which are located at the intersection of the two dashed lines representing CP2X mechanisms. The CP3X mechanisms are represented by $\beta$’s with superscripts corresponding to their right-laying links and subscripts corresponding to their left-laying links.

2.5 Generalization: The Design Space of Folded $N$-Bars

Adding an additional link to the loop adds a dimension to the design space. Thus, four-bar patches are two dimensional and five-bar patches are three dimensional. In general, the design space of a folded $N$-bar contains patches that are $N - 2$ dimensional that join together to make a $N - 1$ dimensional polytope. These higher dimensional patches
are no longer shapes familiar from planar and solid geometry, but their mathematical representations are still valid.

It is possible to construct the design space of folded $N$-bars for any value of $N$. In order to do so, we describe the mathematical conditions that apply to the orientation sets. The following conditions hold on the right orientation set

$$\rho_i \geq 0$$  \hspace{1cm} (2.13)

and

$$\sum_{\rho_i \in M_R} \rho_i = 1$$  \hspace{1cm} (2.14)

Similar requirements hold for the left orientation set

$$\rho_i \leq 0$$  \hspace{1cm} (2.15)

and

$$\sum_{\rho_i \in M_L} -\rho_i = 1$$  \hspace{1cm} (2.16)

The conditions in equations (2.13) and (2.14) are those of standard topological building blocks called *Euclidean simplexes*. Euclidean simplexes come in a number of different forms, 0-simplexes, 1-simplexes, ... $n$-simplexes. Flanders [56] describes them as follows: “A 0-simplex is a single point ($P_0$)” This corresponds to an orientation set with a cardinality of one. Flanders continues “A 1-simplex is a line segment ... with endpoints ($P_0$, $P_1$). A 2-simplex is a closed triangle ... with vertices ($P_0$, $P_1$, $P_2$). A 3-simplex is a tetrahedron with vertices ($P_0$, $P_1$, $P_2$, $P_3$). The n-simplex is the closed convex hull ($P_0$,
..., \( P_n \) of \((n + 1)\) independent points taken in definite order.” The \( n \)-simplex describes the design space geometry of an orientation set with \( n + 1 \) links.

Euclidean simplexes provide intuition for the geometry of the orientation sets. The geometry of each orientation set follows a progression from point to line to triangle to tetrahedron to higher-order closed convex hulls as links are added to it. The geometry of the adjacency patch is related to the Euclidean simplexes for each orientation set. For example, the geometry of a patch that has one right-laying link, \( \rho_1 \), and three left-laying links; \( \rho_2, \rho_3 \) and \( \rho_4 \); combines the geometry of a point (0-simplex) and a triangle (2-simplex) to form a triangle, as shown in Figure 2.11(a). The single point represents the TEOSM equation for the right orientation set, \( \rho_1 = 1 \), and the triangle represents the TEOSM equation for the left orientation set, \( \rho_2 + \rho_3 + \rho_4 = -1 \).

In a different example, the geometry of a patch that has two right-laying links and two left-laying links combines two line segments (1-simplexes) to form a square, as shown in Figure 2.11b. The horizontal line segment represents the TEOSM equation for the right orientation set, \( \rho_1 + \rho_2 = 1 \) and the vertical line represents the TEOSM equation for the left orientation set, \( \rho_3 + \rho_4 = -1 \). The point in the horizontal line labeled 1 represents a right orientation set in which \( \rho_1 = 1 \) and \( \rho_2 = 0 \). Moving along the horizontal line from 1 to 2 decreases \( \rho_1 \) and increases \( \rho_2 \) until at point 2, \( \rho_2 = 1 \) and \( \rho_1 = 0 \). Each point in the simplex represents a possible mixture of lengths in an orientation set.

The geometry of the adjacency set patches for fivebar folded mechanisms can be described in terms of combinations of Euclidean simplexes. As was the case for 4-bars, the \( A \) patch is empty, and the \( B \) patches represent immobile structures in which one
Figure 2.11: Illustration of the combination of Euclidean simplexes to form adjacency set patches. a) Combination of a 0-simplex with a 2-simplex. b) Combination of two 1-simplexes.

orientation set contains only the longest link and the other orientation set contains the other links. The Euclidean simplexes that correspond to orientation sets with one and four links are a point and a tetrahedron, respectively. The combination of a point and a tetrahedron is a tetrahedral patch. Other ways of partitioning the links into orientation sets place two links in one set and three links in the other. The Euclidean simplexes corresponding to orientation sets with two and three links are a line and a triangle, respectively. The combination of the line and triangle results in a wedge-shaped patch. Figure 2.12 shows these different patch shapes and describes the plication of the mechanisms they represent.
Figure 2.12: Schematics of the three basic patches of a fivebar folded mechanism: a) tetrahedral shaped patch type B: $\rho_5 = -(\rho_1 + \rho_2 + \rho_3 + \rho_4)$, b) wedge-shape patch type C: $\rho_4 + \rho_5 = -(\rho_1 + \rho_2 + \rho_3)$, c) wedge-shaped patch type D: $\rho_3 + \rho_5 = -(\rho_1 + \rho_2 + \rho_4)$. The mechanisms represented by patch types B and C have a plication of two and the mechanisms represented by patch type D have a plication of four.
2.6 Parameterizing an Adjacency Set Patch

An adjacency set patch may be parameterized with any point in the patch designated as an origin and a set of \( N - 2 \) independent vectors serving as basis vectors. As an example, consider parameterizing the \( B_2^1 \) patch with the point \( \alpha_2^1 \) as the origin. An arbitrary vector \( R \) in the patch is written in component form and the TEO-SM equations are solved for \( \rho_1 \) and \( \rho_2 \), the coordinates associated with the designated origin. The TEO-SM equations are then substituted into \( R \), to yield a vector expression that incorporates the constraints of the patch.

\[
R = \begin{bmatrix}
\rho_1 \\
\rho_2 \\
\rho_3 \\
\rho_4 \\
\end{bmatrix} = \begin{bmatrix}
1 \\
-1 - \rho_3 - \rho_4 \\
0 \\
0 \\
\end{bmatrix} \begin{bmatrix}
1 \\
-1 + \rho_3 \\
0 \\
0 \\
\end{bmatrix} \begin{bmatrix}
0 \\
-1 \\
1 \\
1 \\
\end{bmatrix} = \alpha_2^1 + \rho_3 \alpha_2^3 + \rho_4 \alpha_2^4
\]

(2.17)

Care must be taken when using a parametric expression such as equation (2.17) to ensure that proper constraints are applied to the parameters. In this case, the parameters \( \rho_3 \) and \( \rho_4 \) vary between 0 and 1, and there is a constraint that their sum also be between 0 and 1. This last constraint can be derived when the constraints on left-laying vectors, equations (2.15) and (2.16), are substituted for \( \rho_2 \).

\[
-1 < \rho_2 < 0
\]

\[
-1 < -1 - \rho_3 - \rho_4 < 0
\]

\[
0 < -\rho_3 - \rho_4 < 1
\]

(2.18)

\[
-1 < r_3 + r_4 < 0
\]
Once a patch has been parameterized it becomes reasonable to compute functions of those parameters. An example of this is provided in the next section.

2.7 Functions on Design Space

Once the TEOSM has been used to construct the design space and the adjacency sets have been parameterized, functions of those parameters may be computed for points in the design space. A function on design space maps the vector, $R$, representing the mechanism’s dimensions, to a real number. Here, we examine a function on the design space that computes a mechanism’s length in its inline position. Other functions that might be of interest are link rotatability, and positions of interference with the substrate.

2.7.1 Example Function: Mechanism Length

The length of the mechanism, along with its width (found using plication), determines the footprint or on-chip area of a mechanism. Thus, the approximate size of a particular mechanism can be estimated from the mechanism’s location in design space. To find the length of the mechanism, the distance between extreme joints is calculated as a function of the mechanism’s location in design space. The nondimensional length, $\lambda$, of the mechanism is found by determining the links $j$ to $k$ that make up the right-oriented path, $P_r$. The links $j$ and $k$ are the links such that the sum of oriented link lengths, $\rho_i$, taken CCW about the mechanism’s loop from $j$ to $k$, inclusive, is a maximum.

$$\lambda = \sum_{\rho_i \in P_r} \rho_i$$ \hspace{1cm} (2.19)

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Because the sum of the links in equation (2.19) is a maximum from link \(j\) to link \(k\), link \(j\) and link \(k\) are each positive, i.e. right-laying. Further, because link \(k + 1\) and link \(j - 1\) each act to reduce the sum, they are negative, i.e. left-laying. Thus, proceeding about the loop, the link \(j\) is the first link in a right adjacency set, and the link \(k + 1\) is the first link in a left adjacency set. Thus, the link \(j\) must be the first link in an adjacency set, and the link \(k + 1\) must be the first link in a left-adjacency set. Thus, the level of complexity of the path depends not on the number of links in the mechanism but in the number of adjacency sets.

Because the oriented paths are identical to the orientation sets in a mechanism with a plication of two (see Figure 2.5c), the length of such mechanisms is half the perimeter. Only one of the adjacency set patches of a four-bar, \(D_{13}^{13}\), has more than two adjacency sets. Because there are four adjacency sets (and only four links), one of the oriented paths consists of three links and the other oriented path consists of a single link. That single link is the largest link of the four and its length is the length of the mechanism. Figure 2.13 shows the non-dimensionalized length of four-bar folded mechanisms in \(D_{13}^{13}\) as a function of their location in the patch. The contours of length are concentric squares about the point \(\beta_{13}^{13}\), a four-bar with all links equal in length. At that point the length function is a minimum, and is equal to 0.5 in non-dimensional units \((l(\beta_{13}^{13}) = p/4)\). The length function increases until it is equal to 1.0 at the patch boundaries \((l = p/2)\). Because all the patches adjacent to patch \(D_{13}^{13}\) have constant length functions of one-half the perimeter, the length function is continuous at the patch boundary.
Figure 2.13: Nondimensionalized length of four-bar folded mechanisms with plication of $Q = 4$
For the purposes of calculating the length function it is convenient to take the point $\beta_{24}^{13}$ as the origin of the patch. An arbitrary point on the patch is located by the expression

$$R = \beta_{24}^{13} + (\alpha_2^3 - \alpha_4^1)\eta_1 + (\alpha_2^1 - \alpha_4^3)\eta_2$$  \hspace{1cm} (2.20)$$

where $\eta_1$ and $\eta_2$ are scalar parameters that multiply the fixed length basis vectors (see Figure 2.13). The constraints on these parameters are that $-1/2 \leq \eta_1 \leq 1/2$, $-1/2 \leq \eta_1 \leq 1/2$, and $\text{abs}(\eta_1) + \text{abs}(\eta_2) < 1/2$. The length, $\lambda$, of the folded four-bar mechanisms represented by the $D_{24}^{13}$ patch can then be written as

$$\lambda(\eta_1, \eta_2) = 0.5 + \text{abs}(\eta_1) + \text{abs}(\eta_2)$$  \hspace{1cm} (2.21)$$

This result can be easily generalized to fivebars with a plication, $Q = 4$. As an example, there is a fivebar adjacency patch which consists of mechanisms with right orientation set $M_R = \{\rho_1, \rho_3\}$ and left orientation set $M_L = \{\rho_2, \rho_4, \rho_5\}$. The difference between these mechanisms and the four-bar mechanisms described by equation (2.20) is that here $\rho_4$ has been replaced by $\rho_4$ and $\rho_5$. Thus, similar equations can be used to parameterize the patch.

$$R = \begin{bmatrix}
1/2 & 1 & 0 \\
-1/2 & -1 & 0 \\
1/2 & +\eta_1 & +\eta_2 \\
-1/2 & 1 & 1 \\
0 & 0 & -1
\end{bmatrix} = \gamma_1 + (\alpha_2^3 - \alpha_4^1)\eta_1 + (\alpha_2^1 - \alpha_4^3)\eta_2 + \alpha_4^4\eta_3$$  \hspace{1cm} (2.22)$$
where $\gamma_1$ is the five-component version of $\beta_{13}^{24}$, i.e. a zero is added as the last component to change a 4-vector to a 5-vector. The vectors multiplying $\eta_1$ and $\eta_2$ have the same names as they had previously, but here they are interpreted as 5-vectors instead of 4-vectors. Lastly, there is a component $\eta_3$ which permits motion in the $\alpha_5^4$ direction, i.e. it permits length to be removed from $\rho_4$ and added to $\rho_5$. The length function, $\lambda$ on this five-bar patch can then be written as

$$\lambda(\eta_1, \eta_2, \eta_3) = 0.5 + \text{abs}(\eta_1) + \text{abs}(\eta_2)$$

(2.23)

The $\eta_3$ component does not change the length function because it does not alter the measure of an adjacency set.

### 2.8 Conclusions

In this chapter, we have derived the Theorem of Equality of Adjacency Set Measures (TEOSM) for folded mechanisms. We have shown the use of the TEOSM in constructing adjacency set patches and in joining them together to construct the design space. A method was illustrated for parameterizing the adjacency set patches. A length function on design space was constructed and its applicability for determining the size of certain MEMS linkages was discussed. The nomenclature and design space construction techniques developed in this chapter provide a novel framework for discussing properties of certain classes of linkages.

In considering a MEMS device as a solution to an engineering problem, questions like ‘Could a MEMS device do ... ?’ naturally arise. Such questions are fundamentally difficult, and are usually answered based on experience and intuition more than analysis.
The critical difficulty is that the question deals not with a specific mechanism that can be analyzed but with a rather broad class of mechanisms with infinite different possibilities. In order to answer this type of question, one must be able to make general conclusions about entire classes of mechanisms. This kind of general conclusion is possible using the mathematical descriptions made possible by the TEOSM.

The constraints expressed by the TEOSM limit its applicability to planar ortho-planar linkages. The typical layout of such linkages is shown in Figure 2.14. Planar ortho-planar mechanisms must be fabricated in plane (shaded in gray) parallel to the substrate (white). They are designed to move normal to the substrate in the region indicated in the figure. The horizontal lines on the plane of fabrication indicate possible positions and orientations for the joint axes. Planar kinematics requires that the joint axes be parallel, and the requirement for ortho-planar motion is that they be parallel to the substrate.

The geometry of general planar mechanisms is modeled using systems of two dimensional vectors in the plane of motion. On the other hand, for ortho-planar planar mechanisms in the their initial position the two-dimensional vectors must be collinear and lie at the intersection of the plane of motion and the plane of fabrication. Since the intersection of the two planes is a line, planar ortho-planar mechanisms inherit some of the symmetries associated with a line. Henderson [57] discusses a number of these symmetries, two of which are significant for the TEOSM. The first is called half-turn symmetry implies that along a line, either direction can be defined as positive. This ambiguity means that some convention must be adopted, such as that the ground link is in the positive direction. The second, self-similarity means that a line looks like a line at any degree of magnification or
Figure 2.14: Schematic of the design layout of a typical ortho-planar planar mechanism. The links have width $W$, and are laid out along a line of length $L$. Several possible joint positions are shown as lines parallel to the substrate plane. The motion of the links of the planar ortho-planar mechanism lies in the plane shown perpendicular to the joint axes and the substrate plane.

scale. The TEOSM incorporates self-similarity by scaling every vector loop representing a mechanism to a standard length. These geometric symmetries or constraints together with the technique of analyzing mechanisms using closed vector loops constitute the fundamental assumptions of the TEOSM.

In later chapters, extensions of the TEOSM for including prismatic joints (Chapter 4), and for spherical mechanisms (Chapter 5) are discussed. The work involved in extending the TEOSM consists of determining to what extent the constraints of the TEOSM as expressed for planar ortho-planar linkages apply to mechanisms with prismatic joints and spherical geometries.
Chapter 3

ANALYSIS OF PLANAR OP MECHANISMS

3.1 Introduction

The design space patches derived in Chapter 2 may be used to analyze groups of OP mechanisms. A function that describes a particular aspect of an OP mechanism may be graphically represented through the use of contour plots. The $x$ and $y$ directions on a contour plot are used to specify the mechanism’s link lengths and the $z$ direction gives the function value. In this chapter, functions are constructed on planar OP four-bars which describe a MEMS device’s tendency to interfere with the substrate, a mechanism’s link’s relative motion (kinematic coefficients), the rotation limits for a mechanism’s motion, and the height that a mechanism could move a specific coupler point.

In the previous chapter, the function that was computed was a length function that did not require the mechanism to move. The functions computed in this chapter all derive the motion of the mechanism. Thus, we review the position analysis of a single-loop planar four-bar mechanism.
3.2 Position Analysis of OP Four-Bars

A single loop OP \( N \)-bar with one ground link has \( N - 3 \) degrees of freedom. Much of what follows may be extended for more complicated mechanisms, but for purposes of illustration, the simplest mobile case, \( N = 4 \), is used. Closed form solutions for the four-bar mechanism may be derived using the law of cosines as illustrated in the following equations using variables which are defined in Figure 3.1.
\[ \delta = \sqrt{\rho_1^2 + \rho_2^2 - 2|\rho_1||\rho_2| \cos(\pi - \theta_2)} \]  \hspace{1cm} (3.1)

\[ \beta = \cos^{-1}\left(\frac{\rho_1^2 + \delta^2 - \rho_2^2}{2|\rho_1|\delta}\right) \]  \hspace{1cm} (3.2)

\[ \psi = \cos^{-1}\left(\frac{\rho_3^2 + \delta^2 - \rho_4^2}{2|\rho_3|\delta}\right) \]  \hspace{1cm} (3.3)

\[ \lambda = \cos^{-1}\left(\frac{\rho_1^2 + \delta^2 - \rho_3^2}{2|\rho_4|\delta}\right) \]  \hspace{1cm} (3.4)

The presence of the substrate means that \(0 \leq \theta_2 \leq \pi\) and hence the solution for \(\theta_3\) and \(\theta_4\) takes on two forms, one leading and one lagging, as shown in Figure 3.1.

For the leading form

\[ \theta_3 = \beta + \pi - \psi \]  \hspace{1cm} (3.5)

\[ \theta_4 = \beta + \pi + \lambda \]  \hspace{1cm} (3.6)

For the lagging form

\[ \theta_3 = \beta + \pi + \psi \]  \hspace{1cm} (3.7)

\[ \theta_4 = \beta + \pi - \lambda \]  \hspace{1cm} (3.8)

In the leading form, the joint between link 3 and link 4 is left of the line labeled \(\delta\), and in the lagging form that joint is to the right of the \(\delta\) line. The particular form that an OP mechanism assumes upon leaving the inline position may be determined by energy considerations when one form results in the system having lower potential energy. The design of the joints may also play a role by limiting the rotation in such a way that only one form is possible.
3.3 Interference of the Substrate with Planar OP MEMS

In considering the aptness of a particular four-bar for moving a coupler point to a standard height, 100 μm, some initial work must be done to demonstrate that motion is possible given the presence of the substrate. Some four-bar designs produce motion that would require that a link pass through the substrate and are not suitable. In the discussion that follows, it is assumed that the input motion is a rotation of link 2 in the direction that rotates it up from the substrate. The details of how such motion is accomplished are given in Chapter 4.

Any interference of the substrate plane may be calculated from the direction of the motion of the output link (link 4). If a rotation of the input link up from the substrate requires that the output link rotate beneath the substrate then the mechanism will not function appropriately.

In Chapter 2, nomenclature was developed for various types of folded four-bars. The designs that make up the $B$ patches are immobile and need no further analysis. The $C$ patches and the $D$ patch are mobile and can be analyzed. A complete discussion of all the design possibilities of mobile four-bars considers the leading and lagging form for each of the mechanisms making up the $C_3^1$, $C_2^4$, and $D_{24}^{13}$ patches.

The rotation direction of the output link may be obtained using small motion approximations to the general equation for a four-bar, which is

$$|ρ_1|e^{iθ_1} + |ρ_2|e^{iθ_2} + |ρ_3|e^{iθ_3} + |ρ_4|e^{iθ_4} = 0$$

and can be written in $x−y$ components as
\[ |\rho_1| \cos(\theta_1) + |\rho_2| \cos(\theta_2) + |\rho_3| \cos(\theta_3) + |\rho_4| \cos(\theta_4) = 0 \quad (3.10) \]
\[ |\rho_1| \sin(\theta_1) + |\rho_2| \sin(\theta_2) + |\rho_3| \sin(\theta_3) + |\rho_4| \sin(\theta_4) = 0 \quad (3.11) \]

A small motion analysis may be performed on equations (3.10) and (3.10) by modeling the angular deflection of a right-laying link as

\[
\cos(\theta) = 1 - \frac{\epsilon^2}{2} \quad (3.12)
\]
\[
\sin(\theta) = \epsilon \quad (3.13)
\]

and the angular deflection of a left-laying link as

\[
\cos(\theta) = \cos(\pi + \epsilon) = -1 + \frac{\epsilon^2}{2} \quad (3.14)
\]
\[
\sin(\theta) = \sin(\pi + \epsilon) = -\epsilon \quad (3.15)
\]

where \( \epsilon \) is understood to represent a small angular deflection. Equation (3.10) simplifies (for small angular displacements) to

\[
\rho_2 \epsilon_2^2 + \rho_3 \epsilon_3^2 + \rho_4 \epsilon_4^2 = 0 \quad (3.16)
\]

and Equation (3.11) simplifies to

\[
\rho_2 \epsilon_2 + \rho_3 \epsilon_3 + \rho_4 \epsilon_4 = 0 \quad (3.17)
\]

where \( \rho_i \) is positive for right-laying links and negative for left-laying links.
Equations (3.16) and (3.17) may be solved simultaneously for \( \epsilon_3/\epsilon_2 \) and \( \epsilon_4/\epsilon_2 \), which are a good approximation to the kinematic coefficients \( h_{32} = \omega_3/\omega_2 \) and \( h_{42} = \omega_4/\omega_2 \), respectively, when the mechanism is near (all angles within 40° for < 5% error) the folded configuration. The kinematic coefficients, \( h_{32} \) and \( h_{42} \), are

\[
h_{32} = -\frac{\rho_2}{\rho_3 + \rho_4} \pm \frac{\sqrt{\rho_1\rho_2\rho_3\rho_4}}{\rho_3(\rho_3 + \rho_4)} \quad (3.18)
\]

\[
h_{32} = -\frac{\rho_2}{\rho_3 + \rho_4} \pm \frac{\sqrt{\rho_1\rho_2\rho_3\rho_4}}{\rho_4(\rho_3 + \rho_4)} \quad (3.19)
\]

The significance of the kinematic coefficient, \( h_{ij} \), is that

- when \( h_{ij} > 1 \), link \( i \) is rotating in the same direction and faster than link \( j \).
- when \( 0 < h_{ij} < 1 \), link \( i \) is rotating in the same direction and slower than link \( j \).
- when \( h_{ij} = 0 \), link \( i \) is not rotating.
- when \( -1 < h_{ij} < 0 \), link \( i \) is rotating in the opposite direction and slower than link \( j \).
- when \( h_{ij} < -1 \), link \( i \) is rotating in the opposite direction and faster than \( j \).
- when \( h_{ij} \) is infinite, link \( j \) is not rotating.

Table 3.1 summarizes information about the kinematic coefficients and the implications for substrate interference for the various mechanism types and gives references to figures with details about specific mechanism types. For each four-bar mechanism patch, two figures are listed for both the leading and lagging form. The first figure shows contour plots of the kinematic coefficients \( h_{32} \) and \( h_{42} \). The second figure gives three
Figure 3.2: Schematic of the $C_3^1$ patch indicating the typical appearance of mechanisms in the a) leading form and b) lagging form. Gray shading indicates that the mechanisms will be unable to move due to interference with the substrate.

different example mechanisms of the three different types for each of the forms on each patch. The different types of mechanisms are classified by certain geometric criteria and are characterized by a particular range of kinematic coefficients. The last column in Table 3.1 states whether the output direction of the mechanism results in interference with the substrate. For patches $C_3^1$ and $C_2^1$, interference results when $h_{42} < 0$. For patch $D_{24}^{13}$, interference results when $h_{42} > 0$. In Figures 3.2, 3.4, and 3.6 which give typical example mechanisms, regions of design space that result in interference for a given form of the mechanism are shaded gray.

3.4 Rotation Limits of Planar OP Linkages

The limits on the rotation of planar OP linkages are defined by a toggle position which prevents further motion or interference with the substrate that removes mobility.
Table 3.1: Kinematic coefficients and substrate interference for folded four-bar mechanisms

<table>
<thead>
<tr>
<th>Mechanism</th>
<th>Geometric Criterion</th>
<th>Kinematic Coefficient</th>
<th>Motion Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Patch Form Types</td>
<td></td>
<td>Coefficient Range</td>
<td></td>
</tr>
<tr>
<td>$C_3^1$ Leading See</td>
<td>$p_3 &gt; p_1$</td>
<td>0 &lt; $h_{32}$ &lt; 1</td>
<td>No interference</td>
</tr>
<tr>
<td>$C_3^1$ Figure 3.2b and</td>
<td>$p_3 = p_1$</td>
<td>$h_{32} = 0$</td>
<td>No interference</td>
</tr>
<tr>
<td>$C_3^1$ Figure 3.3a&amp;b</td>
<td>$p_3 &lt; p_1$</td>
<td>$h_{32} &lt; 0$</td>
<td>No interference</td>
</tr>
<tr>
<td>$C_4^2$ Leading See</td>
<td>$p_4 &gt; p_2$</td>
<td>0 &lt; $h_{32}$ &lt; 1</td>
<td>No interference</td>
</tr>
<tr>
<td>$C_4^2$ Figure 3.4b and</td>
<td>$p_4 = p_2$</td>
<td>$h_{32} = 0$</td>
<td>No interference</td>
</tr>
<tr>
<td>$C_4^2$ Figure 3.5a&amp;b</td>
<td>$p_4 &lt; p_2$</td>
<td>$h_{32} &lt; 0$</td>
<td>No interference</td>
</tr>
<tr>
<td>$D_{24}^{13}$ Leading See</td>
<td>$p_1 &gt; p_2$</td>
<td>$h_{32} &lt; 0$</td>
<td>No interference</td>
</tr>
<tr>
<td>$D_{24}^{13}$ Figure 3.6a and</td>
<td>$p_1 = p_2$</td>
<td>$h_{32} = \infty$</td>
<td>No interference</td>
</tr>
<tr>
<td>$D_{24}^{13}$ Figure 3.7a&amp;b</td>
<td>$p_1 &lt; p_2$</td>
<td>$h_{32} &gt; 1$</td>
<td>Interference</td>
</tr>
<tr>
<td>$D_{24}^{13}$ Figure 3.8a and</td>
<td>$p_1 &gt; p_4$</td>
<td>$h_{32} &gt; 1$</td>
<td>No interference</td>
</tr>
<tr>
<td>$D_{24}^{13}$ Figure 3.9a and</td>
<td>$p_2 &gt; p_3$</td>
<td>$h_{42} &lt; 0$</td>
<td>No interference</td>
</tr>
<tr>
<td>$D_{24}^{13}$ Figure 3.10a and</td>
<td>$p_2 = p_3$</td>
<td>$h_{42} = \infty$</td>
<td>No interference</td>
</tr>
<tr>
<td>$D_{24}^{13}$ Figure 3.11a and</td>
<td>$p_2 &lt; p_3$</td>
<td>$h_{42} &gt; 1$</td>
<td>Interference</td>
</tr>
</tbody>
</table>
Figure 3.3: Kinematic coefficients for the leading form, shown in parts a & b, and lagging form, shown in parts c & d, of mechanisms in the $C_{3}^{1}$ patch at the instant they have left the folded configuration. The kinematic coefficient, $h_{32}$, is shown in parts a & c, and the kinematic coefficient, $h_{42}$, is shown in parts b & d.
Thus, for a four-bar in which the first link is ground and the second link is the input, the third the coupler and the fourth the output link, the input link will not be able to rotate past certain toggle positions or a point in which either the input or output link contacts the substrate. The maximum rotation of the input link depends on the initial configuration of the links, the relative lengths of the links and whether the linkage assumes the leading or the lagging form.

In the absence of external physical stops, a linkage’s rotation limit occurs when two adjacent links are collinear. The condition in which two adjacent links are collinear may be termed a critical rotation condition. In these critical conditions, the links form a triangle, with the two collinear links forming one side of the triangle ($s_1$) and the other two links forming the other sides of the triangle ($s_2$ and $s_3$). The triangle inequality states
Figure 3.5: Kinematic coefficients for the leading form, shown in parts a & b, and lagging form, shown in parts c & d, of mechanisms in the $C_2^4$ patch at the instant they have left the folded configuration. The kinematic coefficient, $h_{32}$, is shown in parts a & c, and the kinematic coefficient, $h_{42}$, is shown in parts b & d.
that the sum of the lengths of any two sides of a triangle is greater than the length of the third side, or

\[ s_1 + s_2 \geq s_3 \]
\[ s_2 + s_3 \geq s_1 \]
\[ s_3 + s_1 \geq s_2 \]

(3.20)

When the triangle inequality is applied to mechanisms in the critical rotation condition, three different inequalities are produced. When the collinear links have the same orientation, \( s_1 = \rho_i + \rho_j \). When collinear links have opposite orientations, \( s_1 = \)
Figure 3.7: Kinematic coefficients for the leading form, shown in parts a & b, and lagging form, shown in parts c & d, of mechanisms in the $D_{24}^{13}$ patch at the instant they have left the folded configuration. The kinematic coefficient, $h_{32}$, is shown in parts a & c, and the kinematic coefficient, $h_{42}$, is shown in parts b & d.
There are four distinct additive combinations:

\[ s_1 = \rho_1 + \rho_2 \]
\[ s_1 = \rho_2 + \rho_3 \]
\[ s_1 = \rho_3 + \rho_4 \]
\[ s_1 = \rho_4 + \rho_1 \]  \hspace{1cm} (3.21-a)

and eight distinct subtractive combinations:

\[ s_1 = \rho_1 - \rho_2 \]
\[ s_1 = \rho_2 - \rho_3 \]
\[ s_1 = \rho_3 - \rho_4 \]
\[ s_1 = \rho_4 - \rho_1 \]
\[ s_1 = \rho_2 - \rho_1 \]  \hspace{1cm} (3.21-b)
\[ s_1 = \rho_3 - \rho_2 \]
\[ s_1 = \rho_4 - \rho_3 \]
\[ s_1 = \rho_1 - \rho_4 \]

Thus, there are twelve critical rotation conditions in all.

Some of the inequalities that can be produced using equations (3.20) and (3.21) state that the sum of lengths of three of the links must be greater than the length of the fourth link. These inequalities are trivial because they are satisfied by all of the mechanisms that are capable of motion. The nontrivial inequalities state that combined length of two links is greater than the combined length of the other two links. These inequalities limit the applicability of critical rotation condition to a specific region of design.
Figure 3.8: The twelve critical rotation positions, or positions that could represent a limit of the mechanism’s motion, which are described in Table 3.2.

The twelve critical rotation conditions are shown in Figure 3.8 and described in Table 3.2 where the nontrivial triangle inequalities are shown, the region of applicability of the critical condition is described, and it is stated which form, leading or lagging, the mechanism takes in the critical condition.

In the first four critical conditions, the collinear links have the same orientation, and only one non-trivial inequality is produced. This one inequality contradicts the TEOSM equation for a single patch, and thus the critical condition is applicable to the other two patches. On each of those two patches, half of the mechanisms satisfy the inequality and half do not. In the last eight critical conditions, the collinear links have opposite orientations and two non-trivial inequalities are produced. These two inequalities contradict the TEOSM equations of two of the three patches that have mobile
<table>
<thead>
<tr>
<th>Condition</th>
<th>Triangle Inequality</th>
<th>Region of Applicability</th>
<th>Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>( r_3 + r_4 &gt; r_1 + r_2 )</td>
<td>( C_2^4 ) and ( D_{24}^{13} )</td>
<td>Leading</td>
</tr>
<tr>
<td>II</td>
<td>( r_1 + r_4 &gt; r_2 + r_3 )</td>
<td>( C_3^1 ) and ( D_{24}^{13} )</td>
<td>Leading</td>
</tr>
<tr>
<td>III</td>
<td>( r_1 + r_2 &gt; r_3 + r_4 )</td>
<td>( C_2^4 ) and ( D_{24}^{13} )</td>
<td>Both</td>
</tr>
<tr>
<td>IV</td>
<td>( r_2 + r_3 &gt; r_1 + r_4 )</td>
<td>( C_3^1 ) and ( D_{24}^{13} )</td>
<td>Leading</td>
</tr>
<tr>
<td>V</td>
<td>( r_1 + r_3 &gt; r_2 + r_4 ) ( r_1 + r_4 &gt; r_2 + r_3 )</td>
<td>( C_3^1 )</td>
<td>Leading</td>
</tr>
<tr>
<td>VI</td>
<td>( r_1 + r_2 &gt; r_3 + r_4 ) ( r_2 + r_4 &gt; r_1 + r_3 )</td>
<td>( C_2^4 )</td>
<td>Lagging</td>
</tr>
<tr>
<td>VII</td>
<td>( r_1 + r_3 &gt; r_2 + r_4 ) ( r_2 + r_3 &gt; r_1 + r_4 )</td>
<td>( C_3^1 )</td>
<td>Lagging</td>
</tr>
<tr>
<td>VIII</td>
<td>( r_2 + r_4 &gt; r_1 + r_3 ) ( r_3 + r_4 &gt; r_1 + r_2 )</td>
<td>( C_2^4 )</td>
<td>Lagging</td>
</tr>
<tr>
<td>IX</td>
<td>( r_2 + r_3 &gt; r_1 + r_4 ) ( r_2 + r_4 &gt; r_1 + r_3 )</td>
<td>( C_3^1 )</td>
<td>Lagging</td>
</tr>
<tr>
<td>X</td>
<td>( r_3 + r_4 &gt; r_1 + r_2 ) ( r_1 + r_3 &gt; r_2 + r_4 )</td>
<td>( C_2^4 )</td>
<td>Lagging</td>
</tr>
<tr>
<td>XI</td>
<td>( r_2 + r_4 &gt; r_1 + r_3 ) ( r_1 + r_4 &gt; r_2 + r_3 )</td>
<td>( C_3^1 )</td>
<td>Both</td>
</tr>
<tr>
<td>XII</td>
<td>( r_1 + r_2 &gt; r_3 + r_4 ) ( r_1 + r_3 &gt; r_2 + r_4 )</td>
<td>( C_2^4 )</td>
<td>Lagging</td>
</tr>
</tbody>
</table>
Table 3.3: Critical rotations as a function on design space

<table>
<thead>
<tr>
<th>Patch</th>
<th>Form</th>
<th>Critical Conditions</th>
<th>Illustration</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_3^1$</td>
<td>Leading</td>
<td>(II), IV, V, XI</td>
<td>Figure 3.9a</td>
</tr>
<tr>
<td></td>
<td>Lagging</td>
<td>IX, XI</td>
<td>Figure 3.9b</td>
</tr>
<tr>
<td>$C_2^4$</td>
<td>Leading</td>
<td>I, III</td>
<td>Figure 3.10a</td>
</tr>
<tr>
<td></td>
<td>Lagging</td>
<td>III, (VI), VIII</td>
<td>Figure 3.10b</td>
</tr>
<tr>
<td>$D_{24}^{13}$</td>
<td>Leading</td>
<td>I, (II), III</td>
<td>Figure 3.11a</td>
</tr>
<tr>
<td></td>
<td>Lagging</td>
<td>III</td>
<td>Figure 3.11b</td>
</tr>
</tbody>
</table>

linkages, leaving a single patch in which the condition applies. In that patch, only a quarter of the mechanisms in the patch satisfy both of the inequalities.

Some of the critical conditions are inapplicable to micro-mechanisms due to interference with the substrate (conditions VII, X, and XII). On the other hand, conditions II and VI occur but do not represent the limit of the motion of the input link (it does represent the position of maximum rotation of the output link).

Once the limiting conditions have been derived and their area of applicability determined, it is convenient to display them on the design space. Table 3.3 lists the critical conditions that occur in the leading and lagging forms of the mechanisms in the $C_3^1$ patch, the $C_2^4$ patch, and the $D_{24}^{13}$ patch. Critical conditions II and VI are given parenthetically because they occur but do not represent end conditions for the motion. Illustrations of the end conditions for the different patches and forms are given in the figures listed in the last column.
Figure 3.9: The limiting conditions on the rotation of the input link in $C^1_3$ patch for mechanisms assuming the a) leading form and b) lagging form. Condition IV: Rotation is limited because link 4 contacts the substrate. Conditions V and IX: Rotation is limited because link 2 contacts the substrate. Condition XI: Rotation is limited because of a toggle position for links 3 and 4. Gray shading indicates that the mechanisms will be unable to move due to interference with the substrate.

Figure 3.10: The limiting conditions on the rotation of the input link in $C^4_2$ patch for mechanisms assuming the a) leading form and b) lagging form. Condition I: Rotation is limited because link 2 contacts the substrate. Condition III: Rotation is limited because of a toggle position for links 3 and 4. Condition VIII: Rotation is limited because link 4 contacts the substrate. Gray shading indicates that the mechanisms will be unable to move due to interference with the substrate.
3.5 Input Rotation Range as a Function on Design Space

Based on the rotation limits described in the previous section it is possible to determine the rotation range of the input link for the mechanisms in the $C_3^1$ patch, the $C_2^4$ patch and the $D_{24}^{13}$ patch.

Table 3.4 gives references to contour plots which indicate the amount of rotation that the input link of an OP four-bar can undergo. Regions in which no input rotation is possible are shaded gray. Regions in which full (180°) rotation of the input is possible are white with the annotation “Full Rotation”. Note that the rotations of right-laying input links are positive and the rotation of left-laying input links are negative.
Table 3.4: Input rotation range as a function on design space

<table>
<thead>
<tr>
<th>Patch</th>
<th>Form</th>
<th>Contour Plot</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_3^1$</td>
<td>Leading</td>
<td>Figure 3.13a</td>
</tr>
<tr>
<td></td>
<td>Lagging</td>
<td>Figure 3.13b</td>
</tr>
<tr>
<td>$C_2^4$</td>
<td>Leading</td>
<td>Figure 3.14a</td>
</tr>
<tr>
<td></td>
<td>Lagging</td>
<td>Figure 3.14b</td>
</tr>
<tr>
<td>$D_{24}^{13}$</td>
<td>Leading</td>
<td>Figure 3.14a</td>
</tr>
<tr>
<td></td>
<td>Lagging</td>
<td>Figure 3.14b</td>
</tr>
</tbody>
</table>

Figure 3.12: Plot of the rotation range of $\theta_2$ for the a) leading form and b) lagging form of mechanisms in the $C_3^1$ patch. Gray shading indicates that the mechanisms will be unable to move due to interference with the substrate.

Figure 3.13: Plot of the rotation range of $\theta_2$ for the a) leading form and b) lagging form of mechanisms in the $C_2^4$ patch. Gray shading indicates that the mechanisms will be unable to move due to interference with the substrate.
3.6 Non-dimensional Height of a Coupler Point as a Function on Design Space

When a planar ortho-planar mechanism is used as a manipulator to lift an object above the substrate, it is important to know the maximum height that it is capable of achieving. Based on the non-dimensionalization in which an orientation set has a length of 1, the maximum height achieved by the center of link 3 (the center was chosen for illustration purposes) has been computed. Table 3.5 gives references to contour plots in which the relative height of a coupler point has been computed as a function on design space.

3.7 Conclusions

This chapter has illustrated the use of the design space in calculating several different functions on design space for OP four-bars. Such functions can be used to choose the optimal OP four-bar mechanism for a particular application. Examples of kinematic
Table 3.5: The relative height of a coupler point as a function on design space

<table>
<thead>
<tr>
<th>Patch</th>
<th>Form</th>
<th>Contour Plot</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{3}^{1}$</td>
<td>Leading</td>
<td>Figure 3.15a</td>
</tr>
<tr>
<td></td>
<td>Lagging</td>
<td>Figure 3.15b</td>
</tr>
<tr>
<td>$C_{2}^{4}$</td>
<td>Leading</td>
<td>Figure 3.16a</td>
</tr>
<tr>
<td></td>
<td>Lagging</td>
<td>Figure 3.16b</td>
</tr>
<tr>
<td>$D_{24}^{13}$</td>
<td>Leading</td>
<td>Figure 3.17a</td>
</tr>
<tr>
<td></td>
<td>Lagging</td>
<td>Figure 3.17b</td>
</tr>
</tbody>
</table>

Figure 3.15: Plot of the maximum height of the center of link 3 for the a) leading form and b) lagging form of mechanisms in the $C_{3}^{1}$ patch. Gray shading indicates that the mechanisms will be unable to move due to interference with the substrate.
Figure 3.16: Plot of the maximum height of the center of link 3 for the a) leading form and b) lagging form of mechanisms in the $C^4_{2}$ patch. Gray shading indicates that the mechanisms will be unable to move due to interference with the substrate.

Figure 3.17: Plot of the maximum height of the center of link 3 for the a) leading form and b) lagging form of mechanisms in the $D^4_{24}$ patch. Gray shading indicates that the mechanisms will be unable to move due to interference with the substrate.
coefficients, interference with the substrate, rotation limits, and the non-dimensional height of a coupler point have been demonstrated.
Chapter 4

PLANAR OP CRANK-SLIDERS

In this chapter, the TEOSM is extended to describe mechanisms which include prismatic joints. The significance of such mechanisms, particularly the crank-slider, and its use in connecting a linear actuator to an ortho-planar mechanism, is described.

4.1 Extending the TEOSM to Linkages with Prismatic Joints

In the previous chapters, the design of ortho-planar mechanisms with revolute joints was investigated using the TEOSM. The TEOSM was derived based on the assumptions that planar OP mechanisms have a folded position and that they are described using closed vector loops. In the mechanisms described previously, a vector represented a continuum of material between two revolute joints and was of a fixed length. By relaxing the requirement that the vector be a fixed length, the vector loops that were used to describe mechanisms with only revolute joints can describe mechanisms with prismatic joints ( sliders). Thus, we use the dimensional form \( \mathbf{r} \) and non-dimensional form \( \mathbf{\rho} \) to represent vectors that have a fixed length, and dimensional form \( \mathbf{s} \) and nondimensional form \( \mathbf{\sigma} \) to describe vectors in which the length is allowed to change.

Figure 4.1 shows a schematic that indicates that \( s \)-vectors could consist of two links that translate with respect to each other in order to shorten or lengthen. Because
the length as well as the angle of the $s$-vector can change, it has one more degree of freedom than the $r$-vector. Thus, if a vector loop consisting of $r$-vectors is modified by replacing $r$-vectors with $s$-vectors, then the vector loop will have one additional degree of freedom for each $r$-vector replaced.

The distinction between $r$-vectors and $s$-vectors is noticeable only when they move out-of-plane. In the folded configuration, which is typically the fabricated position for MEMS, they are subject to the same constraints. The similarity of $s$ vectors and $r$ vectors in the folded configuration means that the design space geometry developed in Chapter 2 is applicable to mechanisms which include prismatic joints.

Although the design space geometry is identical for mechanisms with and without prismatic joints, the process for establishing a correspondence between a catalogue of mechanisms and points in design space is not. In Chapter 2, it was explained that a vector of $N$ real numbers could represent the $N$ links of a mechanism. The sign of a component (positive or negative) indicated the orientation of a corresponding link (right-laying or left-laying). When prismatic joints are allowed, auxiliary information needs to be provided which specifies which of the components are $r$-vectors and which are $s$-vectors (or non-dimensional $\rho$-vectors and $\sigma$-vectors). This information can be specified.

---

Figure 4.1: Schematic representing $s$-vectors, vectors which can change length, and $r$-vectors, vectors which have fixed length. Both $s$-vectors and $r$-vectors have revolute joints at the ends of the vector.
Figure 4.2: Schematic of the three possible configurations for folded threebars. These particular linkages are immobile, but can be transformed into a mobile mechanisms by transforming at least one \(r\)-vector to an \(s\)-vector.

as part of a verbal description of a certain mechanism class or as a binary vector of ones and zeros, where ones represent \(s\)-vectors and zeros represent \(r\)-vectors.

A basic example is the ortho-planar crank-slider, a one-degree-of-freedom mechanism with one prismatic joint. The underlying design space is associated with the zero-degree-of-freedom system of three rigid links and three revolute joints which is modeled with three \(r\)-vectors, \(r_1\), \(r_2\) and \(r_3\). If \(r_1\) is the ground link and the vectors are labeled in a counterclockwise loop, there are three possible inline configurations, which are shown in Figure 4.2. These three configurations are described by the TEOSM equations for a threebar:

\[
A: \quad r_1 = -(r_2 + r_3) = p/2, \tag{4.1}
\]

\[
B: \quad -r_2 = r_3 + r_1 = p/2, \tag{4.2}
\]

\[
C: \quad -r_3 = r_1 + r_2 = p/2. \tag{4.3}
\]

where the letter preceding each equations corresponds to the mechanism labels in Figure 4.2.
Each of these three inline configurations can be converted to a one-degree-of-freedom mechanism model in three different ways, i.e. by changing one of the $r_i$-vectors to a $s_i$-vector. The resulting nine configurations are shown in Figure 4.3. They are labeled $A$-$C$ according to the underlying TEOSM equation, and numbered 1-3 according to the $r$-vector that was converted to an $s$-vector. The mechanisms are shown slightly perturbed from their inline configurations to illustrate the different motions allowed. Mechanisms $A1$, $B1$ and $C1$, are crank-sliders and the $s$-vector represents a slider that can move with respect to the substrate. Mechanisms $A2$, $A3$, $B2$, $B3$, $C2$ and $C3$ are inverted crank-sliders and require two links not in contact with the substrate to slide over each other. The design of such sliders could be challenging in MEMS, and their motion could be problematic and prone to binding.

The original threebar mechanisms $A$, $B$ and $C$ are related by inversion\(^1\). Similarly, the derived mechanisms are related by inversion. Any of the links in a mechanism can be designated as the ground, or reference link. The set of mechanisms that is formed by designating each of the links, in turn, as the ground link is an inversion group. Mechanisms $A1$, $B2$, and $C3$ form one inversion group, mechanisms $B1$, $C2$, and $A3$ form another inversion group, and mechanisms $C1$, $A2$, and $B3$ form a third inversion group.

In mechanisms $B1$, $C1$, and their inversions the $s$-vector must lengthen in order to move out of the folded position. On the other hand, in mechanisms $A1$ and its inversions the $s$-vector must shorten. Mechanisms that require lengthening $s$-vectors are called opening sliders and those that require shortening $s$-vectors are called closing sliders. The following criteria may be used to predict whether a slider will be opening, closing or both:

\(^{1}\text{Recall that inversion means that the link that is designated as the ground link is changed.}\)
Figure 4.3: Schematic of the different one-degree-of-freedom mechanisms that are derived from the zero-degree-of-freedom threebar. The three inline configurations labeled A), B) and C) can be converted to a one-degree-of-freedom mechanism model in three different ways i.e. by changing one of the \(r\)-vectors to a \(s\)-vector. The resulting nine configurations are labeled A-C according to the corresponding threebar, and numbered 1-3 according to the \(r\)-vector that was converted to an \(s\)-vector. The mechanisms are shown slightly perturbed from their folded configuration, thus the sliders have been shorten or lengthen slightly to illustrate the motion of the mechanisms.
**Closing slider:** If one of the orientation sets in the underlying design space consists of a single vector, and if that single vector is an s-vector, it is a closing slider.

**Opening slider:** If one of the orientation sets in the underlying design space consists of a single vector, and the s-vector belongs to the other orientation set, then it is an opening slider.

**Opening and closing slider:** If neither orientation set consists of a single link, then the slider may either open or close.

The s-vector can be physically realized in a number of different ways. For example, the mechanism labeled A1 in Figure 4.3 may be realized in any of the ways shown in Figure 4.4. Thus, a physical realization of these kinematic models may account for other design considerations. The third model shown in Figure 4.4 illustrates that the single s-vector may consist of multiple links, like a telescoping antenna. Such mechanisms have redundant constraints and degrees of freedom but may prove useful in mechanism design.

The examples in Figure 4.3 are sufficient to illustrate how the design space concepts developed in Chapter 2 may be extended to include mechanisms with prismatic joints but do not exhaust the possibilities of such mechanisms. For the remainder of this chapter, the focus will be on the crank-slider mechanism, which is significant in OP mechanism design.

### 4.2 The Significance of the Crank-Slider in OP Mechanism Design

The design of planar linkages with only revolute joints was considered in Chapter 2. Such linkages are generally advantageous because they have considerable design freedom,
Figure 4.4: Sketch indicating the different ways of realizing the kinematic model shown in part A1 of Figure 4.3. Note that the side of the mechanism that is fixed to ground is not specified, nor is the number of links that make up the prismatic link.

they are simple to manufacture (even in MEMS) and the techniques for their synthesis are well developed. However, in the context of ortho-planar MEMS they suffer from one critical drawback. In order to directly actuate an ortho-planar linkage at a revolute joint, it may be necessary to rotate a shaft parallel to the substrate, which is difficult at the micro-level because of the lack of approaches for applying a moment to a shaft. While devices exist that do directly actuate shafts, they rely on either layers underneath or special processing to provide upward thrust. Examples of out-of-plane actuation include micro-mirrors using coiled wires etched on the rotating surface [58], and multi-layer devices in which lower layers are buckled upward to produce the upward thrust to rotate a shaft [59]. The special processing or multiple layers required by these mechanisms make them less than ideal.
An alternative solution involves using MEMS actuators that produce linear motion parallel to the substrate and then transfer the linear motion to rotary motion using a crank-slider. This type of linear motion to rotary motion is very common at the macro-scale; for example, it is used in car engines to transfer the linear motion of the piston to the crankshaft. It has also been previously used in MEMS devices (see for example [60]). Here, a systematic catalogue of crank-sliders is developed for use in OP mechanisms using a design-space formulation.

It was shown in the previous section that there are three different OP crank-slider models which can be readily realized in MEMS, which are shown as configurations $A_1$, $B_1$ and $C_1$ in Figure 4.3. A primary purpose of these devices in OP MEMS may be to transmit an actuating force parallel to the substrate to a link that is rotating out-of-plane. Figure 4.5 shows the input and output ports for the forces for the kinematic model $A_1$, $B_1$ and $C_1$.

The crank-slider mechanisms belonging to the mechanism families $A_1$, $B_1$, and $C_1$ can each be represented by a non-dimensional design vector, $R = [\sigma_1, \rho_2, \rho_3]$. The TEOSM constraint (in non-dimensional form) for the $A_1$ family of mechanisms in their folded position is $\sigma_1 = -(\rho_2 + \rho_3) = 1$. Thus, $R_{A1} = [1, \rho_2, -1 - \rho_2]$, and the folded configuration of a member of the $A_1$ family is specified by $\rho_2$ which varies between $-1$ and $0$. For the $B_1$ family, the TEOSM constraint is $-\rho_2 = \sigma_1 + \rho_3 = 1$. Thus, $R_{B1} = [1 - \rho_3, -1, \rho_3]$, and the folded configuration of a member of the $B_1$ family is specified by $\rho_3$ which varies between $0$ and $1$. For the $C_1$ family, The TEOSM constraint
Figure 4.5: Diagram indicating the location and directions assumed for the input and output forces in the mechanical advantage calculations: a) schematic for kinematic model A1, b) schematic for kinematic model B1 and c) schematic for the kinematic model C1.

$$3 = 1 + 2 = 1.$$ Thus, $R_{C1} = [1 - \rho_2, \rho_2, -1]$, and the folded configuration of a member of the C1 family is specified by $\rho_2$ which varies between 0 and 1.

A crank-slider mechanism (from the A1, B1, or C1 families) is a one-degree-of-freedom mechanism, and its motion can be specified by a single input parameter. Because these devices are used to transform linear motion to rotary motion, the displacement of the slider, $\Delta\sigma_1$, is taken as the input and used to compute aspects of the motion, such as the output rotational displacement and the mechanical advantage.

The output rotational displacement and the mechanical advantage were computed for a family of designs based on the A1 kinematic model. The design of the mechanism from the A1 configuration is specified by $\rho_2$, and the input displacement is specified as a change $\Delta\sigma_1$ from the original $\sigma_1$, and it is negative because $\sigma_1$ is a closing slider. The
results are shown in Figure 4.6, which shows a contour plot of the output angle, $\theta_{out}$, where $\theta_{out}$ is shown in Figure 4.5. Figure 4.7 shows a contour plot of the mechanical advantage, where mechanical advantage is defined as the output force ($F_{out}$) divided by the input force ($F_{in}$), where the forces are defined as shown in Figure 4.5. Figures 4.6 and 4.7 show the dependence of the output angle and the mechanical advantage on a design variable, $\rho_2$, and on a motion variable, $\Delta \sigma_1$. Thus, information about a particular design is contained in the vertical lines in the figures. The shaded portions of the figures are regions that are outside of the motion range of the mechanisms. Thus, one inference that can be made is that, for a given footprint, mechanisms in which the coupler link, $\rho_3$, and the output link, $\rho_2$, are the same length ($\rho_2 = 0.5$), have the largest range of motion for the input slider.
Figure 4.7: Contour plot of the mechanical advantage, \( \frac{F_{\text{out}}}{F_{\text{in}}} \), as a function of the change in the slider length, \( \Delta \sigma_1 \), and the design length, \( \rho_2 \), for crank-slider design A1.

The output displacement and mechanical advantage were also calculated for mechanisms using the B1 kinematic model. The design of the mechanisms is specified by \( \rho_3 \), and their motion by slider displacement \( \Delta \sigma_1 \). The slider displacement is positive because the B1 mechanisms have opening sliders. Figure 4.8 is a contour plot of the angular rotation of the output link as a function of the design parameter \( \rho_3 \) and the slider displacement \( \Delta \sigma_1 \). Figure 4.9 is a contour plot of the mechanical advantage of the B1 mechanisms. As in the previous contour plots, the gray area in the contour plot indicates the limits of the motion for the different designs.

Finally, the mechanical advantage and output rotational displacement were calculated using the C1 kinematic model. The design of the mechanisms is specified by \( \rho_2 \), and their motion by slider displacement \( \Delta \sigma_1 \). The slider displacement is positive because the C1 mechanisms have opening sliders. Figure 4.10 is a contour plot of the angular
Figure 4.8: Contour plot of the output angle, $\theta_{out}$, as a function of the change in the slider length, $\Delta \sigma_1$, and the design length, $\rho_3$, for crank-slider design $B1$.

Figure 4.9: Contour plot of the mechanical advantage, $\frac{F_{out}}{F_{in}}$, as a function of the change in the slider length, $\Delta \sigma_1$, and the design length, $\rho_3$, for crank-slider design $B1$. 
rotation of the output link as a function of the design parameter $\rho_2$ and the slider displacement $\Delta \sigma_1$. Figure [4.11] is a contour plot of the mechanical advantage of the C1 mechanisms. As in the previous contour plots, the gray area in the contour plot indicates the limits of the motion for the different designs. The mechanical advantage of these devices becomes infinite at the maximum output angle, which occurs when the coupler link rotates through $90^\circ$.

4.3 A OP Lifting System Via Crank-Sliders and Thermal Actuators

The rest of this chapter describes an example micropositioning mechanism that illustrates the combination of micro crank-sliders and micro fourbars. The device is called the XZ Micropositioning Mechanism (XZMM) [50], and was designed for positioning optical elements.
Figure 4.11: Contour plot of the mechanical advantage, $\frac{F_{\text{out}}}{F_{\text{in}}}$, as a function of the change in the slider length, $\Delta \sigma_1$, and the design length, $\rho_2$, for crank-slider design C1.

4.3.1 XZMM Description

The XZMM is shown in its folded position in Figure 4.12 and in an out-of-plane position Figure 4.13. It is assumed that the mechanism is intended to lift a micro-sphere weighing 100 $\mu$N to a height of 10 microns. Figure 4.14 shows a schematic of the XZMM which indicates the naming conventions used for the XZMM and the location of the input force and the output force. The crank-slider design is A1 and its design space is parameterized by the nondimensional length $\rho_2$ which varies between 0 and 1. The output link of the crank-slider is the input link of the fourbar, and the links in the fourbar are assumed to have the same length as the output link of the crank-slider. Thus, the design of the XZMM is determined by the selection of $\rho_2$. 

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Figure 4.12: SEM of XZMM in its fabricated position [50]
Figure 4.13: SEM of XZMM in an actuated position

Figure 4.14: Schematic of the XZMM mechanism. Links $s_1$, $r_2$, and $r_3$ are the crank-slider portion of the XZMM. Links $r_3$, $r_4$, $r_5$ and $r_6$ are the four-bar portion of the XZMM. For the example discussed in this section, links $r_3$, $r_4$, $r_5$ and $r_6$ are of equal length.
Some analysis can be performed based on a non-dimensional version of the XZMM. However, because the specifications for the mechanism state a desired vertical displacement, it is possible to determine an appropriate scale for different mechanism designs (different values of \( \rho_2 \)). The maximum height (10 micrometers) of the XZMM platform is achieved when \( \theta_3 = 270^\circ \) if the output link of the slider is shorter than the coupler link of the slider (\( \rho_2 > 0.5 \)). The maximum height is achieved when \( \theta_2 = 90^\circ \) if the output link of the slider is longer than the coupler link of the slider (\( \rho_2 < 0.5 \)). Thus, the nondimensional height, \( \eta = \frac{h}{r_2 + r_4} \), of the XZMM is limited by the smaller of the two nondimensional lengths \( \rho_2 \) and \( \rho_3 \). A scale factor, \( D \), can be found by dividing the specified maximum dimensional height, \( h_{\text{max}} \) (10 micrometers) by the smaller of the two links \( \rho_2 \) and \( \rho_3 \).

\[
D = \frac{h_{\text{max}}}{\min(\rho_2, \rho_3)}
\]  

(4.5)

Multiplying each link length in the non-dimensional design by the scale factor results in a design for a dimensional mechanism that achieves the required height of 10 micrometers, or

\[
r_i = D \rho_i
\]  

(4.6)

The scale factor can also be used to determine the dimensional input distance, \( \Delta s_1 \), at which the required output height is achieved. It is also possible to determine the input force, \( F_{\text{in}} \), required to lift a 100 \( \mu \)N sphere.

### 4.3.2 XZMM Design Calculations

Figure 4.15 shows the scale factor as a function of the non-dimensional design parameter \( \rho_2 \). The dramatic increase in the scale factor as \( \rho_2 \) approaches either 0 or 1 is
Figure 4.15: Plot of the scale factor for the mechanisms as a function of the design parameter $\rho_2$.

due to the relatively small amount of rotation permitted by these designs. Figure 4.16 shows the rotation, $\theta_2$, of the coupler link, $\rho_2$, of the crank-slider as a function of the non-dimensional design parameter $\rho_2$ and the dimensional input displacement $s_1$. The most important qualitative difference demonstrated in this figure is the difference between mechanisms with $\rho_2 < 0.5$ ($\rho_2$ is smaller than $\rho_3$) and those with $\rho_2 > 0.5$ ($\rho_2$ is larger than $\rho_3$). For mechanisms with $\rho_2 < 0.5$, the coupler link $\rho_2$ may undergo a rotation of 180°, while for mechanisms with $\rho_2 > 0.5$ the rotation of the coupler link is less.

Figure 4.17 shows the output angle of the crank-slider, $\theta_3$, as a function of the non-dimensional design parameter $\rho_2$ and the dimensional input displacement $s_1$. The results are similar to those shown in Figure 4.16 in that there are important distinctions between mechanisms with $\rho_2 < 0.5$ and those with $\rho_2 > 0.5$. 
Figure 4.16: Contour plot of the input angle, $\theta_2$, of the crank-slider portion of the XZMM as a function of the design parameter $\rho_2$ and the output displacement $s_1$.

Figure 4.17: Contour plot of the output angle, $\theta_3$, of the crank-slider portion of the XZMM as a function of the design parameter $\rho_2$ and the dimensional output displacement $s_1$. 
Figure 4.18: Contour plot of the height, $h$, of the XZMM as a function of the design parameter $\rho_2$ and the dimensional output displacement $s_1$.

Figure 4.18 shows the dimensional height of the XZMM, $h$, as a function of the non-dimensional design parameter $\rho_2$ and the dimensional input displacement $s_1$. Note that the maximum height (10 micrometers) of the XZMM platform is achieved when $\theta_3 = 270^\circ$ if the output link of the slider is shorter than the coupler link of the slider ($\rho_2 > 0.5$) or when $\theta_2 = 90^\circ$ if the output link of the slider is longer than the coupler link of the slider ($\rho_2 < 0.5$).

Figure 4.19 shows the mechanical advantage of the XZMM as a function of the non-dimensional design parameter $\rho_2$ and the dimensional input displacement $s_1$. Figure 4.20 is very closely related, and shows the input force, $F_{in}$, required to produce an output force of 100 $\mu$N as a function of the non-dimensional design parameter $\rho_2$ and the dimensional output displacement $s_1$. All designs require very large forces at $\Delta s_1 = 0$. The input force required drops to zero as the mechanism approaches the maximum height. The input
Figure 4.19: Contour plot of the mechanical advantage of the XZMM as a function of the design parameter $\rho_2$ and the dimensional output displacement $s_1$.

force becomes negative after the maximum height is reached, implying that weight of the microsphere is pushing in the same direction as the actuator. The implication is that a constraint, such as a physical stop, should be incorporated in the mechanism to prevent over-shooting the maximum height.

Figure 4.21 shows two curves. The solid curve indicates the mechanism footprint for each design; the dashed curve indicates the required input displacement. The input displacement is symmetric about $\rho_2 = 0.5$, the mechanism footprint is close but not quite symmetric. The footprint of mechanisms with $\rho_2 > 0.5$ have smaller footprint than mechanisms with $\rho < 0.5$ with the same input displacement. Both the footprint and the input displacement can be considered indices of merit for the XZMM, with smaller values of each being an indication of a superior design. Weighted linear combinations of the two curves yields a third curve, $w(k) = (k*\text{footprint}) + (1-k)*\text{input displacement}$, 

Figure 4.20: Contour plot of the input force, $F_{in}$, required for the XZMM to lift a microsphere weighing $100 \mu N$ as a function of the design parameter $\rho_2$ and the dimensional output displacement $s_1$.

where $k$ is a weighting factor with $0 \leq k \leq 1$. Plotting the value of $\rho_2$ which gives the minimum point on $w(k)$ for the different values of $k$ gives the Pareto front [61], which is shown in Figure 4.22. The Pareto front gives some indication of the trade-off between the mechanism’s overall length and required input displacement to achieve the specified output displacement. At $k = 0$ no weight is given to the mechanism’s length. At $k = 1$ no weight is given to the required input displacement, and at $k = 0.5$ equal weight is given to mechanism’s length and the required input displacement. The Pareto front indicates that mechanisms with $\rho_2 \geq 0.5$ are preferred, with larger values of $\rho_2$ resulting in larger mechanisms but smaller input displacements. From Figure 4.21 it can be seen that mechanisms with $\rho_2$ slightly larger than 0.5 have both small footprints and require small values of input displacement.
Figure 4.21: Plot showing the length of the mechanism (solid line) and the required input displacement (dashed line) as functions of the design parameter, $\rho_2$.

Figure 4.22: Plot of the nondimensional design parameter, $\rho_2$, which results in the minimum value of the linear combination of mechanism length and input displacement (see Figure 4.21). $k = 0$ means no weight given to the mechanism’s length, $k = 1$ means no weight given to the required input displacement, and $k = 0.5$ means equal weight given to mechanism’s length and the required input displacement.
Once a design for the XZMM has been selected, it is important to choose an actuator capable of supplying the force over an appropriate range of motion. In this example, we consider the actuation of an XZMM with a Thermal In-plane Microactuator (TIM) [62, 63]. The TIM is a simple, reliable, planar actuator that is capable of producing relatively large forces and displacements. Figure 4.23 shows the force-displacement curves of several XZMM designs ($\rho_2 \geq 0.5$) superimposed with the force-displacement curves typical of a TIM. There is little difference among the different XZMM designs and they produce curves that have very large forces near zero displacement and fall to zero force between displacements of -20 and -12. The intersection of the TIM and XZMM curves represents an equilibrium point at which the resistance of the XZMM equals the force supplied by the TIM. The TIM is capable of producing adequate force to balance the XZMM except near zero-displacement where the XZMM forces become very large. The large (infinite) forces required by the XZMM designs at a displacement of zero are due to the poor transmission angle of a completely flat mechanism. The poor transmission angle requires an additional actuation strategy to help the mechanism come out of the in-plane position. Once some initial out-of-plane displacement is provided by some other means, this generic TIM is capable of producing the required force and displacement for a number of the XZMM designs.

We can generalize the XZMM shown in Figure 4.14 by understanding that other OP fourbar mechanisms could be used in series with crank-slider mechanisms. Specifically, the fourbar mechanism will be chosen from one of the $C$ fourbar patches or the $D$ patches [64].

\footnote{There are various strategies for producing this initial motion, including out-of-plane thermal actuators [64], beams with trapped oxide, etc.}
fourbar patch, and the crank slider will be an A1, B1 or C1 crank-slider. Secondly, the relative scale of the orientation sets of the crank-slider and fourbar need not be coupled as in the previous example. The mechanical advantage of the XZMM can be changed by scaling the relative sizes of the fourbar and the crank-slider portions of the XZMM. Increasing the size of the fourbar portion of the XZMM by a factor of 10 maintains the same input distance, multiplies the output displacement by 10, and multiplies the required input force by 10 in order to maintain the same output force. Thus, there is considerably more design flexibility inherent in the XZMM design than has been explored in these examples. However, the work illustrates the applicability of the design space concept to the design of microsystems that incorporate combinations of the crank-sliders and fourbars.
4.4 Conclusions

In this chapter, a method has been developed for applying the TEOSM to linkages with prismatic joints. The utility of crank-sliders in actuating OP mechanisms has been discussed and a specific system for achieving out-of-plane placement of a microsphere has been presented. It has been shown that OP mechanisms can be actuated with a typical MEMS actuator, the Thermal In-plane Micro-actuator (TIM) and that they usually require extra actuation to compensate for poor transmission angles.
Chapter 5

INTRODUCTION TO SPHERICAL OP MECHANISMS

This chapter provides background to aid those familiar with planar kinematics in understanding spherical mechanisms and extends the TEOSM to spherical ortho-planar mechanisms.

5.1 Spherical Mechanism Background

Spherical kinematics is a less commonly understood and more difficult form of mechanism design than planar kinematics. It is considered more general because a plane can be considered a sphere with infinite radius. Thus, planar mechanisms can be considered as limiting cases of spherical mechanisms, and often this fact can be used to check the consistency of mathematical conclusions. The differences between planar and spherical mechanisms require different mathematical techniques as well as an awareness of subtle differences that make planar assumptions invalid.

Spherical trigonometry is discussed as theoretical background for the position analysis of mechanisms. Some common errors that result from incorrect generalizations from planar to spherical geometry are pointed out. Next, the spherical equivalent of
Grashof’s law is discussed and techniques that take the place of the complex vector loop approach of planar kinematics are described.

5.1.1 Spherical Trigonometry

The brief review of spherical trigonometry given here follows Spiegel and Liu [65] and develops analogies between spherical trigonometry and plane trigonometry. The familiar results of plane trigonometry describe relationships between straight lines, angles and triangles on a planar surface. In spherical trigonometry, the surface is no longer flat but is the surface of a sphere. Straight lines and planar figures cannot be drawn on a spherical surface, but there are geometrical features on the spherical surface that have properties mathematically similar to their planar counterparts. A circle in a spherical surface displays many of the same mathematical properties as a line in a plane. Therefore, in place of straight lines, spherical trigonometry is based on circles inscribed on the sphere. Of all the circles that can be drawn on a sphere, great circles are the ones whose radius is the same as the sphere. Each great circle is contained in a plane that intersects the sphere. The normal to that plane that passes through the center of the sphere is the pole of the great circle. Angles between great circles are defined as the dihedral angle formed by the two intersecting planes containing the great circles. Henderson [57] details the similarities and differences between planar and spherical geometries. Here, it will be sufficient to appreciate that there are differences between plane and spherical trigonometry but that similar results can be obtained such as the Law of Sines, the Law

\[\text{Law of Sines} \]

In geography, circles of longitude and the equator are great circles. Circles of latitude other than the equator are not great circles.
of Cosines and relationships applicable to right triangles. The discussion of these laws will be facilitated by notational conventions for distinguishing between dihedral angles, portions of great circles (arcs) and points on great circles.

In this work, Greek letters represent the dihedral angles between the two planes containing intersecting great circles, lower-case roman letters represent circular arcs, and upper-case roman letters represent points. The measurements of arcs and dihedral angles are given in degrees. Referring to the measure of arcs of a spherical triangle in degrees may seem odd, since in planar trigonometry the corresponding quantities are lengths. However, using angular measures instead of arc lengths allows for the derivation of general results that are independent of the radius of the particular sphere. These notational conventions facilitate the description of the spherical equivalents of the Law of Cosines, the Law of Sines and simplified relations for right spherical triangles (spherical triangles in which one of the dihedral angles is ninety degrees). These conventions will also be useful in describing the different parts of spherical mechanisms and their interrelations.

A spherical triangle with great circle arcs $k$, $m$, and $n$ with dihedral angles $\theta$, $\sigma$, and $\xi$ is shown in Figure 5.1. In spherical trigonometry there is a Law of Cosines which relates the three arcs and one of the dihedral angles:

$$
\cos(k) = \cos(m) \cos(n) + \sin(m) \sin(n) \cos(\sigma)
$$

(5.1)
There is also a Law of Cosines which relates the three dihedral angles and one of the arcs:

\[ \cos(\sigma) = -\cos(\theta) \cos(\xi) + \sin(\theta) \sin(\xi) \cos(k) \quad (5.2) \]

The spherical Law of Sines relates two arcs and their opposite two dihedral angles

\[ \frac{\sin(n)}{\sin(\xi)} = \frac{\sin(m)}{\sin(\theta)} = \frac{\sin(k)}{\sin(\sigma)} \quad (5.3) \]

As in planar trigonometry, there are simpler relations which hold if the triangle is a right triangle. Thus, if one of the dihedral angles shown in Figure 5.1, \( \xi \), equals \( \frac{\pi}{2} \), then there are two other dihedral angles, \( \theta \) and \( \sigma \), and the three arcs, \( k, m, n \), which can still vary in magnitude. If any two of these five quantities is known a third can be found using spherical trigonometry. There are ten different equations which describe the
relation between any set of three of the five parts of the right spherical triangle. These ten relations are concisely summarized in two rules known as Napier’s Rules, as described next.

Napier’s Rules are described in relation to the figure formed by arranging the quantities $k$, $m$, $n$, $\theta$ and $\sigma$ around a circle as in Figure 5.2. The prefix ”‘co’” is attached to the hypotenuse $n$ and the angles $\theta$ and $\sigma$ (indicating the complement, i.e. $\frac{\pi}{2} - n$, $\frac{\pi}{2} - \theta$, and $\frac{\pi}{2} - \sigma$). Any one of the parts of this circle can be designated as the middle part, the two neighboring parts are called adjacent parts and the two remaining parts are called opposite parts. Napier’s Rules are

1. The sine of any middle part equals the product of the tangents of the adjacent parts.

2. The sine of any middle part equals the product of the cosines of the opposite parts.

For example, the second rule can be used to find $m$ if $\theta$ and $n$ are known. The middle part is $m$ and the opposite parts are co-$n$ and co-$\theta$. Thus, Napier’s second rule, for this example, is

$$\sin(m) = \cos(co - \theta) \cos(co - n) = \sin(\theta) \sin(n) \quad (5.4)$$

5.1.2 False Analogies Between Planar and Spherical Geometry

Chiang [66] mentions some errors to be avoided when comparing planar and spherical mechanisms. The first of these errors stems from the fact that parallelism does not exist in spherical mechanisms since any two great circles intersect. Thus, while it makes sense in planar mechanisms to reference the angle at $A$ using a line parallel to ground as
in Figure 5.3, in spherical mechanisms, there is no parallel reference line and an attempt to draw one, as in Figure 5.4 is incorrect.

Another related error comes from the fact that pure translations are not possible for spherical mechanisms. Every spherical motion is a rotation. Therefore, the dihedral angle $\phi$ about the axis $n$ in Figure 5.5 does not represent the rotation of link AB from position 1 to position 2. Figure 5.6 indicates how the actual rotation of link AB is found. The rotating $A_1B_1$ through the angle $\phi$ causes the links to lie in the same plane, and the link must then be rotated by an amount $\epsilon$ about the normal to the plane, $E$. The net effect of combining the two rotations can be found as a rotation of $\eta$ about the axis labeled $m$. 
Figure 5.3: In planar mechanisms, it is valid to draw a reference line at $A$ parallel to the line through $A_0$.

Figure 5.4: In spherical mechanisms, all great circles intersect and it is not valid to draw a reference arc at $A$, because there is no great circle through $A$ that does not also intersect the great circle through $A_0$. 
Figure 5.5: Unlike planar mechanisms, the arc $\phi$ does not represent the rotation of the rigid link $AB$ from position 1 to position 2. Figure 5.6 shows the correct angle and axis of rotation.

Figure 5.6: The rotation of the link $AB$ from position 1 to position 2, is a composite of first the rotation $\phi$ about the axis $n$ followed by the rotation $\epsilon$ about the axis $E$. The composition of the two rotations is equal to the rotation $\eta$ about the axis $m$. 
5.1.3 Grashof’s Law for Spherical Mechanisms

Chiang [66] also describes how to apply Grashof’s law to spherical mechanisms. The Grashof rule in planar kinematics is a well-known criterion for determining whether fourbar linkages have a rotatable link. Grashof’s rule states that if the sum of the length of the shortest link and the longest link is less than or equal to the sum of the lengths of the other two links, then the shortest link has full rotation. This rule is valid for spherical fourbars provided that the mechanism is represented correctly. The existence of multiple representations of a spherical fourbar is a result of the fact that any two great circles (fully circular links) intersect at two antipodal (opposite) points on a sphere. In order to obtain correct results from Grashof’s criterion, the joints (intersections of great circles) must be chosen so that the sum of the link lengths (portions of a great circle) are a minimum. In Figure 5.7 a minimal fourbar is shown with link lengths labeled $A_0$, $A$, $B$, and $B_0$. The motion of the mechanism is unaffected if any or all of these points are replaced with their corresponding antipodal points $\hat{A}_0$, $\hat{A}$, $\hat{B}$, and $\hat{B}_0$. Figure 5.8 shows one of the other possible spherical fourbars with identical motion to the mechanism in Figure 5.7. Any fourbar loop can always be reduced to a unique fourbar loop in which the sum of the four link lengths is a minimum. The minimum fourbar loop always moves within one hemisphere, and this is the loop to which the Grashof rule can be applied.

The idea that a joint can be moved to its antipodal position without affecting the motion of the mechanism has important theoretical and practical consequences. First, one can always define a mechanism’s coordinate system so that the x-axis passes through
Figure 5.7: A minimal fourbar loop for the use of Grahof’s Law in spherical mechanisms.

Figure 5.8: A non-minimal fourbar loop that has the same motion as the fourbar shown in Figure 5.7 and uses the joint $\bar{B}$ instead of $B$. 
a joint of ground link, and the other joint locations in the mechanism are all in the hemisphere that contains the x-axis, the z-axis, and the positive y-axis. On the other hand, a mechanism that is modeled using the minimum link lengths may need to be fabricated using another (non-minimal) set of equivalent links in order to avoid interference with the substrate. In MEMS, the substrate (xy plane) represents a barrier to motion. If the motion of a joint has a negative z-component for its entire range of motion, then the z-component of motion of its antipodal point is positive for its entire range of motion and the antipodal joint could move unimpeded.

5.1.4 Loop Closure Equations for Spherical Mechanisms

The TEOSM for planar mechanisms was derived based on geometric considerations and the requirements for loop closure in planar mechanisms. Loop closure equations for spherical mechanisms are needed in order to extend the TEOSM.

Loop closure equations for spherical mechanisms are expressed in terms of a series of rotations instead of vector sums as is done for planar mechanisms. Rotations are used for spherical mechanisms because every motion from one point to another on a sphere can be described by a rotation, $R_p^a$, where $p$ is the axis of rotation and $a$ is the rotation arc about the axis.
The effect of the rotation, $R_p^a$, acting on the unit vector $\hat{A}$ can be described using matrix multiplication (see [67]) as

$$
R_p^a \hat{A} = \begin{bmatrix}
    p_x^2(1-\cos(a))+\cos(a) & p_x p_y (1-\cos(a))-p_z \sin(a) & p_x p_z (1-\cos(a))+p_y \sin(a) \\
    p_y p_x (1-\cos(a))+p_z \sin(a) & p_y^2(1-\cos(a))+\cos(a) & p_y p_z (1-\cos(a))-p_x \sin(a) \\
    p_z p_x (1-\cos(a))-p_y \sin(a) & p_z p_y (1-\cos(a))+p_x \sin(a) & p_z^2(1-\cos(a))+\cos(a)
\end{bmatrix} \begin{bmatrix}
    A_x \\
    A_y \\
    A_z
\end{bmatrix}
$$

(5.5)

The rotation $R_p^a \hat{A}$ sweeps out a great circle arc only when $p$ is orthogonal to the unit vector $\hat{A}$. Due to the convenience of great circles in spherical geometry, links in spherical mechanisms are described using segments of great circles. When a rotation $R_p^a$ describes the geometry of a spherical binary link, $a$ is the length of the link, and $p$ is the link pole. It will be required that the link pole, $p$, satisfy two constraints: first, $p$ is orthogonal to the joint axes on the binary link and second, $p$ is a unit vector.

A procedure for defining the geometry of a spherical linkage may be developed as follows. The joint axis $A_1$ is defined to be the x-axis. The link pole $p_1$ is defined to be the z-axis. The location of the second joint on link 1, $A_2$, is located at a position defined by $R_{p_1}^{a_1} A_1$. The link pole for the second link, $p_2$, is then found by a rotation $R_{A_2}^{p_2} p_1$. Because $A_2$ is orthogonal to $p_1$, the arc traced by the rotation $R_{A_2}^{p_2} p_1$ is a great circle, and insures that $p_2$ is a unit vector orthogonal to $A_2$. The second link is the great circle defined by the rotation from the joint axis $A_2$ to the joint axis $A_3 = R_{p_2}^{a_2} A_2$.

---

2 The Law of Sines, the Law of Cosines and Napier’s Rules apply only to great circles.
Loop closure for an $N$-bar spherical mechanism can then be expressed as a series of rotations, $R_{p_j}^{a_j}$, that satisfy

$$\prod_{j=1}^{N} R_{p_j}^{a_j} \hat{A}_1 = \hat{A}_1$$  \hspace{1cm} (5.6)

where the first joint, $A_1$, is the x-axis, the first link pole, $p_1$, is the z-axis, and succeeding joints are defined by

$$A_{j+1} = R_{p_j}^{a_j} A_j$$  \hspace{1cm} (5.7)

and succeeding link poles are defined by

$$p_{j+1} = R_{A_j p_j}^{\phi_{j+1}} p_j$$  \hspace{1cm} (5.8)

The sequence of rotations that start with the unit vector $A_1$ must return to $A_1$ after traversing the $N$ spherical links with lengths $a_1, a_2, \ldots, a_N$ as shown in Figure 5.9 for $N = 6$. The angle $\phi$ that measures the rotation of the link poles is shown in two places in Figure 5.9. It is shown as the length of the great arc between two link poles and also as the dihedral angle between succeeding links. Thus, just as in planar mechanisms, the spherical loop is specified by a series of link lengths, $a_j$, and angles, $\phi_j$. 
Figure 5.9: A spherical sixbar with link arcs of $a_i$, each of which is a rotation about the pole $p_i$. The rotation $R_{p_i}^{a_i}$ rotates axis $A_i$ to $A_{i+1}$ for $1 \leq i \leq N - 1$. The rotation $R_{p_N}^{a_N}$ rotates axis $A_N$ back to the origin of the loop, $A_1$. 
5.2 Loop Closure For Folded Spherical Mechanisms

Considerable simplification of equation (5.5) is possible for spherical mechanisms that fold, i.e. all their links lie in a plane. The link pole, \( p \) for folded links is the unit vector along the \( z \)-axis, and thus \( R_{z}^{a} \) is

\[
R_{z}^{a} = \begin{bmatrix}
\cos(a) & -\sin(a) & 0 \\
\sin(a) & \cos(a) & 0 \\
0 & 0 & 1
\end{bmatrix}
\]  

(5.9)

Positive values of \( a \) result in CCW rotations in the X-Y plane about the \( z \)-axis. Negative values of \( a \) result in CW rotations in the X-Y plane. Thus, the angle \( a \) is the natural counterpart to the \( r \)-vector in planar mechanisms, and links with a positive value of \( a \) belong to the CCW (positive) orientation set, \( M_{R} \), and links with a negative value of \( a \) belong to the CW (negative) orientation set, \( M_{L} \). Further simplifications are made possible by noting that in the folded position, all linkage geometry lies in a plane and can be modeled very efficiently with complex numbers. Thus, equation (5.9) can be expressed as

\[
R_{z}^{a} = e^{ia}
\]

(5.10)

From equation (5.10) it can be seen that the complicated, non-commutative three-dimensional rotations required to describe general spherical motion simplify to simple, commutative multiplication of complex exponentials in order to describe geometry of
folded spherical mechanisms. Thus, the loop closure equation for general spherical mechanisms, equation (5.6) simplifies to

$$\prod_{j=1}^{N} e^{ia_j} = e^{\sum_{j=1}^{N} a_j} = 1$$  \hspace{1cm} (5.11)

Taking the natural logarithm of equation (5.11) yields

$$\sum_{j=1}^{N} a_j = 2k\pi$$  \hspace{1cm} (5.12)

where $k$ is an integer, $k = 0, \pm 1, \pm 2, \ldots$. When $k = 0$, the result is identical to the loop closure equations (2.6) given for planar mechanisms in Chapter 2. The fact that there are other possibilities, $k \neq 0$, is a result of the fact that loop closure is possible in spherical mechanisms by forming a complete loop around a circle in the plane of fabrication.

### 5.3 The Design Space of Spherical OP Mechanisms

In Chapter 2, the TEOSM was developed to describe the constraints on the geometry of planar ortho-planar linkages. In the first section of this chapter, it was shown how spherical mechanisms differ from planar mechanisms and that some of the fundamental elements of the TEOSM, i.e loop closure, symmetry, and link size limits can be extended to spherical mechanisms.

#### 5.3.1 The TEOSM for Spherical OP Mechanisms

In Chapter 2, the TEOSM was derived for planar OP mechanisms based on loop closure and dimensional similarity. The planar loop closure equation is

$$\sum_{i}^{N} r_i = 0$$  \hspace{1cm} (5.13)
The planar dimensional similarity equation is

\[ \sum_{i}^{N} \text{abs}(r_i) = p \]  

(5.14)

Together, the two relations were used to determine the planar TEOSM equations

\[ \text{measure}(M_R) = \text{measure}(M_L) = p/2 \]  

(5.15)

where \( M_R = r_i|r_i > 0 \) and \( M_L = r_i|r_i < 0 \). It was observed that the planar TEOSM equations could be non-dimensionalized by dividing by \( p/2 \).

There are significant differences between planar ortho-planar mechanisms and spherical ortho-planar mechanisms. Planar OP mechanisms in their folded configuration lie along a line and are nondimensionalized so that the measure of the links in an orientation set is equal to 1. Spherical OP mechanisms in their folded configuration lie along the unit circle and their lengths are expressed as arclengths, \( a_i \). In planar mechanisms, the ground link was taken to be right-laying (positive orientation), and links were classified as right-laying or left-laying by proceeding around the closed vector loop in a CCW direction. In spherical mechanisms, the ground link is taken to have a CCW positive orientation proceeding around the unit circle. The orientation of the links is specified by whether they continue in the CCW direction (positive orientation) around the unit circle or whether they go in the CW direction (negative orientation). For spherical mechanisms, the loop closure equation is not unique, because the loop can be closed by completing a circuit around the unit circle (as opposed to planar mechanisms, which must fold back on themselves). The loop closure equation for spherical mechanisms is:

\[ \sum_{i}^{N} a_i = 2k\pi \]  

(5.16)
where $k = 0, \pm 1, \pm 2, \ldots$. The dimensional similarity equation is

$$\sum_{i}^{N} \text{abs}(a_i) = \lambda \quad (5.17)$$

where $\lambda$ is the perimeter measure (in radians) of the folded spherical mechanism. Unlike planar mechanisms, equations (5.16) and (5.17) are already non-dimensional, and the motion of a spherical four-bar consisting of links with angles $a_1 = \pi/10$, $a_2 = \pi/10$, $a_3 = \pi/10$, $a_4 = \pi/10$ is not a scaled version of the motion of a spherical four-bar consisting of links with angles $a_1 = \pi/2$, $a_2 = \pi/2$, $a_3 = \pi/2$, $a_4 = \pi/2$.

Together, Equations (5.16) and (5.17) can be used to determine the spherical TEOSM equations

$$\text{measure}(M_R) = \lambda/2 + k\pi, \text{measure}(M_L) = \lambda/2 - k\pi \quad (5.18)$$

where $M_R = \{a_i | a_i > 0\}$ and $M_L = \{a_i | a_i < 0\}$. Because the measure of the set, $M_R$ is positive,

$$k \leq \frac{\lambda}{2\pi} \quad (5.19)$$

Equation (5.19) is significant because it implies that the values that $k$ is allowed to assume is dependent on the perimeter, $\lambda$. For example, when $\lambda < 2\pi$, equation (5.19) requires that $k = 0$. Every $2\pi$ increase in $\lambda$ results in the creation of an additional group of adjacency set patches. The geometry of the design space for spherical mechanisms can be understood by analogy to the design space of planar mechanisms. For $k = 0$, the only difference between the equations for the design spaces of spherical and planar mechanisms is that the measure of the orientation sets for spherical mechanisms is scaled by a parameter $\lambda/2$. Thus, when $\lambda/2 = 1$ the equations are identical. Because $\lambda$ can
vary, spherical mechanisms have an additional degree of design freedom when compared with planar mechanisms.

Just as planar sliders can be treated within the framework of the TEOSM by designating some of the $r$-vectors to be $s$-vectors with an additional, translational degree-of-freedom, spherical sliders can be treated within the framework of TEOSM by designating some of the $a$-vectors to be variable-arclength $b$-vectors.

In the next chapter, the design of spherical mechanisms for MEMS is described. The distinctions developed here between planar and spherical mechanisms and the differences between mechanisms with and without sliders are exploited to develop building block mechanisms for MEMS design.
Chapter 6

SPHERICAL MECHANISM DESIGN FOR MEMS

6.1 Introduction

Microelectromechanical Systems (MEMS) with precise and accurate spatial position and orientation control are valuable in a number of applications. In particular, such MEMS would benefit the next generation of spatial light modulators \[46\], which are devices designed to manipulate optical wavefronts, and are used in image projectors, optical switching and adaptive optics. Important factors in the design of spatial light modulators include fine positioning control of a mirror surface, the range of motion of the mirror, and the reflectivity of the mirror. Typically, MEMS spatial light mirrors consist of individually controlled pixels that should fit together to provide a good approximation of a single deformable mirror. These individual pixels may be controlled using simple building block devices designed using spherical kinematics.

Although there are applications that can make use of devices capable of controlled complex motions, the planar fabrication methods usually employed in MEMS complicate the design of such devices. Using spherical kinematics\footnote{Spherical kinematics is concerned with motion of linkages whose joints’ axes intersect at a point.} it is possible to design devices that can move links to a variety of spatial orientations and can convert rotation in...
one plane to rotation in a different plane. In this chapter, we describe the effect of MEMS fabrication constraints on the kinematics and components of the mechanisms. Because the geometric constraints are different from those typically encountered in macro-scale design, the descriptions of the fabrication constraints are intended to facilitate the development of spherical kinematic synthesis techniques specific to MEMS design. Two basic spherical micromechanism building blocks are described in detail, followed by demonstration of their use in creating more complex mechanisms.

The first of the building block devices is the spherical fourbar micromechanism, which is represented in Figure 6.1. The second building block device is the spherical slider-crank micromechanism, which is represented in Figure 6.2. These building blocks are special cases of the general spherical mechanisms which have been described in the literature [68, 49, 66, 69, 70, 71]. Thus, once the special nature of these micromechanisms is understood, their motion can be analyzed using existing theory. Specific features of micro-fabrication that make the design of micro-scale spherical mechanisms different from the design of macro-scale spherical mechanisms include the following fundamentals:

- Mechanisms are fabricated in a planar position
- Mechanisms are fabricated on a flat substrate which can interfere with their desired motion
- Fabrication imposes geometric limitations on the orientations of joint axes and hence on the arc lengths of links

Simple spherical mechanisms which satisfy these design fundamentals can be regarded as building blocks that can be used to create more complex devices.
Figure 6.1: Schematic of a spherical fourbar micromechanism in its fabricated position.

Figure 6.2: Schematic of a spherical slider-crank micromechanism in its fabricated position.
6.1.1 Advantages of Spherical Mechanisms

The spherical mechanism building blocks discussed in this chapter have advantages when considered in relation to the planar mechanisms and the more complex serial and parallel manipulators. Spherical mechanisms may be simpler than planar mechanisms when the design objectives are either to convert an in-plane rotation (perhaps of a gear) to and out-of-plane rotation (perhaps of a mirror), or to produce a spatial rotation about a point. Planar design techniques require that the motion of links be confined to a plane and that all rotations are about an axis perpendicular to that plane. This makes a spatial rotation impossible. Furthermore, to convert an in-plane rotation to an out-of-plane rotation using planar techniques, one must use an in-plane mechanism to convert the rotary motion to linear motion and then another out-of-plane mechanism to convert the linear motion back to rotary motion, as is done in the mechanisms shown in Figure 6.3. Using spherical kinematics it is possible to eliminate the link undergoing linear motion and convert in-plane rotary motion directly to out-of-plane rotary motion. The direct conversion has the advantage of requiring fewer parts and a smaller area to accomplish the same task.

Although spherical mechanisms cannot replicate all the functionality of serial and parallel spatial manipulators, for some applications the spherical building block mechanisms are considerably simpler. There is a niche in which the motions produced by the spherical mechanisms can provide the required motion without the added overhead in complexity and multiplicity of inputs that is often required by robotic manipulators. Furthermore, spherical mechanisms may be at an appropriate level of complexity to address
the critical issues of repeatability and reliability \cite{72}. The building block mechanisms provide opportunities to study the effects of different joint configurations and moment applications within a single-degree-of-freedom device with just a few links.

6.2 Components of Spherical Mechanisms

In spherical mechanisms, each link rotates about a single fixed point in space. The links are connected by revolute joints, and the axis of each revolute joint passes through the fixed point. In a mechanism, some of the joint axes will be fixed in space because they connect to the ground (or reference) link which is immobile. In MEMS devices, the ground link is usually the substrate. Other joint axes, which are not connected to ground, may change their orientation in space subject to the constraints imposed by the linkage geometry. These constraints ensure that the mobile axes always pass through
the fixed point. When modeling the motion of the mechanism, it is typical to model only the projection of the links on the unit sphere. Thus, the links are represented by arcs on the unit sphere, and the joints by points on the unit sphere. This projection is mathematically useful because it means that the links can be described by arc length and permits the use of spherical trigonometry in support of calculations. The greatest value of the unit sphere model is that it allows the designer to neglect the radial distance between the center fixed point and the links or joints. Thus, the actual radial dimensions of the various links can be decided later in the design process, or modified as necessary without affecting the device’s rotational characteristics. The design of the spherical mechanism does need to account for certain limitations of the fabrication process. The fabrication process puts limits on the orientations in which joints may be fabricated. The nature of these fabrication limitations and their implications for the design of links and joints of spherical mechanisms is discussed in the following two sections.

6.2.1 Joint Design

Surface micromachined MEMS are often fabricated by building up alternating layers of structural (polysilicon) and sacrificial (silicon oxide) layers on a substrate. Mechanisms are designed by specifying the presence or absence of material at a particular location in a layer. Typically, the layer thickness is determined by the process and there is no provision for modifying the thickness within the same layer. Thus, the geometry of a particular layer is a two-dimensional pattern accreted to a fixed thickness. Different layers may have distinct two-dimensional layouts, which can result in terrace-like discontinuities between layers. As a result, revolute joints with axes parallel to the substrate have
square axles in holes that are somewhat square with irregular sides. Figure 6.4 shows a hinge attached to the substrate, and Figure 6.5 shows a hinge between two mobile links.

In addition to the square hinges built parallel to the substrate, the MEMS layered fabrication technique also permits pin joints that have axes perpendicular to the substrate as shown in Figure 6.6.

Because typical MEMS fabrication processes cannot produce axles or bearings at significant heights above the substrate, it would be difficult to build joints with axes at an acute angle to the substrate. Thus, MEMS spherical mechanisms can be fabricated with the joints either perpendicular to the substrate or parallel to the substrate. The axes can move to other orientations when actuated, but will start either parallel or perpendicular. This constraint has implications for spherical micromechanism synthesis that deserve further description.

In this chapter, we refer to revolute joints built parallel to the substrate as hinges and to those built perpendicular to the substrate as pin joints. These two terms serve to emphasize that these two categories comprise the catalogue of MEMS revolute joints and that there are significant differences in their implementation. Figure 6.7 is a schematic showing the intersection of the unit sphere with the plane of the substrate. Several possible orientations for hinges are illustrated lying on the circle where the sphere intersects the substrate. The axis of the pin joint is also represented as the line from $C$ to $N$. Further detail on the design of hinges is available [34, 30].
Figure 6.4: SEM of a substrate hinge

Figure 6.5: SEM of a released hinge
Figure 6.6: SEM of a pin joint

Figure 6.7: Schematic of the motion and design region for a MEMS spherical mechanism. All geometry must be fabricated as a thin film on the substrate. Motion is modeled as lying on the sphere. Thus, possible initial positions and orientations are illustrated as lying on the intersection of the sphere with the plane of the substrate. The axis of a pin joint lies along the line from \( C \) to \( N \). The use of a pin joint in spherical MEMS devices makes the device a spherical slider-crank.
6.2.2 Link Design

Once appropriate designs have been selected for the revolute joints in a spherical mechanism, the motion of the mechanism is determined in large part by the arc swept out by the links connecting the joints. In general, the links need to satisfy the assumption of rigidity, i.e. they must be wide and thick enough not to bend and twist, and the links should not interfere with other links. In modeling spherical mechanisms, it is convenient to treat the links as circular arcs of uniform radius and a specified arc length. However, in actual practice, as long as the arc between the joint axes is the specified amount and the link is sufficiently rigid, the actual shape of the link matters little. Thus, there is considerable design freedom in the shape of the link connecting different joints. Figure 6.8 shows examples of differently designed links connecting two joint axes. In each case the relative motion of the axes is constrained in the same way. Clearly, the designer needs to be concerned with how the links will interact with each other and how they may interfere with the substrate. However, choosing a shape that approximates a circular arc may provide a better intuitive sense of the mechanism’s motion.

The issue of links contacting in ways that interfere with the intended motion is particularly significant in MEMS because the substrate (the ground link) represents a barrier to motion. There are three immediate consequences of the presence of the substrate plane. The first consequence is that any link longer than $\pi$ radians which is fabricated parallel to the substrate plane will be incapable of lifting off of the substrate. The second consequence is that pin joints like the one shown in Figure 6.6 must either be connected to the substrate or the tilting of their axis away from normal will result in the
Figure 6.8: Examples of different links which constrain the relative position of joint axes $J_1$ and $J_2$ to the same arc.

edge of the joint interfering with the substrate. The third consequence is that the motion of the mechanism cannot require any part of the device to move below the substrate.

The limitation that pin joints are built at the mechanism’s center implies that they must be connected to the substrate and that there can only be one in a particular mechanism. Thus, if a pin joint is used in a MEMS device, the axis of that joint will be perpendicular to its adjacent joints in the kinematic loop, which implies that the proper way to model the links connecting to the pin joint are as arcs of length $\pi/2$. Chiang [68] states a spherical link of length $\pi/2$ is analogous to a planar link of infinite length i.e a slider. Hence, the inclusion of a pin joint in a spherical fourbar micromechanism creates a spherical slider-crank micromechanism.

Although the links may assume a wide variety of shapes, the fact that they connect joints which are limited to specific orientations places constraints on their arc lengths. For the MEMS fourbars, like the one shown in Figure 6.1, the ground link is $\gamma$, the input link is $\alpha$, the output link is $\beta$ and the coupler link is $\eta$. If all the joints are hinges, all the
links initially lie in the same plane. McCarthy [49] calls mechanisms which can assume a planar configuration *folded linkages* and states that for a mechanism to fold, it must satisfy at least one of the following equations:

\begin{align*}
\gamma - \alpha + \eta - \beta &= 0 \tag{6.1} \\
\gamma - \alpha - \eta + \beta &= 0 \tag{6.2} \\
\gamma + \alpha - \eta - \beta &= 0 \tag{6.3} \\
\gamma + \alpha + \eta + \beta &= 2\pi \tag{6.4}
\end{align*}

### 6.3 Building Block Mechanisms

By appreciating the design constraints inherent in the MEMS fabrication process, one can select appropriate building block devices and compute the appropriate link lengths of the device. In kinematics, choosing the particular link lengths of a mechanism is accomplished through the process of *dimensional synthesis* and is often divided into three distinct types of synthesis problems, position generation, motion generation, and function generation. In position generation problems, the designer computes the link lengths that cause a particular point on the device to pass through certain positions in space. In motion generation, the designer computes the linkage dimensions that cause the coupler link to pass through specified positions and orientations. In function generation, the designer computes the link lengths that result in a particular coordination of the motion of the input link and the output link. Specifics on the synthesis and analysis
of spherical mechanisms may be found in the literature [68, 49]. The adaptation of synthesis techniques to the particular constraints associated with MEMS fabrication is an area for further research.

6.3.1 Spherical Fourbar Micromechanisms

The spherical fourbar micromechanism is one of the building block mechanisms that may prove useful in MEMS design. Figure 6.9 is a scanning electron micrograph (SEM) of an example spherical fourbar micromechanism shown in an actuated position. Spherical four-bars built using only hinges have all their links lying parallel to the substrate when they are fabricated. Upon actuation, the input and output links rotate up from the substrate and carry the coupler through various positions and orientations. The hinged spherical fourbar can thus be used for a variety of position and motion control tasks, and for coordinating the rotations of the input and output links in their out-of-plane rotations.

6.3.2 Spherical Slider-Crank Micromechanisms

The other building block mechanism, the spherical slider-crank is a special case of the spherical fourbar and may be used to transform in-plane rotation into out-of-plane rotation or vice versa. The spherical slider-crank consists of a ground link, a spherical slider and two additional links. The spherical slider is essentially a pin joint and may be small as in Figure 6.10 or it may be large and look more like a ring under a fixed rail as in Figure 6.11. In the initial position of the spherical slider-crank, all the links are parallel to the substrate. As the device is actuated, the spherical slider undergoes a
Figure 6.9: SEM of a spherical fourbar micromechanism. a) The mechanism in its fabricated position. b) The mechanism after actuation.
planar rotation and the other two links rotate up from the substrate to form a spherical triangle \(^2\) with the substrate. Thus, the slider-crank can perform a variety of position and motion tasks, as well as couple in-plane rotation with out-of-plane.

The analysis of the spherical slider-crank is a special case of the spherical four-bar in which the arc between the two joints connected to the substrate is \(\gamma = \pi/2\). The input link, \(\alpha\), also has length \(\pi/2\). Note that in this case the ground link of the mechanism is not parallel to the substrate but is perpendicular to it and lies in the plane defined by the two fixed axes of the device.

### 6.4 Devices using Spherical Kinematic Building Block Mechanisms

The building block devices may be combined in a variety of ways to provide useful motion. We include two examples of the spherical slider-crank micromechanism to illustrate motions that may be accomplished using the building block mechanisms.

\(^2\)A spherical triangle is a triangle drawn on a sphere
Figure 6.11: SEM of a displaced spherical slider-crank micromechanism implemented using a large sliding ring within a fixed rail.
6.4.1 Micro-Helico-Kinematic Platform

In the design of deformable mirrors, it is desirable to cover a portion of a surface as completely as possible with movable mirror pixels. Each pixel should be capable of large vertical travel but should not translate horizontally. The vertical travel permits the deformable mirror to compensate for the distortion in an optical wavefront caused by moving through turbulent air.

One possible pixel design is the Micro Helico-Kinematic Platform (MHKP) [73], shown in Figure 6.12, which is built from three identical spherical slider-crank units with the same center. A platform rests over the joint between the coupler and output links of the slider-crank units. As the input link undergoes in-plane rotation, the platform translates upward and rotates slightly. The advantage of the mechanism is that it provides for quite large vertical translation without any side-to-side translation. The elimination of side-to-side motion means that the device can be arrayed with similar devices and not interfere with their motion.

6.4.2 Bistable Out-of-Plane Mechanism

Bistable mechanisms can robustly hold one of two positions and require energy input only to toggle from one stable position to the other. Bistable mechanisms can be very efficient switching mechanisms. Bistable positioning of a link (perhaps a mirror) may be achieved by combining a planar bistable device, the Young Mechanism [74], and a spherical crank slider to achieve an out-of-plane bistable linkage, as shown in Figures
6.13, 6.14, and 6.15. This device may be also useful in helping to ensure that the link undergoing out-of-plane rotation can return reliably and repeatably to the same position.

6.5 The Actuation of Spherical OP MEMS

The actuation of spherical ortho-planar mechanisms is in many ways similar to the actuation of planar mechanisms. Thus, actuation of spherical mechanisms can be achieved with techniques and devices borrowed from planar in-plane (when the input joint axis is perpendicular to the substrate) and out-of-plane mechanisms (when the input joint axis is parallel to the substrate). As examples of possible actuation methods, we briefly describe devices that have been used by various researchers. The first example [75] achieves rotation about an axis in the substrate due to magnetic forces induced on a bit of permalloy. The permalloy is attached to a link that rotates out-of-plane as shown in
Figure 6.13: SEM of a bistable out-of-plane mechanism in its fabricated position, which is its first stable position.

Figure 6.14: SEM of a bistable out-of-plane mechanism in its second stable position.
Figure 6.15: SEM close-up view of the spherical slider-crank portion of a bistable out-of-plane mechanism in its second stable position.

Figure 6.16 The next three rotate links about axes perpendicular to the substrate. The first of these is the electro-static side-drive micromotor [76]. It has at least four poles on a rotor, which are induced to move by the electro-static charge on nearby stators. The rotor and stators are shown in Figure 6.17. The second device that causes a link to rotate about an axis perpendicular to the plane is the bent-beam rotary drive [77]. The rotary motion is produced by the coordinated movement of two thermal linear actuators which are shown in Figure 6.18. The last device is similar in concept, only the thermal actuators have been replaced by linear electro-static comb drives to make the Sandia MicroEngine [78]. Figure 6.19 shows two comb drives that are used to rotate a gear. The gear is part of a gear train that converts the velocity of the first gear into torque for the final gear. The figure also shows the in-plane rotation of the gears being converted first into the linear motion of a rack and finally into the out-of-plane rotation of the links of a slider-crank.
Figure 6.16: Schematic of a magnetic actuator [75]. The magnetic force on a bit of permalloy on the link produces the out-of-plane rotation.

Figure 6.17: SEM of an electro-static side-drive micromotor [76]. The spokes of the central rotor are electro-statically attracted to the charged stator pads surrounding the rotor.
Figure 6.18: Rotational motion of a link is produced by the coordinated motions of a pair of bent-beam thermal actuators. [77]

Figure 6.19: Rotational motion of a gear is produced by the coordinated motion of a pair of electro-static comb drives [78]. The in-plane rotational motion of the gear is then transformed to the linear motion of a rack and then into the out-of-plane rotation of the links of a slider-crank. Courtesy Sandia National Laboratories, SUMMiTTM Technologies, www.mems.sandia.gov
Thus, the micro spherical four-bar mechanism may be actuated with a magnetic actuator, and the micro spherical slider-crank mechanism may be actuated with a side-drive motor, or a pair of either bent-beam actuators or electro-static comb drive actuators.

### 6.6 Conclusions

Micro spherical mechanisms may be advantageous in the design of mechanisms which require the conversion of in-plane rotation to out-of-plane rotation or spatial rotation about a point. Micro spherical mechanisms have unique constraints which include planar fabrication, substrate interference, and geometric limitations on the orientations of joint axes and hence on arc lengths of links. Links and joints which satisfy these constraints have been demonstrated. Additionally, two building block mechanisms, the spherical four-bar micromechanism and the spherical slider-crank micromechanism have been described and demonstrated. The spherical slider-crank micromechanism can convert in-plane rotation to out-of-plane rotation and the spherical four-bar micromechanism can produce spatial rotations about a point. These two building blocks can be used to develop more complex devices. Two novel devices created using spherical mechanism building blocks are presented. One, the Micro Helico-Kinematic Platform achieves vertical translation with no side-to-side motion. The other allows bistable positioning of a link which rotates out-of-plane.
Chapter 7

OP MECHANISMS WITH SPHERICAL COMPONENTS

The theoretical background provided by the work described in the preceding chapters enables the description of a variety of mechanisms. This chapter describes a number of MEMS devices that have spherical components and whose analysis and design is dependent on the building blocks described in the previous chapters. In particular, the spherical slider-crank and its analysis via the spherical law of cosines is fundamental to many of these mechanisms.

It is proposed to analyze a number of mechanisms with spherical components. The specific mechanisms are the Micro Helico-Kinematic Platform (MHKP), a three-degree-of-freedom platform based on the MHKP, a bistable spherical mechanism, and a multistable version of the three-degree-of-freedom platform.

7.1 Micro Helico-Kinematic Platform

7.1.1 Introduction

There has been interest shown among researchers in optics [79, 80, 81] in designing microelectromechanical systems (MEMS) that achieve motion up from the substrate, in addition to the two dimensions of motion available parallel to the substrate. A number of
authors have described a variety of designs that provide this third dimension of motion. Previous efforts have achieved out-of-plane motion in MEMS devices by adding special surface treatments [29, 28, 27], using electrical or magnetic attraction [30, 31, 32], and using compressive loads to bend a flexure or hinge [33, 34, 35]. The design of systems of flexures, links and hinges belongs to the science of mechanisms, which deals with mechanical methods for transferring or transforming energy or motion. Our studies in mechanism design indicate that a specialized branch of mechanisms, spherical kinematics, may be useful in creating novel surface micromachined MEMS with motion out of the plane of fabrication.

Spherical kinematics is the branch of mechanism analysis that treats mechanisms having axes of rotation that intersect at a single point, and some of the most common in conventional usage are the automotive constant velocity (CV) joints and the wobble plate mechanism sometimes found in rotary fans [68]. This branch of mechanism design has not been explored at the micro-level, and may be unfamiliar to many MEMS designers. In applications requiring out-of-plane motion, spherical mechanisms may be more compact and may have fewer moving parts than more familiar alternatives. Thus, spherical mechanisms may provide useful design alternatives to MEMS designers.

The constraints of surface micromachining are different than encountered in the design of macro-scale spherical mechanisms, in that all links must start parallel to the plane of the substrate. Thus, the introduction of spherical mechanism design in MEMS may include elements of interest not just for MEMS designers but also for researchers of spherical mechanisms.
This section introduces the Micro Helico-Kinematic Platform (MHKP), which is shown in its fabricated position in Figure 7.1 and in a configuration with the platform raised above the substrate in Figure 7.2. The MHKP is a mechanism designed using spherical kinematics to transform a rotational input on the substrate into translation motion of a platform normal to the substrate combined with rotation about the center of the platform. The MHKP has three basic components: the platform, the legs which support the platform, and a slider ring which connects the legs and is the input link where force is applied. The legs and slider ring of the MHKP make spherical crank-sliders, which are the spherical counterparts of the planar crank-slider described in introductory texts on mechanism design [55].

The motion of the spherical crank-slider has several advantages for MEMS design. For example, it produces out-of-plane rotation with an in-plane rotational input. This
allows in-plane actuators to power out-of-plane motion. In addition to the advantages arising from the spherical crank-slider, the MHKP has the further advantage that it may be driven by a number of existing thermal or electrostatic actuators [82, 83, 84] because the input ring does not require full rotation. One specific application under consideration for the MHKP is an optical or infrared tunable filter, [85, 86, 87, 88].

In the sections that follow, a mathematical model of the MHKP is developed, some design considerations are outlined, and a MEMS prototype and its testing is described. The mathematical model is developed in several steps. First, some necessary background in spherical trigonometry is reviewed. Next, a qualitative description of the mechanism and its motion is given. Then, a quantitative position analysis of the mechanism is developed. In the final part of the model description, a force analysis is given based on the methods of virtual work. Based on the mathematical model, three distinct types of
MHKP designs are identified. These design types are briefly explored in the section on design considerations. The MEMS prototype, which belongs to the first type, is described and some limited experimental data is compared with the model predictions.

7.1.2 Mathematical model of the Micro Helico-Kinematic Platform

Mechanism Description

The kinematic analysis of the MHKP relies on certain results, such as Napier’s Rules, from spherical trigonometry, which were reviewed in Chapter 5. A schematic of the mechanism in an arbitrary position is shown in Figure 7.3. The schematic shows three identical spherical crank sliders, which are the heavy black lines, and the platform, which is the shaded gray area. The crank-sliders are represented by identical spherical triangles with vertices labeled \(A_0AB, A'_0A'B',\) and \(A''_0A''B''.\) The vertices \(C, A'_0,\) and \(A''_0\) are fixed to the substrate with a substrate hinge. The vertices \(B, B'\) and \(B''\) are all attached to the input slider ring, which although not shown in the schematic, completely encircles the mechanism. It should be thought of as a ring that joins vertices \(B, B'\) and \(B''\) and has a larger radius than the the circle passing through \(C, A'_0,\) and \(A''_0.\) Thus, although the \(B\) vertices can slide along the substrate toward \(C\) vertices, the \(B\) vertices maintain a fixed arc of 120° from each other. The arc from \(B\) to \(A\) measures \(c\) degrees and is called the input leg because it is attached to the input slider ring. The arc from \(C\) to \(A\) measures \(a\) degrees and is called the output leg because the output rotation of the platform is most easily measured at the point of contact of the platform and this leg. The arcs \(a\) and \(c\) are geometric parameters that are constants for a given mechanism. On
the other hand, the arc between \( C \) and \( B \), which is called the *triangle base arc*, changes as the slider ring is rotated. As the ring is rotated clockwise, the triangle base arc, \( l \), decreases, the output leg rotates counterclockwise off the substrate, the input leg rotates clockwise off the substrate as well as translating toward the output leg, the angle \( CAB \) becomes more acute, and the platform is carried upward on the legs.

The interaction between the platform and legs is complex. The platform has three slots, which are not shown in the schematic, through which the vertices, \( A \), \( A' \), and \( A'' \), and the upper part of the input and output legs protrude. As the \( A \) points move upward, they also move inward, always maintaining a fixed distance from the center of the mechanism. The upward motion of the legs carries the platform upward to the extent allowed by the polar angular arc of the slots. The polar angular arc of the slots is a design parameter called the *straddle arc*, \( p \), and is a constant for a given mechanism. The platform is allowed to slide along the legs to the height where an arc from the input leg to the output leg in a plane parallel to the substrate has a measure equal to the straddle arc, \( p \), of the slot in the platform. The design value of the straddle arc, \( p \), affects how the upward rotation of the legs is transformed into an upward translation of the platform. The slots in the platform constrain the upward motion of the platform by their arc in the polar direction and must accommodate the inward motion of the legs in the radial direction.

The arc along the leg of the spherical crank slider from \( C \) to \( D \) is denoted \( k \) and changes as the platform moves. The azimuthal arc, \( q \), is an arc measured along the great circle on the substrate which passes through the \( C \) and \( B \) points. The arc \( q \) is measured
Figure 7.3: Nomenclature for the Micro Helico-Kinematic Platform

from the point $C$ to the point on the substrate great circle nearest to $D$. The great circle arc from the substrate great circle to the point $D$ is labeled $h$. There is a direct analogy to familiar terms in geography. The substrate great circle is analogous to the equator, the point $C$ is analogous the point on the equator on the prime meridian, the azimuthal arc, $q$, is analogous to the longitude of point $D$ and the arc $h$ is analogous the latitude. It is convenient to refer to $h$ as the arc height and the linear distance from the substrate to the platform, which stays parallel to the substrate, as $H$. The interior dihedral angle of the spherical triangle $ACB$ at $C$ is $\theta$ and the interior dihedral angle of the triangle at $B$ is $\sigma$.

**Position Analysis**

The position analysis of the MHKP describes how the height and rotation of the platform, $H$ and $\phi$, depend on the rotation of the input ring, $\psi$. 

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Figure 7.4: Displacement parameters of the MHKP. Two positions are represented, the initial position with points labeled $A_0, B_0, D_0,$ and $E_0$; and an arbitrary position with points labeled $A, B, D,$ and $E$. The angular height, $h$, and the linear height, $H$, are shown. The input rotation, $\psi$, and the output rotation, $\phi$, are also shown. Additionally, the radial displacement of the legs with respect to the platform, $S$, is shown.

The position analysis is described in three parts: first, expressions for the initial position of the platform and slider ring are derived. Next, the geometric limiting conditions on the motion of the slider input ring are described. Finally, equations are derived that apply to the mechanism at any arbitrary position of the mechanism. In the discussion that follows it is convenient to indicate the initial value of a parameter that changes as the mechanism moves with a subscripted zero. Thus, the initial position of the parameter $l$ is $l_0$.

**The Initial Position** The displacement angles $\psi$ and $\phi$ are the change from the initial position of the arcs $l$ and $q$, respectively.
Figure 7.5: Schematic of the MHKP

Thus, the input angle $\psi$ can be expressed as

$$\psi = l_0 - l$$  \hspace{1cm} (7.1)$$

and the output angle $\phi$ can be expressed as

$$\phi = q_0 - q$$  \hspace{1cm} (7.2)$$

The initial arc between the fixed vertex $C$ and the sliding vertex $B_0$ is $l_0$ and is equal to the sum of the two leg angles, $a$ and $c$, and also the sum of $k_0$, $p$ and $m_0$, which are respectively, the distance from $C$ to the left side the slot in the platform, the straddle arc of the platform, and the distance from the right side of the slot in the platform to $B$.

$$l_0 = a + c = k_0 + p + m_0$$  \hspace{1cm} (7.3)$$
In the initial position, the arc \( q \) is equal to \( k_0 \). In that same position, the Law of Sines dictates that the ratio of \( \sin(a) \) to \( \sin(c) \) equals the ratio of \( \sin(k_0) \) to \( \sin(m_0) \), or
\[
\frac{\sin(a)}{\sin(c)} = \frac{\sin(k_0)}{\sin(m_0)} \tag{7.4}
\]
By using equation (7.3) to eliminate \( m_0 \) from equation (7.4) and solving for \( k_0 \) an expression can be found for the initial value \( k_0 \) and \( q_0 \),
\[
q_0 = k_0 = \tan^{-1}\left(\frac{\sin(a + c - p)\sin(a)}{\sin(c) + \cos(a + c - p)\sin(a)}\right) \tag{7.5}
\]
which completes the derivations necessary to determine the initial or reference position for the platform, \( q_0 \), and for the slider ring, \( l_0 \).

The Range of Motion  There are three possible limiting conditions on the input rotation for different types of MHKP designs. Figure 7.6 shows the different types of MHKP designs in their initial positions in parts 1A, 2A and 3A, and their final positions in parts 1B, 2B and 3B.

In the first type of MHKP design, the difference in arc between link \( a \) and \( c \) is less than the straddle arc, \( p \). As point \( B \) on the input slider ring is rotated toward the fixed pivot at \( C \), when the distance between them is less than the straddle arc \( l < p \), the platform comes to rest on the substrate. When the arc \( l \) between \( C \) and \( B \), has values less than \( p \), the motion of the platform is decoupled with the slider. Thus, the minimum allowed value of \( l \) is denoted \( l_f \) and is equal to \( p \).

In the second type of MHKP design, link \( c \) is larger than \( a \), such that \( c - a > p \), and link \( a \) rotates \( 180^\circ \) when the slider ring is rotated to the maximum possible extent.
Figure 7.6: Limiting conditions on the input rotation of different types of MHKP. The light colored link is the output leg, the dark colored link is the input leg, and the black square represents the platform. Type I mechanisms are characterized by \(|a - c| \leq p\). The initial position of a type I mechanism is shown in 1a) and the final position after \(B\) has been slid the maximum possible amount toward \(C\) is shown in 1b). In type II mechanisms, the input leg is longer than the output leg and \(c - a > p\). The initial position of a type II mechanisms is shown in 2a) and the final position in 2b). In the final position, the input leg is atop the output leg. In type III) mechanisms, the output leg is longer than the input leg and \(a - c > p\). The initial position of a type III mechanism is shown in 3a) and the final position in 3b). In the final position the output leg is atop the input leg.

When the slider ring has been rotated its full amount, link \(c\) has been pushed up and over link \(a\) and both are again lying parallel to the substrate. In the final position, \(l_f = c - a\).

In the third type of MHKP design, link \(a\) is larger than \(c\), such that \(a - c > p\), and link \(c\) rotates \(180^\circ\) when the slider ring is rotated to the maximum possible extent. When the slider ring has been rotated its full amount, link \(c\) has been pushed underneath link \(a\) and both are again lying parallel to the substrate. In the final position, \(l_f = a - c\).

The input range, \(\psi_r\), is equal to the difference between \(l_0\) and \(l_f\),

\[
\psi_r = l_0 - l_f
\] (7.6)
An immediate consequence of the limiting conditions is that the range of motion of the input, $\psi_r$, is never more than twice the arc of the shortest link.

Thus, values of the output height, $H$, and rotation, $\phi$, can be found for any value of $\psi$ between 0 and $\psi_r$ by the use of spherical trigonometry.

Table 7.1 summarizes the distinctions between the different types of MHKP designs.

<table>
<thead>
<tr>
<th>Type</th>
<th>Geometric Criterion</th>
<th>Input Range, $\psi_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$</td>
<td>a - c</td>
</tr>
<tr>
<td>II</td>
<td>$c - a &gt; p$</td>
<td>$2a$</td>
</tr>
<tr>
<td>III</td>
<td>$a - c &gt; p$</td>
<td>$2c$</td>
</tr>
</tbody>
</table>

At an Arbitrary Position of the Input  The position of the MHKP can be found at an arbitrary input rotation using equations from spherical trigonometry. In a few cases, more than one trigonometric expression is available for determining a parameter. Thus, there are expressions for finding $\sigma$ and $\kappa$ using both the Law of Cosines and the Law of Sines. For the position analysis, the expressions using the Law of Cosines are preferred since the range of the inverse cosine function matches the values that the parameters can assume. However, expressions using the Law of Sines are also included here because derivatives of the expressions used in the position analysis are used in the force analysis, and the derivatives of the Law of Sines expressions are simpler.
The angles $\theta$ and $\sigma$ of Figure 7.5 can be found using the spherical Law of Cosines for arcs, as

$$\cos(\theta) = \frac{\cos(c) - \cos(a) \cos(l)}{\sin(a) \sin(l)}$$  \hspace{1cm} (7.7)

and

$$\cos(\sigma) = \frac{\cos(a) - \cos(c) \cos(l)}{\sin(c) \sin(l)}$$  \hspace{1cm} (7.8)

or alternatively,

$$\sin(\sigma) = \frac{\sin(a)}{\sin(c)} \sin(\theta)$$  \hspace{1cm} (7.9)

To complete the position analysis, consider the triangle formed by drawing a great circle arc parallel\(^1\) to $BE$ such that it passes through point $D$. This parallel arc is shown as a dashed arc in Figure 7.5. The resulting spherical triangle can then be analyzed as a necessary preliminary to finding the arc height, $h$, of the platform. The angle $\xi$ is found using spherical Law of Cosines for angles, as

$$\cos(\xi) = -\cos(\theta) \cos(\sigma) + \sin(\theta) \sin(\sigma) \cos(n)$$  \hspace{1cm} (7.10)

where $n = l - p$ is the length of the base of the spherical triangle.

The arc $k$ is found by using the spherical Law of Cosines for angles as

$$\cos(k) = \frac{\cos(\sigma) + \cos(\xi) \cos(\theta)}{\sin(\xi) \sin(\theta)}$$  \hspace{1cm} (7.11)

\(^1\)Strictly speaking, two great circles on a sphere cannot be parallel in the sense of not intersecting. Two great circles can be parallel in the sense that they both form the same angle $\sigma$ with a third great circle.
or alternatively,

\[ \sin(k) = \frac{\sin(n)}{\sin(\xi)} \sin(\sigma) \]  

(7.12)

Finally, the arc \( h \) from the substrate to the platform is found by using Napier’s rules on the right triangle formed by the altitude \( h \), the base portion \( q \) and the side \( k \).

\[ \sin(h) = \sin(k) \sin(\theta) \]  

(7.13)

The output height, \( H \), of the platform above the substrate is then given by

\[ H = R \sin(h) \]  

(7.14)

where \( R \) is the radius of the sphere of motion of the spherical crank-sliders.

The other output parameter, \( \phi \), was defined previously in equation (7.2) in terms of the azimuthal angle, \( q \), at its initial position, which given in equation (7.5), and at an arbitrary position which is found by using Napier’s rules on the right triangle formed by sides \( k \), \( q \) and \( h \), as

\[ \sin(q) = \cot(\theta) \tan(h) \]  

(7.15)

As observed previously, points on the legs move inward as they move upward. In contrast, the platform cannot move to accommodate all three sliders. Thus, the radial width of the slots in the platform must accommodate the inward tendency of the legs. The amount that the slider legs translate inward, \( S \), at a given arc height is given by

\[ S = R(1 - \cos(h)) \]  

(7.16)
The radial width of the platform slots must be equal to or greater than the value of $S$ at the maximum arc height of the platform.

**Design Considerations**

Insight into the design and motion of the MHKP can be gained by plotting the motion of the point $D$, which is the point of contact between the platform and the output leg. Diagrams of the motions of the point $D$ (and hence the height and rotation of the platform) are given for several designs specified by different values of the input leg, $c$, the output leg, $a$, and the straddle arc, $p$. The values of these parameters were chosen for illustrative purposes and are adequate to give some design guidance. Figure 7.7 indicates the motion of point $D$ for a number of MHKP designs with a small straddle arc, $p = 1^\circ$, and other dimensions as specified in Table 7.2. Figure 7.8 indicates the motion of point $D$ for a number of MHKP designs with larger straddle arc, $p = 20^\circ$, and the other dimensions as specified in Table 7.3. The different paths are labeled by a number near the zenith of the platform’s motion. The start position of each path is indicated by an asterix, *, and the end of each path is marked by a circle, o. At even increments, $0.1\psi_r$, of input motion a cross is placed on the path, to assist in visualizing the relationship between input and output motions. The origin of the coordinate system is directly beneath the center of the platform and is the point that all the axes of the spherical mechanism’s joints pass through. The x-axis in the figure is collinear with the rotational axis of joint $C$ in Figure 7.5.

In Figures 7.7 and 7.8 all the mechanism’s paths originate to the right side of the diagram. The type I designs, the paths numbered 1, 5, and 9, all terminate at the
Table 7.2: MHKP designs with straddle arc, \( p = 1^\circ \), used in Figure 7.7

<table>
<thead>
<tr>
<th>Number</th>
<th>Leg arc, ( a )</th>
<th>Leg arc, ( c )</th>
<th>Design Type</th>
<th>Input range ( \psi_r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
<td>1</td>
<td>39</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>40</td>
<td>2</td>
<td>40</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>80</td>
<td>2</td>
<td>60</td>
</tr>
<tr>
<td>4</td>
<td>40</td>
<td>20</td>
<td>3</td>
<td>40</td>
</tr>
<tr>
<td>5</td>
<td>40</td>
<td>40</td>
<td>1</td>
<td>79</td>
</tr>
<tr>
<td>6</td>
<td>40</td>
<td>80</td>
<td>2</td>
<td>80</td>
</tr>
<tr>
<td>7</td>
<td>80</td>
<td>20</td>
<td>3</td>
<td>40</td>
</tr>
<tr>
<td>8</td>
<td>80</td>
<td>40</td>
<td>3</td>
<td>80</td>
</tr>
<tr>
<td>9</td>
<td>80</td>
<td>80</td>
<td>1</td>
<td>159</td>
</tr>
</tbody>
</table>

x-axis. The type II designs, the paths numbered 2, 3, and 6, terminate to the left side of the diagram, and the type III designs, paths numbered 4, 7, and 8, terminate on the right side of the diagram. One can conclude that designs of type II will have the greatest platform rotation and designs of type III will have the least platform rotation for a given input rotation. The Type III designs exhibit very little platform rotation and their paths terminate very close to where they originated. The type II mechanisms exhibit nearly circular paths, whose motion is little influenced by the length of link \( c \), as evidenced by the similarity in paths 2 and 3.

The mechanisms with a straddle arc, \( p = 20^\circ \), differ from the mechanisms with the same leg lengths but smaller straddle arc. In Figure 7.8 most of the mechanisms exhibit qualitatively different motions. Most dramatically, mechanism designs 2 and 4, have changed to type I mechanisms, and their paths now terminate at the x-axis. All the mechanisms have their maximum height reduced. The type III mechanisms evidence
more platform rotation with the larger straddle arc. Most of the mechanisms show a similar pattern of rise from their initial position up to their apex point. The paths show their most significant qualitative differences on between the apex and the terminal points. For example, in mechanism 9, the platform drops very abruptly in the last 10% of the input rotation. Mechanism 2 shows a change in the platform rotation direction after its apex point and mechanism 4 exhibits a gradual, almost linear drop from its apex point. All this suggests that the mechanisms may exhibit more path variability and be more suitable for optimization in the region of motion between the apex and the terminal point.

**Force Analysis**

Virtual work methods can be used to model the output force neglecting friction effects. The input is a torque applied to the slider ring and the outputs are a vertical force located at the center of the platform and a torque applied by the platform in the direction of the rotation of the platform.
Figure 7.7: The path of the point $D$ on the platform for several different MHKP designs with straddle arc $p = 1^\circ$.

Figure 7.8: The path of the point $D$ on the platform with several different MHKP designs with straddle arc $p = 20^\circ$.
Thus, at equilibrium the virtual work, $\delta W$, done by a torque applied at the slider is assumed to balance the work done by the force and torque on the platform. The displacement coordinate associated with the input torque, $T_{in}$ is the input rotation, $\psi$. The displacement associated with the output force, $F_{out}$, is the height of the platform, $H$, and the displacement coordinate associated with the output torque, $T_{out}$, is the output rotation, $\phi$. Thus, the equilibrium condition is

$$\delta W = T_{in} \delta \psi + F_{out} \delta H + T_{out} \delta \phi = 0 \quad (7.17)$$

where $\delta \psi = -\delta l$, or

$$-T_{in} \delta l + F_{out} \frac{dH}{dl} \delta l + T_{out} \frac{d\phi}{dl} \delta l = 0 \quad (7.18)$$

The arc $l$ is chosen as the generalized coordinate, and $d\phi/dl = -dq/dl$, resulting in

$$T_{in} = F_{out} \frac{dH}{dl} - T_{out} \frac{dq}{dl} \quad (7.19)$$

The kinematic coefficients $dH/dl$ and $dq/dl$ can be found by computing the derivatives of the position equations (7.7)-(7.15) with respect to the generalized coordinate $l$, or

$$\frac{dH}{dl} = R \left( \cos(h) \frac{dh}{dl} \right) \quad (7.20)$$

$$\frac{dq}{dl} = -\csc(\theta) \sin(h) \frac{dh}{dl} + \cos(\theta) \sec(h) \frac{dh}{dl} \quad (7.21)$$
where,

\[
\frac{d\theta}{dl} = \frac{1}{\sin(\theta)} \left( \frac{\cos(c) - \cos(a) \cos(l)}{(\sin(a) \sin(l))} \cot(l) - \cot(a) \right) \tag{7.22}
\]

\[
\frac{d\sigma}{dl} = \frac{\cos(\theta) \sin(a) d\theta}{\cos(\sigma) \sin(c) dl} \tag{7.23}
\]

\[
\frac{d\xi}{dl} = -\frac{A \frac{d\theta}{dl} + B \frac{d\sigma}{dl} + C}{\sin(\xi)} \tag{7.24}
\]

\[
\frac{dk}{dl} = \frac{D \frac{d\xi}{dl} + E \frac{d\sigma}{dl} + F}{\cos(k) \sin(\xi)} \tag{7.25}
\]

\[
\frac{dh}{dl} = \frac{1}{\cos(h)} \left( \cos(k) \sin(\theta) \frac{dk}{dl} + \sin(k) \cos(\theta) \frac{d\theta}{dl} \right) \tag{7.26}
\]

and

\[
A = \sin(\theta) \cos(\sigma) + \cos(\theta) \sin(\sigma) \cos(n) \tag{7.27}
\]

\[
B = \cos(\theta) \sin(\sigma) + \sin(\theta) \cos(\sigma) \cos(n) \tag{7.28}
\]

\[
C = -\sin(\theta) \sin(\sigma) \sin(n) \tag{7.29}
\]

\[
D = \frac{-\sin(n) \sin(\sigma) \cos(\xi)}{\sin(\xi)} \tag{7.30}
\]
\[ E = \sin(n) \cos(\sigma) \quad (7.31) \]

\[ F = \sin(\sigma) \cos(n) \quad (7.32) \]

The input torque, \( T_{in} \), required to produce the output force, \( F_{out} \), and output torque, \( T_{out} \), can be calculated by equation (7.19) for arbitrary position and dimensions. The output resistance that must be overcome by the input torque can be due to any combination of the two outputs. Thus, in some applications most of the resistance may be due to rotational resistance and in others it may be due to translational resistance in the vertical direction.

**Prototype Testing**

A macro-scale proof-of-concept prototype was fabricated in polypropelene to validate the design. The macro model had input and output legs of equal length and a very small straddle arc. As predicted in the model, the output rotation, \( \phi \), was approximately half the input rotation, \( \psi \) and the output height was very close to the height of the apex of the spherical triangle, point \( A \).

The MKHP was also fabricated using the MUMPS (Multi-User MEMS ProcesS) \[89\] with the dimensions listed in Table 7.4. The micrographs shown in Figures 7.1 and 7.2 are images of one of the prototypes. The slider ring and legs were patterned in the first releasable layer of polysilicon (POLY1) and the platform was fabricated in the second releasable layer (POLY2). The joints between different links in the spherical crank
slider were scissor hinges [34] that appeared to have minimal clearances. The sacrificial layers of the MHKP were removed and the mechanism was tested. It was shown to be capable of lifting the platform off the substrate as shown in Figure 7.2. The motion of the platform off from the substrate was not measured quantitatively. However, it was observed that the vertical translation occurred in distinct stages. First, little vertical movement occurred as the rotation of the slider $B$ acted to take up the clearance in the joints. Next, larger vertical translations were seen for very small input rotation. The motion of the platform then became less sensitive to the displacement of the slider as the platform is in the neighborhood of its maximum height. Finally, as the input motion is continued the height of the platform decreased. The motion of the prototype was qualitatively similar to design 5 shown in Figure 7.8 and described in Table 3.

Figure 7.9 shows a plot of the model predictions and measured values from the prototype of the input rotation, $\psi$, and the output rotation, $\phi$. These values are measured in parallel to the substrate and suffer no distortion as the image plane is also parallel to

![Table 7.4: Dimensions of a prototype Micro Helico-Kinematic Platform](image)

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>43°</td>
</tr>
<tr>
<td>$c$</td>
<td>43°</td>
</tr>
<tr>
<td>$p$</td>
<td>26°</td>
</tr>
<tr>
<td>$R$</td>
<td>175µm</td>
</tr>
<tr>
<td>$l_0$</td>
<td>86°</td>
</tr>
<tr>
<td>$l_f$</td>
<td>45°</td>
</tr>
<tr>
<td>$k_0$</td>
<td>30°</td>
</tr>
</tbody>
</table>
the substrate. In general, the measured and predicted values match well, though it is noted that in some instances the value of the output rotation, $\phi$, is higher than predicted.

The fact that the output rotations appear to be slightly higher than predicted in a few instances corresponds well to the observation that the platform experienced some stick-slip behavior.

### 7.1.3 Conclusions

The Micro Helico-Kinematic Platform introduces the micro spherical crank-slider and incorporates it in the design of a mechanism which simultaneously lifts and rotates a platform. A theoretical model was developed describing displacement and force relations for the mechanism. The model was used to develop design considerations including the distinction of three distinct types of MHKP designs. A prototype mechanism was built which permitted some verification of the position analysis portion of the model.
7.2 Spherical Bistable Micromechanism

7.2.1 Introduction

There is a need for accurate, low power mechanisms for the out-of-plane positioning of microelectromechanical system (MEMS). Such mechanisms are useful in mirror arrays [90] and in erectable structures [35]. One possible means of achieving these accurate, low power mechanisms is to develop out-of-plane bistable mechanisms. Several different design concepts for bistable mechanisms have been identified [91] including mechanisms composed of rigid and compliant links [92, 93, 74], bucking structures [94, 95, 96], and braking or latching devices [97, 98]. Buckling [99] and latching devices [35] have been used to position out-of-plane mechanisms.

However, out-of-plane compliant bistable mechanisms are somewhat challenging [74] because the devices are fabricated in-plane and the elasticity of the compliant segments tends to cause them to return to the plane of fabrication.

The mechanism described in this section combines two recent advances in MEMS design in a unique way to provide a device that achieves bistable out-of-plane positioning through the use of compliant mechanisms. One of the advances is the adaption of the spherical slider-crank [68] to the micro-level, where it is useful for transforming an in-plane input-rotation to an out-of-plane output rotation [100]. The other advance is the discovery that an extension to a known planar bistable compliant mechanism, the Young Mechanism [74, 101], can provide the input motion needed for the micro spherical slider-crank. The input to the combined spherical slider-crank-Young Mechanism system is
Figure 7.10: Scanning Electron Micrograph (SEM) of a Spherical Bistable Mechanism (SBM) in its fabricated position, which is its first stable position. The mechanism is made by the combination of a Young Mechanism [74, 101] and a spherical slider-crank [68].

Another extension of the device. The bistability of the Young Mechanism provides two stable positions for the output link of the spherical slider-crank.

The combination of the Young Mechanism and the spherical slider-crank is called the Spherical Bistable Micromechanism (SBM) and a surface micromachined prototype (fabricated using the MUMPS process [102]) is shown in its first stable equilibrium position (the as fabricated position) in Figure 7.10 with all links parallel to the substrate. A schematic view is shown in Figure 7.11. The SBM is shown in its second stable equilibrium position in Figures 7.12 and 7.13.

The SBM avoids the difficulty in achieving a stable out-of-plane position for a compliant mechanism by keeping the motion of the compliant portion of the device (the
Figure 7.11: Schematic of a Spherical Bistable Mechanism (SBM) in its fabricated position. The Young Mechanism portion of the device is shaded gray, and the spherical slider-crank portion is white.

Figure 7.12: SEM of a SBM in its second stable position
Young Mechanism) planar. The out-of-plane motion is achieved by virtue of the spherical slider-crank’s ability to transform an in-plane rotation into an out-of-plane rotation.

This section describes the geometry of the SBM and provides equations for obtaining motion and performance characteristics. Analysis of the device requires background into two different specialties, compliant mechanisms and spherical trigonometry.

**Compliant Mechanisms**

Compliant mechanisms are mechanisms that gain some or all of their motion from the deflection of flexible members [103]. Flexible members are advantageous in that their motion is precise and that they can store energy. On the other hand, the analysis of compliant mechanisms is, in general, more difficult than the analysis of rigid-link mechanisms. For example, the position analysis of a rigid-link mechanism requires algebraic
equations, while the complete position analysis (in which the location of every point in
the segment is specified) of a compliant mechanism involves differential equations. Fortunately, the complete analysis of compliant mechanisms is not always required. An approximation technique called the Pseudo-Rigid-Body Model (PRBM) allows the determination of the relative positions of the endpoints of various compliant segments without precise modeling of the location of interior points. PRBMs also allow the computation of the amount of force required to produce the desired deflections. The idea of a PRBM is to model the compliant segment with rigid links and joints in a way that closely approximates the motion of the compliant segment. Two PRBMs that are pertinent to the motion of the Young Mechanism are the cantilever beam with a force at the free end, and the small-length flexural pivot.

In these two PRBMs, the flexible segment is modeled by placing a revolute joint, the characteristic pivot, at a specified distance, the characteristic radius, from the free end. The bending of the segment is modeled by the rotation, Θ, of the characteristic pivot. The resistance of the flexible segment to bending is modeled with a torsional spring at the characteristic pivot with a stiffness, K. As the segment bends, the position of the beam end is specified by the coordinates (a, b), where a is the coordinate along the direction of the undeflected segment, and b is the coordinate in the direction perpendicular to the undeflected segment.
Figure 7.14 shows a schematic of a cantilever beam with a force at the free end, $F$, and its pseudo-rigid-body model. The model parameters are

\[
\begin{align*}
a &= l[1 - \gamma(1 - \cos \Theta)] \\
b &= \gamma l \sin \Theta \\
K &= \frac{\gamma K_\theta EI}{l} \\
\gamma &\approx 0.85 \\
K_\theta &\approx \pi \gamma
\end{align*}
\]  

(7.33)

The approximate values given for $\gamma$, $c_\theta$, and $K_\theta$ are most appropriate when the applied force is perpendicular to the undeflected segment. More accurate approximations are given in [103] for other loading conditions. The maximum stress in the segment occurs at the fixed end and is given by

\[
\sigma_{max} = \pm \frac{P(a + nb)c}{I} - \frac{nP}{A}
\]  

(7.34)

where $P$ is the component of the applied force, $F$, in the direction perpendicular to the undeflected segment, and $nP$ is the component of the applied force in the direction parallel to the undeflected segment.

Figure 7.15 shows a schematic of a small-length flexural pivot and its pseudo-rigid-body model. The small-length flexural pivot is a flexible segment which is small in comparison to a rigid segment to which it is attached such that $l << L$ and $(EI)_L << (EI)_L$. The characteristic pivot is located at the center of the flexible beam. The model parameters are

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Figure 7.14: Schematic of a) a cantilever beam undergoing a large deflection and b) the Pseudo-rigid-body model equivalent \[103\].

\[
a = \frac{l}{2} + \left( L + \frac{1}{2} \right) \cos \Theta \\
b = \left( L + \frac{1}{2} \right) \sin \Theta \\
K = \frac{EI}{l}
\]  
(7.35)

The maximum stress in the small-length flexural pivot occurs at the fixed end and is given by

\[
\sigma_{\text{max}} = \frac{Mc}{I}
\]  
(7.36)

The maximum strain for both models is related to the maximum stress and is given by

\[
\epsilon_{\text{max}} = \frac{\sigma_{\text{max}}}{E}
\]  
(7.37)

where \( E \) is Young Modulus (or modulus of elasticity).

These two PRBMs allow the compliant portion of the SBM to be analyzed as a four-bar mechanism with torsional springs on two of the joints as shown in Figure 7.16.

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Figure 7.15: Schematic of a) a small-length flexural pivot undergoing a large deflection and b) the Pseudo-rigid-body model equivalent [103]

Figure 7.16: Illustration of a) a Young Mechanism, b) its pseudo-rigid-body model, and c) parameters for its position analysis (adapted from [74])
Analysis of the spherical slider-crank portion of the SBM requires some background on spherical mechanisms.

**Spherical Mechanisms**

Spherical mechanisms are linkages that have the property that every link in the system rotates about the same fixed point [49]. A common method for visualizing their motion represents the links in a spherical mechanism as arcs inscribed on a unit sphere. Any two links in a spherical mechanism are joined with pin (or revolute) joint which permits rotation about an axis in space that passes through the fixed point. In an SBM, the fixed point may be either of the Young Mechanism’s two pin joints.

There are numerous possible approaches for describing the motion of spherical mechanisms [68] [49]. In this section, we use an approach based on spherical trigonometry which was reviewed in Chapter 5.

The spherical law of cosines is useful in the position analysis of spherical slider-crank portion of the SBM. The background given on compliant mechanisms and spherical mechanisms allows for the position and energy analysis of the SBM.

**7.2.2 Position Analysis**

The position analysis of the SBM is divided into two parts, the Young Mechanism and the spherical slider-crank. The Young Mechanism portion of the SBM can be analyzed using the PRBM as a four-bar with two torsional springs, as was shown in Figure 7.16. The analysis of a four-bar can be found in texts on planar mechanisms (see
for example [101] and [103]) and may be derived using the law of cosines from planar trigonometry using the angles labeled in Figure 7.16:

\[ \delta = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\pi - \theta_2)} \]  
\[ \beta = \cos^{-1}\left( \frac{r_1^2 + \delta^2 - r_2^2}{2r_1\delta} \right) \]  
\[ \psi = \cos^{-1}\left( \frac{r_3^2 + \delta^2 - r_4^2}{2r_3\delta} \right) \]  
\[ \lambda = \cos^{-1}\left( \frac{r_4^2 + \delta^2 - r_3^2}{2r_4\delta} \right) \]  

For \( 0 \leq \theta_2 \leq \pi \), \( \theta_3 \) and \( \theta_4 \) are given by

\[ \theta_3 = \beta + \pi - \psi \]  
\[ \theta_4 = \beta + \pi + \lambda \]

and for \( \pi \leq \theta_2 \leq 2\pi \), \( \theta_3 \) and \( \theta_4 \) are given by

\[ \theta_3 = -\beta + \pi - \psi \]  
\[ \theta_4 = -\beta + \pi + \lambda \]

The orientations \( \theta_5 \) and \( \theta_6 \) of links \( a_5 \) and \( a_6 \) in the spherical slider-crank portion of the mechanism can be determined based on the spherical triangle formed by links \( a_5 \), \( a_6 \), and the arc length \( s_7 \) between the fixed pivot \( D \) and the rotational slider \( C \) as shown in Figure 7.17. In the fabricated position of the SBM, the arc length of \( \widehat{DC} \) is given by

\[ s_7 = a_5 + a_6. \]

The change in \( \widehat{DC} \) as the input link \( r_2 \) rotates is \( \Delta S_7 \) and is equal to \( \Delta \theta_2 = \theta_{20} - \theta_2 \), where \( \theta_{20} \) is the original orientation of the pseudo link labeled \( r_2 \) in Figure 7.16. Thus,
Using the spherical law of cosines, expressions using $\theta_5$ and $\theta_6$ can be found. An expression in which $\theta_5$ is the only unknown is given by

$$\cos(a_6) = \cos(a_5) \cos(S_7) + \sin(a_5) \sin(S_7) \cos(\theta_5)$$

(7.48)

which can be solved for $\theta_5$ as

$$\theta_5 = \cos^{-1} \left( \frac{\cos(a_6) - \cos(a_5) \cos(S_7)}{\sin(a_5) \sin(S_7)} \right)$$

(7.49)

An expression in which $\theta_6$ is the only unknown is given by

$$\cos(a_5) = \cos(a_6) \cos(S_7) + \sin(a_6) \sin(S_7) \cos(\theta_6)$$

(7.50)
which can be solved for $\theta_6$ as

$$\theta_6 = \cos^{-1} \left( \frac{\cos(a_5) - \cos(a_6) \cos(S_7)}{\sin(a_6) \sin(S_7)} \right)$$

(7.51)

Sustituting equation (7.47) into equation (7.51) gives the angle of the spherical mechanism output, $\theta_6$, in terms of the Young Mechanism input, $\theta_2$.

The motion of the spherical slider-crank output link depends on the distance and angle between its joints but not on its shape. Thus, both of the links shown in Figure 7.18 can be modeled by the forgoing equations and the output link can take a shape that is most suited to a given application.

7.2.3 Energy Analysis

The input force is applied on link $r_2$ as shown in Figure 7.16. The potential energy, $W$, stored in the SBM’s flexible segments can be estimated as a function of $\theta_2$ using the pseudo-rigid body model as

$$W(\theta_2) = \frac{1}{2} (K_A \psi_A^2 + K_B \psi_B^2)$$

(7.52)
where $\psi_A$ and $\psi_B$ are defined by

$$\psi_A = (\theta_2 - \theta_{20}) - (\theta_3 - \theta_{30}) \quad (7.53)$$

and

$$\psi_B = (\theta_4 - \theta_{40}) - (\theta_3 - \theta_{30}) \quad (7.54)$$

where the spring constants $K_A$ and $K_B$ are calculated using the PRBM, as

$$K_A = \frac{EI}{l_s} \quad (7.55)$$

$$K_B = 2.25 \frac{EI}{l_4} \quad (7.56)$$

The values of $\theta_2$ for which the potential energy, $W$, is a local minimum are the stable equilibrium points for the mechanism. In between the two local minima there is a local maximum, which is the unstable equilibrium point. The input torque, $T_{in}$ required to actuate the mechanism can be found as the derivative of the potential energy with respect to $\theta_2$, or

$$T_{in} = \frac{dW}{d\theta_2} = K_A\psi_A(1 - h_{32}) + K_B\psi_B(h_{42} - h_{32}) \quad (7.57)$$

where $h_{32}$ and $h_{42}$ are kinematic coefficients

$$h_{32} = \frac{d\theta_3}{d\theta_2} = \frac{r_2 \sin(\theta_4 - \theta_2)}{r_3 \sin(\theta_3 - \theta_4)} \quad (7.58)$$

$$h_{42} = \frac{d\theta_4}{d\theta_2} = \frac{r_2 \sin(\theta_3 - \theta_2)}{r_4 \sin(\theta_4 - \theta_3)} \quad (7.59)$$

The joints in the spherical slider-crank are not compliant and so do not enter into the calculation of potential energy. On the other hand, because the spherical slider-crank has a poor transmission angle ($\approx 180^\circ$) in the fabricated position, the SBM mechanism
can be more difficult to actuate than a Young Mechanism alone. It may be helpful to include an auxiliary actuation method to insure that the links in the spherical crank-slider portion of the mechanism lift from the substrate.

7.2.4 Prototype Testing

The polysilicon ($E \approx 169$ MPa) prototype mechanism shown in Figure 7.10 was fabricated using the MUMPS process and has the dimensions of $r_1 = 100 \mu m$, $r_2 = 250 \mu m$, $r_3 = 176 \mu m$, $r_1 = 250 \mu m$. The original orientations of links 2 and 4 are $\theta_{20} = 73^\circ$ and $\theta_{40} = 53^\circ$. The lengths of the compliant segments, $l_s$ and $l_4$, are $30 \mu m$ and $295 \mu m$, respectively. The bending moments of inertia for the compliant segments are $I_2 = I_4 = 3.3 (\mu m)^4$. The spherical mechanism links have a radius of $140 \mu m$ and arc lengths $a_5 = a_6 = 75^\circ$. Because links $a_5$ and $a_6$ have the same nominal arc length, the spherical triangle is isosceles and angles $\theta_5$ and $\theta_6$ are equal.

Figure 7.19 shows a plot of the rotation parameters of the mechanism, $\theta_2$, $\theta_3$, $\theta_4$, $\theta_5$, $\theta_6$, and $S_7$ as functions of the magnitude of the change in the input angle $|\Delta \theta_2|$. The stable equilibrium positions of the mechanism are marked with ‘o’s and the unstable equilibrium position of the mechanism is marked with an ‘x’. Note that most of the rotation of links $a_6$ and $a_5$ occurs within the first 30 degrees of rotation of $\theta_2$. This implies that the ratio of output motion, $\theta_6$, to input motion, $\theta_2$ is much smaller near the second equilibrium position than it is near the first equilibrium position. This results in finer control of the output motion near the second equilibrium position and it is possible to design for a precise orientation of link 6 in the second equilibrium position.
Figure 7.19: The rotation of the links in the SBM as a function of the input rotation $|\Delta \theta_2|$. 

Figure 7.20 shows angular measurements from the second stable position. The motion of the input link, $\Delta \theta_2$ was measured as $79^\circ$ and the motion and $S_7$ was measured as $72^\circ$. These values compare well with the predicted values of $\Delta \theta_2 = 80.6^\circ$ and $S_7 = 69.34^\circ$. Precise measurements were not available for the out-of-plane rotation and psuedo-link rotation variables. However, in Figure 7.20 the link $a_5$ appears to be slightly less than vertical which agrees well with the predicted value of $79.3^\circ$ for $\theta_5$.

Figure 7.21 shows the potential energy curve for the silicon prototype and Figure 7.22 shows the input torque required to actuate the device. Note that the input torque curve (Figure 7.22) is the derivative of the potential energy curve (Figure 7.21).

Figure 7.23 shows the calculated strain in the flexures. A design goal is to maintain the strain magnitude below $1.05 \times 10^{-2}$ to avoid fracture [74].
Figure 7.20: A top view of the second stable equilibrium position, where $|\Delta \theta_2|$ is measured as 79° and $S_7$ is measured as 72°.

Figure 7.21: The total potential energy stored in the compliant segments of the SBM as a function of the input rotation $|\Delta \theta_2|$.
Figure 7.22: The input torque required to hold the SBM in equilibrium at a given value of the input rotation $|\Delta \theta_2|$.

Figure 7.23: The strain in the compliant segments in the SBM as a function of the input rotation $|\Delta \theta_2|$.
7.2.5 Conclusions

This section has discussed the design of a novel device for the bistable positioning of an out-of-plane link, such as a micro-mirror. The combination of bistability with spherical mechanism design results in several advantageous features, which include: two stable positions that require power only in transitioning from one position to the other, robustness against small disturbances, and an output link with a stable out-of-plane orientation that can be achieved with great precision. The equations for position, potential energy, input torque and maximum stress have been presented. The devices have been fabricated using the MUMPS surface micromachining process and bistable behavior has been demonstrated.

7.3 Three-degree-of-freedom Platform

7.3.1 Introduction

A new mechanism can be derived from the MHKP by decoupling the input spherical crank-sliders. This new mechanism is called the three-degree-of-freedom platform (3DOFP) and a schematic of it in its fabricated position is shown in Figure 7.24. The three degrees of freedom are manifested in three input parameters (the rotation of three different spherical slider-crank) and three output parameters (the height of the platform, and magnitude and direction of the platform tilt). Because the platform can tilt away from the horizontal, the system could be used as an orientable mirror in 3-D optics applications [106].
The 3DOFP can be analyzed by considering each spherical slider-crank independently, and then analyzing their combined effect on the position and orientation of the platform. Each spherical slider crank consists of a rotational slider, a coupler link, $k_i$, and an output link, $m_i$. Each output link is attached to the substrate and has an output extension link, $q_i$, attached to it. At the end of each output extension link is an output point $\overline{x}_i$, that is constrained to move inside of a slot in the platform. Each spherical slider-crank is analyzed independently to determine the location of $x_i$ and then the $x_i$ can be used to determine the location of the platform center $P$ and the orientation of a normal vector $\hat{n}p$ to the platform.

### 7.3.2 Position Analysis of the 3DOFP’s Spherical Slider-Cranks

The 3DOFP has three rotational inputs, $\psi_1$, $\psi_2$ and $\psi_3$. The value of an input rotation, $\psi_i$, can be used to determine the orientations of the links in the $i^{th}$ spherical slider-crank and the location of an output point $\overline{x}_i$ as shown in Figure 7.25.

The input, $\psi_i$, controls the arclength of one side, $n_i$, of a spherical triangle. In the fabricated position, the length of the controlled side, $n_{i0}$, is equal to the sum of the lengths of the other two sides of the spherical triangle, $m_i$ and $k_i$, or

$$n_{i0} = m_i + k_i$$  \hspace{1cm} (7.60)

The length of the controlled side, $n_i$, can be expressed as a function of the input rotation as

$$n_i = n_{i0} - \psi_i$$  \hspace{1cm} (7.61)
Figure 7.24: A schematic of the three-degree-of-freedom platform in its fabricated position showing the center of the platform, $P$, the three input rotations, $\psi_i, i = 1, 2, 3$, the nomenclature for the links, and the locations of hinges.
Figure 7.25: A schematic of one of the leg assemblies of the three-degree-of-freedom platform showing the assembly in an actuated position with the links, angles and the coupler point $x_i$ labeled.
The dihedral angles, $\sigma_i$, $\xi_i$, and $\theta_i$, formed by the sides of the spherical triangle may then be calculated from the spherical law of cosines as

$$\sigma_i = \cos^{-1} \left( \frac{\cos(m_i) - \cos(k_i) \cos(n_i)}{\sin(k_i) \sin(n_i)} \right) \quad (7.62)$$

$$\theta_i = \cos^{-1} \left( \frac{\cos(k_i) - \cos(m_i) \cos(n_i)}{\sin(m_i) \sin(n_i)} \right) \quad (7.63)$$

$$\xi_i = \cos^{-1} \left( \frac{\cos(n_i) - \cos(k_i) \cos(m_i)}{\sin(k_i) \sin(m_i)} \right) \quad (7.64)$$

where $\sigma_i$ is the output angle (link $m_i$ is attached to the ground), $\theta_i$ is the coupler angle (link $k_i$ is the coupler link), and $\xi_i$ is the transmission angle that determines the aptness of the force transmission from link $k_i$ to $m_i$.

The output extension arm, $q_i$, is attached to the output link. The position of the coupler point, $\vec{x}_i$, on the tip of the coupler arm is given by using Napier’s rules to solve the right spherical triangle formed by $q_i$ at the angle $\sigma$ with the substrate as shown in Figure 7.26. The adjacent side of the right spherical triangle, $g_i$, and the opposite side, $h_i$, are convenient for expressing the position of $\vec{x}_i$ in spherical coordinates. The adjacent side $g_i$ is the azimuthal angle of the output point with respect to the fixed corner of the spherical triangle, and the opposite side $h_i$ is the elevation angle of the output point with respect to the substrate.

$$\tan(g_i) = \tan(q_i) \cos(\sigma_i) \quad (7.65)$$

$$\sin(h_i) = \sin(q_i) \sin(\sigma_i) \quad (7.66)$$
The location of the output point, $\vec{x}_i$, can then be expressed in terms of the components $x_{ix}$, $x_{iy}$, and $x_{iz}$.

\[
\begin{align*}
x_{ix} &= \cos(h_i) \cos(g_i + \phi_i) \\
x_{iy} &= \cos(h_i) \sin(g_i + \phi_i) \\
x_{iz} &= \sin(h_i)
\end{align*}
\]

where $\phi_1 = 0^\circ$, $\phi_2 = 120^\circ$, and $\phi_3 = 240^\circ$.

Once the coordinates of all three $x_i$ are known, the position of the center of the platform $P$ and the orientation of the normal to the platform $\vec{n}_p$ can be found.
7.3.3 Position Analysis of the 3DOFP’s Platform

The coupler point $x_i$ is constrained to travel in a slot in the platform. The three slots in the platform are straight lines that are radially arrayed around the center of the platform, $P$, with 120° between them as shown in Figure 7.27.

As an input angle, $\psi_i$, increases, the corresponding output point, $x_i$, tends to move closer to the center of the platform $P$. The position of $P$ relative to the output points, $x_i$, can be solved in closed form using the geometric relationships illustrated in Figures 7.28 and 7.29 which are based on the lines $\bar{P}_i$ which connect point $P$ with the points $x_i$ at 120° angles to each other.

The locus of points that can form a 120° angle with the points $x_1$ and $x_2$ lies on a circle, $c_1$, that contains the point $P$. Similarly, the pairs $\{x_2, x_3\}$ and $\{x_3, x_1\}$ and
Figure 7.28: A schematic indicating quantities used to calculate the location of the platform center $P$.

Figure 7.29: A schematic indicating additional quantities used to calculate the location of the platform center $P$. 
the angle constraint define circles, $c_2$ and $c_3$, that contain the point $P$. These circles all intersect at a single common point, which is the point $P$ that is sought.

The equation for a particular circle (i.e. a circle in bipolar form) that forms an angle $\beta$ with two points $\{-a,0\}$ and $\{a,0\}$ is given \[65\] by

$$x^2 + (y - a \cot(\beta))^2 = a^2 \csc^2(\beta)$$

(7.70)

where $x-y$ coordinate frame is located at the midpoint of the segment between the two points. The $x$ coordinate is parallel to the line segment that joins the two points and the $y$-coordinate is perpendicular to that segment. The circle described by equation (7.70) has its center at a distance $a \cot(\beta)$ away from the midpoint of the two reference points along a line perpendicular to the segment connecting the reference points. The circle has a radius given by $a \csc(\beta)$.

The circles, $c_i$, are located in the same plane as the platform, which is determined by the three points $\vec{x}_i$. Thus, the circles and the platform have the same normal vector. The unit normal to the platform, $\hat{n}p$ is determined by the cross product

$$\hat{n}p = \frac{\vec{d}_{21} \times \vec{d}_{32}}{|\vec{d}_{21} \times \vec{d}_{32}|}$$

(7.71)

where $\vec{d}_{21} = \vec{x}_2 - \vec{x}_1$ and $\vec{d}_{32} = \vec{x}_3 - \vec{x}_2$ are two of the sides of the triangle formed by the points $\vec{x}_1$, $\vec{x}_2$ and $\vec{x}_3$. The third side is given by $\vec{d}_{13} = \vec{x}_1 - \vec{x}_3$. The location of the center of the circle $c_i$ is given by

$$\vec{o}_i = \vec{A}_i + \frac{|\vec{d}_{ji}|}{2} \cot(120^\circ) \hat{e}_i$$

(7.72)

where $\vec{o}_i$ is the coordinate for the center of the circle $c_i$, $\vec{A}_i$ is the midpoint of the segment between $\vec{x}_i$ and $x_{i+1}$ and $\hat{e}_i$ is a unit vector in a direction perpendicular to $\hat{n}p$ and the
triangle side $d_{ji}$ (where $j = i + 1$ for $i = 1, 2$ and $j = 1$ for $i = 3$) or

$$\hat{e}_i = \frac{\hat{n}_p \times d_{ji}}{|\hat{n}_p \times d_{ji}|} \quad (7.73)$$

The radii, $r_i$, of the circles $c_i$ are given by

$$r_i = \frac{|d_{ji}|}{2} \csc(120^\circ) \quad (7.74)$$

The point, $P$, lies at the intersection of the three circles, $c_i$. In general, two non-concentric circles intersect at two points. One of the intersection points is an $x_i$ point and the other is the point $P$. Since the point $P$ is in the interior of the triangle there is not any ambiguity about which of the intersection points is the point $P$. The points $o_i$, $o_j$ and $P$ form a triangle with sides $l_{ji}$, $r_i$, and $r_j$. Thus, using the law of cosines the angle $\delta_{ji}$ that $r_i$ makes with $l_{ji}$ can be found as

$$\delta_{ji} = \cos^{-1}\left(\frac{|l_{ji}|^2 + r_i^2 - r_j^2}{2|l_{ji}|r_i}\right) \quad (7.75)$$

The calculation of $\delta_{ji}$ allows the position of $P$ to be found as

$$\vec{p}_{ji} = \vec{o}_i + r_i \cos(\delta_{ji})\vec{c}_u + r_i \sin(\delta_{ji})\vec{c}_n \quad (7.76)$$

where $\vec{c}_u$ is a unit vector parallel with the segment from $o_i$ to $o_j$, and $\vec{c}_n$ is a unit vector perpendicular to $\vec{c}_u$ and $\hat{n}_p$. Equations for $\vec{c}_u$ and $\vec{c}_n$ are given by

$$\vec{c}_u = \frac{l_{ji}}{|l_{ji}|} \quad (7.77)$$

$$\vec{c}_n = \hat{n}_p \times \vec{c}_u \quad (7.78)$$

There are three different $\vec{p}_{ji}$ that can be calculated for each set of $\vec{x}_i$ and each gives the correct location of the point $\vec{p}$. The redundancy of the solution method provides
a check on the correctness of the calculations. The length $\vec{p}_i$ is important because the platform slots do not extend to the center of the platform. Thus, the platform slots act as stops on the motion of the input sliders which results in a minimum permissible value for $p_i$.

### 7.3.4 Model Predictions for a Three-degree-of-freedom Platform

The model described above was used to predict the motion of a 3DOFP with $m_i = k_i = 30^\circ$, $q_i = 90^\circ$ and a slot constraint that prevents $p_i \leq \frac{1}{3}p_{i0}$, where $p_{i0}$ is the value of $p_i$ as fabricated.

Figure 7.30 shows a top view and Figure 7.31 shows a side view of a random sampling of the locus of points that can be reached by the center of the platform, $P$, and the $\vec{x}_i$. To illustrate the limitations placed by the slot geometry, the platform locations that require $p_i \leq \frac{1}{3}p_{i0}$ are colored gray. In the top view (Figure 7.30), the circular arcs traced by the $x_i$ appear to be straight lines. In the side view (Figure 7.31), the curved paths of the $x_i$ are evident. In Figure 7.30 the locus of points that can be occupied by $P$ has three distinct lobes, which are positioned $180^\circ$ away from the arcs formed by the $x_i$. This indicates that as a slider-crank is actuated, it tends to move the platform’s center away from the slider-crank. The side view shows that the greatest range of motion in the $x-y$ plane occurs at mid-range displacements in the $z$-direction.

Figure 7.32 shows a top view of a unit hemisphere. Points on the hemisphere represent the different orientations for the platform unit normal. The points on the hemisphere are colored in accordance with the relative frequency of platform normal assuming within one degree of a particular orientation given a large random sampling
Figure 7.30: A top view of a random sampling of points (i.e. points generated from random values of the inputs $\psi_i$) that can be reached by the center of the platform, $P$, and the spherical slider-crank output points, $\bar{x}_i$. 
Figure 7.31: A side view of a random sampling of points that can be reached by the center of the platform, $P$, and the spherical slider-crank output points, $\bar{x}_i$. Darker colored areas indicate more frequent occurrence of a particular orientation. As a result of the three different input directions there is a slight bias towards tilting in the direction inline with the path of an output point.

Figure 7.33 plots the radial distance from the $z$-axis versus the tilt of the platform from the horizontal. The two quantities are strongly and nonlinearly correlated. Thus, tilting the platform cannot be achieved without some movement of the platform center away from the $z$-axis, and in general the more tilt required, the larger the required movement away from the $z$-axis.

Figure 7.34 shows the magnitude of the platform tilt versus the direction of tilt for randomly distributed input values. The platform can achieve a tilt magnitude of
Figure 7.32: A top view of a unit hemisphere which represents the different possible directions for the normal to the platform. Areas on the hemisphere are colored in accordance with the relative frequency of platform normal assuming a particular direction given a random sampling of input values.
Figure 7.33: The radial distance from the $z$-axis versus the tilt of the platform from the horizontal. The two quantities are strongly and nonlinearly correlated.
Figure 7.34: The magnitude of the platform tilt versus the direction of tilt for randomly distributed input values.

approximately 40° in any direction and up to 50° in preferred directions corresponding to the paths of the output points $x_i$.

7.3.5 A Multistable Three-degree-of-freedom Platform

A bistable device can be attached to each of the input spherical slider-crank of the 3DOFP in a manner similar to that discussed in 7.2. The resulting mechanism has a multistable platform with two-stable positions for each input slider, yielding a total of eight stable positions for the platform. For the same 3DOFP model as described in the previous section, and bistable devices (Young Mechanisms) which induce stable input positions of $\psi_{io} = 0°$ and $\psi_{if} = 30°$, the eight output positions are detailed in Table 7.5.
Table 7.5: Stable positions of a Multistable 3DOFP

<table>
<thead>
<tr>
<th>Position Number</th>
<th>$\psi_1$ (deg)</th>
<th>$\psi_2$ (deg)</th>
<th>$\psi_3$ (deg)</th>
<th>$\xi$ (deg)</th>
<th>$\eta$ (deg)</th>
<th>z</th>
<th>r</th>
<th>$\theta$ (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>30</td>
<td>60</td>
<td>42.6</td>
<td>0.34</td>
<td>0.13</td>
<td>60</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>30</td>
<td>0</td>
<td>-60</td>
<td>42.6</td>
<td>0.34</td>
<td>0.13</td>
<td>-60</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>30</td>
<td>30</td>
<td>0</td>
<td>35.7</td>
<td>0.75</td>
<td>0.04</td>
<td>180</td>
</tr>
<tr>
<td>5</td>
<td>30</td>
<td>0</td>
<td>0</td>
<td>180</td>
<td>42.6</td>
<td>0.34</td>
<td>0.13</td>
<td>180</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
<td>0</td>
<td>30</td>
<td>120</td>
<td>35.7</td>
<td>0.75</td>
<td>0.04</td>
<td>-60</td>
</tr>
<tr>
<td>7</td>
<td>30</td>
<td>30</td>
<td>0</td>
<td>-120</td>
<td>35.7</td>
<td>0.75</td>
<td>0.04</td>
<td>60</td>
</tr>
<tr>
<td>8</td>
<td>30</td>
<td>30</td>
<td>30</td>
<td>0</td>
<td>0</td>
<td>0.89</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

and shown graphically in Figure 7.35. The table gives the input angles, and the resulting tilt parameters of the platform, $\xi$ is the direction that the platform tilts, and $\eta$ is the magnitude of the tilt in that direction. The location of the platform center is given in cylindrical coordinates, where $r$ is the distance of the platform center from the $z$ axis, and $\theta$ is the angle formed by the vector that measures the platform center’s displacement from the $z$ axis and the $x$ axis.

In Figure 7.35 and Table 7.5, the platform positions are at the same height when a single input is at the second stable position. Likewise, the platform positions are at the same height when two of the inputs are at the second stable position. This implies that the translation of the platform in the $z$-direction is strongly dependent on the average of the input values $\psi_i$. The magnitude of the platform tilt appears to be strongly related to how skewed an input value is compared with the other two inputs. Thus, when all the inputs are the same value, the platform is not tilted. When one input value is different from the other two, the platform tilts in the plane of motion of the input $x_i$. When the
Figure 7.35: A schematic of the multistable 3DOFP. The eight stable positions for the platform center are numbered. The direction of the normal to the platform is also shown for each position. The paths for the output points of the spherical slider-cranks are also shown.
different input is larger than the other two, the platform tilts away from the different input. When the different input is smaller than the other two, the platform tilts toward the different input.

7.3.6 Conclusions

In this section, a Three-degree-of-freedom Platform has been presented which may be useful in optical applications. A mathematical model for its position analysis has been presented and a multistable version of the device has been described.

7.4 Conclusions

In this chapter, descriptions and mathematical models have been presented for three novel MEMS devices. Prototypes of the first two devices have been fabricated and tested and show good agreement with the proposed models. The devices show that spherical kinematics can be used to produce a MEMS device, the MHKP, that has vertical travel with no side-to-side motion and a device, the SBM, that exhibits bistable out-of-plane motion. Model results predict that a 3DOFP will be able to move a platform with large amounts of vertical travel, significant tilt in any direction and minimal side to side motion.
Chapter 8

CONCLUSIONS

In this chapter, conclusions are provided, the contributions of the research are summarized and recommendations for future work are given.

A method for representing the design space of ortho-planar mechanisms has been developed. The method is based on the Theorem of Equality of Orientation Set Measures (TEOSM) which allows mechanisms to be represented by points in an abstract space. The method was developed specifically for single loop planar folded mechanisms with revolute joints, and later extended to mechanisms with prismatic joints and to spherical folded mechanisms. Functions which assign a value to each point in design space can be used to describe classes of mechanisms and evaluate their utility for MEMS design.

It is also shown that spherical mechanisms have characteristics that may be useful in MEMS design. These characteristics include spatial positioning of a link and the ability to convert rotation about an axis perpendicular to the substrate to rotation about an axis parallel to the substrate. Spherical kinematics has been used to develop three novel mechanisms, the Micro Helico-Kinematic Platform (MHKP), the Spherical Bistable Mechanism (SBM), and the Three-degree-of-freedom Platform (3DOFP). Mathematical models of these devices have been developed and MEMS prototypes have been designed.
and fabricated. Testing of the MHKP and the SBM tends to validate their mathematical models.

8.1 Contributions

The fundamentals of ortho-planar mechanisms described in this work can be the basis for future work on a broad range of topics. The fundamentals are important because they permit a mathematical approach to the MEMS design and enable the avoidance of problematic mechanisms and the identification of promising ones. For example, in chapter 3, numerous mechanisms were identified as problematic because of their tendency to interfere with the substrate. On the other hand, the spherical slider-crank, described in Chapter 6, is identified as a particularly useful and promising mechanism, and several ingenious mechanisms were developed using it, as is described in Chapter 7. The primary contributions of this research are as follows:

- A theorem (TEOSM) that describes the geometric fundamentals of ortho-planar planar and spherical mechanisms was developed. The TEOSM incorporates the principles of loop closure, non-trivial link lengths, the planar initial position, and dimensional similarity to describe the design space of single-loop planar and spherical OP mechanisms. It was shown that these principles can be applied to planar and spherical mechanisms with prismatic joints.

- The TEOSM can be used to construct the design space of planar OP four-bars and crank-sliders. This was demonstrated and functions describing corresponding mechanisms were computed and graphed.
• The utility of spherical mechanisms in MEMS was described and demonstrated through the use of building block mechanisms, the spherical four-bar and the spherical slider-crank, which can be incorporated into more complicated systems. The basics of these mechanisms were described and demonstrated.

• The development of several MEMS devices utilizing spherical kinematics including the Micro Helico-Kinematic Platform, the Spherical Bistable Mechanism, and the Three-degree-of-freedom Platform. The devices were described and closed-form kinematic models were developed that describe their motion.

### 8.2 Suggestions for Future Work

Possible future directions for research in this area include the use of the TEOSM as the basis for computer algorithms that aid non-specialists in the design of OP MEMS devices. Given the importance of precision and repeatability in MEMS devices, it may also be important to extend the TEOSM to compliant OP mechanisms. As part of that extension, much work could be done in order to develop compliant spherical mechanisms. Such work could lead to the extension of the TEOSM to multi-loop and spatial OP mechanisms and the investigation of the use of OP mechanisms in other domains besides MEMS, such as deployable structures.
Bibliography


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