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Fraction Multiplication and Division Image Change in Pre-Service Elementary Teachers

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FRACTION MULTIPLICATION AND DIVISION

IMAGE CHANGE IN PRE-SERVICE

ELEMENTARY TEACHERS

by

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As chair of the candidate's graduate committee, I have read the dissertation of Jennifer J. Cluff in its final form and have found that (1) its format, citations, and bibliographical style are consistent and acceptable and fulfill university and department style requirements; (2) its illustrative materials including figures, tables, and charts are in place; and (3) the final manuscript is satisfactory to the graduate committee and is ready for submission to the university library.

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ABSTRACT

FRACTION MULTIPLICATION AND DIVISION IMAGE CHANGE IN PRE-SERVICE ELEMENTARY TEACHERS

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This study investigated three pre-service elementary teachers' understanding of fractions and fraction multiplication and division. The motivation for this study was lack of conceptual understanding of fractions and fraction multiplication and division. Pre-service elementary teachers were chosen because teachers are the conduit of information for their students. The subjects were followed through the fractions unit in a mathematics methods course for pre-service elementary teachers at Brigham Young University.

Each subject volunteered to participate and were interviewed and videotaped throughout the study, and they also provided copies of all work done in the fractions unit in the course. The data is presented as three case studies, each beginning with a

discussion of the subject's math history and prior understanding of fractions. Then the case studies discuss the subject's change in understanding of fractions, fraction multiplication, and fraction division. Finally, at the end of each case study, a discussion of the subject's conceptual understanding is discussed.

Each participant showed a deepened conceptual understanding of fractions, fraction multiplication, and fraction division. The subjects' prior knowledge of fractions and fraction multiplication and division did affect their growth of understanding. Each participant had unique levels of growth and inhibitors to growth of understanding. At the times of most growth of understanding, the subjects' inhibitors of growth were also the most evident.

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INTRODUCTION

Mathematics education deals both with what constitutes mathematics and how mathematics is learned and taught. Romberg and Kaput (1999) describe teaching mathematics "as teaching students to use mathematics to build and communicate ideas, to use it as a powerful analytic and problem-solving tool, and to be fascinated by the patterns it embodies and exposes" (p. 16). They also emphasize that mathematics should be an "experience from which students derive enjoyment and earn confidence" (p. 16). The teaching of mathematics should bring the subject alive for the students. Teachers are the lynch-pin to connect their students with the richness of the subject of mathematics.

Teachers of mathematics are heavily influenced by what they have learned and been taught. "Watching teachers and paying attention to their own experiences, they develop ideas about the teacher's role, form beliefs about 'what works' in teaching math, and acquire a repertoire of strategies and scripts for teaching specific content" (Ball, 1988, p. 40). The new teacher develops understanding of what it means to teach mathematics through their own experiences in mathematics classrooms.

Thus, pre-service teachers do not arrive at teacher education programs as *tabula rasa*, but rather have preconceived notions of what mathematics education is all about. Once pre-service teachers have arrived at the beginning of teacher preparation courses, they "have already clocked more than 2000 hours in a specialized 'apprenticeship of observation'" (Lortie, 1975, as paraphrased in Ball et al, 2001, p. 437). This apprenticeship has "not only instilled in them traditional images of teaching and learning but [has also] shaped their understanding of mathematics" (Ball, 1988, as paraphrased in Ball et al, 2001). Given that what teachers understand about mathematics is the mathematics they will teach, the mathematics learned in elementary and high school becomes a significant component of their preparation for teaching (Ball et al, 2001).

The experience of most mathematics students is to spend their time in a classroom "where mathematics is no more that a set of arbitrary rules and procedures to be memorized," (Ball et al, 2001, p. 434). Students in most classrooms experience "instruction that delivers knowledge in a prepackaged form rather than in a form that encourages students to construct their own knowledge, and that instruction rarely provides students with structured learning experiences to help them acquire essential conceptual and procedural knowledge," (Armstrong & Bezuk, 1995, paraphrase of Behr, 1988, p. 85-6). The pre-service teacher learns that the way to teach mathematics is by giving the same prepackaged knowledge they learned, and lacks essential conceptual and procedural knowledge that they can pass to their students.

"Teachers must understand concepts and procedures themselves in order to select and construct fruitful tasks and activities for their pupils, as well as to flexibly interpret and appraise pupils' ideas" (Ball , 1988, p. 43). The teacher of mathematics has need of knowledge of the subject as well as knowledge about mathematics, i.e. the teacher needs to have knowledge about the nature of mathematics and an understanding of what it means to do mathematics (Borko et all, 1992). Simply put, it is not enough to have successfully completed mathematics courses, it is also necessary to have experience with and reflect upon what it means to know mathematics.

To know a subject means “getting inside it and seeing how things work, how things are related to each other, and why they work like they do” (Hiebert et al, 1997, p. 2). In mathematics this translates to having understanding of how to compute solutions, why a particular method of computation works and gives the correct solution, and how the different concepts of mathematics are connected to each other. The more connections that can be established between ideas the better a person understands the mathematics. This process of coming to know mathematics involves reflection and reasoning about the subject. The learner of mathematics should have opportunities to reflect and communicate about mathematics with peers and teachers, (Hiebert et al, 1997). Teachers of mathematics are the front line in providing students with opportunities to gain understanding of mathematics. The mathematics teacher is the one who can create an atmosphere of learning which includes reflection and communication about what is being taught. This will lead the student to know why a method of computation works, because the student will have had opportunities to make the connections. In short, the mathematics teacher provides opportunities for the student to reason and make conclusions about the mathematics.

“If teachers are to be successful in leading their students to reason…, they must be able to reason…themselves” (Sowder et al., 1998, p. 151). In order to teach someone the process of reasoning and making connections, the teacher should know how to reason and make connections also. To help students gain understanding of mathematics, the teacher should have understanding of the nature of mathematics. This understanding comes as connections between computations and explanations of computations are made. This is true of all areas of mathematics, including the study of fractions.

“We know that teachers and most other adults…have a limited understanding of the meaning of multiplication and division of fractions” (Armstrong and Bezuk, 1995, p. 87). Nancy Mack (1990), in a literature review of students’ understanding of fractions, characterized this understanding as reasoning by “knowledge of rote procedures, which are often incorrect, rather than by [knowledge of] the concepts underlying the procedures” (p. 17). This knowledge exhibited by students is that which pre-service teachers have learned. Teacher knowledge about multiplication and division of fractions is limited, learned largely by rote procedure, without conceptual understanding (Armstrong and Bezuk, 1995). For teachers to teach their students more than just the rote procedures, they “must first approach these topics themselves in ways that are very different from all of their previous experiences with mathematics,” (Armstrong and Bezuk, 1995, p. 87-88).

In order to overcome these limitations teachers need to completely reform their ideas about multiplication and division of fractions and how to teach them (Armstrong and Bezuk, 1995). The teachers need to find a way to teach multiplication and division of fractions conceptually. However, this may pose a problem, because the teacher may not be aware that a conceptual base for multiplication and division of fractions even exists. Nothing from their previous mathematics experiences may have suggested that there are conceptual underpinnings for multiplication and division of fractions (Armstrong and Bezuk, 1995). It is to this end of helping teachers know the conceptual underpinnings of fraction arithmetic and similar mathematical topics, that special mathematics courses for prospective elementary school teachers have been developed.

The Mathematics Education Department of Brigham Young University has developed two sequential courses which investigate the concepts of elementary school mathematics. The second of these courses includes the study of multiplication and division of fractions. This unit is designed to help students gain conceptual understanding of multiplication and division of fractions, as well as connections between the algorithms they learned and why they work. This promotes the learning of concepts in the pre-service teachers, which leads to opportunities for the pre-service teachers' future students. "It [is] hard to override a rule-based education" (Armstrong and Bezuk, 1995, p. 93), but it is important to do so, so that the students of the future are not subject to the same lack of conceptual understanding.

This research study will focus on the developing conceptual knowledge of multiplication and division of fractions in pre-service elementary teachers enrolled in a mathematics for elementary school teachers course at Brigham Young University. The investigation will focus on how the pre-service teachers' images and understanding of what fractions are and of operations on fractions deepen and expand as a result of their experience in the course.

CONCEPTUAL FRAMEWORK

In this chapter, I will address several areas of research related to teachers' knowledge of multiplication and division of fractions. I will begin with a discussion of teachers' subject matter knowledge and why it is important. Next, I will discuss what is known about teacher knowledge of multiplication and division of fractions. Finally, I will discuss what constitutes an understanding of multiplication and division of fractions.

TEACHERS' SUBJECT MATTER KNOWLEDGE

> "Students learn mathematics through the experiences that teachers provide. Thus, students' understanding of mathematics, their ability to use it to solve problems, and their confidence in, and disposition toward mathematics are all shaped by the teaching they encounter in school. The improvement of mathematics education for all students requires effective mathematics teaching in all classrooms" (NCTM, 2000, p. 16-17).

This is the opening quote in the National Council of Teachers of Mathematics' (NCTM) new Principles and Standards for School Mathematics (2000), under the heading of the Teaching Principle. This statement suggests that what kind of learning takes place in a classroom is orchestrated by teachers and what teachers know about the subject being taught. The teacher is the person who most influences the learning of his or her classroom. This section will discuss what subject matter knowledge teachers bring to the classroom and how that subject matter knowledge influences what students learn in the classroom.

In mathematics education research the NCTM Standards are often quoted and used to implement good teaching. This is in part because they develop a set of guidelines for what should be taught in mathematics courses, based on research and the best current thinking. As such the NCTM standards not only list what should be taught but some overriding principles for *how* that mathematics should be taught, including what has been labeled the *Teaching Principle.*

The *Teaching Principle* describes characteristics of a good teacher. For teachers "to be effective, [they] must know and understand deeply the mathematics they are teaching and be able to draw on that knowledge with flexibility in their teaching tasks," (NCTM, 2000, p. 17). Teachers need to be able to perform the mathematics well, understand the concepts of the mathematics, and be able to help their students learn this deep understanding of mathematics. "Teachers need to understand the big ideas of mathematics and be able to represent mathematics as a coherent and connected enterprise," (Schifter 1999, Ma 1999; as paraphrased by NCTM, 2000, p. 17). This knowledge could be described as "profound understanding of fundamental mathematics," (Ma 1999, as quoted by NCTM, 2000, p. 17). Teachers should have the mathematics understanding necessary to be able to select tasks that will help students do mathematics and reflect on the mathematics. These tasks that are chosen help students understand the mathematics and build bridges between what the students know and the new information being taught (NCTM, 2000).

In research on prospective teachers, Ball (1988) discusses what pre-service teachers know and their understanding about mathematics. The study was conducted with pre-service elementary teachers in a general methods course. The project was intended to explore what prospective teachers assumptions are about the teaching process, specifically with mathematics. The pre-service teachers "were surprised to discover how crucial subject matter knowledge was when they tried to teach the concept…to another person" (Ball, 1988, p. 43). It was obvious that knowledge about mathematics is necessary to help another learn it. However, it is necessary to go beyond the surface learning of mathematics in order to teach.

"Teachers must understand concepts and procedures themselves in order to select and construct fruitful tasks and activities for their pupils, as well as to flexibly interpret and appraise pupils ideas" (Ball, 1988, p. 43). In the teaching experiment described by Ball (1988), the teachers were confronted with the distinction between knowledge necessary for a learner and that of a teacher. One student described this understanding as "[knowing] your subject matter well enough to be able to play around with it…If you know your subject matter well, it is easier to find different explanations and examples" (p. 44). Having only a limited knowledge of mathematics will limit the teacher's effectiveness in the classroom. The ability to "play around" with the mathematics will allow for diverse approaches to teaching the mathematics.

These ideas of flexibility in teaching and understanding concepts relate to the perspective of what it means to be a constructivist teacher. "A constructivist perspective [of education] holds that children's learning of subject matter is the product of an interaction between what they are taught and what they bring to any learning situation"

(Ball, 1988,p. 40). This means from a teaching perspective that teachers' learning of subject matter is an interaction between their experiences in education and subject matter courses and what they bring with them into these courses.

> "By the time [teachers] begin their professional education, [they] have already clocked more than 2000 hours in a specialized 'apprenticeship of observation', which not only has instilled traditional images of teaching and learning but also has shaped their understanding of mathematics. Because this understanding of mathematics is the mathematics they will teach, what they have learned about the subject matter in elementary and high school turns out to be a significant component of their preparation for teaching" (Ball, et al 2001, p. 437).

This prior apprenticeship in teaching would be classified as informal knowledge of what it means to teach mathematics. This informal knowledge must be taken into account, in order to expand the teachers' knowledge about what it means to teach mathematics (Ball, 1988).

So the critical question becomes how we expand on this subject matter knowledge and knowledge of what it means to teach mathematics. Ball et al (2001) describe research on teacher subject matter knowledge which concluded that the number of advanced mathematics courses helped to a point, but it was the mathematics methods courses that contributed more to pupil performance. The teachers already know much of the procedural mathematics; it now comes to knowing the subject matter well enough to "play around with it."

In a study described by Ball et al (2001) of teachers' knowledge of rational numbers it was found that "even when the teachers were able to provide computationally sound solutions to problems, they were unable to provide pedagogically sound explanations for their students," (p. 447). This suggests that teachers need to learn *pedagogical content knowledge* (Shulman, 1986). This pedagogical content knowledge is a link between content and pedagogy. This knowledge includes "things like what topics children find interesting or difficult and what the representations are that are most useful for teaching a specific idea. Pedagogical content knowledge is a unique kind of knowledge that intertwines content with aspects of teaching and learning" (Ball et al, 2001, p. 448). Teachers need to not only know the subject matter, but they need to know in what areas students might have difficulty and what bridges will help the students in overcoming these difficulties, (NCTM, 2000).

However, in order to have pedagogical content knowledge teachers need to understand the content. The important point is not that the teachers have taken mathematics courses, but rather "whether and how teachers are able to use mathematical knowledge in the course of their work," (Ball et al, 2001, p. 450). Ball et al summarize the conclusions of a study done by Sowder, Philipp, Armstrong, and Schappelle "that as the teachers' content knowledge increased and deepened, the teachers were more willing to try new mathematics with their students, saw their students as more capable mathematically, encouraged and expected more conceptual explanations of material…and tended to probe students' thinking more often" (p. 450). In other words, as the teachers develop and strengthen their own content knowledge, their pedagogical content knowledge is strengthened, and they are able to help build bridges of understanding for their students.

In this section we have discussed what subject matter knowledge teachers bring to the classroom and how that affects the classroom. This is of importance because what teachers know about the subject they are teaching constrain the kind of learning that takes place in classroom. The teacher is the one who is the strongest influence for what is learned in their classroom. The mathematical content knowledge a teacher has strongly influences their ability to bridge between new knowledge and students' prior knowledge, and to anticipate and alleviate difficulties a student may have.

TEACHER KNOWLEDGE OF MULTIPLICATION AND DIVISION OF FRACTIONS

As I have already discussed in this paper, content knowledge is important for teachers to help their students understand concepts. This content knowledge allows the teachers to be effective in helping their students understand mathematics. This knowledge helps the teacher to build bridges of understanding in their students and help them overcome difficulties. This portion of the paper will discuss teachers' knowledge of multiplication and division of fractions and how this relates to practice.

Armstong and Bezuk note that "we know that teachers and most other adults in our country have a limited understanding of the meaning of multiplication and division of fractions. This should not come as a surprise" (1995, p. 87). Teachers are a product of the education system in which they matriculated, which has been one of rote learning of rules related to multiplication and division of fractions. Armstrong and Bezuk conducted a study designed to help teachers develop a deeper understanding of multiplication and division of fractions. One of the questions they asked was, "If teachers don't understand how to teach division or multiplication of fractions conceptually, why don't they find a way to learn?" (p. 91). Their research concluded that teachers might not know that a conceptual base for multiplication and division of fractions exists (Armstrong & Bezuk, 1995).

Eisenhart, Borko, Brown, Underhill, Jones, and Agard (1993) discuss lack of conceptual understanding in the case of one pre-service elementary teacher. The teacher, named Ms. Daniels for this study, was observed during her student teaching experiences, of which there were four parts. Ms. Daniels recognized the difference between procedural knowledge and conceptual knowledge and believed both were necessary for understanding of mathematics, but it was difficult for her to explain her ideas about how to teach for conceptual understanding. Ms. Daniels understood that procedural knowledge dealt with mastery of computational skills. She described conceptual knowledge as "using your brain—your thinking skills—at a much higher level" (p. 15). The authors of the study concluded that "Ms. Daniels was more confident in her arithmetic [or procedural] skills than she was in her conceptual knowledge, and that she could not complete conceptual explanations for common topics in the elementary and middle school curriculum" (p. 17). Ms. Daniels' content and pedagogical knowledge limited her ability to explain how she would teach for conceptual knowledge and she actually taught for conceptual knowledge rarely. Ms. Daniels was unable to teach for conceptual knowledge because she herself lacked the conceptual understanding.

A companion paper to Eisenhart et al (1993) is a paper by Borko, Eisenhart, Brown, Underhill, Jones, and Agard (1992) in which the focus of the study was on Ms. Daniels' teaching episodes. The episode involved multiplication and division of fractions. Ms. Daniels set up the problems and was teaching students how to compute the answers without any instruction in the concepts of multiplication and division. One student asked why the invert and multiply rule works for division of fractions and why no such rule is used for multiplication. Ms. Daniels then proceeded to set up a situation that would illustrate the concepts behind the invert and multiply rule. Ms. Daniels ran into difficulty here because instead of a problem that modeled division, she presented a problem that modeled multiplication. Halfway through the problem, Ms. Daniels realized her mistake and stopped. Ms. Daniels did not explain the mistake she made, instead she told her students:

> "Well, I am just trying to show you so you can visualize what happens when you divide fractions, but it is kind of hard to see. We'll just use our rule for right now and let me see if I can think of a different way of explaining it to you. OK? But for right now, just invert the second number and then multiply" (Borko et al, 1992, p. 198).

While Ms. Daniels attempted to provide conceptual understanding, she stopped because she was using the wrong illustration. Then to compound the problem, she said to her students that it is difficult to see why the invert and multiply rule works and to just follow the rule. She did not revisit the problem the next day, so the students were left with the impression that the reasons for the invert and multiply rule are mysteries.

Ms. Daniels in this study (Borko et al, 1992) believed that good mathematics consisted of computational ability and conceptual ability. She believed that in order for a student to know mathematics, the student must be proficient with the calculations and rules, as well as be able to reason about why the mathematics works. However, Ms. Daniels was not able to teach the reasoning skills to her students, because she did not have them. She had a strong computational background in mathematics, and was able to teach this to the students. Ms. Daniels mathematical background was strong and she had taken several upper division mathematics courses, prior to becoming an elementary education major. However, these mathematics courses were only helpful to a point. She was unable to bridge between what was sufficient for her success in mathematics to what would enable the success of her students, which became apparent in interviews during her pre-service education (Borko, et al, 1992).

Borko, et al (1992) identified difficulties Ms. Daniels had with fraction division concepts during her student teaching and at the conclusion of her methods for teaching courses. Ms. Daniels was asked to explain fraction division and did so relying upon applications and visual representations These descriptions were global and when Ms. Daniels was asked to clarify her explanations, she was often unable to respond. But when she did respond, her illustrations evidenced the limits of her knowledge of fraction division. These illustrations contained applications showing or suggesting multiplication of fractions (p. 208). Although Ms. Daniels appeared to be using the information learned in her methods course, her recollection of the information and explanations was only partial and she was unable to construct complete or appropriate explanations (p. 208). Ms. Daniels explanations for fraction division showed she was drawing on the algorithm to explain the process and her recall of problem situations from her methods course show her limited understanding of both fraction multiplication and fraction division (p. 209). Ms. Daniels was considered to highly trained in mathematics through several upper division mathematics courses, however her ability to explain and understanding the concept of fraction division was limited to the algorithm, lacking deep understanding of the concept and evidencing a lack of understanding of fraction multiplication.

Another study which examined pre-service teachers' understanding of fraction division was Ball (1990). This study investigated nineteen prospective elementary and secondary teachers' understanding of division in three contexts; division with fractions, division by zero, and division with algebraic equations. In the fraction division portion participants were asked to solve $1\frac{3}{4} \div \frac{1}{2}$ and to give a real-world situation for the problem. Seventeen of the participants were able to calculate the division correctly, while only five of the nineteen participants could give an appropriate representation. (The most common mistake in the representations was to show division by two instead of one-half.) Most of the participants had "significant difficulty with the meaning of division of fractions, [which] indicated a narrow understanding of division" (p. 140). The participants most often considered division in terms of sharing only (which works well for whole number division), forming a certain number of equal parts. But the sharing model of division corresponds less easily to fraction division than does the measurement interpretation of division (p. 140). (See *What Constitutes Understanding of Multiplication and Division of Fractions* section in this chapter to learn more about the sharing and measurement models of division.) Recognizing this one-sided view of division helps explain why making meaning for fraction division was difficult for the participants of the study. The participants were unable to explain the process of fraction division, although most could compute the process

The representations for $1\frac{3}{4} \div \frac{1}{2}$ in this study (Ball, 1990) showed that a few participants were able to correctly explain the process of fraction division, but these were an exception. The most common error of the representations was to show division by two instead of one-half. Of the participants that showed division by 2, most of them were unable to reconcile the difference in the representation answer with the computational answer. This evidenced the teachers were unable to identify what division by one-half meant in practice. Of the participants who were unable to provide a representation, two recognized the conceptual problem—i.e. they initially represented division by two and identified the discrepancy—however, they were unable to provide a correct representation. The others who were unable to provide a representation "seemed to think that trying to relate $1\frac{3}{4} \div \frac{1}{2}$ to a concrete situation was not a feasible task—that $1\frac{3}{4} \div \frac{1}{2}$ could not be represented in real-world terms" (p. 136). This evidences that the participants' knowledge of division was more memorization than conceptual understanding (p. 141), or in other words, the participants lacked images of what fraction division is.

A third study of fraction operations is reported by Ma (1999). This particular study investigated understanding of elementary mathematics in U.S. and Chinese Teachers, including fraction division. The participants were asked to compute and give a story problem to represent $1\frac{3}{4} \div \frac{1}{2}$. Only 43% were able to successfully calculate the answer and, among the twenty-three participants, six could not provide a story problem and sixteen provided a story problem which was inaccurate. One teacher explained in order to solve the problem, the fractions needed to be changed so that there were no mixed numbers and the denominators were the same. This changed the problem to $\frac{7}{4} \div \frac{2}{4}$, which changed the fraction division to whole number division, i.e. how many twos are in seven. Although this representation was accurate, this participant lacked confidence in her computation. Other teachers attempted an explanation of the division but were unable to be successful because they could not remember the algorithm correctly or at all. These misrepresentations evidence the lack of understanding of fraction division among the teachers in the study.

The teachers also wrote story problems for $1\frac{3}{4} \div \frac{1}{2}$ (Ma, 1999). Ten of the participants wrote problems using division by two instead of one-half, the stories showed one and three-fourths being shared between two groups. This discrepancy went unnoticed by the teachers who gave it. Six of the teachers wrote stories showing multiplication by one-half, e.g.

> "Probably the easiest would be pies, with this small number. It is to use the typical pie for fractions. You would have a whole pie and three quarters of it like someone stole a piece there somewhere. But you would happen to divide it into fourths and then have to take one-half of the total" (p. 65).

This error evidenced that the teachers not only had difficulty with fraction division, they had difficulty with fraction multiplication (p. 65). Two of the teachers confused division by one-half, division by two, and multiplication by one-half. Of the remaining teachers two were unable to provide a story at all and one was able to provide a conceptually correct representation, although she used people as the objects which are difficult to cut in half. The teachers' ability to give a representation evidences their conceptual understanding of fraction division, which was weak. The study also evidences those who had strong procedural knowledge were impeded by this knowledge as they tried to develop the conceptual knowledge to write a story problem.

These four studies evidence prospective and practicing teachers lack the conceptual understanding of fraction multiplication and division. We learn from the first two studies (Borko, et al, 1992, and Eisenhart, et al, 1993) that a high number of academic courses does not guarantee prospective teachers will acquire pedagogical content knowledge, "Academic courses, as they are currently taught, do not do a particularly good job of fostering such knowledge" (Borko, et al, 1992, p. 219). Ball (1990) and Ma (1999) show that prospective and practicing teachers have difficulty representing fraction division in story problems which shows they are unable to explain the concept of division. Ma further evidences the practicing teachers lack conceptual understanding of fraction multiplication and fractions themselves. Without these concepts, it is difficult to teach for understanding of multiplication and division of fractions. Students are left without any understanding besides a possible grasp of the rules. These rules come from an advanced understanding of multiplication and division of fractions. Armstrong and Bezuk (1995) in a paraphrase of Kieran (1988) state "premature formalism leads to symbolic knowledge that children cannot connect to the real world, resulting in a virtual elimination of any possibility for children to develop number sense about fractions and operations on fractions" (p. 86). So, without the conceptual understanding of multiplication and division of fractions, the procedural understanding becomes a stumbling block for students. Thus, it is important to teach conceptual understanding along with procedural understanding. In order to teach both conceptual understanding and procedural knowledge in tandem, the teacher must know both.

WHAT CONSTITUTES UNDERSTANDING OF FRACTION MULTIPLICATION AND DIVISION

"Understanding the multiplication [and division] of fractions involves understanding ideas about fractions and understanding ideas about multiplication [and division]" (Mack, 1998, p. 34). In order to understand the multiplication and division of fractions, one must understand what fractions are, what multiplication and division mean, and the connections between these two ideas. In this section I will address what it means to understand fractions and multiplication and division of fractions. In looking at this understanding I will be looking at the images of fractions which promote understanding.

Images are the mental visualization of the concepts and operations of mathematics. These images can enhance our ability to work with fractions and fraction operations. The images used in fraction work can help us to reason about what fractions are and what the operations mean. The images we have of fractions and fraction operations may limit or enhance our ability to expand our understanding of fraction and fraction operations.

In the National Council of Teachers of Mathematics (NCTM) 2002 yearbook, Smith discusses the development of students' knowledge of fractions. Smith states there are two broad phases of development: the first is to make meaning for fractions by linking quotients to divided quantities and the second is to explore the mathematical properties of fractions as numbers. Thus students first learn what fractions are and then learn how to perform arithmetic operations on them.

In the first stage of understanding, Smith (2002) suggests that the learning of what fractions are is not difficult once students can partition. Partitioning is the idea of subdividing a unit (the whole) into subunits of equal size. (For example, a cookie that is cut into four equal size pieces has been subdivided into four subunits.) The students can then take a collection of the subdivided pieces (by iterating one of the pieces) and express this as a fraction (i.e. three of the four pieces of the cookie is “three-fourths” of the cookie, written as $3/4$). Even though partitioning helps in the understanding of fractions, there may be some challenges to understand partitioning. The key is to grasp the idea that fractions name the relationship between the collections of parts and the whole, not the size of the whole or its parts (Smith, 2002). Smith suggests that students need practice with partitioning of wholes into many different sized pieces in order to bring understanding of partitioning.

Siebert and Gaskin (in press) discuss the power that comes from learning partitioning and iterating of the whole in understanding fractions. They claim that the images of partitioning and iterating are powerful for the following reasons:

> First, they make explicit the actions children can perform on quantities to produce, compare, and operate on fractional parts…. Second, these images provide ways for students to justify their fraction reasoning…. Because these two images provide ways to reason and talk about fractions, they can enable children to develop robust meanings for fractions and fraction operations. (p. 3).

This process of partitioning and iterating keeps the referent whole for the fractions relevant, “because the images of iterating and partitioning make explicit the referent whole from which the fraction is created or compared” (p. 7). The fraction amount is based upon the referent whole, not on the number of pieces or parts they comprise. Thus the understanding of fractions is made more complete through the practice of partitioning the whole to find a “unit” fraction (i.e. subdividing the whole into six pieces and one of the pieces is “one-sixth” the whole and is a unit fraction) and then iterating to create other parts of the whole (i.e. iterating the “one-sixth” five times to produce “five-sixths”).

Both Smith (2002) and Siebert and Gaskin (in press) suggest that the key to understanding fractions comes from practice with partitioning and iterating. These two processes use the referent whole as the basis for developing fractions. The referent whole is a necessary link for fraction understanding, because it allows for reasoning about what the fraction means. This helps students to understand fractions that are less than one and fractions of size greater than one. Because the students know the referent whole, eight-fifths becomes understandable, and the students are able to connect the idea of the fraction to their prior knowledge of quantities (Siebert and Gaskin, in press). So, in essence understanding of fractions comes as the concepts of iterating, partitioning, and understanding what the fraction means in relation to the referent whole are learned and strengthened.

Once students have made meaning for fractions, they are then ready to move to the second stage which explores the mathematical properties of fractions as numbers (Smith, 2002). In this second stage the exploration of multiplication and division of fractions occurs. Students have learned what the fractional quantity means and then are able to combine two or more quantities to make new quantities. Acquiring understanding of multiplication and division of fractions involves at least two aspects. The first aspect (after understanding of fractions is attained) in understanding how to multiply and divide fractions is first to understand what it means to multiply and divide.

Multiplication is most simply described as "fancy", or efficient, counting. For example, three multiplied by four (written 3×4) means the total number in three groups of size four. The first number in the problem (in the United States) is the number of groups, while the second is the size of the groups. The first number can be seen as an operator telling how many copies of the second number to combine. So to multiply $3 \times \frac{1}{4}$ means to find how much there is in three groups (or copies) of size one-fourth and the answer is three-fourths. In multiplying two fractional quantities like $\frac{2}{3} \times \frac{4}{5}$ the question asked is how much is two-thirds a group of size four-fifths. Here again, the first number can be seen as an operator telling how many copies of the second number to combine, but in this case we are taking a fractional quantity of the group instead of a whole number quantity. This idea of fraction multiplication, i.e. $\frac{a}{b} \times \frac{c}{d}$ as being "*a*-*b*ths" of a group of size "*c*-*d*ths", is an extension of the concept of whole number multiplication. Having the understanding of whole number multiplication and what fractions are makes it possible to make a bridge between whole number multiplication and fraction multiplication, because students first have knowledge of what fractions are in relation to the referent whole.

The ideas of partitioning and iterating and understanding what fractions are in relation to the referent whole allow the students to find the solution to $\frac{2}{3} \times \frac{4}{5}$ (Siebert and

Gaskin, in press). The students' knowledge of what a fraction is in relation to the referent whole makes it clear that the two-thirds of a whole are two of the unit fraction of one-third of the same whole. In fraction multiplication, the operation $\frac{2}{3}\times\frac{4}{5}$ is performed by first identifying four-fifths. Next the four-fifths is partitioned into thirds, or three equal pieces, to identify one-third of four-fifths (four-fifteenths). Then, after identifying one-third of four-fifths, the one-third is iterated twice to obtain two-thirds of four-fifths. This gives a solution of eight-fifteenths of the whole, the same referent whole for four-fifths. The solution of $\frac{2}{3}\times\frac{4}{5}$ as $\frac{8}{15}$ means eight pieces of size one-fifteenth of the whole is two-thirds of a group of size four-fifths. It is important that the referent whole is kept in mind in order to make sense of what the answer means.

Understanding the multiplication of fractions requires that students understand the concept of what fractions are and the concept of what it means to multiply. The same can be said of division of fractions. Students must first understand the concept of what fractions are and the concept of what it means to divide. The concept of division, "at its foundation, has to do with forming groups [with] two kinds of groupings …possible" (Ball, 1990b, p. 452). These two types of groupings formed from division are measurement and sharing division. In the problem of $a \div b$, measurement division asks the question of how many groups of size *b* are in a group of size *a*. Sharing division interprets the problem as how large will the group be if *a* things are shared equally among *b* groups (Sinicrope, Mick, and Kolb, 2002; Ball, 1990). This understanding of the two types of division for whole numbers and fractions provide support for understanding of fraction division.

As in whole number division, there are two types of groupings formed in fraction division: measurement and sharing. However, to understand fraction division, extensions of whole number division must be made. Looking at measurement division, as described above where $a \div b$ means how many group of size *b* are in a groups of size *a*, an adjustment for $\frac{a}{b} \div \frac{c}{d}$ must be made. Now the division is determining how many groups of size $\frac{c}{d}$ are in a group of size $\frac{a}{b}$. In order to make sense of the division, it is necessary to understand what the fraction $\frac{a}{b}$ means in reference to the whole and how to interpret $\frac{c}{d}$ and its referent whole. The referent whole here is the same for both fractions. However, the solution to $\frac{a}{b} \div \frac{c}{d}$ has a different referent whole, which is the group size. For example, in the problem of $\frac{5}{8} \div \frac{2}{3}$, measurement division would interpret this as how many groups of size two-thirds of the whole are in five-eighths of the same whole or how many groups of size two-thirds of the whole will cover a group of size five-eighths of the whole. The answer is there are fifteen sixteenths groups of size two-thirds (where the referent whole is groups of size two-thirds) or it will take fifteen-sixteenths of the whole to cover five-eighths of the whole. An example of a story problem using $\frac{5}{8} \div \frac{2}{3}$ is: Derek has $\frac{5}{8}$ cups of tropical punch concentrate; it takes $\frac{2}{3}$ cups of concentrate to make one pitcher of tropical punch; how many pitchers of tropical punch can he make?. In the measurement case of division, the referent whole for the answer is the divisor (the second number in the operation). The extension of whole number measurement division to measurement division for fractions can be made by expanding the meaning for whole number division to include what the referent whole is for each fraction in the problem, including the solution.

Having looked at the transition from whole number measurement division to fraction measurement division, I will now do the same for sharing division. As with measurement division, an adjustment must be made to transition from $a \div b$ (which for sharing means how large will each group be if *a* things are shared equally among *b* groups) to $\frac{a}{b} \div \frac{c}{d}$ in sharing division. Again this transition is made through understanding the division for whole numbers and identifying what each fraction in the process represents, by identifying its referent whole. For $\frac{a}{b} \div \frac{c}{d}$ we want to know if a group of size $\frac{a}{b}$ was shared among $\frac{c}{d}$ of a group, how large is the group size. The referent whole for $\frac{a}{b}$ is the same as the referent whole for the solution, but the referent whole for $\frac{c}{d}$ is the group size. For example the problem of $\frac{5}{8} \div \frac{2}{3}$ is how large is the group if two-thirds of the group is five-eighths of the whole, which is fifteen-sixteenths of the whole. Here the solution of fifteen-sixteenths has the same referent whole as five-eighths and the referent whole for two-thirds is the size of the group (Siebert, 2002). An example of a story problem using $\frac{5}{8} \div \frac{2}{3}$ is: Alex is printing out copies of his novel to give to friends to look over before he sends it to a publisher; he manages to get $\frac{5}{8}$ copies of his novel printed with the $\frac{2}{3}$ ream of paper left in his printer; how many copies of his novel can he print on one ream of paper? (Alex can print fifteen-sixteenths of his novel on one ream of paper). The referent whole for five-eighths and fifteen-sixteenths is the novel and the referent whole for two-thirds is the ream of paper, or the group size. The understanding of sharing division for fractions is built from the concepts of whole number division and fractions. Identifying the original number of objects, how many groups receive objects, and how many objects are in each group is what sharing division means. For fractions the number of objects is a portion of a whole number, the divisor is a fractional quantity of the number of groups, and the solution is the size of each group—a fraction with the same referent whole as the fractional quantity the problem began with.

Once again, the bridge between understanding the arithmetic operation on fractions can be built from the understanding of the arithmetic operation on whole numbers together with understanding of what fractions are. Building upon what it means to divide two whole numbers, from the sharing and measurement perspectives, and what fractions are, in relation to the referent whole, allows students to make meaning of the results of division of fractions (Siebert, 2002).

Building understanding for division of fractions is done like understanding of multiplication of fractions. Students first must understand that fractions are quantities and what the quantity represents. Students must also understand multiplication and division in terms of whole numbers. Once students can make sense of fractions and have made sense of multiplication and division of whole numbers, the students can then build bridges to the understanding of multiplication and division of fractions (Siebert, 2002).

Sáenz-Ludlow (1995) reports a study of one student and their progress through the two stages of fraction understanding. The student is a third-grade girl when the research begins, and is considered by her teacher to be a very capable student. She was willing to overcome challenging or difficult questions through talking and reflecting on her solutions. The study was designed to integrate the use of the student's prior whole-number knowledge to build understanding of fractions. The study first investigated and built what the student knew about whole numbers and their operations. Then the study moved to building knowledge of fractions as quantities. Finally the study built knowledge of fraction operations. The study reports that the student was able to build her understanding of fractions from her prior knowledge of whole numbers and her strong conceptualization of units.

Sáenz-Ludlow's (1995) study supports the idea of building upon understanding of whole number multiplication and division to understand multiplication and division of fractions, along with building understanding of what fractions are. Mack (1990) also discusses the need to build upon students' prior knowledge of what fractions are and how operate on them. The students of Mack's study were able to build upon what they knew to develop strong understanding of what fractions are. The students could use their informal knowledge and build upon it to give meaning to formal procedures and symbols.

In this section I have addressed what it means to understand fractions and the multiplication and division of fractions. This understanding is found through understanding the concepts of fractions and fraction multiplication and division. Key concepts in understanding fractions are iterating, partitioning, and understanding what the fraction means in relation to the referent whole. The concept of multiplication of fractions is a connection between the concept of fractions and the concept of whole number multiplication, which extends whole number multiplication to fractions. And the concept of fraction division is also an extension of the concept of whole number division. There are two types of division, measurement and sharing, with different concepts. Since the concepts of fraction multiplication and division are based upon concepts of fractions and whole number multiplication and division, it is necessary to develop these concepts first. These concepts are then used to build bridges of understanding, or connections, to the concepts of fraction multiplication and division.

CONCLUSION

In this chapter I have discussed several areas of research related to teachers' knowledge of multiplication and division of fractions. I began with a discussion of teachers' subject matter knowledge and why it is important. Then I discussed aspects of teacher knowledge in relation to multiplication and division of fractions. Finally, I discussed what it means to understand fractions and multiplication and division of fractions.

From this discussion, we learn that it is not enough for teachers to have taken mathematics courses; they must also know how to use the knowledge they have learned in teaching. Specifically, teachers need to know what fractions are and how to multiply and divide them. Teachers need to have conceptual understanding of fractions and fraction operations, rather than just procedural knowledge, to help their students gain understanding of fractions and fraction operations. Because of this need to have conceptual understanding to teach fractions and fraction understanding, this research study will focus on the developing of this conceptual knowledge. Development of this conceptual knowledge will follow the pattern described by Petrie (1981) in Ball (1989) of conceptual change. “He argues that conceptual change—instances when individuals come to think or see differently—may involve one of the following: changes in meaning, changes in perception, [or] changes in methodology [and]…is viewed as part of the continuity of growth” (p. 5). This research will investigate this “continuity of growth” in pre-service elementary teachers through their experience in the *Mathematics for Elementary School Teachers* course which includes the investigation of fraction multiplication and division at Brigham Young University. The investigation will focus on how the pre-service teachers’ images of what fractions are and their images of operations on fractions change as a result of their experience in the course. I will research the following question: How do the images and concepts of fractions and fraction multiplication and division deepen and expand in pre-service elementary teachers during the *Concepts of Mathematics* course for elementary teachers?

RESEARCH METHODOLOGY

SETTING

The data in this study has been collected in a mathematics education course for pre-service elementary teachers during winter semester of 2005. The course, *Concepts of Mathematics*, is the second in a sequence of two mathematics classes required for pre-service elementary teachers. The course is designed to involve the students in a "concept –oriented exploration of rational numbers and proportional reasoning…in relation to children's learning" (Brigham Young University Undergraduate Catalog, 2004). As part of the course, the pre-service teachers have been introduced (or reintroduced) to the concepts of fractions and fraction operations. The course first looked at what fractions are through the images of iterating and partitioning. Next the course investigated addition and subtraction of fractions. Then the course investigated multiplication and division of fractions. The course has involved the pre-service teachers in an exploration of these concepts, how children think about these concepts, and ways in which students may learn and explore these concepts.

In the discussion in the conceptual framework I argued that it is not enough for teachers to have taken mathematics courses involving fractions and their operations, but that they need to have conceptual understanding of fractions and their operations. The *Concepts of Mathematics* course is designed to promote conceptual understanding of fractions and multiplication and division of fractions and that is why I chose this particular class to investigate my research questions. As I have mentioned before, teachers need to not only know the subject matter, but they need to know what areas students might have difficulty and what bridges will help the students in overcoming these difficulties, (NCTM, 2000). The teachers have need of not only mathematics for their own understanding, but how to "play around" with the mathematics in order to help their students build the bridges of understanding. Therefore, this study has investigated how pre-service teachers go about gaining this kind of knowledge to help their students. This has been done through investigation of students' images about fraction operations and how they have changed as a result of this course.

I have studied how pre-service teachers expand their knowledge of fractions and multiplication and division of fractions. In Sáenz-Ludlow's (1995) study the student was able to build her knowledge of fractions and fraction operations through interaction with the instructor and reflection on her own responses to questions. The *Concepts of Mathematics* course is designed to provide this type of interaction with the instructor and fellow students, as well as reflection through writing about the day-to-day activities in class. Students are given tasks and questions, which they work on in small groups and then discuss their findings in whole-class discussion.

SUBJECTS

The students enrolled in the *Concepts of Mathematics* course are elementary education majors in their second semester of a four-semester elementary education program. The *Concepts of Mathematics* course is a required course during this semester of the program.

All of the students in this course have taken the first course, *Basic Concepts of Mathematics*, as a pre-requisite to this course. Most of the students have already experienced the types of investigations that have taken place in this course. For the most part, the students have a desire to learn mathematics in such a way to help their students understand mathematics. The students have a varied background with their own learning of fractions and multiplication and division of fractions which influences their perspective and approach to the content and the classroom setting in different ways. Their perspectives and approaches are evidenced as they participate in the various forms of class discussions.

Because the students have already had experience in gaining conceptual understanding of basic mathematics in the previous course, they are more focused on the concepts themselves rather than the procedures of how to learn conceptually. This made the setting ideal to study how the conceptual knowledge and teacher knowledge of fractions is expanded.

DATA SOURCES

Data have been collected over the course of the winter 2005 semester. Three subjects were selected to participate in this study. These students were selected because they volunteered. Each student was unique and diverse in their background and growth as a result of their participation in the course. The images of fractions and fraction multiplication and division in the three subjects were varied, giving multiple perspectives on the changes in images of these ideas. Data were collected from multiple sources. Initial data were collected with a pre-assessment (see appendix A) to determine what the subjects' images of fractions and multiplication and division of fractions was. The second set of data was collected in the classroom. The structure of the *Concepts of Mathematics* course was for the students to participate in groups. The three subjects were grouped together and their classroom experience was videotaped. The researcher also took extensive notes of the classroom experience. Along with the videotapes of their classroom experience, copies of their homework, class work, and journals were obtained to gain insight into their experience with fractions. A third set of data collected consisted of weekly interviews, with each subject participating in eight. Each subject was interviewed one-on-one to gain additional insight into their classroom experiences. All interviews were videotaped and the interviewer took notes of the interviews.

ANALYSIS

The analysis of the data was conducted as three individual case studies, one for each subject. Each of the case studies reports the subject's prior understanding of fractions and fraction multiplication and division and then discusses the subject's change in understanding as a result of the course. Each case study is divided into four sections: *History, Understanding of Fractions, Fraction Multiplication,* and *Fraction Division.*

Each participant in the study began with an initial questionnaire (Appendix A) which provided information about their understanding of fractions and fraction multiplication and division. The questionnaire was followed with an interview to determine more of the subject's mathematics history and their knowledge of fractions.

The questionnaire and the interview, along with journal responses provided the data for the *History* section of the case studies.

The data for the remaining three sections was provided through class work, homework, journals, and interviews. Each subject's class work and homework was investigated for commonalities. The homework was similar to the class work. Typically each class session, an in class worksheet was given, then the students were given a homework set which provided for more work with the concepts covered on the in class assignment. Because the class work and the homework were related closely, common responses on each gave stronger evidence of the subject's understanding.

Each subject's class experience, journals, and interview responses were also analyzed. The journals and the interviews were designed so that the subject could explain her ideas and her experiences with fractions and fraction operations. Each of these gave an expanded picture of the subject's understanding of fractions and fraction multiplication and division. In the journals, the subject's were asked to share their experiences in class and their change in understanding. The interviews also gave the subject opportunity to expound on their learning experiences. Both the journals and the interviews discussed class work, class experiences, and homework. The interviews also discussed further the concepts covered in the class. These two data sets were used to provide explanation of the subject's understanding of fractions and fraction multiplication and division.

The analysis of the data went through two main stages. The preliminary analysis was done during the classroom and interview phases. During the classroom time, I followed the subjects' experiences and made notes about their experiences. Then I reviewed the notes and their homework and journals to identify interview questions and topics. The interviews were directed to provide more information about the subject's experience in the classroom and with the concepts. Insights, comments, questions, and etc. the subject had during the classroom experience and with their work were further investigated. These interviews then drove the primary analysis of the data.

In writing the case studies, each section of the case studies was reviewed individually, i.e. the data analysis for the subject's history was done with the focus only on the history portion of the data. I first reviewed the tapes of the interviews and my notes of the interview. In this review, I was looking for change in the subject's understanding of the concepts. I took note of any situation in which the subject's understanding deepened or where they resisted change in their understanding. From this review, I then read the subject's journal which pertained to the particular interviews. The journal helped clarify the key parts of the interviews. Then I would return to the interviews to further clarify the journals. The interview was the prime data collected, but the interview was built around the class experience, class work, homework, and the journals.

After identifying situations where the subject's understanding was deepened, the new understanding was resisted, and difficulties the subject had in the learning process; I looked at the corresponding homework, class experiences, and class work. I used this data to support the conclusions I had drawn from the journals and interviews. After finding the support, I then returned to the interviews to verify my conclusions. This process was repeated as frequently as needed to draw conclusions about the subject's change in understanding.

The last stage of analysis was review of the case studies. After writing the case studies, they were reviewed. Then the case studies were revised to provide a better explanation of the subjects' experiences in the course. This allowed for a stronger understanding of how the subjects' images and concepts of fractions and fraction multiplication and division deepened and expanded during the course of the study. Finally the conclusion section of each case study was written to discuss the conceptual change that occurred over the course of the study.

RESULTS

This research study has focused on the developing conceptual knowledge of multiplication and division of fractions in pre-service elementary teachers enrolled in a mathematics for elementary school teachers course at Brigham Young University. The investigation has focused on how the pre-service teachers' images of what fractions are and their images of operations on fractions have changed as a result of their experiences in the course.

The results of this study will be presented as three individual case studies, one for each subject. The case studies will report the individual subject's experience as a part of this study. Each case study will begin with a brief history of the subject's math experiences and their prior knowledge of fractions and fraction multiplication and division. Next the case study will report the subject's knowledge of fractions and fractions as iterating and partitioning. Then the case study will report the subject's experience with fraction multiplication. Then the subject's experience with fraction division will be discussed. Finally a conclusion will be written discussing the conceptual change that took place in the study for each individual.

CASE STUDIES

GRACE

History

In the prerequisite course to *Concepts of Mathematics*, Grace had been introduced to the investigative model of learning that was used in the course. She had been taught to make inquiries into learning and been exposed to conceptual learning. So her descriptions of how and what she learned came more easily for her. She was able to articulate her ideas well.

Grace has a strong background of success in traditional school mathematics and enjoyed her experiences there. Grace learned mathematics through algorithms and was able to use the algorithms to complete her work. She learned fraction multiplication through these algorithms without any explanations as to why these worked. In the initial assessment (Appendix A), Grace stated that she didn't know how to draw a picture to show $\frac{2}{3}\times\frac{4}{7}$, but when she saw the multiplication, she associated this with area, as she learned in the prerequisite course. She then tried to use the area model to show the multiplication of the two fractions. She placed $\frac{2}{3}$ on one side of her rectangle and $\frac{4}{7}$ on an adjacent side. This was as far as she could go. She did not relate the fraction to a referent whole, so her picture showed the length of each side being the fraction. This limited her progress and she was unable to finish her picture to explain the operation.

For fraction division, Grace was able to draw a picture and give an explanation for $3\div\frac{1}{2}$, but had difficulty with $\frac{1}{8}\div\frac{7}{9}$. In her explanation for $3\div\frac{1}{2}$ she stated "if I was to divide by 2, I would be cutting the piece (3) in half, but since I'm dividing by $\frac{1}{2}$, 3 must double." Grace's picture showed two boxes of length 3 next to each other. Grace used the idea that dividing by 2 is the opposite of dividing by $\frac{1}{2}$. This explanation showed the application of the division algorithm to a picture. Grace was able to illustrate this division well when she was dividing a whole number by a unit fraction, but was unable to show a picture or give an explanation other than the algorithm when dividing by fractions other than unit fractions. In the second problem of $\frac{1}{8} \div \frac{7}{9}$, Grace stated that all she visualized was the algorithm when performing the operation, she would change this to a multiplication problem of $\frac{1}{8} \times \frac{9}{7}$. She was unable to offer any picture to represent the division or explain why the division problem could be changed to a multiplication problem in this situation.

Even though Grace had difficulty in giving explanations for why fraction multiplication and division worked, she was able to determine what a single fraction meant, i.e. $\frac{2}{5}$ as asked on the questionnaire. She drew a bar and divided it into five equal pieces, with each piece being a fifth of the bar; taking two of those pieces would be two-fifths of the bar. This is a strong sense of what a fraction is. In an interview, Grace stated that she learned about what fractions are through fraction bars. She has a strong connection between the manipulative (fraction bar) and what the symbol of a fraction means. For Grace, fractions are always connected to a concrete example using the fraction.

Grace's history with fractions is strongly algorithmic. She is able to explain what an individual fraction means and she is able to perform the fraction operations. In the interview, Grace expressed a desire to know why the fraction algorithms worked, because she lacked this understanding. This is evident from her questionnaire. Grace stated "most of the time I just think of the common algorithms used to solve the multiplication or division problems. I don't usually think of visual pictures associated with the numbers. So I had to really think about pictures that would explain." The pictures that Grace developed were accurate for specific instances only, were incomplete, or nonexistent. Grace came to the course with limited images of fraction multiplication and division.

Understanding of Fractions

The first day of class, each student in the class was given a worksheet in which they worked with Cuisenaire rods (Appendix B.1). The worksheet asked the students to determine new fractions from old ones. Grace was able to complete this assignment with ease. In the first question she needed to find the rod which had a value of 1 given the dark green rod having a value of $\frac{3}{4}$. Grace determined which rod was one-third of the dark green rod (the red rod). Then she put four red rods together and determined this is the same length as the brown rod, making the brown rod 1. She had similar reasoning with the remainder of the problems. This shows that Grace knew how to make new fractions from old ones. Grace used the idea of partitioning to determine the unit fraction. Then she would iterate the unit fraction to build or determine other fraction values.

The idea of partitioning was natural for Grace, although she did not know it by this name. Grace stated that partitioning was more natural for her; she could start with the whole and break it into smaller pieces. This made sense to Grace. However, the iterating concept, by itself, was not comfortable. Grace's question about iterating was how one knows that the unit fraction iterated the appropriate number of times would make the whole. (She did use iterating with the Cuisenaire rods, but did not recognize that she had.) For example given a piece called "one-sixth" if iterated six times, what was the guarantee that the result would be the whole.

Although iterating by itself was uncomfortable to use and accept, Grace was able to use the two processes together. Grace understood the process of iteration, but was not comfortable accepting that the final result obtained was the whole. This stemmed from her questioning that the result of iterating a unit fraction the required number of times actually gave the referent whole, i.e. what guarantee did she have that result was the unit whole. She did however use the process of iterating in conjunction with partitioning. This was especially useful in building and understanding fractions larger than one. Grace would partition the whole and determine the unit fraction and then iterate the unit fraction sufficiently to obtain the desired fraction. Always for Grace she needed to start with some quantity, either the whole or designated amount, partition to determine the unit fraction, and then she would use iteration.

Understanding fractions through iterating and partitioning was only one part the class investigated. The students also studied equivalent fractions, fractions as decimals, and the relationship between whole number division and fractions. Grace's ability to understand these different concepts was facilitated by her understanding of the fraction.

Grace states that drawing the pictures in class didn't always help her understand the processes taking place. However, Grace was able to gain understanding by interpreting fractions differently. In her journal to the instructor she wrote "In class you said to think of every number as a quantity in reference to the whole rather than just a symbol to help us picture or visualize the operation. That helped a great deal because I often would just think of the symbol rather than the quantity in relation to the whole and that could have been why I was having a hard time visualizing the operation." Grace gave an example of this in an interview by stating that $\frac{1}{4}$ is not just a number [symbol] but it is a quantity in relation to a whole. The whole world of fractions changed for Grace because of this idea, e.g. her inability to draw a picture for multiplication.

Using the idea of a fraction being a quantity in relation to a referent whole allowed Grace to be successful in using pictures to illustrate finding equivalent fractions and fractions as decimals. This came about because Grace would represent her fractions as a part of a referent whole and then could identify equivalent fractions or parts of a power of ten. She was also able to use the pictures to determine the relationship between whole number division and fractions.

In determining the relationship between whole number division and fractions, this was the class's first foray into measurement and sharing division. The class was asked to give a problem situation for each type of division using the problem of $3 \div 4$. For Grace, she was able to work with the sharing type of division easily, but measurement was not as easy to understand. Identifying what the group was for measurement in the problem $3 \div 4$ took Grace some time. Measurement division became clear to her during a class discussion of the homework. Grace learned the a major difference in measurement and sharing division is the whole, what the fraction answer refers to. In the problem $3 \div 4$, Grace identified for sharing the 3 represents the quantity started with and the 4 represents the number of groups 3 is "split among." The answer then represents each group receiving three-fourths of one whole. For measurement, the 3 is still the initial quantity but the four represents how much each group will hold. Here the answer identifies there are three-fourths of one group filled. Connecting the division problems to her understanding of what a fraction is.

Grace was able to use this key to unlock the world of fraction understanding. Grace did this by expanding her definition of what a fraction is. At the beginning of this course, Grace interpreted fractions as a number and could represent them with pictures. She could also use the ideas of partitioning and iterating to build fractions. However, because she learned this key to understanding fractions, she was able to gain greater understanding of fraction concepts, as illustrated below.

Fraction Multiplication

At the time in the course in which fraction multiplication was discussed, there were other topics also discussed. The instructor had introduced a video clip which depicted a young girl working on a math concept. Up to this particular topic, the girl had been taught conceptually first, and then computationally. During this particular instruction the girl was taught the computation. The video shows a follow up to the instruction where the girl was asked to do the computation. The girl struggled with the computation and had difficulty remembering the procedures. The girl described that previously she would have been able to fall back on the concepts to develop the procedure, but she couldn't because she didn't have that training. This situation is mentioned, because it changed Grace's viewpoint about conceptual understanding.

Up to this point Grace relied strongly on the algorithms. She felt comfortable and successful with them. Also, the story problems and illustrations were a frustration for her and she struggled with their importance. However, Grace had an epiphany because of the video clip. She was shown and realized the importance of teaching the concepts before the algorithm. This changed her approach to the class.

At the beginning of the multiplication portion of the course, the instructor identified the norm for multiplication. When performing the multiplication of $a \times b$, this was read and computed as *a* groups of *b* things. (However, in looking at the problem of $a \times b$ Grace saw this as identical to $b \times a$ because the answers were the same. During the interview process Grace identified this, but also accepted that because of her work with the norm and in class there was a difference between $a \times b$ and $b \times a$, although she may not specifically know what that is.) Grace readily accepted this norm, because she stated she already used the norm and it was her own, but she struggled initially with the norm in context. The class was working on the problem $\frac{3}{4} \times 6$ using the pattern blocks. Another member of her group placed six groups of $\frac{3}{4}$ on the table and computed the answer. This was a point of confusion for Grace. She knew the answer was correct because she had applied the algorithm, however the representation bothered her. It took her a moment to determine why this was confusing. Instead of the $\frac{3}{4}$ acting on the 6, the problem was reversed. The group member had not followed the norm. After the instructor discussed the norm again, Grace strengthened her argument about the representation. The reason

Grace could not identify the problem with the representation at first is she sees $\frac{3}{4} \times 6$ and $6 \times \frac{3}{4}$ as the same problem, because the solution is the same. She had to review the norm and identify how to use the norm in the context of the pattern blocks. Grace further learned the difference between $a \times b$ and $b \times a$ through story problems.

After Grace's epiphany, her attitude about story problems changed. Grace was able to recognize how the story problems were of benefit to her, especially in multiplication of fractions. But the big change was not recognizing the help of story problems, but how she has come to rely on her pictures. Before this course, Grace was unable to accurately draw a representation of fraction multiplication. She now uses picture representations easily and frequently. She still relies on the algorithm for multiplication, but she can draw the picture representing the algorithm. This picture then allows her to develop story problems which are examples of the picture. Without this picture, she has difficulty writing story problems for fraction multiplication.

Grace's experiences with fraction multiplication in this course evidence her ability to learn the connections between what she already knows-the algorithm-and the concepts she is learning. She identified that learning the algorithms first has made her number sense of fractions shaky, because she thinks in the rules of the algorithms first, before the concepts. But, working with the pictures she is able to identify why the multiplication works the way it does and also identify and use the "cross-canceling" rule of fraction multiplication. (The cross canceling rule is to divide the same number out of the top of one fraction and the denominator of the other, which simplifies the multiplication taking place.) An example of this is the problem of $\frac{1}{4} \times \frac{8}{3}$ from a class work assignment (Appendix B.4). The students were asked to draw a picture to represent the multiplication and solve. Most of the representations in the group and class were done by drawing eight one-third pieces and partitioning each third into fourths, giving thirty-two one-twelfth pieces. Then one one-fourth piece was taken from each one-third giving eight one-twelfth piece. This fraction was then reduced to two-thirds. However, Grace identified that she had eight one-third pieces that she could put into four groups, each of size two-thirds. She then took one of these groups giving an answer of two-thirds. When asked about this method she identified that the top number of the second fraction gave the number of pieces and if the number of pieces could be divided by the denominator of the first fraction, this was the most efficient method, otherwise she would use the first method.

Fraction Division

Prior to this course, Grace's experiences with fraction division were only procedural, based on the algorithm. As part of this course, Grace was asked to represent fraction division through pictures and models to deepen understanding. Grace was able to do this, after some struggle. She could provide pictures to explain and write story problems illustrating the division. However at the end of this study, Grace was not confident in her ability to explain to someone else how fraction division works.

In the beginning, Grace had a difficult time discerning between the two types of division. She overcame this frustration by diagramming each problem before she solved it. She would write a statement identifying what the total was, how many groups there were, and what the group size was. She would then use the definitions of sharing and measurement to set up the problems. This was especially useful with story problems. From this Grace was able to have success with the two types of division.

At the end of this study, Grace still believed in the algorithm. She trusted that the algorithm worked and performed the division. But, she was not as secure in her implementation of the algorithm. As part of the homework, Grace would draw pictures to represent the division, and then she would double check the picture with the algorithm. She had an interesting situation occur while doing her homework.

> On one of the problems, I was solving it using a picture and came to what I thought was a correct answer. But then I cheated and checked my answer using the algorithm, and I got a different answer. I sat there and couldn't figure out another way to draw my picture to solve the problem. It got kind of annoying. Then I realized that my picture was right and I had simplified my fraction in the algorithm wrong. So my picture was right all along.

This experience taught Grace that the pictures are more trustworthy than her computations. She decided that she needed to trust her pictures more and not rely on the algorithm as much.

Grace gained a stronger understanding of fraction division as a result of this course. Before the course, all she knew about fraction division was the algorithm. She knew the algorithm worked, but not why it worked or what it meant. She trusted the algorithm implicitly, there was no question. Now Grace has a deeper understanding of what it means to divide fractions because of pictures and the development and solving of story problems. She has also strayed from her blind faith of the algorithm to rely more on pictures, because she knows the pictures do work and sometimes her computations are inaccurate. Although Grace has strayed from her blind faith, she still has to remind herself to rely on her illustrations of the work and not on the algorithm.

Conclusion

At the beginning of this study, Grace had knowledge of the algorithms for fraction multiplication and division. She had limited knowledge of the concept of multiplication for whole numbers as shown in her attempt to illustrate fraction multiplication, but did not have the same conceptual understanding of fraction multiplication. She was unable to offer any explanation regarding fraction division other than an illustration of how the algorithm applies when dividing a whole number by a unit fraction. This evidenced Grace's knowledge of the algorithms and her limited or non existent conceptual understanding of fraction multiplication and division.

Although Grace lacked the conceptual understanding of fraction division and multiplication, she did have some conceptual understanding of fractions prior to this study. Grace was able to create a fraction from a whole and draw a picture to illustrate the process. This was also evidenced in the initial class assignment (Appendix B.1). Grace was able to use the ideas of partitioning and partitioning and iterating together to create fractions. However, at this time Grace did not recognize or use the concept of each fraction being a quantity in relation to a referent whole, as evidenced in her journal writings.

During the fraction exploration section of this course, Grace demonstrated her conceptual understanding of fractions and expanded her images and conceptual understanding. Evidence of this expansion came in her recognition of each fraction being a quantity in relation to a whole. From this she was able to visualize the operations taking place with fractions; namely fractions as decimals, equivalent fractions, and whole number division with fractional answers. Her prior image of fractions was limited to fraction bars, but at the end of this study she could describe the operations taking place because she recognized the referent whole for the fractions. Being able to describe the operations is evidence of deepened conceptual understanding of fractions, and being able to "visualize" the operations is evidence of expanded images of fractions.

Recognizing the referent whole for each fraction also expanded Grace's understanding of fraction multiplication. She used this concept to interpret fraction multiplication through pictures. This evidenced her deepened understanding of the concept and an expansion of her image of fraction multiplication, because in the beginning she could offer no explanation and her illustration was incorrect. This identification of the referent whole and her ability to illustrate the multiplication also allowed Grace to identify and explain the concepts behind the "cross-canceling" rule.

Also during the fraction multiplication sequence, Grace learned the difficulties associated with learning algorithms first and then the concepts. Grace's background was algorithmically based and she had had success, which caused her to rely heavily on the algorithms. This was a stumbling block for her, because her reliance on the algorithms impeded her reliance on concepts to explain fractions and fraction operations. She was able to recognize conceptual understanding was necessary for long term success in students, which prompted an increased determination to learn the concepts herself.

As Grace focused on the concepts with a stronger commitment, she still returned to the algorithms to verify the accuracy of her work and explanations of the processes, as she explained in her journal. This was evidenced in her fraction division studies. During this portion of the study, Grace learned how to explain the division process through pictures and story problems, but she double checked her work with the algorithm. After one episode, described in the previous section, where her algorithm and her explanation of division did not agree, Grace learned her conceptual explanation of fraction division was more correct than her algorithmic answers. This situation helped her to further recognize her need to know the concepts of fractions and fraction multiplication and division and rely on the concepts to perform the operations.

During this sequence on fraction division, Grace demonstrated conceptual understanding and evidenced images of fraction division she did not previously have. Grace was able to illustrate and explain fraction division without the algorithm. She was able to identify the two types of division and explain the processes taking place in each type. And she was able to write and solve story problems illustrating each type of division. Prior to this course, she was unable to do any of this. This evidenced a deepened conceptual understanding and expansion of her image of fraction division. By self-admission we know she still has limited understanding of the fraction division concepts, but it is evident she has some understanding.

ELIZABETH

History

In the prerequisite course to *Concepts of Mathematics*, Elizabeth had not been introduced to the investigative model of learning that was used in the course. The section she was in had not been taught to develop explanations of the learning taking place, however she did learn more of the concepts and "whys" of mathematics. Elizabeth had difficulties explaining her ideas and understanding of fractions.

In the interview process Elizabeth stated she was uncomfortable with fractions. She did see the fractions themselves as portions of a pie, always from concrete examples. She can see the symbols from the concrete examples. Elizabeth sees numbers, including fractions, first in concrete situations and then sees them symbolically. She first visualizes an example and then the number relating to the example. For instance given a number such as *three*, Elizabeth recognizes this as a collection of three objects and then she assigns the symbol *3*.

Elizabeth's math history was in a traditional, algorithmically driven curriculum. She states that her math experience up to Calculus was okay, but she was not as fast computationally as others. She sensed this as a weakness in herself, because of the way mathematics was taught. In the initial interview, Elizabeth stated she would like to help her students not only do well in the mathematics, but to understand why the mathematics works the way it does. Essentially, she would like her students to be proficient with the algorithms and also know why they work.

Elizabeth's responses to the questionnaire (Appendix A) were strongly algorithmic. When asked to explain $\frac{2}{5}$, Elizabeth's response was that she thought of it as 40% and something that is nearly $\frac{1}{2}$. This evidences knowledge of percentages and ordering of fractional quantities. In the interview process, she stated that she saw fractions as the shaded part of a pie. From this and her response on the questionnaire, Elizabeth seems to have a good understanding of fractional quantities and can express them in multiple forms. Although Elizabeth demonstrated her understanding of fractional quantities, she struggled to verbalize her ideas.

Elizabeth's ability to explain what she understood fraction multiplication and division to mean was done in two different ways. There are two multiplication and two division problems on the questionnaire. Elizabeth explained one multiplication problem concretely, and the other she attempted to explain algorithmically. The first question was a whole number multiplied by a fractional quantity ($2 \times \frac{4}{7}$). Elizabeth saw this operation as $\frac{4}{7}$ two times. She saw this multiplication as a counting problem or groups of something. This is how she visualized whole numbers, which idea she used to explain the operation. However on the second multiplication problem ($\frac{2}{3} \times \frac{4}{7}$), involving two fractional quantities, she was unable to show anything but an incorrect algorithm of the procedure. She drew two double ended arrows showing that 2 and 7 should be multiplied and 3 and 4 should be multiplied. This response shows that not only was she unable to explain what multiplication was for two fractional quantities, she had the algorithm incorrect as well.

As with multiplication, there were two division problems, one involving a whole number divided by a fraction ($3 \div \frac{1}{2}$) and the other dividing two fractional quantities ($\frac{1}{8} \div \frac{7}{9}$). In the first problem, Elizabeth rewrote the problem and stated that $3 \div \frac{1}{2}$ was the same as $\frac{3}{2}$ or 1.5. Here Elizabeth was unable to demonstrate what fraction division means. In the second problem she stated "I visualize $\frac{1}{8}$ being divided $\frac{7}{9}$ times." She then restates this using pictures to represent $\frac{1}{8}$ and $\frac{7}{9}$. From this we learn that initially, Elizabeth doesn't know, or can't explain, what division of fractions means.

Elizabeth came with this type of understanding of fractions and fraction multiplication and division. In her own words she explains how she works with multiplication and division of fractions. "Based on my answers [to the questionnaire], I think of multiplication and division of fractions based on the steps I was taught. Rather than the process being visual, I think of the steps past teachers have taught me to come to an answer." This statement and her responses during the first interview and to the questionnaire show that Elizabeth only remembers the algorithms when working with fraction multiplication and division, and what she remembers is inaccurate.

Understanding of Fractions

The first homework assignment (Appendix C.1) asked students to identify why a drawing represents a particular fraction. (At this time, the students had not been introduced to the partitioning and iterating vocabulary.) The first fraction represented was $\frac{1}{4}$. Elizabeth explained the drawing was $\frac{1}{4}$ because the bar was evenly divided into four equal pieces and one of those pieces was shaded making $\frac{1}{4}$. The second fraction was $\frac{5}{3}$. Elizabeth first stated this was $1\frac{2}{3}$. Then she drew a picture representing this. She did not draw five one-third pieces. This shows that Elizabeth was able to partition whole numbers into units. In the interview process she stated that partitioning was what she naturally used to represent fractions.

Partitioning and iterating were ideas that Elizabeth was able to grasp, although she had never, previously, thought that fractions could be represented in two different ways. For Elizabeth, partitioning was more natural, however, iterating made "more sense" for "complex" fractions (fractions where the numerator is larger than the denominator). The second problem on the homework also demonstrated her understanding of complex fractions.

In the beginning of the course, Elizabeth always saw complex fractions as a whole plus a fraction (a mixed number). Elizabeth would automatically convert the complex fraction into the mixed number in her mind and work in that format. However as part of the course, the students were instructed to see all fractions $\frac{a}{b}$ as *a* one-*b*th pieces. Elizabeth describes this as an "a ha" moment in the course. It was not that she didn't know this, but having it specifically mentioned helped her to find a bridge between complex fractions and mixed numbers. This idea also helped Elizabeth to expand her understanding of fractions.

Elizabeth was able to identify the significance of each of the parts *a* and *b* in the fraction $\frac{a}{b}$. She stated that she knew this, but was not consciously aware of it. She identified that the *b* represents the number of equal pieces a whole has been partitioned into. She identified *a* as the number times the unit "one-*b*th" had been iterated. She did not use the terms iterating and partitioning to describe this, but she did use the concepts.

As part of the course, Elizabeth also investigated other aspects of fractions including equivalent fractions, fractions as decimals, and the relationship between whole number division and fractions. Elizabeth classifies herself as a visual learner, which was helpful to her in understanding the fractions. Illustrating the fractions facilitated her ability to complete the work. The illustrations made computing equivalent fractions simple. In her own words "[the illustrations] helped me to understand what it exactly means to 'simplify' or 'reduce' a fraction." Before this she had been taught to find equivalent fractions symbolically only and didn't have a complete understanding of the process taking place. She could compute them, but didn't understand why multiplying or dividing the top and bottom of the fraction by the same number produced an equivalent fraction.

Elizabeth's understanding of the relationship between whole number division and fractions was also strengthened and expanded as a result of the class work and environment. The class was asked to explain why $3 \div 4 = \frac{3}{4}$ from a measurement and from a sharing perspective, using story problems. Elizabeth knew that the statement was true, however prior to the class, she didn't know why it worked and that there are two types of division. The instructor outlined the differences between sharing and measurement, which Elizabeth was able to understand and apply to whole number division with non fractional answers. She had more difficulty with the measurement problems than with the sharing problems. For the problem above, Elizabeth was able to easily identify a sharing problem and illustrate it. The measurement problems were more difficult to determine. Both of these types of problems became easier as she drew pictures to explain. Elizabeth states she was able to complete the homework assignment

(Appendix C.2), although she struggled with it. As she began each problem she stated what she was looking for at the top, allowing for her to organize her thoughts.

Elizabeth expanded her images of what fractions are. She deepened her knowledge of what each part of a fraction means and how the two parts work together. She learned much more of the underlying concepts of fractions through illustrating, explanations of her work, class discussions, and reflection of her work. While she knew how to do the procedures of making fractions, finding equivalent fractions, writing fractions as decimals, and whole number division; she now has a conceptual basis for these ideas.

Fraction Multiplication

The multiplication portion of the course was started with a discussion of what $a \times b$ means. The instructor identified $a \times b$ to mean *a* groups of *b* things and established this for the class norm. The students in the class were then asked to work on a list of fraction multiplication problems using pattern blocks (Appendix B.3). One of the problems was $\frac{3}{4} \times 6$. Elizabeth's group went to work on this problem and one of the group members set up six groups of size three-fourths. Elizabeth agreed with this representation; however another member of the group did not. The instructor recognized the class had confusion over how the norm was implemented with the pattern blocks, and gave further instruction on this. After this clarification, Elizabeth recognized why the first representation was incorrect and was then showed three-fourths groups of size six. The next problem was $\frac{4}{3} \times 1\frac{1}{2}$. The group member, who gave the first representation, repeated the error with this problem. At this step, Elizabeth was able to identify the error and correctly represent the problem. Elizabeth understood how the multiplication was represented and how the norm was used.

Prior to this class, Elizabeth was only saw fraction multiplication in terms of the algorithm, which was not always correct. However, she is now able to give picture representations of the multiplication process. This is especially helpful to her, because she is a visual learner. She needs to see the process to understand. Elizabeth was able to strengthen her understanding of fraction multiplication through story problems.

Elizabeth did not have a favorable experience with story problems prior to this course and did not like them. However, as part of the course in writing story problems for fraction multiplication, she expanded her understanding of fraction multiplication. She states "[the story problems] helped me to see [I] must start with the whole, take a fraction of that and then a fraction of that," (meaning in the problem $\frac{4}{3}\times 1\frac{1}{2}$, $1\frac{1}{2}$ is a fraction of the whole and then take $\frac{4}{3}$ of that). The story problems allowed her to identify the referent whole in fraction multiplication, giving the solution meaning.

Elizabeth's experience with fraction multiplication in this course provided her with a stronger conceptual understanding of fraction multiplication. She was able to identify the process taking place in multiplication, illustrate the process, and write problems which used the process. These abilities evidence Elizabeth can use fraction multiplication in multiple ways, an attribute of understanding mathematics (Hiebert et al, 1997).

Fraction Division

Elizabeth's ability to perform fraction division prior to this course was weak based upon the questionnaire (Appendix A). However, in the classroom situation she was able to perform the division correctly with the algorithm and with pictures. In the final interview, Elizabeth identified fraction division as her weakest area for this course. Elizabeth found that drawing pictures of the mathematics was very helpful for her understanding including fraction division. In her own words, Elizabeth explains her experience in the course:

> In Math Ed 306, I have been having difficulties fully grasping the concept of fraction division. The first assignment…took me hours and I didn't even answer all of the questions because I couldn't.… [The next class time] my understanding of fraction division did improve a little. Once we were handed a worksheet with fraction story problems, the division made more sense…I was able to understand what the problems were asking; then, my only problem was trying to figure out how to answer the problems. With practice, I am sure my understanding of fraction division will improve. I do hope that we spend more time in this area because I am in need of more practice and discussion (from Elizabeth's class journal).

Elizabeth was aware of her struggles. She states she was able to learn more about what fraction division means and to understand it better. She can perform the work, but she still needs extensive help from her notes. She still struggled with this at the end of the study.

Unlike her change in understanding of fractions and fraction multiplication, Elizabeth's experience with fraction division is hard to determine based upon the data collected. She was unclear in her explanations of how her knowledge changed and her responses in journals were vague. From her statements her experience with fraction division was difficult for her, but it was also beneficial. She felt she was able to learn more about fraction division and she learned about the development of the division algorithm. She has had experience with why the invert and multiply rule works. Specifically, Elizabeth has some understanding, as opposed to none in the beginning, as to why fractions are inverted. The lesson on the invert and multiply rule actually strengthened her understanding of multiplication.

Conclusion

Prior to this study, Elizabeth did not have much depth of conceptual knowledge of fractions and fraction multiplication and division beyond algorithms, and even her use of the algorithms was often incorrect. Elizabeth demonstrated her image of a fraction was as a decimal or some percentage. During her initial coursework she showed evidence of not relating each fraction in terms of a collection of unit fractions, namely when changing "complex" fractions into mixed numbers immediately. Elizabeth also showed limited knowledge of the process of multiplication. She was able to explain multiplication involving a whole number as a counting process, but for multiplication of fractional quantities the only explanation offered was an incorrect algorithm. And for fraction division she could not offer any explanation or picture for the process. From this evidence and her own words it is clear Elizabeth thought of fractions only in terms of the steps she was taught, a strictly procedural understanding.

From this inauspicious beginning, Elizabeth evidenced growth in her understanding of fraction concepts and her images of fractions and fraction operations. During the fraction understanding sequence of the course, she showed an ability to perform operations involving iterating and partitioning, and from the work with these processes, Elizabeth was able to have deepened understanding of the concept of fractions. This understanding was evidenced when she recognized and was able to explain each fraction $\frac{a}{b}$ as *a* one-*b*th pieces. This understanding allowed her more flexibility in working with fractions, evidence of understanding (Hiebert, et al 1997).

Also in this sequence of learning fractions, Elizabeth was able to expand her images of fractions. This expansion came through learning how to visualize equivalent fractions. She was asked to illustrate the process of simplifying fractions, and by so doing was able to connect her procedural knowledge to conceptual knowledge. She proclaims herself as a visual learner and being able to illustrate the process through pictures strengthened her understanding and expanded her images of fractions.

As a result of Elizabeth's participation in the fraction multiplication sequence, she developed meaning for fraction multiplication, which translates to conceptual understanding and image expansion. Elizabeth explained her image of fraction multiplication of $\frac{a}{b} \times \frac{c}{d}$ as starting with a whole, taking *c*-*d*ths of that and then taking *a*-*b*ths of that. This was a change from her initial incorrect algorithm for fraction multiplication. Elizabeth demonstrated her deepened conceptual knowledge and image expansion further through solving and writing and solving story problems for fraction multiplication.

Elizabeth also demonstrated a deepened understanding of fraction division. Prior to the course, she was unable to give any explanation and was unsure of the computation. At the end of the fraction division sequence, she was able to identify the different types of division and give a limited explanation of the process taking place. This shows change in her conceptual understanding.

Although there is evidence of change in her conceptual understanding in fractions and fraction multiplication and division, Elizabeth struggled to provide explanation of her understanding. Never before had she been asked to explain her mathematical ideas and processes. During this course she evidenced growth in this area, but her limitation in this area also impeded her growth in understanding of concepts and expansion of her images. She was able to perform the work conceptually but limited in her ability to explain what she had done. At the end of the study she could explain well what fraction multiplication means and evidenced a strong image of this concept. She could also explain what a fraction was, after extended questions and responses. But her ability to explain fraction division was severely limited as evidenced in her journals. Although she difficulty in explaining her ideas, she did evidence growth in her explanations in the interviews and in some of her journals.

HANNAH

History

Hannah's mathematics history was rather untraditional. She began in a traditional schooling experience. She describes herself as being slow in mathematics, in the first grade. In order to work on the mathematics, she needed to count with her fingers and was told she couldn't use her fingers. As a result of this, she was unable to work the mathematics and would try to think through the mathematics but couldn't. This was the beginning of her troubles, but not the end.

In her third grade year, the mathematics program was a series of worksheets. The teacher would give an explanation of the worksheets and the students were then expected to do the work. Hannah was unable to grasp all of the ideas in this first explanation; therefore she would approach the teacher for additional help. The teacher would instruct her to return and try again, without further instruction. Hannah would struggle through and often stayed in at recess to work on her mathematics worksheets. Hannah was approximately a month into this situation when the teacher informed her parents that she thought Hannah was mentally handicapped. Hannah's parents immediately transferred her to a charter school. It was in this charter school that she was able to have success in mathematics. Hannah did well in the charter school environment because she had a teacher who believed she could do the work, and she was able to. This positive experience continued until sixth grade where she struggled, but was able to succeed with her parents' help.

In junior high and high school, Hannah's mathematics experience continued to be positive. In Hannah's Geometry and Calculus courses, she was taught using a mixture of discovery and traditional methods. This helped make the mathematics meaningful for her. She states that she cared about the mathematics because "they [the teachers] made it apply to me and showed me how to do the work on paper." She was able to understand the mathematics both conceptually and computationally.

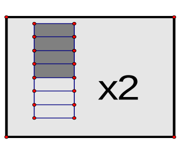
Hannah had a variety of experiences in mathematics throughout her kindergarten through high school years. She had periods of frustration and discouragement, but also strong periods of success. When she determined to pursue an elementary education degree, she thought of mathematics as something she would have to teach her students, but she was “scared to teach it.”

In the prerequisite course to *Concepts of Mathematics*, Hannah became more confident in her ability to teach mathematics and she became more excited about the mathematics. Her section of the prerequisite course was formatted similarly to the *Concepts of Mathematics* course. The section she was enrolled in used investigative methods which helped to flesh out the concepts of basic mathematics. The students discussed the ideas they were investigating in small groups and whole class discussions. Through this method of investigation and reflection, Hannah was able to develop a stronger understanding of the mathematics involved which led to an increase in her confidence and a decrease in her fear.

In the interview process, Hannah stated that she enjoyed fractions—“some parts more than others.” Hannah thought of fractions in two ways: in concrete examples, like a portion of a pie and as a symbol, depending on the context in which the fraction is presented. For example, if the fraction is presented around other numbers and symbols she only thinks of the fraction as a symbol, and does not necessarily relate the symbol to a portion of a pie. Hannah’s two representations of fractions are disjoint, meaning she interprets them as two separate ideas. These two ideas are correlated in her mind, but the correlation is minor and only evidenced after investigation into the correlation. Usually

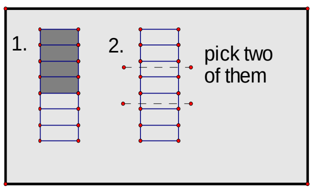
Hannah interprets the fraction concretely first and then with some work will interpret the fraction symbolically, but they are still separate ideas.

Hannah's responses to the questionnaire reflect her mathematics experiences and ideas about fractions. In the first question to interpret $\frac{2}{5}$, Hannah drew a box, sectioned the box into five equal pieces and colored in two of the pieces. Then she stated she thought of the fraction as slightly less than one-half. Both of these interpretations show that Hannah gives concrete examples first and has a good understanding of what the fraction means. She sees the fraction in a contextual setting which gives the fraction meaning.

Hannah's responses to the multiplication and division questions show that Hannah understands what the operations of multiplication and division mean in general and how fractions are developed from division, in the symbolic sense. In the problem of $2\times\frac{4}{7}$, Hannah interprets this problem as $\frac{4}{7}+\frac{4}{7}$, showing that multiplication by two (a whole number) is really two of the objects added together. She states that in performing the operations of multiplication and addition she thinks of them symbolically first and then she drew a picture of $\frac{4}{7}$ and next to it wrote $\times 2$, then she showed a picture example of the answer. 

In her picture, she did not show the multiplication, or addition, taking place only a pictorial of $\frac{4}{7}$ and the answer.

Hannah continues to see the operation of multiplication and addition symbolically in the second problem ($\frac{2}{3} \times \frac{4}{7}$). The question asks the respondent to show a picture representing the operation. Hannah first drew a picture representing $\frac{4}{7}$, then states "take the $\frac{4}{7}$, divide in 3" (and gives a picture of $\frac{4}{7}$ divided in three parts) "pick 2 of them."



From this picture, we see that Hannah is not able to determine the answer, which she stated "[I] don't really get [the] exact answer from this [process] just a picture, I use approximations to figure it out," This picture shows that Hannah identifies multiplication by two-thirds as partitioning the whole picture into thirds and taking two of them. However, she did not subdivide each one-seventh piece into thirds; therefore she was unable to identify the answer from the picture. In this instance, she was unable to correctly show a picture of the operation of fraction multiplication. Her statement following the picture evidences that the picture is secondary to the symbolic answer for the operations. Hannah's responses to this multiplication problem show she did not recognize the symbolic operation as a picture and had difficulty representing multiplication of fractions through illustrations.

Hannah continues to use symbolic representations for the division problems, but in division the problems are immediately rearranged to become multiplication problems. They are not interpreted as division problems. In the first question of $3 \div \frac{1}{2}$, Hannah drew three circles to represent three and then wrote $\times 2$ to show the operation. Her caption to the problem is “what, if divided in half would make three.” So, the problem changes from $3 \div \frac{1}{2}$ to $3 \div \frac{1}{2} = x \to x \div 2 = 3$. This change of the division problem to multiplication identifies Hannah knew that division by one-half was the inverse operation of multiplying by two. Whether this shows understanding of the division process taking place, or the awareness is from application of the division algorithm of inverting and multiplying is not known. However, the second division problem sheds more light on the situation.

In the second division problem, Hannah is asked how she would visualize the operation $\frac{1}{8} \div \frac{7}{9}$. Here Hannah drew a picture of seven blocks and stated she would take one-eighth of nine of them, which isn’t very clear. However, to the side of the problem she has written $\frac{1}{8} \times \frac{9}{7} = \frac{9}{56}$. This evidences that Hannah had used the division algorithm for fractions to determine the solution and then drew a picture to show the multiplication. Both this second problem and the first problem show that Hannah reinterpreted division problems into multiplication problems and then solves them accordingly. These problems are symbolically interpreted and then a concrete example, or picture, is drawn. Hannah’s reinterpretations of the division to multiplication are accurate, but she does not explain if she uses the algorithm to make the change or some other way. In the subsequent discussion, it will be shown that Hannah was not necessarily using the algorithm, as she reinterprets many problems into something more easily accessible for her.

Hannah’s history and responses to the questionnaire show a varied understanding of fractions and fraction multiplication and division. Hannah states “I love fractions, but often struggle with multiplying and dividing them." She was able to work within a structure that may be confusing and difficult and still found success and joy in the work she was performing.

Understanding of Fractions

Hannah's images of fractions and what a fraction means were solid before she began the course. She could draw concrete examples of what those fractions are. Through the assignments in class and her homework she also evidenced this (see Appendix B.1 and Appendix C.1 for problems). For Hannah, partitioning and iterating were easy concepts to understand. She already used the partitioning and iterating to build fractions. Hannah also identified that each fraction is compared to a referent whole. When given any fraction, Hannah immediately compared the fraction to a whole, i.e. it was less than a whole, greater than a whole (and by how much), or it was the whole. She identified that in working with fractions wholes are being made, and without the whole fractions cannot be understood.

Not only did the class build fractions, but they examined the operations of finding equivalent fractions, changing fractions to decimals, and the relationship between whole number division and fraction operations. Hannah was able to use pictures to explain the process of finding equivalent fractions, but only after a bit of a struggle. Her discovery of what the illustration meant allowed her to identify how the algorithm worked, i.e. divide the top and bottom of the fraction by the same number. She was able to identify where this divisor number came from in her picture. For Hannah this is important because her images of fractions as symbols and fractions as a picture are divided. She was able to find the connection from a picture to the symbolic interpretation. Also, she was able to show why some fractions are "irreducible" from her pictures.

Hannah's experiences with the relationship between whole number division and fractions were more difficult. This was the first time Hannah had been introduced to the two types of division. She was often puzzled with which type was which. However, she didn't really believe in the importance in learning the two types of division. At this point in the course, she felt that it was interesting, but would not be useful for her students to learn the differences.

Hannah was asked to explain why $3 \div 4 = \frac{3}{4}$ from the measurement and sharing division, during the interview. Hannah was able to remember the differences between the two types after prompts of what they were from the interviewer. She began with a sharing problem. The problem began with Hannah writing 3/4 on the board. She was experiencing some confusion because she had eliminated the division sign. She forgot what the four was supposed to represent for sharing and what the three represented. Eventually she was able to work through the problem and explain the problem. However, measurement presented a different issue altogether.

For the measurement process, Hannah explained she would write the problem "the other way around." She started with four people and stated "if each [of the four] person[s] get $\frac{3}{4}$ of a pie, how many pies do I need." This statement shows the multiplication problem of four groups of size $\frac{3}{4}$, which showed that instead of writing a different type of division problem, she changed it to multiplication. This was something Hannah was not aware that she did.

Hannah's tendency to do this was also evident on the homework (Appendix C.2). Hannah's measurement problems for (fractional answers) she produced were actually sharing problems and her sharing problems were multiplication problems. For example the problem $9 \div 4$, for measurement Hannah stated "there are four people and nine pies, how many pies does each person get?". Where nine was the original amount, four was the number of groups and the answer $\frac{9}{4}$ was the group size, a sharing division problem. Using the same problem and writing a division problem for sharing, Hannah wrote "If each person got $\frac{9}{4}$ of a piece of a pie, and you had nine pies, how many people would get pies?". This was a multiplication problem for $9 \times \frac{9}{4}$, not division. For non fractional answers, Hannah switched her sharing and measurement problems. At this time, Hannah was unable to identify which type was which, without guided practice.

Hannah's knowledge of fractions deepened in some aspects, and stalled in others. She was able to incorporate the ideas of iterating and partitioning into her lexicon. She also expanded and developed stronger images of equivalent fractions. But in the relationship between whole number division and fractions, Hannah had difficulty in her conceptual explanations, without help.

Fraction Multiplication

To begin the fraction multiplication section of the course, the instructor discussed a norm for the statement $a \times b$. The norm stated $a \times b$ means a groups of b objects. The class discussed this idea and came to a consensus they would accept this norm. Hannah voiced no objections to this norm, when two others in the class did. Hannah thought the norm was the right way to view $a \times b$. Then, along with the group-and the class, Hannah proceeded to work on fraction multiplication problems.

The class was given a worksheet with fraction multiplication problems on it and was instructed to illustrate the multiplication using pattern blocks (Appendix B.3). The first problem was $\frac{3}{4} \times 6$. Hannah quickly set up a pattern block representation showing six groups of size three-fourths. Hannah was convinced this worked because she knew the answer to be correct. However, another member of her group did not agree. The member pointed out that the representation showed $6 \times \frac{3}{4}$ instead of $\frac{3}{4} \times 6$. Hannah did not recognize the difference. At this point, the group received further clarification of the norm from the instructor. Hannah seemed to understand what the norm was and her group members were convinced. However, on the next problem ($\frac{4}{3} \times 6$) Hannah set up a pattern block representation showing six groups of size four-thirds. This she saw this as following the norm, which it wasn't. The group was able to show Hannah how the norm should be represented, helping her to see the change that needed to be made. Hannah was beginning to accept the actual norm and with further help from the instructor was convinced. This is evidenced in Hannah's homework for that day (for homework questions see Appendix C.3). In her illustration of $\frac{3}{2} \times \frac{1}{4}$, Hannah drew one-fourth of a whole, then she drew three-halves of that one-fourth. Hannah was able to identify which number was the group and which number acted on the group.

Another part of the fraction multiplication portion of the course involved writing story problems which used fraction multiplication. Hannah was able to write these problems successfully, because she understood how the norm worked. She stated that because she knew the norm the story problems were easier to develop. Hannah enjoyed story problems because she saw numbers first in concrete situations, such as a story problem. Hannah found it easier to work with numbers in the concrete setting, so being able to write the story problems enhanced her understanding of fraction multiplication.

Prior to this course, Hannah was able to compute fraction multiplication successfully and give a limited explanation symbolically to explain her procedure. As a result of this course, Hannah is now able to explain fraction multiplication through pictures and story problems, which have strengthened her conceptual understanding.

Fraction Division

Hannah's understanding of fraction division did change as a result of her experiences in the course. In the questionnaire, Hannah only showed her ability to divide fractions based upon the algorithm, and how division relates to multiplications. Hannah stated in the interviews that she has always struggled with fraction division, but the algorithm helped her. Hannah showed progress in that by the end of the fraction division segment, she was able to work with fraction division separate from the algorithm through pictures and story problem examples. However, Hannah still rewrites the fraction division problems to suit herself.

Hannah's experience with the concept of fraction division was difficult, but she maintained a positive attitude. She described her initial experience with story problems as being unable to develop her own story problems and was grateful for the examples shown in class. From these examples she was able to then develop her own.

Hannah's ability to compute sharing and measurement problems grew from her first experience in the course. Hannah was able to correctly compute and write story problems for each type. However, she stated she relied heavily upon her notes to clarify what each type was. She was aware of the two types and what they are but she would forget which is which. After a review of her notes and some practice, she was able to work without the notes in determining if the division is measurement or sharing. She also stated she is hopeful with practice her reliance on her notes would become nonexistent. She stated that working on fraction division in this way has helped her understand more and when she does understand her frustration is alleviated; and for the most part she was able to understand.

In her class work, Hannah showed how she is able to determine which type of division is taking place using the process outlined by the instructor. She has written statements explaining what each fraction represents; what is the original amount, what is the size of the group, and how many are in a group. This process allowed Hannah to correctly compute the problem and write story and solve story problems for measurement and sharing division. This process also helped Hannah determine what type of division problem was shown. (An interesting note: Hannah stated that illustrating both types of division was easier when the fractions were both less than one, however measurement was more difficult than sharing. But when at least one of the fractions was greater than one, measurement was easier to illustrate than sharing, and she had difficulties showing the sharing in these cases.)

During the course, Hannah's interpretation of the division problems showed a weakness in her understanding of fraction division. In the fraction exploration part of the course involving whole number division and fractions, she would write one division problem and the second problem would be a multiplication problem. Hannah was able to overcome this problem in the fraction division section, and showed this by writing two division problems, one for sharing and one for measurement. However, Hannah said the reason she had difficulty in this situation is because she immediately rewrites the division problem to understand what it is asking of her, as discussed in the section *Understanding of Fractions.* Through the use of story problems and pictures to illustrate the division Hannah is able to envision the process taking place. But Hannah also became aware of another issue in rewriting the fraction division problems.

In working on her homework, and subsequent class review, Hannah became aware of her misinterpretation of story problems involving fraction division. The students were asked to solve the story problem and then write a number sentence illustrating the division. Hannah would correctly interpret the story problem with pictures to explain her number sentence, but her number sentence was backwards (i.e. if the number sentence was supposed to be $a \div b$, she wrote $b \div a$). This would change her answers. She became aware of this through the pictures she used to solve the problem, because the picture did not answer the problem asked in the question. Hannah would correct her sentence and then be able to work. (An example from her work (see Appendix B.5) shows this. The problem was:

> Tana has enough books to fill 1 1/2 of the 8-shelf floor-to-ceiling bookshelves in her apartment. The apartment is starting to look cluttered, however, so she decides to box up some of her books to make room for photographs, CDs, DVDs, and photo albums. She figures that when will only

have room for books on 7/8 of one bookshelf. How much of her entire book collection can she leave on the book shelf?

Hannah interprets this problem initially as $1\frac{1}{2} \div \frac{7}{8}$ and tries to solve the problem. However, from her drawing she realized the problem is $\frac{7}{8} \div 1\frac{1}{2}$. She was able to correct her work from her picture and then proceed. She ran into this difficulty multiple times. She stated she was able to understand more from the explanations in class "but the light hasn't quite come on in my head with that concept so that it is crystal clear" (from class journal).

Hannah showed a deepened understanding of the concept of fraction division. AT the end of the research study, she was able to work with both types of division using drawings and story problems. Prior to and during the study she stated she was an algorithm user without an understanding of the process. Hannah has deepened her understanding of fraction division as a result of this course. Hannah's tendency to rewrite division problems posed a problem for her that she has become aware of and, at the end of this study, showed progress in overcoming. Hannah was able to work successfully with fraction division in the course and showed progress in her understanding.

Conclusion

Prior to this course, Hannah showed conceptual understanding of what a fraction is and strong procedural understanding of fraction multiplication and division. Hannah was able illustrate what a fraction is and in her first interviews recognized fractions as being less than one, equal to one, or greater than one. This evidences connections to the referent whole, an important part in conceptual understanding of fractions. Hannah also showed conceptual understanding of whole number multiplication, but could not give an appropriate representation of fraction multiplication. Hannah's responses to the division questions on the initial questionnaire (Appendix A) evidence her conceptual understanding of division being limited to the algorithm.

During the fraction understanding portion of the course, Hannah evidenced strong understanding of partioning and iterating. She also expanded her image of fractions relating to the referent whole. Before she saw each fraction compared to one, she expanded this image to include each fraction being a portion of one. This was evidenced by identifying with fractions wholes are being made and without the whole, fractions cannot be understood.

Also during this sequence, Hannah was able to expand her images and deepened her understanding of other concepts relating to fraction understanding. Hannah used illustrations to explain equivalent fractions and from this was able to identify why the algorithm worked. Before this course, she used procedures to compute equivalent fractions, but her illustrations allowed Hannah to understand the process of finding equivalent fractions, evidence of deepened understanding.

Hannah's experience in the fraction multiplication sequence also deepened her understanding of the concepts. After some struggle in understanding the norm and what multiplication meant, she showed a deepened understanding of multiplication by being able to explain what it means to multiply fractions. At the beginning of the course, all Hannah could do was draw an incorrect picture to demonstrate fraction multiplication. At the end of this sequence, she was able to illustrate correctly fraction multiplication in pictures, oral and written explanations, and in story problems. This newfound understanding also made fraction multiplication accessible to Hannah because she could then use concrete examples to explain the process. This ability also shows an expansion of Hannah's images of fraction multiplication.

For fraction division, Hannah showed large levels of growth. In her first contact with the two types of division (during the fraction understanding sequence), Hannah was unconvinced of the importance of the differences between measurement and sharing. At the end of the fraction division sequence, she could identify both types of division—with small reference to her notes, perform and write both types of division problems, and explain the differences between the two types. She could also illustrate the processes involved in the different types of division. This evidenced growth in her conceptual understanding and expansion of her image of fraction division.

Although Hannah was able to perform all the work at each part of the course, she did struggle, in varying degrees, with fraction multiplication and fraction division. Hannah exhibited a tendency to rewrite problems to make them more accessible. She also would switch the operator and operand in her problems. This tendency to rewrite problems was evidenced on the initial questionnaire (Appendix A) in her responses to the fraction division questions. This revision of problems caused her difficulties in computing division problems (when asked to represent $a \div b$ using both measurement and sharing, she would write one division problem and one multiplication problem). She showed evidence of overcoming this difficulty at the end of this study by successfully showing both types of division.

Hannah's tendency to switch the operator and operand in multiplication and division problems was also evidenced on the initial questionnaire (Appendix A) and throughout the study. This tendency inhibited Hannah's ability to have success with fraction multiplication representations and made it difficult for her to recognize why her representations were incorrect. She was able to overcome this limitation in fraction multiplication through help from her group mates and the instructor. This was further evidenced in her story problem work. In working with fraction division this tendency was further shown. She was able to identify the error as she worked with illustrations to solve the story problems and write the correct number sentence for the division. In talking with her previous teacher, it was identified that in the prior course this order was identified as being insignificant for multiplication, which contributed to the limitation Hannah experienced.

CONCLUSION

From the conceptual framework (chapter 2) we learn that teachers need to know what fractions are and how to multiply and divide them. Also, teachers need to have conceptual understanding of fractions and fraction operations, rather than just procedural knowledge, to help their students gain understanding of fractions and fraction operations. Because of this need to have conceptual understanding to teach fractions and fraction operations, this research study has focused on the developing of this conceptual knowledge. The development of this conceptual knowledge was studied under the pattern discussed by Ball (1989) in a paraphrase of Petrie (1981) in that conceptual change is change in meaning, changes in perception, or changes in methodology and is viewed as part of a continuity of growth. This research study investigated this "continuity of growth" in pre-service elementary teachers through their experience in the *Concepts of Mathematics* course at Brigham Young University. The investigation focused on how the pre-service teachers' images of fractions are and their images of operations of fractions changed as a result of their experience in this course.

This study investigated how the images and concepts of fractions and fraction multiplication and division deepen and expand in the pre-service elementary teachers. This was done by following three students in the course throughout the unit on fractions. These participants responded to an initial questionnaire to illustrate prior understanding, participated in weekly one-on-one interviews, provided copies of all work from the course, and were videotaped during class time.

Each of these participants showed a deepening and expansion of their images and conceptual understanding of fractions and fraction multiplication and division during this study. They each exhibited stronger understanding of what fractions are through multiple representations. According to Hiebert et al (1997) this is one aspect of knowing mathematics. The participants were able to “get inside” and see “how [fractions] work, how [fractions] are related to each other, and why they work like they do” (Hiebert et al, 1997, p. 2). However, each participant evidenced different levels at which they were able to do this. Their prior knowledge of fractions did affect their ability to strengthen their understanding. Grace was able to expand upon her image of fractions from fraction bars to include a fraction as a quantity in reference to a whole. Elizabeth was able to identify what a fraction means in relation to the whole prior to this course and could use symbolic representations for the fractions. Her understanding was strengthened when she used illustrations to represent the fractions, which allowed her to identify and understand the processes taking place in renaming fractions, changing fractions to decimals and vice versa, and the relationship between whole number division and fractions. Hannah was able to strengthen her connections between concrete examples and symbolic representations, which prior to the course she identified as two distinct ideas.

Each participant also showed growth of understanding in fraction multiplication. Grace’s perception of the need for conceptual understanding in students changed in this portion of the class, causing her to change her focus in the course. This focus change caused Grace to rely more on her pictures to explain the processes of fraction multiplication instead of solely on the algorithm. Elizabeth's prior knowledge of fraction multiplication was to an algorithm that was often incorrectly used. At the end of this study, Elizabeth was able to identify and explain the process taking place in fraction multiplication, illustrate the process, and write problems which used the process. Hannah saw fraction multiplication symbolically and could draw an illustration to represent it, but the symbolic interpretation and the illustration were disjoint prior to this course. At the end of this study, Hannah was able to bridge these two interpretations together, thereby expanding her image of the concept.

Fraction division was the most difficult concept investigated in this study, but, proportionally, showed the largest growth in understanding for the participants. Grace knew fraction division algorithmically only, which allowed her to reinterpret the problem in terms of multiplication. She was unable to identify the process of division without relying on changing it to multiplication prior to this course. At the end of the study, Grace was able to illustrate the process of fraction division through pictures and word problems. Grace also learned that, although the algorithm is infallible, her use of the algorithm was not. She identified that her pictures were more correct and came to rely on these more. Elizabeth had a weak understanding of fraction division prior to this course and she did strengthen this understanding through the course. However, she had constant struggles and confusion. She was able to follow the discussions in class and work with her group, but she struggled with the homework. The story problems helped Elizabeth interpret the division process, but, at the end of the study, she still had to reference her notes frequently and extensively to complete the work. Hannah's prior experience with fraction division was to reinterpret the division as multiplication. At the end of the study, Hannah was able to illustrate the division as division as a result of working with the pattern blocks and story problems. She still needed an initial prompt at the beginning of each session from her notes about the two types of division, but she was able to remember quickly and do the work. Hannah was able to write a statement to determine which type of division was being used, use a picture to illustrate the division, and use the illustration to perform the operation.

Although each participant showed a deepening of understanding, they each had factors which inhibited their progress. Grace relied heavily on algorithms to complete her work in the course. She would do the work and then double check with the algorithm or she would use the algorithm first to determine the answer and work backwards. She believes the algorithm to be infallible and believed the ability to use the algorithm was a sign of strong understanding. However, Grace learned in the course that students who are taught conceptually first and the algorithm second are able to understand the process and complete the work more successfully than those who are taught the algorithm first. Also, Grace learned that her implementation of the algorithm was fallible. Both of these experiences helped Grace to identify the impediment her belief in the algorithm caused her and to work to overcome this impediment.

For Elizabeth, this course was the first mathematics course in which she was asked to discuss verbally her understanding and explain her reasoning. This presented a problem for her, because she had difficulty explaining her thought process and identifying why the process worked. She was able to do the computation work, but was limited in her ability to explain the work. She would often be frustrated as she tried to explain her ideas during the interview process and her limitation was also evidenced in her weekly journals. However, she did grow stronger and more confident in this area and was able to give clearer explanations of her ideas as the study progressed.

Hannah was a dominant participant in the course. Because of her math history and her experience in the prerequisite course, she wanted to understand the concepts and she knew that she could have success. However, she evidenced an interesting phenomenon. She would transpose the operator and the operand in the multiplication and division problems. In multiplication, the first evidence of this was in her use of the norm. She agreed with the norm, and then went in opposition to it on her work. After explanation from the group, she was able to identify what the norm meant in context and work with it. At first glance, this seemed to be an isolated incident. However, during the division problems she would also switch the operator and the operand (i.e. if the problem illustrated $a \div b$ she would write and solve it as $b \div a$). This presented a problem for her because her computations would be wrong. But, through her illustrations of the problem, she was able to identify this impediment and could rewrite the symbolic representation correctly and do the work.

Each participant in this study evidenced a change in their understanding of fractions and fraction multiplication and division. Their images and concepts of fractions and fraction multiplication deepened and expanded during their experience in the *Concepts of Mathematics* course. The participants evidenced a stronger ability to understand fractions and fraction multiplication and division in accordance to what Mack (1998) and Hiebert et al (1997) identify as understanding mathematics. In that understanding of fractions and fraction multiplication and division is the ability to understand ideas about these ideas and to use the ideas flexibly and in multiple ways.

This stronger understanding will translate to a better learning experience for their future students because the participants will be able to help their students understand and build bridges between what the students know and the new information being taught.

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APPENDIX A

QUESTIONNAIRE

Please respond to the following questions completely. Explain your reasoning.

1. What do you think of when you see: $\frac{2}{5}$?
2. When performing this operation $2\times\frac{4}{7}$ what kind of pictures do you have in mind?
3. What kind of picture would you draw to show this operation: $\frac{2}{3}\times\frac{4}{7}$?
4. What kind of picture would you draw to show this operation: $3\div\frac{1}{2}$?
5. What do you visualize to help you perform this operation: $\frac{1}{8}\div\frac{7}{9}$?
6. Based upon the answers to the above questions, how do you think of multiplication and division of fractions?

APPENDIX B

APPENDIX B.1

What's My Size?

Use Cuisenaire rods to answer the following questions. Children who have not yet learned how to simplify, add, subtract, multiply or divide fractions can solve these problems just by reasoning with the rods. Try to do the same. Your final answer will emerge from the way you arrange the rods, i.e., you should be able to see your answer in the way you set up your Cuisenaire rods.

For each problem below, draw a picture to illustrate you answer, and be sure you can explain how your answer can be seen in your picture.

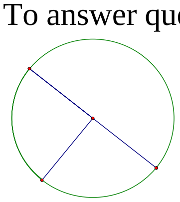
1. If dark green is ¾, what color of rod has a value of 1?
2. If blue is 3/2, what color of rod has a value of 1?
3. If purple is 2, what is the value of black?
4. If brown is 2/3, what is the value of light green?

APPENDIX B.2

Fraction Representations and Conceptions

1. Draw a picture of 1/5. Explain how you know it is 1/5 from both an iterating and a partitioning perspective.
2. Draw a picture of 2/3. Explain how you know it is 2/3 from both an iterating and a partitioning perspective.

To answer questions 3 and 4 use the picture below:



(where the two quarter pieces are shaded.)

3. A child drew the above picture to show 2/3. Is the child right? Why or why not?
4. In your opinion, what does the child think 2/3 means?

APPENDIX B.3

Multiplying Fractions (in class)

Use pattern block to do each of the following.

1. $\frac{2}{3}\times\frac{1}{2}$
2. $\frac{3}{4}\times 6$
3. $\frac{4}{3}\times 6$
4. $\frac{4}{3}\times 1\frac{1}{2}$
5. $\frac{2}{3}\times\left(\frac{3}{2}\times 4\right)$

APPENDIX B.4

Multiplying Fractions

Solve the following multiplication problems using pictures.

1. $\frac{1}{4}\times\frac{1}{6}$
2. $\frac{1}{4}\times\frac{8}{3}$
3. $\frac{2}{3}\times\frac{3}{5}$
4. $\frac{3}{5}\times\frac{2}{3}$
5. $\frac{2}{3}\times 2\frac{1}{4}$

APPENDIX B.5

Measurement Division Story Problems

Solve each of the story problems below using images. After solving the problems, write a division number sentence that models the story problem.

1. Derek has 5/3 cups of tropical punch concentrate. It takes 1/2 cup of concentrate to make one pitcher of tropical punch. How many pitchers of tropical punch can he make?
2. Allan has 2 pages to write for his philosophy class. If he works at home (where he is easily distracted), he thinks he can probably write 2/3 of a page in an hour. How long will it take him to write the entire two pages?
3. Lyndsey goes running at her local community center when it's too cold to run outside. Each lap of the track at the community center is 1/6 of a mile long. If Lyndsey decides to sprint the last 1/4 mile of her run, how many laps will she sprint?
4. Tana has enough books to fill 1 1/2 of the 8-shelpf floor-to-ceiling bookshelves in her apartment. The apartment is starting to look cluttered, however, so she decides to box up some of her books to make room for photographs, CDs, DVDs, and other photo albums. She figures that she will only have room for books on 7/8 of one bookshelf. How much of her entire book collection can she leave on the bookshelves?

APPENDIX C

APPENDIX C.1

Fraction Images

14. Draw a picture that represents $\frac{1}{4}$. Then explain how you know that you picture represents $\frac{1}{4}$.
15. Draw a picture that represents $\frac{5}{3}$. Then explain how you know that you picture represents $\frac{5}{3}$.

APPENDIX C.2

Measurement and Sharing

1. Make up two division story problems for $15 \div 3 = 5$ that involve the measurement model of division. You do not need to compute the answer.
2. Make up two division story problems for $15 \div 3 = 5$ that involve the sharing model of division. You do not need to compute the answer.

For problems 3 and 4 do the following:

   a. Write a story problem that involves the measurement model of division.
   b. Use a picture to compute the answer to your story problem (using the measurement model of division). Explain your reasoning. Be sure that you can see the answer from your picture.
   c. Write a story problem that involves the sharing model of division.
   d. Use a picture to compute the answer to your story problem (using the sharing model of division). Explain your reasoning. Be sure that you can see the answer from your picture.

3. $5 \div 8$
4. $9 \div 4$

APPENDIX C.3

Multiplying Fractions

A. Think about how we reasoned about multiplication of fractions in class as you answer the questions below.

1. In the space below shade in 3/8 of four circles in two different ways. Explain your reasoning for each way. How do your methods differ? According to your reasoning what is 3/8 of 4?
2. In the space below, shade in **3/4 of 2 rectangles.** Draw two more rectangles and use this set, each representing one, to show **two 3/4**. Use your drawings to explain why 3/4□2 = 2□3/4.
3. Use a rectangle, draw and shade in **3/4 of 2/3**. Label the 2/3 in your picture. What is it 2/3 of?
4. Use the same picture as in #3 and this time label the 3/4. What is it 3/4 of?
5. Now draw and shade in **3/2 of 1/4**. Label the 1/4 in your picture. What is it 1/4 of?
6. Use the same picture as in #5 and this time label the 3/2. What is it 3/2 of?

B. Write a story problem for each of the following multiplication number sentences. Then solve the problems using pictures. In clued a written explanation of your solution. Do not use any algorithms in your solution, or even to check your answer.

1. 1/2÷1/4
2. 3/4÷2/5
3. 1 1/3÷1 1/2