Gravitational Descendants and the Moduli Space of Higher Spin Curves

Tyler J. Jarvis
jarvis@math.byu.edu

Takashi Kimura

Arkady Vaintrob

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GRAVITATIONAL DESCENDANTS AND THE MODULI SPACE OF HIGHER SPIN CURVES

TYLER J. JARVIS, TAKASHI KIMURA, AND ARKADY VAINTROB

Abstract. The purpose of this note is introduce a new axiom (called the Descent Axiom) in the theory of r-spin cohomological field theories. This axiom explains the origin of gravitational descendants in this theory. Furthermore, the Descent Axiom immediately implies the Vanishing Axiom, explicating the latter (which has no *a priori* analog in the theory of Gromov-Witten invariants), in terms of the multiplicativity of the virtual class. We prove that the Descent Axiom holds in the convex case, and consequently in genus zero.

0. Introduction

In [8], the notion of an r-spin cohomological field theory was introduced for each integer $r \geq 2$. This is a particular realization of a cohomological field theory (CohFT) in the sense of Kontsevich-Manin [12]. Its construction was motivated by drawing an analogy [9] with the Gromov-Witten invariants of a smooth, projective variety V.

The analog of the moduli space $\overline{\mathcal{M}}_{g,n}(V)$ of stable maps into V is $\overline{\mathcal{M}}_{g,n}^{1/r}$, the moduli space of stable r-spin curves [6, 7]. The moduli space $\overline{\mathcal{M}}_{g,n}^{1/r}$ is the disjoint union of $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ where $\mathbf{m} = (m_1, \dots, m_n)$ and $m_i = 0, \dots, r-1$ for all $i = 1, \dots, n$. Suppose that $r \geq 2$ is prime. For any n-tuple of integers $\mathbf{m}$, $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ is a compactification of the moduli space of (connected) Riemann surfaces Σ of genus g with n marked points, $(p_1, \dots, p_n)$, together with a holomorphic line bundle $\mathcal{E}$ over the curve whose r-th tensor power is $\mathcal{K}$, the canonical bundle of Σ, twisted by $\mathcal{O}(-\sum_{i=1}^n m_i p_i)$, together with a bundle isomorphism between $\mathcal{E}^{\otimes r}$ and $\mathcal{K} \otimes \mathcal{O}(-\sum_{i=1}^n m_i p_i)$. When r is not prime then one must include, in addition, d-th roots for all d dividing r and suitable compatibility morphisms. Furthermore, when the m_i are permitted to range over all non-negative integers, any two spaces $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ and $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}'}$ are canonically isomorphic if the n-tuples $\mathbf{m}$ and $\mathbf{m}'$ are componentwise equal mod r. Therefore, one may restrict without loss of generality to the

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space $\overline{\mathcal{M}}^{1/r}_{g,n}$. In particular, the space $\overline{\mathcal{M}}^{1/r}_{g,n}$ is a smooth stack which is a ramified cover of the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$.

There is an analog $c^{1/r}$ of the cap product of the virtual fundamental class of $\overline{\mathcal{M}}_{g,n}(V)$ with the Gromov-Witten classes on $\overline{\mathcal{M}}_{g,n}(V)$ in the r-spin CohFT called a *virtual class* on $\overline{\mathcal{M}}^{1/r}_{g,n}$. Following the ideas of Witten [16, 17], we proved in [8] that if $c^{1/r}$ satisfied certain axioms then it would give rise to a CohFT. The class $c^{1/r}$ was constructed for all genus when $r = 2$ and in genus zero for all $r \geq 2$.

One can form the large phase space potential $\Phi(\mathbf{t})$ associated to the r-spin CohFT which is a generating function for the correlators (including the gravitational descendants) of the theory. It was proved in [8] that for all $r \geq 2$, the part of the potential corresponding to genus zero solves (a semi-classical limit of) the r-th Gelfand-Dickey (or KdV_r) integrable hierarchy following the ideas of Witten [16, 17]. Witten conjectured that this correspondence should extend to all genera. When $r = 2$, this conjecture reduces to Kontsevich's result [11] relating the usual KdV_2 hierarchy to intersection numbers on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves [18].

While most of the axioms satisfied by $c^{1/r}$ have analogs in the theory of Gromov-Witten invariants, there is one axiom satisfied by $c^{1/r}$, called the Vanishing Axiom, which appears to have no analog in the theory of Gromov-Witten invariants. The Vanishing Axiom states that $c^{1/r}$ vanishes on $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$ if $\mathbf{m} = (m_1, \dots, m_n)$ and $m_i = r - 1$ for some i and $0 \leq m_j \leq r - 1$ for all j. Since this axiom does not immediately appear to have the form of a factorization identity, it does not appear to have an analog in the theory of Gromov-Witten invariants.

There is another puzzling feature in the r-spin CohFT. Although the class $c^{1/r}(\mathbf{m})$ is defined on $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$ where $\mathbf{m} = (m_1, \dots, m_n)$ and $m_i = 0, \dots, r-1$ for all $i = 1, \dots, n$, the axioms of the virtual class are such that one may extend its definition to $c^{1/r}(\widetilde{\mathbf{m}})$ in $H^\bullet(\overline{\mathcal{M}}^{1/r}_{g,n})$ for arbitrary n-tuples of nonnegative integers $\widetilde{\mathbf{m}} = (\tilde{m}_1, \dots, \tilde{m}_n)$ under the identification of $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$ with $\overline{\mathcal{M}}^{1/r,\mathbf{m}'}_{g,n}$ whenever the n-tuples of integers $\mathbf{m}$ and $\mathbf{m}'$ are equivalent (componentwise) mod r. Furthermore, it is simple to see that the usual construction of the virtual class in genus zero, extends straightforwardly to yield a class $c^{1/r}(\widetilde{\mathbf{m}})$ for arbitrary nonnegative n-tuples $\widetilde{\mathbf{m}}$. It is natural to ask what role these additional (infinite number of) classes $c^{1/r}(\widetilde{\mathbf{m}})$ play in the r-spin CohFT.

Finally, there is a broader issue that permeates both the theory of Gromov-Witten invariants and the r-spin CohFT. In the case of r-spin CohFT (the Gromov-Witten case is analogous), the gravitational descendants are correlators which are integrals of products of tautological ψ_i classes on $\overline{\mathcal{M}}^{1/r}_{g,n}$ (where $i = 1, \dots, n$) with $c^{1/r}$. From a purely geometric perspective, the appearance of the ψ_i classes in the correlators is rather ad hoc. The real question is: *Where do these ψ classes come from?*

All of these issues can be simultaneously addressed if one assumes that $c^{1/r}$ satisfies a new axiom called *the Descent Axiom*.

Descent Axiom 1.9 . *Let $r \geq 2$ be an integer, and $\mathbf{m} = (m_1, \ldots, m_n)$ be an n-tuple of integers such that either $m_i \geq 0$ for all $i = 1, \ldots, n$, or there exists an integer $1 \leq j \leq n$ such that $m_i \geq 0$ for all $i \neq j$ and $m_j = -1$. Let $c^{1/r}(\mathbf{m})$ denote the virtual class on $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$. Let $r\boldsymbol{\delta}_i$ denote an n-tuple of integers which is r in the ith position and zero in all others then*

$$c^{1/r}(\mathbf{m} + r\boldsymbol{\delta}_i) = -\tilde{\psi}_i(\mathbf{m})c^{1/r}(\mathbf{m})$$

The Vanishing Axiom can be shown to follow from properties of the class $\tilde{\psi}_i(\mathbf{m}_i)$.

The Descent Axiom implies that if one considers a generating function of intersection numbers obtained by integrating ONLY $c^{1/r}(\mathbf{m})$ for *all* $\mathbf{m}$ with $m_i \geq 0$ against the fundamental class of $\overline{\mathcal{M}}_{g,n}^{1/r}$ then one obtains, after a simple variable redefinition, the usual large phase space potential of the r-spin CohFT! In other words, the Descent Axiom explains geometrically why the ψ classes appear in the usual definition of gravitational descendants.

We will prove that the Descent Axiom holds in the convex case (and, hence, in genus zero) using the multiplicativity property of the top Chern class. The Vanishing Axiom can then be interpreted as following from a kind of factorization property of $c^{1/r}$.

Finally, it is worth pointing out that all of the ideas in this note generalize to the moduli space of case of Gromov-Witten invariants through the introduction of the moduli space of stable r-spin maps into V [10].

In the first section of this note, we briefly review the properties of the moduli space of stable r-spin curves $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ and the tautological cohomology classes ψ_i and $\tilde{\psi}_i(\mathbf{m})$ associated to tautological line bundles over $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$. We then recall the Vanishing Axiom and prove that the Descent Axiom implies the Vanishing Axiom by using a universal relation between these classes.

In the second section, we show that the Descent Axiom implies that the usual large phase space potential function of an r-spin CohFT $\Phi(\mathbf{t})$, containing both $c^{1/r}$ and the ψ_i classes, agrees (after a simple variable redefinition) with the generating function for correlators associated to the virtual classes $c^{1/r}(\mathbf{m})$ where $\mathbf{m} = (m_1, \ldots, m_n)$ are arbitrary, nonnegative n-tuples.

In the third section, we recall the construction of the virtual class $c^{1/r}$ in the convex case and prove that the Descent Axiom holds in this case.

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1. The Moduli Stack of r-spin Curves

We will use the notation and conventions of [8], which we briefly review here. Complete details and proofs may be found in [8] and in [6].

1.1. **Definitions and Basic Properties.**

Definition 1.1. A *prestable curve* is a reduced, complete, algebraic curve with at worst nodes as singularities.

Definition 1.2. Let $(X, p_1, \ldots, p_n)$ be a prestable, n-pointed, algebraic curve, let $\mathcal{K}$ be a rank-one, torsion-free sheaf, and let $\mathbf{m} = (m_1, \ldots, m_n)$ be a collection of integers. A d-th root of $\mathcal{K}$ of type $\mathbf{m}$ on X is a pair $(\mathcal{E}, c)$ of a rank-one, torsion-free sheaf $\mathcal{E}$ and an $\mathcal{O}_X$-module homomorphism $c : \mathcal{E}^{\otimes d} \to \mathcal{K} \otimes \mathcal{O}_X(-\sum_{i=1}^n m_i p_i)$ with the following properties:

- $d \cdot \deg \mathcal{E} = \deg \mathcal{K} - \sum m_i$
- c is an isomorphism on the locus of X where $\mathcal{E}$ is locally free
- for every point $p \in X$ where $\mathcal{E}$ is not free, the length of the cokernel of b at p is $d - 1$.

Although the condition on the cokernel seems strange, it turns out to be a very natural one (see [6]). Indeed, the construction of r-spin curves can also be done [1] in terms of line bundles on the "twisted curves" of Abramovich and Vistoli, and in their formulation, the cokernel condition amounts exactly to the condition that the local (orbifold) index of the curve at each node or marked point must divide d—a redundant condition in that formulation.

For any d-th root $(\mathcal{E}, c)$ of $\mathcal{K}$ of type $\mathbf{m}$, and for any $\mathbf{m}'$ congruent to $\mathbf{m}$ (mod d), we can construct a unique d-th root $(\mathcal{E}', b')$ of type $\mathbf{m}'$ simply by taking $\mathcal{E}' = \mathcal{E} \otimes \mathcal{O}(1/d \sum (m_i - m_i')p_i)$. Consequently, the moduli of curves with d-th roots of a bundle $\mathcal{K}$ of type $\mathbf{m}$ is canonically isomorphic to the moduli of curves with d-th roots of type $\mathbf{m}'$.

Definition 1.3. Let $(X, p_1, \ldots, p_n)$ be a prestable, n-pointed curve. An *r-spin structure on X of type* $\mathbf{m} = (m_1, \ldots, m_n)$ is a pair $(\{\mathcal{E}_d\}, \{c_{d,d'}\})$ of a set of sheaves and a set of homomorphisms as follows. The set of sheaves consists of a rank-one, torsion-free sheaf $\mathcal{E}_d$ on X for every divisor d of r; and the set of homomorphisms consists of an $\mathcal{O}_X$-module homomorphism $c_{d,d'} : \mathcal{E}_d^{\otimes d/d'} \longrightarrow \mathcal{E}_{d'}$ for every pair of divisors d', d of r, such that d' divides d. These sheaves and homomorphisms must satisfy the following conditions:

- The sheaf $\mathcal{E}_1$ is isomorphic to the canonical (dualizing) sheaf ω_X.
- For each divisor d of r and each divisor d' of d, the homomorphism $c_{d,d'}$ makes $(\mathcal{E}_d, c_{d,d'})$ into a d/d'-th root of $\mathcal{E}_{d'}$ of type $\mathbf{m}'$, where $\mathbf{m}' = (m_1', \ldots, m_n')$ is the reduction of $\mathbf{m}$ modulo d/d' (i.e. $0 \leq m_i' < d/d'$ and $m_i \equiv m_i'$ (mod $d)/d'$).
- The homomorphisms $\{c_{d,d'}\}$ are compatible. That is, for any integers d dividing e dividing f dividing r, we have $c_{f,d} = c_{e,d} \circ c_{f,e}$.

If r is prime, then an r-spin structure is simply an r-th root of ω_X. Even when d is not prime, if the root $\mathcal{E}_d$ is locally free, then for every divisor d' of d, the sheaf $\mathcal{E}_{d'}$ is uniquely determined, up to an automorphism of $\mathcal{E}_{d'}$. In particular, if $\mathbf{m}'$ satisfies the conditions $\mathbf{m}' \equiv \mathbf{m} \pmod{d'}$ and $0 \leq m'_i < d'$, then the sheaf $\mathcal{E}_{d'}$ is isomorphic to $\mathcal{E}_d^{\otimes d/d'} \otimes \mathcal{O}\left(\frac{1}{d'}\sum(m_i - m'_i)p_i\right)$.

Every r-spin structure on a smooth curve X is determined, up to isomorphism, by a choice of a line bundle $\mathcal{E}_r$, such that $\mathcal{E}_r^{\otimes r} \cong \omega_X(-\sum m_i p_i)$. In particular, if X has no automorphisms, then the set of isomorphism classes of r-spin structures (if non-empty) of type $\mathbf{m}$ on X is a principal homogeneous space for the group of r-torsion points of the Jacobian of X. Thus there are r^{2g} such isomorphism classes.

Definition 1.4. A *stable, n-pointed, r-spin curve of genus g and type* $\mathbf{m}$ is a stable, n-pointed curve of genus g with an r-spin structure of type $\mathbf{m}$. For any $\mathbf{m} = (m_1, \dots, m_n)$, the stack of connected, stable, n-pointed r-spin curves of genus g and type $\mathbf{m}$ is denoted by $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$.

The disjoint union $\coprod_{\substack{\mathbf{m} \\ 0 \leq m_i < r}} \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ is denoted by $\overline{\mathcal{M}}_{g,n}^{1/r}$.

Remark 1.5. No information is lost by restricting $\mathbf{m}$ to the range $0 \leq m_i \leq r-1$, since when $\mathbf{m} \equiv \mathbf{m}' \pmod{r}$ every r-spin curve of type $\mathbf{m}$ naturally gives an r-spin curve of type $\mathbf{m}'$ simply by

$$\mathcal{E}_d \mapsto \mathcal{E}_d \otimes \mathcal{O}\Big(\sum \frac{m_i - {m'}_i}{d} p_i\Big).$$

Thus $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ is canonically isomorphic to $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}'}$. Since one of our main goals in this note is to relate the virtual classes for different values of $\mathbf{m}$, including those outside the range $0 \leq m_i \leq r-1$, we will always consider $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}'}$ to be equal, via this canonical isomorphism, to the appropriate component of $\overline{\mathcal{M}}_{g,n}^{1/r}$.

The stack $\overline{\mathcal{M}}_{g,n}^{1/r}$ is a smooth Deligne-Mumford stack, finite over $\overline{\mathcal{M}}_{g,n}$, with a projective, coarse moduli space. For $g > 1$ the spaces $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ are irreducible if $\gcd(r, m_1, \dots, m_n)$ is odd, and they are the disjoint union of two irreducible components if $\gcd(r, m_1, \dots, m_n)$ is even. When the genus g is zero, the coarse moduli space $\overline{M}_{0,n}^{1/r,\mathbf{m}}$ is either empty (if r does not divide $2 + \sum m_i$), or is canonically isomorphic to the moduli space $\overline{\mathcal{M}}_{0,n}$. Note, however, that this isomorphism does not give an isomorphism of stacks, since the automorphisms of elements of $\overline{\mathcal{M}}_{0,n}^{1/r,\mathbf{m}}$ vary differently from the way that automorphisms of the underlying curves vary. In any case, $\overline{\mathcal{M}}_{0,n}^{1/r,\mathbf{m}}$ is always irreducible.

When the genus is one, the stack $\overline{\mathcal{M}}_{1,n}^{1/r,\mathbf{m}}$ is the disjoint union of d irreducible components, where d is the number of divisors of $\gcd(r, m_1, \dots, m_n)$.

Throughout this paper we will denote the forgetful morphism by $p : \overline{\mathcal{M}}^{1/r}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$, and the universal curve by $\pi : \mathcal{C}^{1/r}_{g,n} \longrightarrow \overline{\mathcal{M}}^{1/r}_{g,n}$. As in the case of the moduli space of stable curves, the universal curve possesses canonical sections $\sigma_i : \overline{\mathcal{M}}^{1/r}_{g,n} \longrightarrow \mathcal{C}^{1/r}_{g,n}$ for $i = 1, \dots, n$. Unlike the case of stable curves, however, the universal curve $\mathcal{C}^{1/r,\mathbf{m}}_{g,n} \longrightarrow \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$ is not obtained by considering $(n+1)$-pointed r-spin curves. The universal curve $\mathcal{C}^{1/r,\mathbf{m}}_{g,n}$ is birationally equivalent to $\overline{\mathcal{M}}^{1/r,(m_1,m_2,\dots,m_n,0)}_{g,n+1}$, but they are not isomorphic.

1.2. **Natural Cohomology Classes on $\overline{\mathcal{M}}^{1/r}_{g,n}$.** All of the usual classes on $\overline{\mathcal{M}}_{g,n}$ pull back to classes on $\overline{\mathcal{M}}^{1/r}_{g,n}$. We will abuse notation and use the same symbol for these classes regardless of whether they are on $\overline{\mathcal{M}}_{g,n}$ or $\overline{\mathcal{M}}^{1/r}_{g,n}$. In particular, for each i with $0 \leq i \leq n$ we have the classes $\psi_i := \sigma_i^*(\omega)$, and for each positive integer j, we have the class λ_j, which is the degree-j term in the Chern polynomial of the K-theoretic pushforward* of ω:

$$c_t(R\pi_*\omega) = 1 + \lambda_1 t + \lambda_2 t^2 + \dots.$$

In addition to the classes pulled back from $\overline{\mathcal{M}}_{g,n}$, there are other natural cohomology classes on $\overline{\mathcal{M}}^{1/r}_{g,n}$. These include classes arising from $\mathcal{E}_r$, the r-th root in the universal r-spin structure on the universal curve over $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$:

$$\tilde{\psi}_i(\mathbf{m}) := \tilde{\psi}_i := \sigma_i^*(\mathcal{E}_r),$$

and μ_j is the degree-j term in the Chern character of the pushforward of $\mathcal{E}_r$:

$$ch(R\pi_*\mathcal{E}_r) = 1 + \mu_1 t + \mu_2 t^2 + \dots.$$

In [8], we established some relationships between these classes. Especially useful for the current work is the following.

Proposition 1.6 (Proposition 2.2 of [8]). *The line bundles $\sigma_i^*\omega$ and $\sigma_i^*(\mathcal{E}_r)$ on the stack $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$ are related by*

$$r\sigma_i^*(\mathcal{E}_r) \cong (m_i+1)\sigma_i^*(\omega).$$

Therefore, in $Pic\,\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n} \otimes \mathbb{Q}$ and in $H^2(\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}, \mathbb{Q})$ we have

$$\tilde{\psi}_i(\mathbf{m}) = \frac{m_i+1}{r}\psi_i. \tag{1}$$

It is important to note that these relations hold for all choices of $\mathbf{m} = (m_1, \dots, m_n)$, with no restriction on the range of the m_i.

An immediate corollary is the following.

Corollary 1.7. *If $\mathbf{m} = (m_1, \dots, m_n)$, and if $m_i = -1$ for some i, then we have*

$$\tilde{\psi}_i(\mathbf{m}) = 0.$$

*In [8] we used the topologists' notation $\pi_!$ for the pushforward that we are calling $R\pi_*$ here.

1.3. **The Virtual Class and the Vanishing Axiom.** In [8, §4.1] we give axioms for a so-called *virtual class* $c^{1/r}$, where for each $\mathbf{m} = (m_1, \dots, m_n)$ with $0 \le m_i \le r-1$ the virtual class $c^{1/r}(\mathbf{m})$ is a cohomology class in $H^{2D}(\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}, \mathbb{Q})$. Here the dimension D is given by

$$D = \frac{1}{r}\Big((r-2)(g-1) + \sum_{i=1}^{n} m_i\Big). \tag{2}$$

The class $c^{1/r}$ plays the role of Gromov-Witten classes (capped with the virtual fundamental class of the moduli space of stable maps) in the theory of Gromov-Witten invariants of a smooth, projective variety.

Although the axioms stated in [8, §4.1] are only for $\mathbf{m}$ in the range $0 \le m_i \le r-1$, these axioms make sense for all non-negative choices of m_i, and indeed, the constructions of $c^{1/r}$ given in [8] for $g = 0$ (and all $r > 1$) satisfy the axioms for all choices of non-negative m_i. Even better, these constructions make sense and satisfy the axioms in the case that one (but not more) of the m_i is equal to -1, and the rest are non-negative. The axioms are generally inconsistent in the case that two or more of the m_i are negative, although there may be some special cases with two or more negative m's, where the virtual class exists.

One of the axioms that the virtual class must satisfy is the vanishing axiom [8, §4.1, Axiom 4]:

Vanishing Axiom 1.8. If $\mathbf{m} = (m_1, \dots, m_n)$ has at least one i such that $m_i = r-1$, and if all m_j are non-negative, then $c^{1/r}(\mathbf{m}) = 0$.

This axiom seems very strange, in that it has no clear counterpart in Gromov-Witten theory, but it is a straightforward consequence of the Descent Axiom, which is satisfied by the constructions of [8] for $g = 0$.

Descent Axiom 1.9. *Let $r \ge 2$ be an integer, and $\mathbf{m} = (m_1, \dots, m_n)$ be an n-tuple of integers such that either $m_i \ge 0$ for all $i = 1, \dots, n$, or there exists an integer $1 \le j \le n$ such that $m_i \ge 0$ for all $i \ne j$ and $m_j = -1$. Let $c^{1/r}(\mathbf{m})$ denote the virtual class on $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$. Let $r\boldsymbol{\delta}_i$ denote an n-tuple of integers which is r in the ith position and zero in all others then*

$$c^{1/r}(\mathbf{m} + r\boldsymbol{\delta}_i) = -\tilde{\psi}_i(\mathbf{m})c^{1/r}(\mathbf{m})$$

An immediate result is the following proposition.

Proposition 1.10. *The Descent Axiom (1.9) implies the Vanishing Axiom (1.8).*

Proof. If $\mathbf{m}$ has all m_j non-negative and some $m_i = r-1$, then $\mathbf{m}' := \mathbf{m} - r\boldsymbol{\delta}_i$ meets the conditions for the Descent Axiom to apply, so we have

$$c^{1/r}(\mathbf{m}) = -\tilde{\psi}_i(\mathbf{m} - r\boldsymbol{\delta}_i)c^{1/r}(\mathbf{m} - r\boldsymbol{\delta}_i).$$

But by Corollary 1.7, we have

$$\tilde{\psi}_i(\mathbf{m} - r\boldsymbol{\delta}_i) = 0.$$

□

2. The Origin of Gravitational Descendants

In this section, we show that the Descent Axiom gives a geometric explanation of the usual definition of gravitational descendants in an r-spin CohFT.

Let $\mathbf{m} := (m_1, \dots, m_n)$ consist of integers m_i such that $0 \leq m_i \leq r-1$ for all $i = 1, \dots, n$. Let a_i be nonnegative integers for all $i = 1, \dots, n$. Recall that the *n-point correlators of genus g* are defined [8, 16] by the formula

$$\langle \tau_{a_1,m_1} \cdots \tau_{a_n,m_n} \rangle_g := r^{1-g} \int_{[\overline{\mathcal{M}}^{1/r}_{g,n}]} c^{1/r}(\mathbf{m}) \prod_{i=1}^{n} \psi_i^{a_i}. \tag{3}$$

These correlators assemble into *the large phase space potential function* $\Phi(\mathbf{t})$ in $\lambda^{-2}\mathbb{Q}[[\mathbf{t}, \lambda^2]]$, where $\Phi(\mathbf{t}) := \sum_{g=0}^{\infty} \Phi_g(\mathbf{t})\lambda^{2g-2}$ and

$$\Phi_g(\mathbf{t}) := \langle \exp(\mathbf{t} \cdot \boldsymbol{\tau}) \rangle_g,$$

where λ and $\mathbf{t} := \{t_a^m\}$ (for nonnegative integers a and integers $0 \leq m \leq r-1$) are formal parameters. Furthermore, we define $\mathbf{t}\cdot\boldsymbol{\tau} := \sum_{a=0}^{\infty} \sum_{m=0}^{r-1} t_a^m \tau_{a,m}$.

The *small phase space potential* is the generating function $\chi(\mathbf{x}) := \Phi(\mathbf{t})$ where $\mathbf{x} := (x^1, \dots, x^{r-1})$ and, on the right hand side, we have set $x^m := t_0^m$ and $t_a^m := 0$ for all $a > 0$.

The function $\chi(\mathbf{x})$ is a generating function for correlators which do not contain any ψ classes. Such correlators are the analogs, in the r-spin CohFT of the Gromov-Witten invariants of a smooth, projective variety. Correlators where a_i is nonzero for some i are called *the gravitational descendants*.

Unlike the case of Gromov-Witten invariants, however, the only nonzero terms in $\chi(\mathbf{x})$ come from intersection numbers on $\overline{\mathcal{M}}^{1/r}_{0,n}$, for dimensional reasons (although higher contributions are present in $\Phi(\mathbf{t})$). In some ways, the potential $\chi(\mathbf{x})$ behaves as though the r-spin CohFT corresponded to Gromov-Witten invariants of a variety with fractional dimensional cohomology classes [8, 9].

As mentioned in the introduction, the appearance of the ψ classes in the definition of the gravitational descendants seems rather mysterious from a purely geometric perspective. It is also unnatural to restrict only to intersection numbers of $c^{1/r}(\mathbf{m})$ on $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$ corresponding to n-tuples of nonnegative integers $\mathbf{m} = (m_1, \dots, m_n)$ where $m_i \leq r-1$ for all $i = 1, \dots, n$.

On the other hand, it is geometrically more natural to consider the correlators

$$\langle \tilde{\tau}_{\tilde{m}_1} \cdots \tilde{\tau}_{\tilde{m}_n} \rangle_g := r^{1-g} \int_{[\overline{\mathcal{M}}^{1/r}_{g,n}]} c^{1/r}(\widetilde{\mathbf{m}}) \tag{4}$$

where $\widetilde{\mathbf{m}} := (\tilde{m}_1, \dots, \tilde{m}_n)$ are any n-tuple of nonnegative integers. They assemble into a potential function $\tilde{\chi}(\tilde{\mathbf{t}}) := \sum_{g=0}^{\infty} \tilde{\chi}_g(\tilde{\mathbf{t}})\lambda^{2g-2}$ in $\lambda^{-2}\mathbb{Q}[[\tilde{\mathbf{t}}, \lambda^2]]$

where

$$\tilde{\chi}_g(\tilde{\mathbf{t}}) := \langle \exp(\tilde{\mathbf{t}} \cdot \tilde{\boldsymbol{\tau}}) \rangle_g.$$

Here $\tilde{\mathbf{t}} := \{\tilde{t}^{\tilde{m}}\}_{\tilde{m}\geq 0}$ and λ are formal parameters and we define $\tilde{\mathbf{t}} \cdot \tilde{\boldsymbol{\tau}} = \sum_{\tilde{m}\geq 0} \tilde{t}^{\tilde{m}} \tilde{\tau}_{\tilde{m}}$. If one sets $x^m := \tilde{t}^m$ for all $0 \leq m \leq r-1$, and if one sets $\tilde{t}^{\tilde{m}} := 0$ otherwise, then the function $\tilde{\chi}(\tilde{\mathbf{t}})$ clearly is equal to $\chi(\mathbf{x})$ where $\mathbf{x} = (x^1, \ldots, x^{r-1})$. Remarkably, the functions $\tilde{\chi}(\tilde{\mathbf{t}})$ and $\Phi(\mathbf{t})$ are the same.

Proposition 2.1. *Let $\mathbf{a} := (a_1, \ldots, a_n)$ be an n-tuple of nonnegative integers and $\mathbf{m} := (m_1, \ldots, m_n)$ be integers such that $0 \leq m_i \leq r-1$ for all $i = 1, \ldots, n$. Let $\widetilde{\mathbf{m}} = (\tilde{m}_1, \ldots, \tilde{m}_n) = r\mathbf{a} + \mathbf{m}$. The following equation holds*

$$c^{1/r}(\widetilde{\mathbf{m}}) = c^{1/r}(\mathbf{m}) \prod_{i=1}^{n} \left(\left(\frac{-\psi_i}{r} \right)^{a_i} [r(a_i - 1) + m_i + 1]_r \right)$$

where for all $0 \leq m \leq r-1$,

$$[r(a-1) + m + 1]_r := \prod_{i=1}^{a} (r(a-i) + m + 1)$$

if $a \geq 1$. If $a = 0$ then we define $[r(0-1) + m + 1]_r := 1$. Furthermore, $c^{1/r}(\widetilde{\mathbf{m}}) = 0$ if, for some $i = 1, \ldots, n$, we have that $\tilde{m}_i = a_i r - 1$ for some nonnegative a_i.

Proof. This follows from repeated application of the Descent Axiom and Equation 1. □

Corollary 2.2. *Let $\mathbf{t} := \{t_a^m\}$ where $a \geq 0$ and $0 \leq m \leq r-1$. Let $\tilde{\mathbf{t}} := \{\tilde{t}^{\tilde{m}}\}$ where $\tilde{m}$ runs over the nonnegative integers then*

$$\tilde{\chi}(\tilde{\mathbf{t}}) = \Phi(\mathbf{t})$$

under the identification

$$\tilde{t}^{ar+m} = \frac{(-1)^a r^a}{[r(a-1) + m + 1]_r} t_a^m$$

where a and m are integers such that $a \geq 0$ and $0 \leq m \leq r-1$.

Remark 2.3. This Corollary suggests that the proper geometric definition of gravitational descendants should be given by Equation 4 rather than the usual definition given in Equation 3.

It is also amusing to note that the equations of the KdV$_r$ (r-th Gelfand-Dickey) hierarchy [16, 8] simplify somewhat when written in terms of the variables $\tilde{t}^{ar+m}$ instead of the t_a^m. The KdV$_r$ equations are, for all nonnegative integers a and m,

$$i\sqrt{r} \left(a + \frac{m+1}{r} \right) \frac{\partial Q}{\partial \tilde{t}^{ar+m}} = [Q_+^{a+\frac{m+1}{r}}, Q]$$

where

$$Q := D^r - \sum_{m=0}^{r-2} u_m(x) D^m$$

and $D = \frac{i}{\sqrt{r}} \frac{\partial}{\partial x}$.

Whereas, the usual equations, written in the t_a^m, are

$$i[ar+m+1]_r \frac{\partial Q}{\partial t_a^m} = (-1)^a r^{a+(1/2)} [Q_+^{a+\frac{m+1}{r}}, Q].$$

3. The Convex Case

In this section, we prove that the descent property of the r-spin virtual class is satisfied in the convex case.

Recall that an r-spin structure on a family of curves $\pi : X \to T$ is said to be *convex* if the r-th root sheaf $\mathcal{E}_r$ satisfies

$$\pi_* \mathcal{E}_r = 0.$$

In particular, on a family of irreducible curves of genus g with n marked points, an r-spin structure of type $\mathbf{m}$ with $|\mathbf{m}| := \sum_{i=1}^n m_i > 2g-2$ is always convex since $\mathcal{E}_r$ in this case has negative degree on each fiber X_t. (This is not true for families with reducible curves because they may have components of positive genus on which $|\mathbf{m}|$ is too small.)

Theorem 3.1. *If the universal r-spin structure on a (connected) component of $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ is convex then the Descent Axiom (1.9) holds.*

Proof. The convexity axiom of an r-spin virtual class (cf. [8]) gives that

$$c^{1/r}(\mathbf{m}) = c_D(R\pi_*\mathcal{E}_r) = (-1)^D c_D(R^1\pi_*\mathcal{E}_r),$$

where

$$D = \frac{1}{r}\left((r-2)(g-1) + \sum_{i=1}^n m_i\right),$$

and $\mathcal{E}_r$ is the r-th root sheaf of the universal r-spin structure.

The r-th root sheaf $\mathcal{E}'_r$ of the universal r-spin structure of type $\mathbf{m}' = \mathbf{m} + r\boldsymbol{\delta}_i$ is isomorphic to $\mathcal{E}_r(-p_i) = \mathcal{E}_r \otimes \mathcal{O}(-p_i)$ under the identification

$$\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}'} = \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}.$$

Since $\mathcal{E}'_r$ is a subsheaf of $\mathcal{E}_r$, it is also convex and

$$D' = \operatorname{rk} R^1\pi_*\mathcal{E}'_r = 1 + \operatorname{rk} R^1\pi_*\mathcal{E}_r = D + 1.$$

Therefore,

$$\begin{aligned} c^{1/r}(\mathbf{m} + r\boldsymbol{\delta}_i) &= (-1)^{D'} c_{D'}(R^1\pi_*\mathcal{E}_r(-p_i)) \\ &= -(-1)^D c_D(R^1\pi_*\mathcal{E}_r) c_1(p_i^*\mathcal{E}_r) = -c^{1/r}(\mathbf{m})\tilde{\psi}_i(\mathbf{m}), \end{aligned}$$

where, in the second equality, we have used the following simple lemma. □

Lemma 3.2. *Let $\pi : X \to T$ be a flat family of curves, $q : T \to X$ be a section of π, and $\mathcal{E}$ be a coherent sheaf on X. Assume that the image $q(T)$ of q is disjoint from the singular locus of π and the restriction of $\mathcal{E}$ to $q(T)$ is locally free of rank one.*

If $\pi_*\mathcal{E} = 0$ then

$$c_{top}(R^1\pi_*\mathcal{E}') = c_1(q^*\mathcal{E})c_{top}(R^1\pi_*\mathcal{E}),$$

where $\mathcal{E}' = \mathcal{E}(-q)$ and c_{top} denotes the top Chern class of a vector bundle.

Proof. Consider the exact sequence of sheaves on X

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}|_q \to 0 \ .$$

Since $\pi_*\mathcal{E} = 0$ and $R^1\pi_*(\mathcal{E}|_q) = 0$, the corresponding long exact sequence for the functor $R\pi_*$ gives a short exact sequence of bundles on T

$$0 \to q^*\mathcal{E} \to R^1\pi_*\mathcal{E}' \to R^1\pi_*\mathcal{E} \to 0.$$

Now the statement of the lemma follows from multiplicativity of the total Chern class. □

In [8, Proposition 4.4] we proved that in the genus zero case, the universal r-spin structure is convex for all r and $\mathbf{m}$, such that $m_i \geq -1$ for all i and at most one m_i equal to -1. This yields the following.

Corollary 3.3. *The Descent Axiom (1.9) holds in the case $g = 0$.*

Remark 3.4. In [15], a class c on $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ was constructed in the cases where $\mathbf{m} = (m_1, \ldots, m_n)$ has all m_i nonnegative. The class c was shown to have the correct dimension and some of the axioms of [8] for a spin virtual class were verified. Furthermore, c was shown to satisfy the Descent Axiom whenever it is defined. However, this does not imply the Vanishing Axiom, since the class c is not defined when any m_i is negative.

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Department of Mathematics, Brigham Young University, Provo, UT 84602, USA
E-mail address: `jarvis@math.byu.edu`

Department of Mathematics, 111 Cummington Street, Boston University, Boston, MA 02215, USA
E-mail address: `kimura@math.bu.edu`

Department of Mathematics, University of Oregon, Eugene, OR 97403, USA
E-mail address: `vaintrob@math.uoregon.edu`