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STATISTICAL PROPERTIES OF THOMPSON'S GROUP AND RANDOM PSEUDO MANIFOLDS

by

Benjamin M. Woodruff

A dissertation submitted to the faculty of

Brigham Young University

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics

Brigham Young University

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BRIGHAM YOUNG UNIVERSITY

GRADUATE COMMITTEE APPROVAL

of a dissertation submitted by

Benjamin M. Woodruff

This dissertation has been read by each member of the following graduate committee and by majority vote has been found to be satisfactory.

Date	James W. Cannon, Chair
Date	David Grant Wright
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As chair of the candidate's graduate committee, I have read the dissertation of Benjamin M. Woodruff in its final form and have found that (1) its format, citations, and bibliographical style are consistent and acceptable and fulfill university and department style requirements; (2) its illustrative materials including figures, tables, and charts are in place; and (3) the final manuscript is satisfactory to the graduate committee and is ready for submission to the university library.

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ABSTRACT

STATISTICAL PROPERTIES OF THOMPSON'S GROUP
AND RANDOM PSEUDO MANIFOLDS

Benjamin M. Woodruff
Department of Mathematics
Doctor of Philosophy

The first part of our work is a statistical and geometric study of properties of Thompson's Group F. We enumerate the number of elements of F which are represented by a reduced pair of n-caret trees, and give asymptotic estimates. We also discuss the effects on word length and number of carets of right multiplication by a standard generator x0 or x1. We enumerate the average number of carets along the left edge of an n-caret tree, and use an Euler transformation to make some conjectures relating to right multiplication by a generator. We describe a computer algorithm which produces Fordham's Table, and discuss using the computer algorithm to find a corresponding Fordham's Table for different generating sets for F. We expound upon the work of Cleary and Taback by completely classifying dead end elements

of Thompson's group, and use the classification to discuss the spread of dead end elements and describe interesting elements we call deep roots. We discuss how deep roots may aid in answering the amenability problem for Thompson's group.

The second part of our work deals with random facet pairings of simplexes. We show that a random endpoint pairings of segments most often results in a disconnected one-manifold, and relate this to a game called "The Human Knot." When the dimension of the simplexes is greater than 1, however, a random facet pairing most often results in a connected pseudo manifold. This result can be stated in terms of graph theory as follows. Most regular multi graphs are connected, as long as the common valence is at least three.

ACKNOWLEDGMENTS

I would like to thank my advisor Jim Cannon for the many hours he has spent helping me with this work, as well as for his encouragement and support. In addition I would like to thank Luke Henderson, Sharleen De Gaston, and John Bankhead for the time they spent over the summer of 2003 as this project began. The ideas they brought helped to fuel some of the results contained herein. I will try to make mention of the results which stemmed from their work.

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Part I

Statistical Properties of Thompson's Group

In 1965, Thompson's group F was used to find the first example of a finitely presented infinite simple group. Since then, F has appeared in other branches of mathematics, as well as computer science. In particular F is one of the simplest non-trivial examples of a diagram group (see [9]) and it appears almost anywhere there are group actions on binary trees. Thompson's group appears in homotopy theory and splitting homotopy idempotents. Other places where Thompson's group has been used include: proper isometric actions on Hilbert space, symmetries of the modular tower of genus zero real stable curves, the first example of an FP_∞ finitely generated group, and the list goes on. For many years, mathematicians have been trying to solve the amenability problem for Thompson's group F. The history of amenability dates back to the trying to find an ideal measure. The Hahn-Banach Theorem was invented to prove that Abelian groups are amenable. The Banach-Tarski paradox is directly related to amenability. Thompson's group is arguably the most famous group for which the amenability problem is still a question. Answering the amenability problem for Thompson's group should provide new insights into the study of amenability and the geometrical structure of F. Our work is a statistical and geometric study of properties of Thompson's Group F. Some of these properties may help in understanding the geometry of Thompson's group as it relates to amenability.

In this dissertation we prove the following:

- The number of elements $N_n$of Thompson's Group F represented by a reduced n-caret tree pair is (Theorem 2.8)

$$N_n = \sum_{k=1}^{\lceil n/2 \rceil} \sum_{i=0}^{k} 2^{n-2k+1} \binom{n-1}{n-2k+1} C_{n-k} (-1)^i \binom{k}{i} C_{n-i}$$

where C_n denotes the number of n-caret trees. This gives a type of growth function for F, and aids in understanding the geometric growth of F as it relates to caret tree pairs.

- The growth function N_n is bounded above by $(8+4\sqrt{3})^n \approx (14.93)^n$ (Theorem 2.18). This bounds the exponential growth rate of N_n, and shows that most random n-caret tree pairs are not reduced (Corollary 2.19). We conjecture an asymptotic estimate of $(8+4\sqrt{3})^n/n^3$ (Conjecture 2.22).

- We enumerate the number of n-caret trees that have a specified set of exposed carets (Theorem 2.30). The counting scheme is highly dependent on the positions of the exposed carets. Surprisingly, Theorems 2.7 and 2.28 show that the number of n-caret trees that can be paired with a given n-caret tree and result in a reduced tree pair depends only on the number n of carets and the number k of exposed carets. This fact is central to enumerating N_n.

- The average number of L_L's in an n-caret tree approaches two as $n \to \infty$ (Theorem 2.31 joint with Sharleen De Gaston). This is directly related to multiplication by the generator x_0 of Thompson's Group F, as the number of L_L's is precisely the number of times we may multiply by x_0 without increasing the number of carets in a reduced tree pair.

- We give a set of conditions on tree pairs which completely characterizes dead end elements in Thompson's Group F (Theorem 4.3). This expounds upon the work of Cleary and Taback in [5], by providing a converse to Theorem 4.1 in their paper.

In addition, we clarify and discuss the following:

- The classification of dead end elements allows us to discuss the spread of dead end elements, as well as define equivalence classes and minimal dead end elements (Section 4.2, in particular Lemma 4.4 and Corollary 4.8). In addition, we describe elements we call k-deep roots (Definition 4.11) which have the geometric property that along a path of length k toward the origin, multiplication by all but one generator reduces word length. Deep roots give an extension of the dead end phenomenon by providing a path of elements where the only way to leave the path is to first reduce word length. We also use the characterization of dead ends to define equivalence classes and minimal deep roots. This is further discussed in Section 4.3.

- We wrote a computer algorithm (Section 3.2) to generate Fordham's Table (Table 5), with the intent to use this algorithm to find a corresponding table using a different generating set for F. Two features of the standard two generator presentation which make it ideal for study are the existence of normal forms and Fordham's Table. The generating set we explored includes an additional generator y_1 which is the mirror image of x_1. We give a set of normal forms using this new generating set. We discuss attempts at finding a corresponding Fordham's Table in 3.3. Using this generating set allows one to immediately see a $\mathbb{Z} \times \mathbb{Z}$ in the Cayley graph of F at every element of F. In addition, the presentation for F becomes shorter with this new generator, and, letting $y_0 = x_0^{-1}$, interchanging x and y is equivalent to mirror imaging the domain and range trees of a reduced tree pair.

1 Introduction

This section is intended to give the background needed to follow our work. We will assume some familiarity with Cayley graphs. The reader is encouraged to see [3] for an introduction to Cayley graphs.

In section 1.1 we review the definition of Thompson's group F and illustrate how F is equivalent to the set of reduced ordered pairs of rooted ordered binary n-caret trees. We introduce stacked tree pairs to discuss reduction. We also illustrate multiplication in F, and motivate drawing tree pairs from right to left.

In section 1.2 we introduce the generators $x_0, x_1, \ldots$, and give two standard presentations for F. We also discuss a convenient set of normal forms.

In section 1.3 we define amenability for a finitely presented group and illustrate the definition by showing that $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ are amenable. Using a token passing argument, we show that $\mathbb{Z} * \mathbb{Z}$ is nonamenable. We conclude by discussing the amenability problem for Thompson's Group F and its relation to the token passing argument.

1.1 Thompson's Group

Richard Thompson first defined Thompson's group F in 1965. He used F to construct the first example of a finitely presented infinite simple group. Often F is defined to be the set of automorphisms f of $[0, 1]$ which satisfy the following.

1. f is piecewise linear (finiteness is automatically implied).

2. $f(0) = 0$ and $f(1) = 1$ (f is said to be orientation preserving).

3. If f is differentiable at x, then $f'(x) = 2^k$ for some integer k.

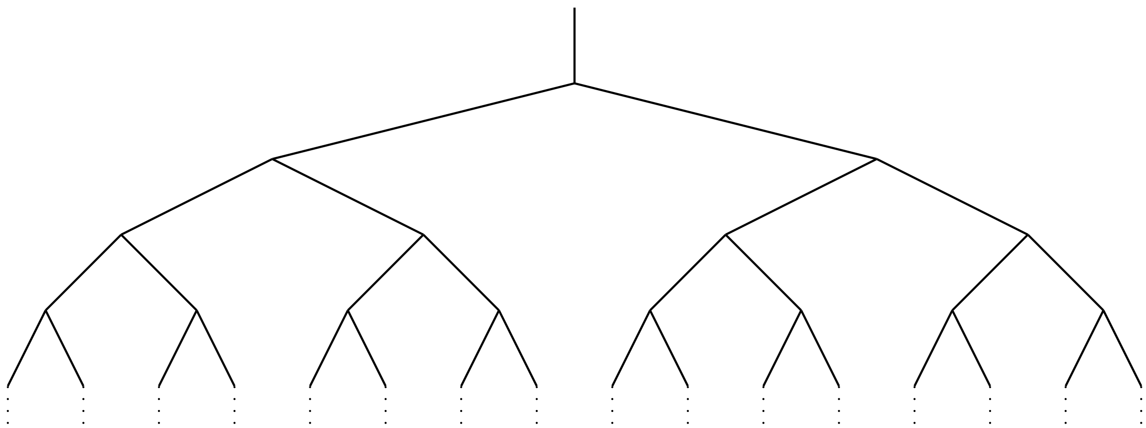

Figure 1: The infinite binary tree T

4. If f is not differentiable at x, then x is a dyadic rational, i.e., $x \in \mathbb{Z}[\frac{1}{2}] = \{\frac{p}{2^q} | p, q \in \mathbb{Z}\}$.

This description makes F a group with function composition as multiplication. Checking associativity is a trivial consequence of the associativity of function composition. Since Thompson's original definition, various alternative methods of defining F have been provided in the literature. For example, F can be described using pairs of caret trees in [4], as a diagram group in [9], or using forest diagrams in [1]. We introduce a new description using "tree functions" (section 2.3) which is particularly suited for comparing consecutive carets in the caret tree pair description. While all the descriptions are essentially equivalent, each method of defining F has its own advantages. An excellent detailed introduction to Thompson's group F using caret tree pairs is given in [4]. In this section we provide some introductory details (many without proof), particularly when our development differs from that of [4].

A rooted ordered n-caret binary tree T_0 (henceforth called a caret tree, where n will be specified if needed) is a connected subgraph of the infinite rooted ordered binary tree T (see Figure 1), where there are n interior vertices of degree 3, and

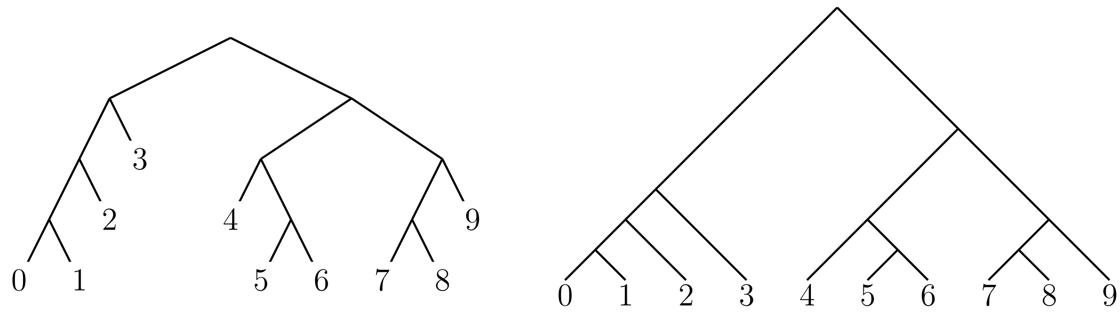

Figure 2: Number the leaves according to the infix ordering.

$n+2$ vertices of degree one (called pendant vertices). The pendant vertex at the top is sometimes omitted. The other $n+1$ pendant vertices are called the leaves of T_0 (sometimes we refer to the vertex as well as its edge as a leaf). If we extend the edges of T_0 so that all the leaves lie on a horizontal line, we get a natural numbering of the leaves from left to right, starting with zero (see Figure 2). This numbering scheme is referred to as the infix ordering. Alternatively, we walk counter clockwise around the outside of the tree, and number the leaves as we pass them.

An interior vertex, together with the two edges leading away from the root, is called a caret. There is a similar natural numbering of the carets (or interior vertices). Again after extending the edges of T_0 so that all the leaves lie on a horizontal line, we number the gaps between adjacent leaves. Each gap between leaves can be followed upwards till the gap is capped off by an interior vertex. We number the gaps (and hence interior vertices or carets) from left to right, starting at 1 (see Figure 3). Note that in [8], the numbering of carets begins with zero just as the leaves. Our convention differs so that both leaves and carets end with the number n.

An n-caret tree T_0 gives rise to a partition of $[0,1]$ into $n+1$ intervals of the form $[\frac{p_1}{2^{q_1}}, \frac{p_2}{2^{q_2}}]$. We use the word partition loosely, as the endpoints of neighboring intervals coincide. To see how this partitioning is accomplished, we first associate

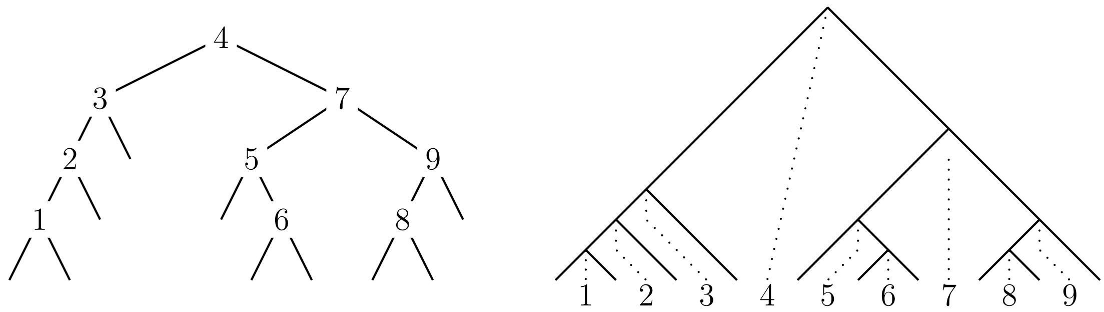

Figure 3: Number the carets using the gaps between leaves.

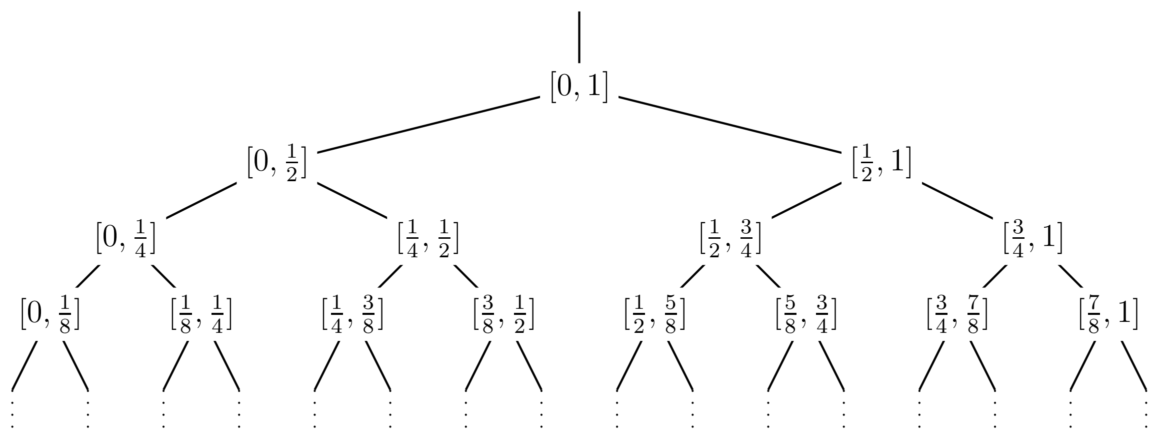

Figure 4: The infinite binary tree T subdivides $[0, 1]$

with every vertex of the infinite tree T a label. We label the root $[0, 1]$, and then for a vertex labeled $[a, b]$, we label the left child $[a, \frac{a+b}{2}]$ and the right child $[\frac{a+b}{2}, b]$ (see Figure 4). As T_0 is a subgraph of T, we take as our partition the intervals A_i, where A_i is the label on leaf i for $i \in \{0, \ldots, n\}$ (see Figure 5).

If (D, R) is an ordered pair of n-caret trees (where n is the same for both trees), we let A_i and B_i denote the partitions of $[0, 1]$ associated with D and R, respectively. We get a function $f : [0, 1] \to [0, 1]$ by mapping A_i to B_i affinely, preserving orientation (see Figures 6 and 7). It follows that f is an element of Thompson's group F, and that every $g \in F$ can be obtained in this manner. We get a map φ from the set of ordered caret tree pairs to F. As shown in Figure 7, φ is not a bijection, so we say that (D_1, R_1) and (D_2, R_2) are equivalent if their image

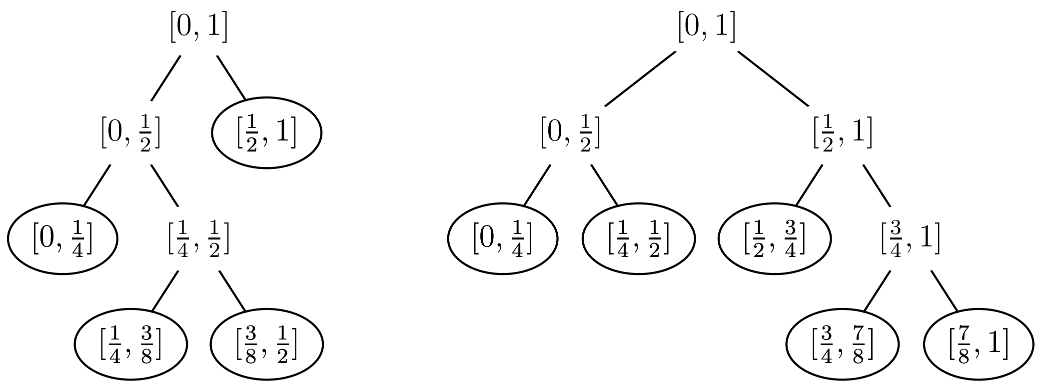

Figure 5: Partitions of $[0,1]$. The 3-caret tree gives the partition $A_0 = [0, \frac{1}{4}], A_1 = [\frac{1}{4}, \frac{3}{8}], A_2 = [\frac{3}{8}, \frac{1}{2}], A_3 = [\frac{1}{2}, 1]$. The four caret tree gives the partition $A_0 = [0, \frac{1}{4}], A_1 = [\frac{1}{4}, \frac{1}{2}], A_2 = [\frac{1}{2}, \frac{3}{4}], A_3 = [\frac{3}{4}, \frac{7}{8}], A_4 = [\frac{7}{8}, 1]$.

is the same under φ. This equivalence relation gives the following standard result.

Theorem 1.1. *The set of equivalence classes of ordered pairs of caret trees is in bijective correspondence with the elements of Thompson's group F.*

We will say that an ordered pair of n-caret trees is reduced if every equivalent caret tree pair has at least n-carets. The preceding theorem is then strengthened by the following theorem.

Theorem 1.2. *Every equivalence class of ordered caret tree pairs contains a unique reduced ordered caret tree pair.*

It is worth describing a process for finding the reduced ordered pair of caret trees in any equivalence class. To describe the reduction process, we consider the "stacked tree" representation (or similarly the forest diagram description in [1]). For the ordered pair (D, R), sometimes written $R \leftarrow D$ for reasons explained shortly, we form the stacked tree diagram by extending the leaves of both trees to the same horizontal line, and then placing the mirror image of D under R (see Figure 8). Any diamond (i.e, opposing pair of carets) where the stacked tree pair meets is of

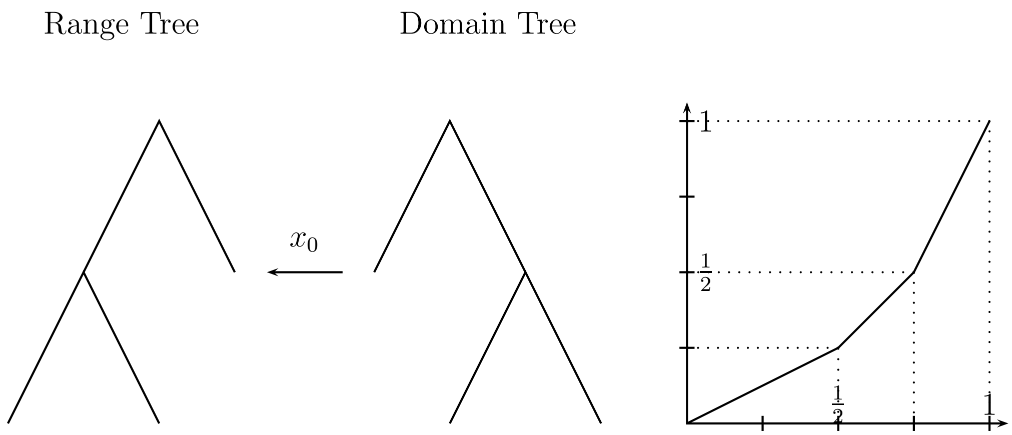

Figure 6: The tree pair on the left gives the element of Thompson's group F on the right, using the partition of $[0, 1]$ from the right tree for the domain. This element is called x_0.

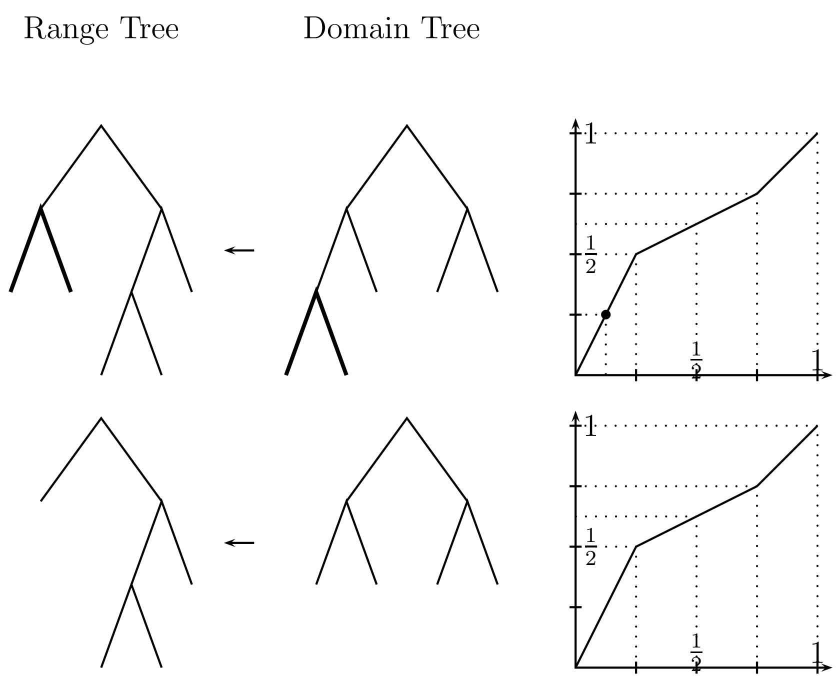

Figure 7: These tree pairs represent the same element of Thompson's group. Removing the first caret from both the domain and range trees of the top pair, gives the bottom pair. The effect of removing the "extra" caret is illustrated in the graphs on the right.

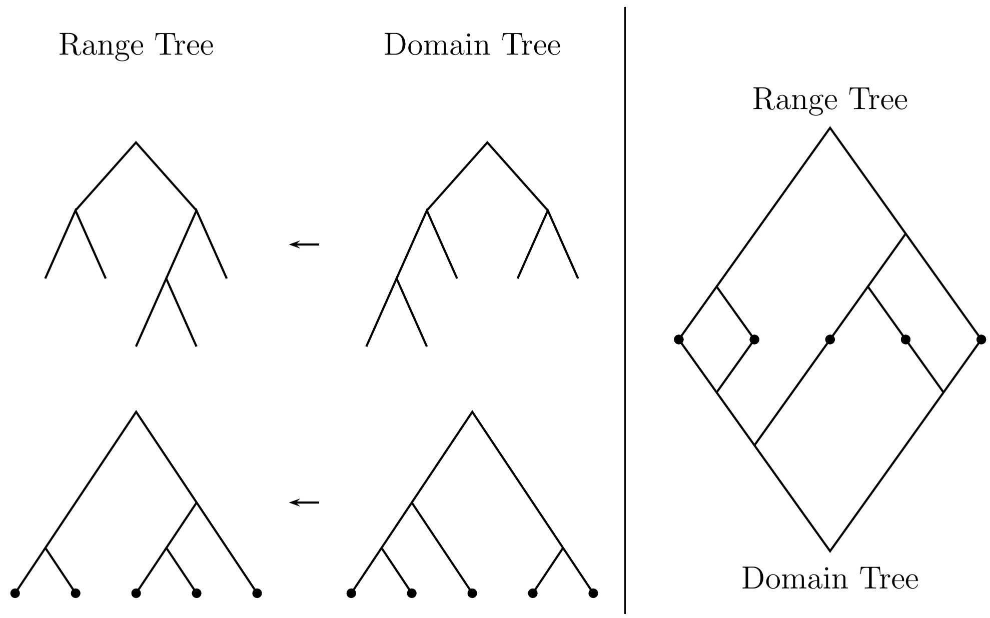

Figure 8: The stacked tree diagram is formed by first extending the leaves to the same horizontal line. Then the domain tree is stacked underneath the range tree.

particular interest.

Theorem 1.3. *An ordered pair of caret trees* (D, R) *is reduced if and only if there are no diamonds in its stacked tree representation. The reduced pair of caret trees in the equivalence class of* (D, R) *is found by collapsing any diamond to a point, and repeating the collapsing until the resultant pair* (D', R') *contains no diamonds in its stacked tree representation.*

Figures 9 and 10 show the reduction process for a few ordered tree pairs. Figure 9 shows the reduction process for the equivalent pairs shown in Figure 7. Removing a diamond does not change the function associated with the original ordered tree pair. Note that we can insert diamonds into a stacked tree diagram, without leaving the equivalence class of ordered tree pairs. It will be useful later on to be able to insert diamonds as needed.

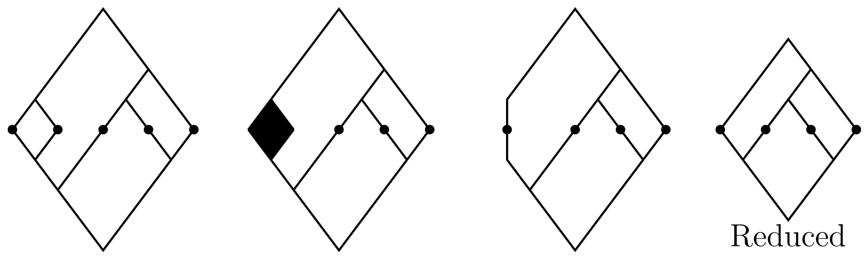

Figure 9: A Simple Reduction. We remove diamonds from a stacked tree diagram to obtain the reduced tree pair.

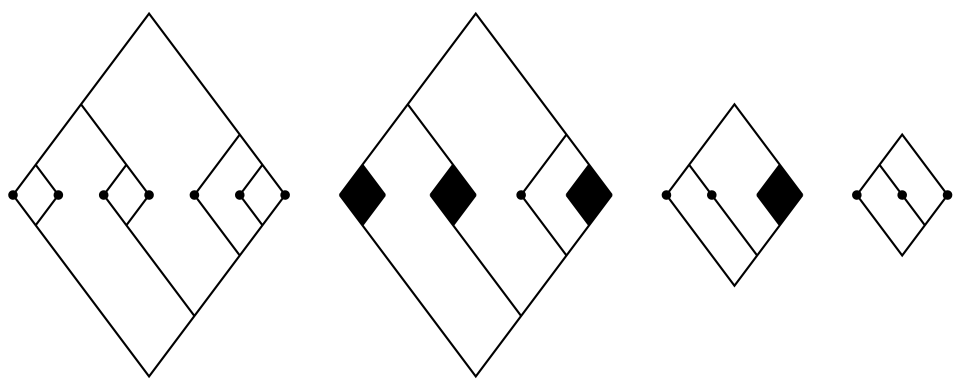

Figure 10: A More Complex Reduction. We successively remove diamonds from the stacked tree diagrams to obtain the reduced tree pair.

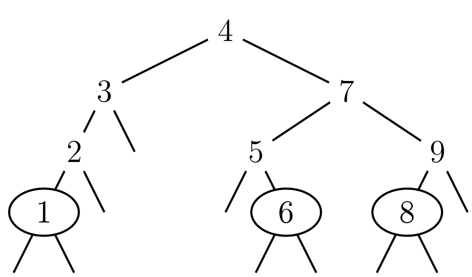

Figure 11: The exposed carets are 1, 6, and 8, since they have no carets underneath them.

The preceding process gives a visual way of recognizing reduced pairs of caret trees, yet is time consuming to draw. Given a tree T_0, we say caret k is exposed (or free) if both vertices below it are leaves. If we think of a caret as the parent of the carets below it, then a caret is exposed if it has no children (see Figure 11). Theorem 1.3 is rewritten in this form.

Theorem 1.4. *Suppose (D, R) is an (ordered) pair of n-caret trees, where the sets of indices of exposed carets of D and R are $I = \{i_1, i_2, \ldots, i_s\}$ and $J = \{j_1, j_2, \ldots, j_t\}$, respectively. Then (D, R) is reduced if and only if I and J are disjoint. In other words, (D, R) is reduced if and only if, for each $k \in \{0, \ldots, n\}$, caret k is exposed in D implies caret k is not exposed in R. The reduced pair of caret trees in the equivalence class of (D, R) is found by removing caret i from both D and R if caret i is exposed in both trees, and continuing to remove carets until the resultant pair is reduced. See Figure 12.*

Every equivalence class is associated with an integer n, given by the number of carets in either tree of the reduced tree pair. The empty pair $(\emptyset, \emptyset)$ corresponds to the identity element of Thompson's group F.

We now describe multiplication in Thompson's Group F. The bijection between elements of F and reduced pairs of caret trees induces a group structure on the set of reduced pairs of caret trees. To describe the induced multiplication, recall that a pair of caret trees (D, R) gives a function f defined on the partitions associated to D and R. The product of two caret tree pairs (D, R) and (R, T) is easily seen to be the tree pair (D, T) (where reduction may be required). The range partition of (D, R) is the same as the domain partition of (R, T), hence the composition of the functions f and g associated with (D, R) and (R, T), respectively, is formed by bypassing the common partition. It is convenient to write a pair (D, R) in the form

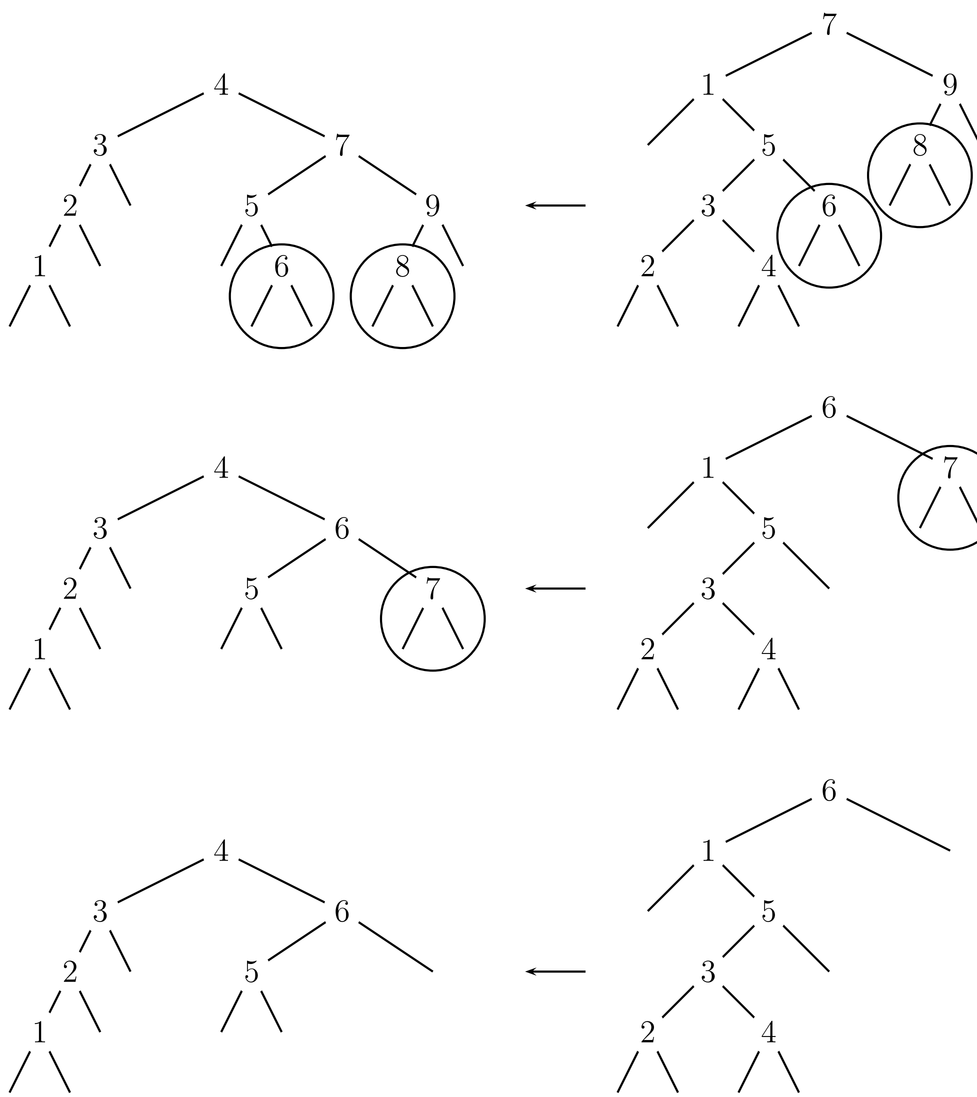

Figure 12: The sets of exposed carets are $\{2, 4, 6, 8\}$ in the domain tree, and $\{1, 6, 8\}$ in the range tree. After removing carets 6 and 8 from both trees (and renumbering), caret 7 is exposed in both trees. The final tree pair is reduced.

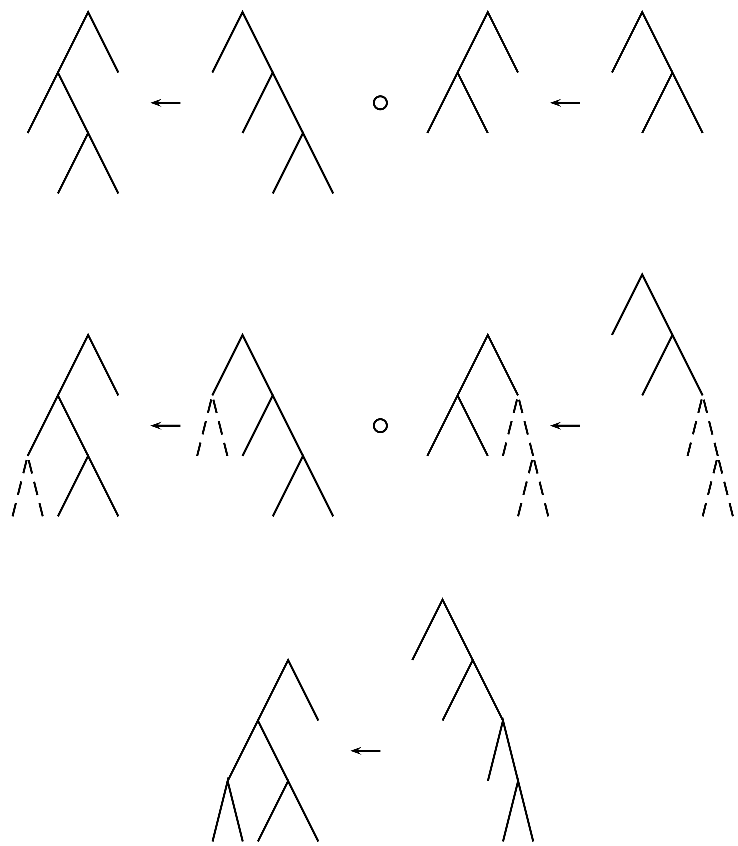

Figure 13: To multiply trees, we first equate the middle trees. The product is then formed by bypassing the common middle trees. In this case, the product was reduced, however, in general we may have to reduce the final pair.

$R \leftarrow D$. The reason for this notation is that the product of two pairs of caret trees $R_1 \leftarrow D_1$ and $R_2 \leftarrow D_2$ (when $D_1 = R_2$), is then formed by erasing the common middle, i.e.,

$$(R_1 \leftarrow D_1) \circ (R_2 \leftarrow D_2) = R_1 \leftarrow D_2.$$

The notation is designed to mimic function composition, where the composite of $f : Y \to Z$ and $g : X \to Y$ is a map $f \circ g : X \to Z$ which passes through the common middle space.

For two pairs of caret trees $R_1 \leftarrow D_1$ and $R_2 \leftarrow D_2$ where $D_1 \neq R_2$, we choose, in each equivalence class, non-reduced pairs $R_1' \leftarrow D_1'$ and $R_2' \leftarrow D_2'$, respectively, where $D_1' = R_2'$. This can be done by repeatedly inserting diamonds into each pair (i.e., hanging extra carets from the same leaf of a pair) until the middle trees are the same. This is always possible since any two finite partitions may be refined to be the same partition. The product is then

$$(R_1 \leftarrow D_1) \circ (R_2 \leftarrow D_2) = (R_1' \leftarrow D_1') \circ (R_2' \leftarrow D_2') = R_1' \leftarrow D_2'.$$

The final pair $R_1' \leftarrow D_2'$ represents an equivalence class, and so after reduction gives the desired reduced pair of n-caret trees. The choice of which non-reduced elements to pick can be done by inserting extra carets to D_1 and R_2 until they match up, and then repeating the insertion of carets to R_1 and D_2 so that the resultant tree pairs represent the correct equivalence classes. The main idea is that if we can't compose tree pairs (functions) because the domain and range don't match up, then we change the domain and range until they do match up. The product of two tree pairs is then the tree pair which bypasses the common middle pair (see Figure 13 for an example).

It is worth emphasizing that the tree pair $R_1' \leftarrow D_2'$ may not be reduced. Later it will be important to know when the multiplication results in a reduced pair. Notice that the product of $R \leftarrow D$ and $D \leftarrow R$ is just $R \leftarrow R$ which reduces to $\emptyset \leftarrow \emptyset$. This shows that the inverse of a tree pair is found by turning the arrow around.

1.2 Presentations and Normal Forms

We now give two presentations for Thompson's group F, as well as describe a convenient set of normal forms.

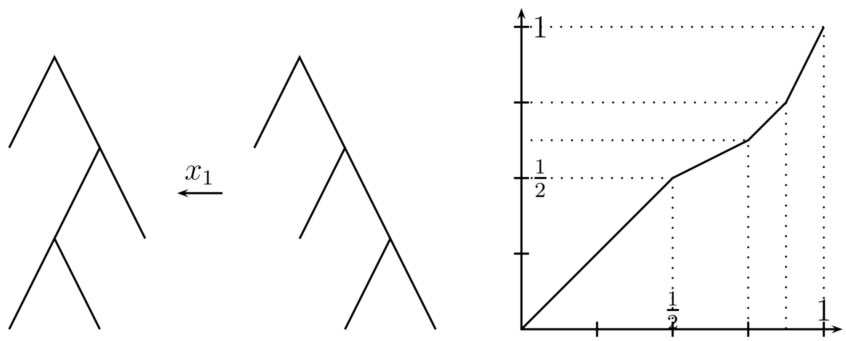

Figure 14: The generator x_1.

We described Thompson's group F as the set of reduced pairs of caret trees, with multiplication described in a manner similar to function composition. Two special elements x_0 and x_1 (also referred to as a and b or A and B) will be central the remainder of our work (see Figures 6 and 14). Given an element $w = (D, R) \in F$, the products wx_0 and wx_1 are formed by rotations described in Figure 15. We define the elements $x_{k+1} = x_0^{-k} x_1 x_0^k$ for $k \geq 1$, and similar computations show that wx_k is a clockwise rotation at the kth caret along the right side of the domain tree D in w, provided there is a left child of the the kth caret along the right side of D. The inverse of each x_k acts on w by a counterclockwise rotation at the kth caret along the right side of the domain tree D of w, again provided there is a right child of the kth caret along the right side.

Thompson's group F has two standard presentations in the literature,

$$F_\infty = \left\langle x_0, x_1, x_2, \ldots | x_{k+1} = x_i^{-1} x_k x_i \text{ for } 0 \leq i < k < \infty \right\rangle$$

$$F_2 = \left\langle x_0, x_1 | x_1^{-1} x_2 x_1 = x_3, x_1^{-1} x_3 x_1 = x_4 \right\rangle$$

(recall that x_k is defined in terms of x_0 and x_1). The infinite presentation says that geometrically F is generated by all rotations along the right side of the domain tree. The relators say that a clockwise rotation at depth $k+1$ is accomplished in three steps. First, rotate counter-clockwise at depth $i \leq k-1$ (which moves the $(k+1)$st

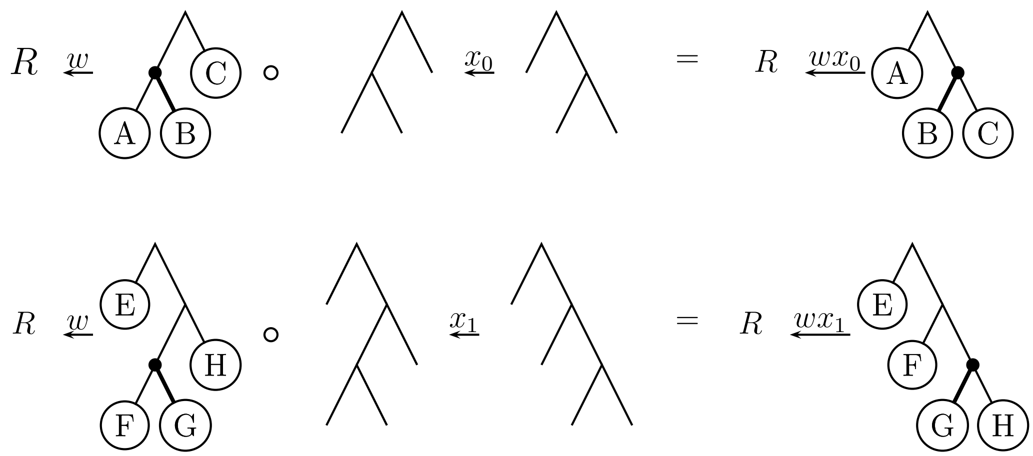

Figure 15: The rotational effects of applying x_0 and x_1. Capital letters represent (possibly empty) subtrees. For the pair $w = (D, R)$, where the left child D is not a leaf, the product wx_0 is formed by rotating clockwise the edge above subtree B so that it is on the right side of the root. The product wx_1 is formed similarly by rotating the edge above subtree G. The rotated edges are marked by thicker lines.

right side caret to the kth right side caret). Second, rotate clockwise at depth k. Third, undo the counter clockwise rotation at depth i by a clockwise rotation at depth i. The beauty of this presentation is that there are no additional relators needed other than these obvious ones. By definition of x_k, F is generated by x_0 and x_1. Just two of the relators generate all the other relators. The finite presentation allows us to form a Cayley graph, discuss the growth function of F, and examine various other group properties of F with respect to this two generator presentation. More details can be found in [4] and [5].

There are a convenient set of normal forms for elements of Thompson's group F, when using the infinite presentation. Any element of Thompson's group has a unique representation of the form

$$x_{i_1}^{r_1} x_{i_2}^{r_2} \cdots x_{i_k}^{r_k} x_{j_l}^{-s_l} \cdots x_{j_2}^{-s_2} x_{j_1}^{-s_1},$$

where $r_i, s_i > 0$, $i_1 < i_2 < \cdots i_k, j_1 < j_2 < \cdots j_l$. To get uniqueness, we require that when both x_i and x_i^{-1} appear, so does x_{i+1} or x_{i+1}^{-1}.

The exponents $r_1, r_2, \ldots, r_k, -s_1, -s_2, \ldots, -s_j$ are easily computed from a reduced tree pair, by computing what are called the exponents of the leaves. At each leaf, construct the longest possible upwards path to the origin made entirely of left edges, excluding left edges which reach the rightmost side of the tree. The number of edges in this path is called the exponent of that leaf. If w is represented as the reduced n-caret tree pair (D, R), then r_p is the exponent of the i_pth leaf of R, and s_q is the exponent of the j_qth leaf of D. Many examples and calculations are given in [4]. In section 3.3, we will introduce a new set of normal forms for a different generating set.

1.3 Amenability

In this section we review a definition of amenability for finitely presented groups. Using this definition, we show $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ are amenable and use a token passing argument to show that $\mathbb{Z} * \mathbb{Z}$ is non-amenable. We then discuss the amenability problem for Thompson's group F as well as describe our goals for answering the amenability problem for F.

Let G be a group. Let $G = \langle C | R \rangle$ be a finite presentation for G, with C symmetric, which means that if $g \in C$, then $g^{-1} \in C$. Let $\Gamma(G, C) = \Gamma$ be the Cayley graph of G, with respect to the generating set C. Let $S \subset G$ be a subset of G. We define the border of S to be $Bord(S, C) = \{x \in G - S | xg \in S \text{ for some } g \in C\}$. That is, $Bord(S, C)$ is the set of all elements of G which are distance one from S with respect to the generating set C. For finite S, the ratio $|Bord(S, C)|/|S|$ (where

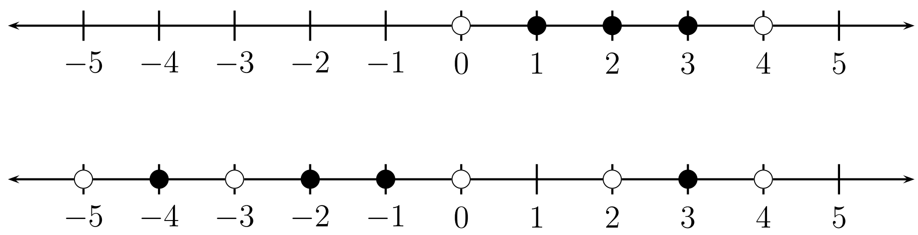

Figure 16: The Cayley graph of $\mathbb{Z}$ can be represented as a number line, where group elements correspond to the integers. Closed dots represent the set S_3 in the top line, and the set $\{-4,-2,-1,3\}$ in the bottom line. Open dots represent the border of each set.

$|X|$ denotes the cardinality of X) compares the size of S with its border. Let

$$\lambda(G,C) = \inf_{S\subset G, |S|<\infty} \frac{|Bord(S,C)|}{|S|}.$$

If $\lambda(G,C)=0$, then G is said to be amenable. If $\lambda(G,C)>0$, then G is said to be non-amenable. Note that our definition of amenability depends on the choice of generators for G. It can be shown that if $\lambda(G,C)=0$ for some finite set of generators C, then $\lambda(G,C')=0$ for any finite set of generators C' (see [7]). There is an equivalent analytic definition of amenability. The definition we gave only works for finitely presented groups. The following examples illustrate the calculations, and motivate some of our further work.

Consider the integers $\mathbb{Z}\cong\langle x|\rangle$. Let $S_n=\{1,\dots,n\}$. Then $Bord(S_n)=\{0,n+1\}$. Hence

$$\lambda(\mathbb{Z},\{x\}) = \inf_{S\subset G, |S|<\infty} \frac{|Bord(S,\{x\})|}{|S|} \le \lim_{n\to\infty}\frac{2}{n}=0,$$

so $\mathbb{Z}$ is amenable. Figure 16 shows the Cayley graph of $\mathbb{Z}$, as well as illustrates the border of various sets.

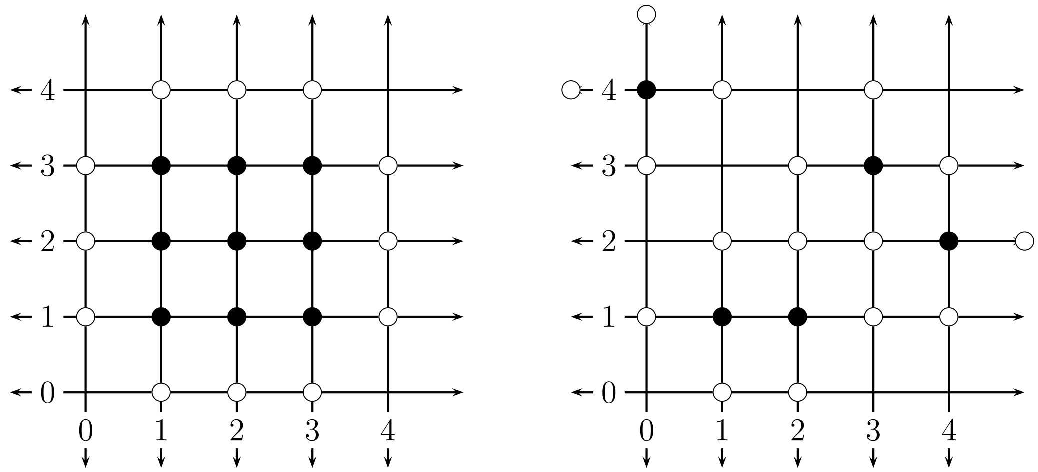

Figure 17: The Cayley graph of $\mathbb{Z} \times \mathbb{Z}$ is represented above, where group elements correspond to the integer lattice. Closed dots represent the set S_3 on the left and the set $\{(0,4),(1,1),(2,1),(3,3),(4,2)\}$ on the right. Open dots represent the border of each set.

Now consider the group $\mathbb{Z} \times \mathbb{Z} \cong \langle x, y | x^{-1}y^{-1}xy \rangle$, the integer lattice in the plane (see Figure 17). Let $S_n = \{1, \ldots, n\} \times \{1, \ldots, n\}$. Then $Bord(S_n) = \{0, n+1\} \times \{1, \ldots, n\} \cup \{1, \ldots, n\} \times \{0, n+1\}$. Hence

$$\lambda(\mathbb{Z} \times \mathbb{Z}, \{x, y\}) = \inf_{S \subset G, |S| < \infty} \frac{|Bord(S, \{x, y\})|}{|S|} \leq \lim_{n \to \infty} \frac{4n}{n^2} = 0,$$

so $\mathbb{Z} \times \mathbb{Z}$ is amenable.

As these two examples suggest, proving that a group is amenable often requires finding a collection of sets S_n where $|Bord(S_n, C)|/|S_n| \to 0$ as $n \to \infty$. Such sets S_n we call Følner sets, and their existence is, by very definition, equivalent to amenability. It is often more difficult to prove nonamenability than amenability, because we must show the size of the border of every set is at least a fixed percentage of the size of the set. The following example is crucial in our study.

Let $G \cong \mathbb{Z} * \mathbb{Z} = \langle x, y | \rangle$ be the free group on two generators. We will show that G is non-amenable. The idea of the proof is the motivation behind our work. Place on

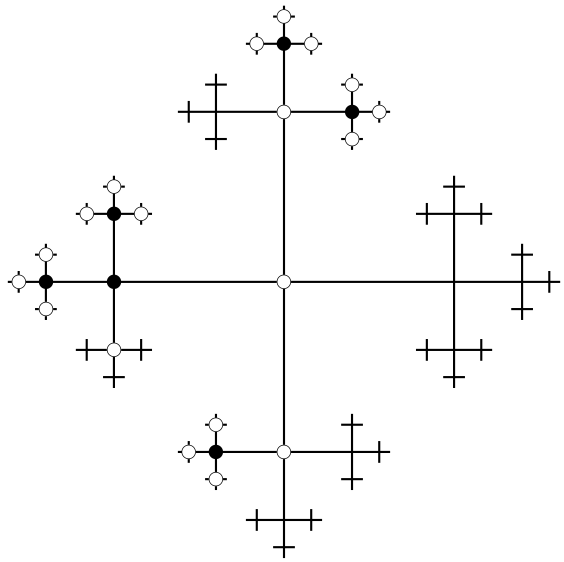

Figure 18: The Cayley graph of $\mathbb{Z} * \mathbb{Z}$ is represented above by an infinite valence 4 tree, where group elements correspond to the intersection of crossing lines. The identity element is in the center. Closed dots represent a set S, and open dots represent the border of S.

every element of a finite set $S \subset G$ exactly two tokens. We show that we can always pass every token to an element of $Bord(S, \{x, y\})$ in a such a way that no element of $Bord(S, \{x, y\})$ has more than one token on it. Hence $|Bord(S, \{x, y\})| \geq 2|S|$ for each finite $S \subset G$. This shows that $\lambda(\mathbb{Z} * \mathbb{Z}, \{x, y\}) \geq 2$, which shows that $\mathbb{Z} * \mathbb{Z}$ is non-amenable. A representation of the Cayley graph of $\mathbb{Z} * \mathbb{Z}$ is shown in Figure 18.

For $x \in G$, let $|x|$ denote word length. Given a finite set $S \subset \mathbb{Z} * \mathbb{Z}$, $|S| = n$, we order the elements $s_1, s_2, \ldots, s_n$ so that $|s_i| \leq |s_j|$ whenever $i \leq j$. Excluding the identity, every element in $\mathbb{Z} * \mathbb{Z}$ has exactly three adjacent elements with greater word length, and one adjacent element with smaller word length. Beginning with s_1, we pass the two tokens on s_1 to two of the adjacent elements of greater word length. Then s_2 has at most 3 tokens on it, since it could have received at most one token from s_1, so we pass the tokens on s_2 to distinct adjacent elements with greater word length. We repeat this passing for all s_k, where at each stage we always pass at most one token to each of the adjacent elements of greater word length. Since each non-identity element is adjacent to only one word of smaller word length, it is impossible for more than one token to be passed onto any element. Hence, there are never more than 3 tokens on an element of S, and so it is always possible to pass all the tokens on s_k to adjacent elements of G by sending at most one to each adjacent element with greater word length. Once we have passed the tokens off each s_k, we find that every token is now on an element of $Bord(S, \{x, y\})$, and that there is at most one token on each $x \in Bord(S, \{x, y\})$. Hence $\mathbb{Z} * \mathbb{Z}$ is non-amenable since $|Bord(S, \{x, y\})| \geq 2|S|$ for every finite $S \subset \mathbb{Z} * \mathbb{Z}$.

The free group on two generators is central to the study of amenability. It can be shown that if $\mathbb{Z}*\mathbb{Z}$ is a subgroup of G, then G is non-amenable. Thompson's group F does not contain $\mathbb{Z}*\mathbb{Z}$ as a subgroup, but it admits many properties similar to non-amenable groups. It also admits some properties characteristic of amenable groups. For over 30 years, mathematicians have worked to solve the amenability problem for Thompson's group. Mathematicians originally studied the amenability problem for F as it was known to be a counterexample to one of two conjectures. Other examples have since been found to disprove both conjectures, but the amenability problem for F still puzzles mathematicians. Thompson's group is perhaps the most famous group for which the amenability problem is still unresolved. It is hoped that a solution to the amenability problem will shed new light on the nature of amenability, as well as provide interesting geometric properties of F which may have application in many branches of mathematics. Cannon, Floyd, and Perry provide more motivation for the amenability problem in [4], as well as give proofs of many of the facts mentioned above.

One of our goals was to gather enough facts which would allow us to prove one of the following two mutually exclusive conditions.

1. Thompson's Group F behaves statistically enough like the free group $\mathbb{Z} * \mathbb{Z}$ so that we can successfully modify the token passing argument of non-amenability and apply it to show F is non-amenable.

2. There are obstructions to the token passing argument from which we can extract Følner sets in F.

We showed that Thompson's group is statistically not equivalent to random pair of caret trees in Section 2.1. In Section 2.5, we show that most n-caret trees have only two carets on the left edge of the tree, excluding the root. This leads to a conjecture about the average number of multiplications by x_0 and x_1 before an increase in word length is statistically likely to occur. Section 4 describes various collections of sets which cause obstructions to the token passing argument. We describe how dead end elements occur in clusters. We define k-deep roots, and we show that these deep roots appear in clusters as well. Our work gathers facts that support both of the conditions above, and more work is required to answer the amenability problem for F.

Occasionally, I will draw a Cayley graph vertically, with the origin at the base, and the remaining elements arranged on horizontal lines corresponding to their word length. Figure 19 shows vertical representations of the Cayley graphs of $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$, and $\mathbb{Z} * \mathbb{Z}$, using this convention. The growth function of a group is then given by counting the number of elements on each horizontal line in the Cayley graph. The token passing argument used to prove the nonamenability of $\mathbb{Z} * \mathbb{Z}$ amounts

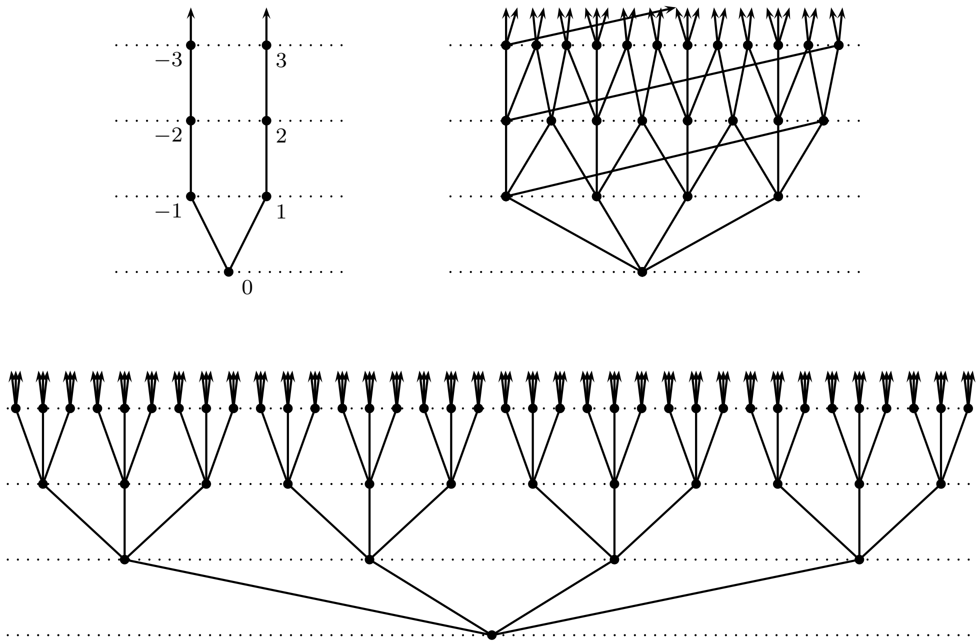

Figure 19: The Cayley graphs of $\mathbb{Z}, \mathbb{Z} \times \mathbb{Z}$, and $\mathbb{Z} * \mathbb{Z}$ are represented vertically. Group elements are placed along horizontal lines (shown as dotted lines) corresponding to their word length.

to always being able to pass tokens upwards in this vertical representation of the Cayley graph. Notice that when we pass tokens upwards in $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$, tokens begin to stockpile because either there is only one path to follow, or the upward paths from the origin eventually merge. Amenability (for groups with more than one generator) is in some sense a measure of how much merging occurs in these upward paths. If the paths merge too often, then the group is amenable. The Cayley graph of Thompson's group F looks like the Cayley graph of $\mathbb{Z} \times \mathbb{Z}$ for words with length less than five, but upwards paths begin to merge after crossing the fifth horizontal line.

Blake Fordham discovered certain elements in Thompson's group F, called dead end elements, which have the property that $|w\alpha| \leq |w|$ for any generator $\alpha \in$

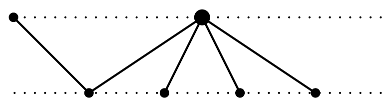

Figure 20: This depicts the local Cayley graph structure (drawn vertically) of a dead end element of Thompson's group F. The larger dot at the top represents the dead end.

$\{x_0^{\pm 1}, x_1^{\pm 1}\}$. In other words, any tokens which reach a dead end element must first be passed back toward the identity before they can be passed further outwards. All upward paths which reach a dead end element must turn around before continuing on an upwards course. Sean Cleary and Jennifer Taback showed that for every dead end element w, there is a pair of generators α, β with $|w\alpha\beta| = |w|$ and $w\alpha\beta$ is not a dead end element (see [5] p. 2837). Hence we can always, in some sense, bypass dead end elements by going around them. However, passing tokens to $w\alpha\beta$ may lead to further problems as there may be more dead end words nearby. Figure 20 depicts the local Cayley graph structure of dead end element of Thompson's group F.

I have improved upon the results of Cleary and Taback by completely classifying dead end elements. In addition, I will show that the existence of a dead end element means that there is a cluster of dead end elements "nearby". Also, the classification of dead end elements reveals some interesting elements I call k-deep roots. Along a specific shortest path to the identity from a k-deep root w, every element x on the path within distance k of w satisfies $|x\alpha| < |x|$ for all but one generator α. The local picture of a 3-deep root is shown in Figure 21. I will show that there are k-deep roots for every $k \geq 1$. These deep roots pose a threat to token passing, as the underlying idea of token passing is to pass upwards when possible. Passing

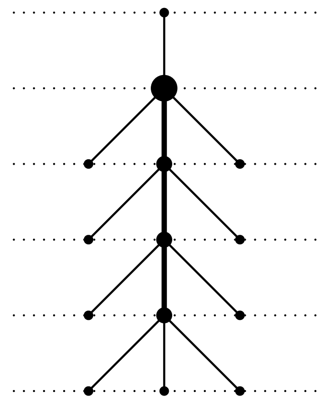

Figure 21: This depicts the local Cayley graph structure (drawn vertically) of a 3-deep root (the large dot at the top represents the root). The darkened path of length three shows the path (or "root") which heads back to the identity.

upwards along a path to the deep root creates a large stockpile of tokens at the deep root. A future study of the deep root structure should examine how many distinct paths to the indentity share the same "downwards" property. This could fatten up the deep roots, and cause an even larger token passing obstruction. Such a study would require patience, and perhaps would be more feasible with the aid of a computer.

2 Enumerating Thompson's Group by Number of Carets

In this section we prove the following key results:

- The number of elements $N_n$of Thompson's Group F represented by a reduced n-caret tree pair is (Theorem 2.8)

$$N_n = \sum_{k=1}^{\lceil n/2 \rceil} \sum_{i=0}^{k} 2^{n-2k+1} \binom{n-1}{n-2k+1} C_{n-k} (-1)^i \binom{k}{i} C_{n-i}$$

where C_n denotes the number of n-caret trees. This gives a type of growth function for F, and aids in understanding the geometric growth of F as it relates to caret tree pairs.

- The growth function N_n is bounded above by $(8+4\sqrt{3})^n \approx (14.93)^n$ (Theorem 2.18). This bounds the exponential growth rate of N_n, and shows that most random n-caret tree pairs are not reduced (Corollary 2.19). We conjecture an asymptotic estimate of $(8+4\sqrt{3})^n/n^3$ (Conjecture 2.22).

- Theorem 2.30 enumerates the number of n-caret trees that have a specified set of exposed carets. The counting scheme is highly dependent on the positions of the exposed carets. Surprisingly, Theorems 2.7 and 2.28 show that the number of n-caret trees that can be paired with a given n-caret tree and result in a reduced tree pair depends only on the number n of carets and the number k of exposed carets. This fact is central to enumerating N_n.

- The average number of L_L's in an n-caret tree approaches two as $n \to \infty$ (Theorem 2.31 joint with Sharleen De Gaston). This is directly related to multiplication by the generator x_0 of Thompson's Group F, as the number of L_L's is precisely the number of times we may multiply by x_0 without increasing the number of carets in a reduced tree pair.

In section 2.1, we count the number N_n of elements of Thompson's group F represented by a reduced pair of n-caret trees. In section 2.2 we show that a random pair of n-caret trees will result in a reduced pair with probability approaching 0 as $n \to \infty$. We give an upper bound for N_n and conjecture precise asymptotic estimates for N_n. We define tree functions in section 2.3. Let $B(n,k)$ represent the number of n-caret trees that can be paired with an n-caret tree with k exposed carets to get a reduced pair. We use tree functions to show that $B(n,k)$ depends only on n and k. In section 2.4, we use generating function techniques to count the number of n-caret trees with a prescribed set of exposed carets. In section 2.5 we enumerate the average number of L_L's in an n-caret tree, and relate the count to multiplication by x_0.

2.1 The Number of Reduced Pairs of n-caret Trees

The growth function of Thompson's group F is currently unknown, and an area of active research. The main goal of this section is to enumerate the number of elements of Thompson's group F that are represented by a reduced pair of n-caret trees. Much of this work began as a summer research project with John Bankhead, Jim Cannon*, Sharleen De Gaston and Luke Henderson. Sharleen De Gaston suggested the following approach.

1. Enumerate the trees with n-carets.

2. Enumerate the possible configurations of exposed carets.

3. Enumerate the trees having each possible configuration.

4. Match those pairs that share no exposed caret.

Her suggestion is precisely what we did, despite our initial feeling that this would prove too difficult to accomplish. The results of this section show that the number N_n of elements of Thompson's group F represented by a reduced pair of n-caret $(n \geq 1)$ trees is (see Theorem 2.8)

$$N_n = \sum_{k=1}^{\lceil n/2 \rceil} \sum_{i=0}^{k} 2^{n-2k+1} \binom{n-1}{n-2k+1} C_{n-k} (-1)^i \binom{k}{i} C_{n-i}$$

*This research was supported by NSF research grant DMS-10104030.

n	0	1	2	3	4	5	6	7	8	9	10
N_n	1	0	2	14	108	930	8700	86598	904176	9804516	109624536

Table 1: The Number N_n of Elements of Thompson's Group F: N_n is the number of elements of Thompson's Group F represented by a reduced pair of n-caret trees.

where C_n denotes the number of n-caret trees. Table 1 gives the first few values of N_n.

Lemma 2.1. *The number of n-caret trees is given by the Catalan number*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. This is a well known result, with various proofs (see [12] for a huge collection of sets that the Catalan numbers enumerate). The following proof will be instructive later. We add a root to each binary tree, i.e., we insert a single edge at the base vertex of each tree. Consider the set $C[n]$ of n-caret rooted binary trees. Given an n-caret binary tree, select one of the $2n+1$ edges ($2n$ edges come from the $2n$ carets, and the remaining edge is the root above the root caret). Then pick either the left or right side of that edge, and on that side draw a new edge from the middle of the chosen edge. This constructs an $(n+1)$-caret tree. Since there were $2n+1$ edges, there are $2(2n+1)$ ways to construct an $(n+1)$-caret tree from a given n-caret tree. Similarly we construct an n-caret trees from an $(n+1)$-caret tree by removing one of the $n+2$ leaves. Consider the following bipartite multi-graph. The vertices are the sets $C[n]$ of n-caret trees and $C[n+1]$ of $(n+1)$-caret trees. For each n-caret tree T, we create $2(2n+1)$ edges with vertices at T and the $(n+1)$-caret tree formed by inserting a leaf. Note that if S is an $(n+1)$-caret tree, then the number of edges leaving S is $n+2$, since the removal of any of its leaves gives the n-caret tree which formed S. By counting the number of edges in our bipartite multi-graph, we see that $2(2n+1)C_n = (n+2)C_{n+1}$, which is the recurrence of the Catalan numbers (see Lemma 2.2). □

The following well known facts are stated for reference purposes. The last fact is proved using Stirling's formula.

Lemma 2.2. *The Catalan numbers satisfy the recurrence* $2(2n{+}1)C_n = (n{+}2)C_{n+1}$, *and hence*

1. $\dfrac{C_{n+1}}{C_n} = \dfrac{2(2n+1)}{n+2} < 4$,
2. $\lim\limits_{n\to\infty} \dfrac{C_n}{4^n} = 0$,
3. $\lim\limits_{n\to\infty} \dfrac{C_n n^{3/2}}{4^n} = 1$, *and*
4. $\lim\limits_{n\to\infty} \dfrac{C_n}{(4-\epsilon)^n} = \infty$ *for small* $\epsilon > 0$.

Having enumerated the number of n-caret trees, we now explore the possible configurations of exposed carets.

Lemma 2.3. *Among n-caret trees, the number having exactly one exposed caret, and that caret in position i, is* $\binom{n-1}{i-1}$.

Proof. The number of exposed carets is 1 plus the number of splitting carets in the tree (a splitting caret has both a left and a right child). Consequently, if there is only one exposed caret, then each caret has at most one child, as there are no splitting carets. An n-caret tree with one exposed caret is thus determined by a sequence of $n-1$ choices of left or right. As we have decided to number our carets beginning with one, the position of the final exposed caret is equal to the one more than the number times we choose right. There are clearly $\binom{n-1}{j}$ ways of choosing right j times. Hence, the number of n-caret trees with one exposed caret in position i is is the number of ways of choosing right $i-1$ times, which is $\binom{n-1}{i-1}$. ☐

Lemma 2.4. *1. A set $I = \{i_1, \ldots, i_k\}, (1 \leq i_1 < i_2 < \cdots < i_k \leq n)$, of indices is a possible configuration of exposed carets in an n-caret tree if and only if $i_j - i_{j-1} \geq 2$ for all j. Note that the largest possible value for k is the ceiling function $k = \lceil n/2 \rceil$.*

2. Given the number n of carets, the number of possible configurations of length 1 is $A(n,1) = \binom{n}{1}$. The number of possible configurations of length 2 is $A(n,2) = \binom{n-1}{2}$. In general, the number of possible configurations of length j is $A(n,j) = \binom{n-j+1}{j}$.

3. The total number a_n of possible configurations of exposed carets for a given n satisfies the initial conditions $a_1 = 1$, $a_2 = 2$, and the recurrence relation $a_{n+2} = 1 + a_n + a_{n+1}$.

4. The difference $b_{n+1} = a_{n+1} - a_n$ is the $(n+1)$st Fibonacci number.

Proof. If caret i is exposed, then leaves $i-1$ and i are edges of caret i. Since leaf i is an edge of caret i, it cannot be an edge of caret $i+1$, hence caret $i+1$ cannot be exposed since both its edges would then be leaves. This shows that two consecutive carets cannot both be exposed.

Since there are n carets which could be exposed, we can pick $\binom{n}{1}$ to be exposed to get a possible configuration of exposed carets of length one. The number of possible configurations of exposed carets of length j is the same as the number of binary strings of length j where no two consecutive digits are 1. To count these, begin with $n-j$ 0's. Between each pair of 0's is a possible place to put a 1, as well as on either end of the zeros. Hence there are $(n-j-1)+2 = n-j+1$ choices of where to place a 1. This gives $\binom{n-j+1}{j}$ possible configurations of length j.

We check that $a_1 = 1$ and $a_2 = 2$. The recurrence comes by counting the number of possible configurations of exposed carets which caret 1 exposed and those without caret 1 exposed. Suppose there are $n+2$ carets. If caret 1 is exposed, then either it is the only exposed caret (which contributes the 1 to the recurrence) or there are other exposed carets. Since caret 2 cannot be exposed when caret 1 is exposed, there are precisely a_n possible configurations of exposed carets for carets 3 through $n+2$. If caret 1 is not exposed, then there are a_{n+1} possible configurations of exposed carets for carets 2 through $n+2$. This gives $a_{n+2} = 1 + a_n + a_{n+1}$.

The Fibonacci numbers satisfy the recurrence $f_2 = 1, f_3 = 2, f_{n+2} - f_{n+1} - f_n = 0$. Note that $b_2 = 1, b_3 = 2$, and

$$\begin{aligned}
b_{n+2} - b_{n+1} - b_n &= a_{n+2} - a_{n+1} - (a_{n+1} - a_n) - (a_n - a_{n-1}) \\
&= 1 + a_{n+1} + a_n - a_{n+1} - a_{n+1} + a_{n-1} \\
&= 1 + a_n + a_{n-1} - a_{n+1} \\
&= a_{n+1} - a_{n+1} = 0
\end{aligned}$$

Hence b_n satisfies the Fibonacci recurrence and initial conditions, and so b_n is a Fibonacci number. □

Corollary 2.5. *The number of n-caret trees having at least a prescribed set $I = \{i_1, \ldots, i_k\}, (1 \leq i_1 < i_2 < \cdots < i_k \leq n)$ of carets exposed (and possibly others as well) is given by the Catalan number C_{n-k}.*

Proof. We form all possible n-caret trees having at least these k carets exposed in the following way. Take an arbitrary tree T having $n-k$ carets. Note that T has at least $k-1$ carets, and at least k leaves. Numbering the leaves of T from left to right, beginning with 0, hang k new carets from leaves $i_1 - 1$, $i_2 - 2$, $i_3 - 3$, etc. This forms an n-caret tree with a set of exposed carets that contains I.

This process gives a bijection between the set of n-caret trees with at least the corresponding set of exposed carets I and $(n-k)$-caret trees. The inverse process is to simply remove the k exposed carets. The result then follows from Lemma 2.1. □

Theorem 2.6 (Henderson's Observation)**.** *The number $T(n,k)$ of n-caret trees with exactly k exposed carets is given by the formula*

$$T(n,k) = 2^{n-2k+1}\binom{n-1}{n-2k+1}C_{k-1}.$$

Proof. There are two approaches to prove this. The first is to note the recurrence relation

$$T(n,k) = \frac{n-2k+2}{k}T(n-1,k-1) + 2T(n-1,k),$$

and then proceed by induction.

The second approach is more combinatorial in flavor. We first show that $T(2k-1,k) = C_{k-1}$. Removing the k exposed carets from a $(2k-1)$-caret tree gives a $(k-1)$-caret tree. Similarly, adding a caret at each exposed leaf of a $(k-1)$-caret tree gives a $(2k-1)$-caret tree with exactly k exposed carets. This gives a bijection between $(k-1)$-caret trees and $(2k-1)$-caret trees with exactly k exposed carets. We will call a $(2k-1)$-caret tree with k exposed carets a k-structure tree. Note that in order to have k exposed carets, a tree must contain at least $2k-1$ carets.

We now construct a bipartite multi-graph as done in Lemma 2.1. The vertices correspond to the set of n-caret trees with k exposed carets and to the set of k-structure trees. The edges are described below.

From the k-structure trees, we now reconstruct all possible n-caret trees with k exposed carets by inserting carets one at a time. Pick a $(2k-1)$-caret k-structure tree. Pick one of its carets. Insert a new caret above the chosen caret by drawing a leaf either to the left or to the right of the segment coming into the top of the chosen caret (it is important to include the root above the top caret). This process does not introduce any new exposed carets, though it may change the position of the exposed carets. There are C_{k-1} ways to choose a k-structure tree, $2k-1$ ways to pick a caret, and 2 ways to pick a side. Now repeat the above insertion procedure. There are $2k$ ways to pick a caret, and 2 ways to pick a side. Repeat this procedure $n-2k+1$ times, giving an n-caret tree with k exposed carets. Note that we made

$$2^{n-2k+1}(2k-1)(2k)(2k+1)\cdots(n-1)C_{k-1}$$

choices. In our bipartite multi-graph, each sequence of choices will constitute an edge between a k-structure tree and an n-caret tree with k exposed carets. We stop at $(n-1)$ since $(n-1)-(2k-2)=n-2k+1$.

Any n-caret tree with k exposed carets has a unique underlying structure which come from one of the k-structure trees. Remove from the n-caret tree every leaf which is not an edge of an exposed caret. The resulting tree has k exposed carets, $2k$ leaves, and hence $2k-1$ carets. Since there are $n-2k+1$ leaves which are not an edge of an exposed caret, any permutation of these $n-2k+1$ leaves represents an order in which the leaves can be removed to get to the k-structure tree. Similarly, each permutation represents an order in which we could insert leaves to construct the n-caret tree from its k-structure tree. This shows that each n-caret tree in our bipartite multi-graph is the endpoint of precisely $(n-2k+1)!$ edges. Hence, by counting edges in our multi-graph, we see that

$$(n-2k+1)!T(n,k)=2^{n-2k+1}(2k-1)(2k)(2k+1)\cdots(n-1)C_{k-1}.$$

Isolating $T(n,k)$ and noticing the binomial coefficient completes the proof. □

Luke Henderson's observation caused us to examine the number of n-caret trees that can be paired with a given n-caret tree with a prescribed set of exposed carets, and result in a reduced tree pair. Due to Lemma 2.3, we suspected initially that this count would depend on the position of the exposed carets. To our surprise, we discovered that number of exposed carets, not the position, determined the number of reduced pairs we could form. The shortness of the following proof should not detract from the fact that this theorem is of critical importance to all our counts involving Thompson's group.

Theorem 2.7. *Given an admissible and prescribed set* $I = \{i_1, \ldots, i_k\}, (0 \leq i_1 < i_2 < \cdots < i_k \leq n-1)$ *of exposed carets, the number* $B(n,k)$ *of* n*-caret trees that have none of these carets exposed depends only on* n *and* k *and is given by the formula*

$$B(n,k) = \binom{k}{0} C_n - \binom{k}{1} \cdot C_{n-1} + \binom{k}{2} \cdot C_{n-2} - \binom{k}{3} \cdot C_{n-3} + \cdots = \sum_{i=0}^{k} (-1)^i \binom{k}{i} C_{n-i}.$$

Proof. This is a standard inclusion-exclusion argument, and the interested reader may see [12], pg. 65, for a detailed treatment of the inclusion-exclusion principle. We provide a proof here for completeness.

Following the notation found in [12] on pg. 65, let A_n be the set of n-caret trees. We need to define a set S_n of properties that the n-caret trees can have. We say an n-caret tree has property j if caret i_j is exposed, and let $S_n = \{i_1, \ldots, i_k\}$. We then want to know how many n-caret trees have none of the properties in S_n, and compute $f_=(\emptyset) = \sum_{T \subset S_n} (-1)^{|T|} f_\geq(T)$, where $f_\geq(T)$ gives the number of n-caret trees having at least the carets in T exposed. If $|T| = i$, we have $f_\geq(T) = C_{n-i}$ by Corollary 2.5. The number of subsets T of S_n with $|T| = i$ is $\binom{k}{i}$. Hence $f_=(\emptyset) = \sum_{i=0}^{k} (-1)^{|T|} \binom{k}{i} C_{n-i}$. ∎

We are now prepared to enumerate the number of elements of Thompson's group F which are represented by a reduced pair of n-carets trees.

Theorem 2.8. *The number N_n of elements of Thompson's group that are represented by a reduced pair of n-caret trees is given by the formula*

$$N_n = T(n,1) \cdot B(n,1) + T(n,2) \cdot B(n,2) + \cdots = \sum_{k=1}^{\lceil n/2 \rceil} T(n,k)B(n,k).$$

Proof. The number of n-caret trees with k exposed carets is given by $T(n,k)$ in Theorem 2.6. To form a reduced pair of caret trees, an n-caret tree with k exposed carets can be paired with an n-caret tree that has none of these k carets exposed. By Theorem 2.7 there are $B(n,k)$ caret trees that have none of these k carets exposed. Hence there are $T(n,k)B(n,k)$ distinct reduced pairs of n-caret trees with k exposed carets. The proof is complete after summing over all k, noticing that there are at most $\lceil n/2 \rceil$ exposed carets in an n-caret tree, and using the bijection between reduced pairs of n-caret trees and elements of Thompson's group F. □

2.2 Asymptotics and Probabilities

Having enumerated the number N_n of elements of Thompson's group F represented by a reduced pair of n-caret trees, we now ask with what probability will an ordered pair of n-caret trees result in a reduced pair, where each tree is chosen at random from all n-caret trees. Initially we suspected this probability would be zero, but the following lemma made us hopeful that a fixed positive percentage of random pairs would be reduced.

Lemma 2.9. *At the root, a caret tree may branch to the left side only, or it may branch to the right side only, or it may branch to both sides. Asymptotically,* 1/4 *branch only to the left,* 1/4 *branch only to the right, and* 1/2 *branch to both sides.*

Proof. The trees that branch only to the left can be formed by hanging an arbitrary $(n-1)$-caret tree from the left leaf of a root caret. By Lemma 2.2, asymptotically $C_n/C_{n+1} \to 1/4$. Thus 1/4 split to the left, and by symmetry 1/4 split to the right. □

Our intuition was that there would not be a lot of branching, hopefully increasing the probability that a random pair of caret trees is reduced. The probability that a random pair of n-caret trees is reduced is given by the ratio N_n/C_n^2, since there are C_n^2 possible ordered pairs of n-caret trees. Using the estimate $C_n \approx 16^n$, we shall show that

$$\lim_{n\to\infty} \frac{N_n}{C_n^2} = 0 \tag{2.1}$$

by showing

$$N_n < (8 + 4\sqrt{3})^n \approx (14.9282\ldots)^n. \tag{2.2}$$

In addition we show

$$\lim_{n\to\infty} \frac{N_n}{(8+4\sqrt{3})^n} = 0, \tag{2.3}$$

which implies that N_n grows exponentially at a rate no greater than $8+4\sqrt{3}$. We conjecture that

$$\lim_{n\to\infty} \frac{N_n}{(8+4\sqrt{3}-\epsilon)^n} = \infty \tag{2.4}$$

for small $\epsilon > 0$. Using Stirling's formula estimates and other simplifications, we conjecture that $\lim_{n\to\infty} n^3 N_n/(8+4\sqrt{3})^n$ is a constant. This last limit is left as an open problem.

2.2.1 Random Pairs of Caret Trees

We begin by showing $N_n < (8+4\sqrt{5})^n \approx (16.9442\ldots)^n$. This estimate is clearly useless, but the proof of this estimate will motivate the remainder of the work in this section. We acknowledge the obvious fact that $N_n < C_n^2 < 16^n$. The weaker estimate that we give has an instructive proof.

Lemma 2.10. *For all $n > 0$, $N_n < (8+4\sqrt{5})^n$.*

Proof. The proof follows from a sequence of overestimates, including $C_n < 4^n$, and from repeated use of the binomial theorem.

$$N_n = \sum_{k=1}^{\lceil n/2 \rceil} T(n,k)B(n,k) \tag{2.5}$$

$$= \sum_{k=1}^{\lceil n/2 \rceil} 2^{n-2k+1} \binom{n-1}{n-2k+1} C_{k-1} \sum_{i=0}^{k} (-1)^i \binom{k}{i} C_{n-i} \tag{2.6}$$

$$< \sum_{k=1}^{\lceil n/2 \rceil} 2^{n-2k+1} \binom{n-1}{2(k-1)} 4^{k-1} \sum_{i=0}^{k} \binom{k}{i} 4^{n-i} \tag{2.7}$$

$$= \sum_{k=1}^{\lceil n/2 \rceil} 2^{n-2k+1} \binom{n-1}{2(k-1)} 4^{k-1} 4^{n-k} \sum_{i=0}^{k} \binom{k}{i} 4^{k-i} \tag{2.8}$$

$$= 4^{n-1} 2^{n+1} \sum_{k=1}^{\lceil n/2 \rceil} 4^{-k} \binom{n-1}{2(k-1)} (1+4)^k \tag{2.9}$$

$$= \frac{8^n}{2} \sum_{k=1}^{\lceil n/2 \rceil} \binom{n-1}{2(k-1)} \left(\frac{5}{4}\right)^k \tag{2.10}$$

$$= \frac{8^n}{2}\frac{5}{4} \sum_{k=1}^{\lceil n/2 \rceil} \binom{n-1}{2(k-1)} \left(\sqrt{\frac{5}{4}}\right)^{2(k-1)} \tag{2.11}$$

$$< \frac{8^n}{2}\frac{5}{4} \sum_{i=0}^{n-1} \binom{n-1}{i} \left(\sqrt{\frac{5}{4}}\right)^{i} \tag{2.12}$$

$$= \frac{8^n}{2}\frac{5}{4} \left(1+\frac{\sqrt{5}}{2}\right)^{n-1} \tag{2.13}$$

$$< \left(8+4\sqrt{5}\right)^n \approx (16.94)^n \tag{2.14}$$

$\square$

The first inequality uses an absolute value overestimate to remove the alternating sum. To prove $N_n < (8+4\sqrt{3})^n \approx (14.92)^n$, we shall show that we may replace the inequality

$$B(n,k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} C_{n-i} < \sum_{i=0}^{k} \binom{k}{i} 4^{n-i} = 5^k 4^{n-k} \tag{2.15}$$

with the stronger inequality

$$B(n,k) \leq 3^k C_{n-k} \tag{2.16}$$

for $k \in \{1, 2, \ldots, \lceil n/2 \rceil\}$. Inequality (2.16) is a subtle inequality which would follow easily if it were permissible to replace all the C_k terms with 4^k and leave the $(-1)^k$ in the alternating sum of (2.15). Unfortunately, we shall have to work much harder. If we assume (2.16), then lines (2.10) through (2.14) remain unchanged after replacing each 5 with a 3. Hence, in order to prove $N_n < (8+4\sqrt{3})^n$, it suffices to prove (2.16). Inequality (2.16) is equivalent to

$$\frac{B(n,k)}{3^k C_{n-k}} \leq 1. \tag{2.17}$$

The definition of $B(n,k)$ required $1 \leq k \leq \lceil n/2 \rceil$. It is valuable to define $B(n,k)$ for all k with $0 \leq k \leq n$ as the sum $B(n,k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} C_{n-i}$. As long as $n \geq 2$, it can be shown (using Lemma 2.13 and Observation 2.19) that $B(n,k) > 0$ for $0 \leq k \leq n$. The only problematic term is $B(1,1) = 0$. We will assume from now on that $n \geq 2$.

When $k = 0$ the inequality (2.17) is an equality. Hence, it suffices to prove

$$1 = \frac{B(n,0)}{C_n} \geq \frac{B(n,1)}{3^1 C_{n-1}} \geq \cdots \geq \frac{B(n, \lceil n/2 \rceil)}{3^{\lceil n/2 \rceil} C_{n-\lceil n/2 \rceil}}.$$

In brief, it is enough to show, for $k \in \{1, 2, \dots, \lceil n/2 \rceil\}$,

$$\frac{B(n, k-1)}{3^{k-1} C_{n-(k-1)}} \geq \frac{B(n, k)}{3^k C_{n-k}}.$$

This can also be written as

$$B(n, k-1) \geq \frac{C_{n-(k-1)} B(n, k)}{3 C_{n-k}}.$$

Since $C_{n-(k-1)}/C_{n-k} \leq 4$ by Lemma 2.2, inequality (2.16), and consequently $N_n < (8 + 4\sqrt{3})^n$, will follow immediately once we prove

$$\frac{B(n, k+1)}{B(n, k)} \leq \frac{3}{4} \tag{2.18}$$

for $k \in \{1, 2, \dots, \lceil n/2 \rceil\}$ and large n. We will see (Corollary 2.17) that (2.18) is true for $0 \leq k \leq n-1$ and $n \geq 8$.

Observation 2.11. The $B(n, k)$ terms satisfy the following recurrence relation (for $0 \leq k \leq n-1$)

$$B(n, k+1) = B(n, k) - B(n-1, k) \tag{2.19}$$

This is a standard result in the study of finite difference equations. It follows immediately from the binomial identity $\binom{k+1}{i} = \binom{k}{i-1} + \binom{k}{i}$. This observation allows us to visually write the $B(n, k)$ terms in a triangle, called the difference triangle (or difference array) of the Catalan numbers (see Table 2).

As we are interested in the ratio $\dfrac{B(n, k+1)}{B(n, k)}$, we introduce the notation

$$R_K(n, k) = \frac{B(n, k+1)}{B(n, k)}. \tag{2.20}$$

The "K" means increment the second variable k. The following lemma relates $R_K(n, k)$ to the following ratios, obtained by incrementing the first variable or both

$B(n,k)$	$n=0$	1	2	3	4	5	6	7
$k=0$	1	1	2	5	14	42	132	429
1		0	1	3	9	28	90	297
2			1	2	6	19	62	207
3				1	4	13	43	145
4					3	9	30	102
5						6	21	72
6							15	51
7								36

Table 2: The numbers $B(n,k)$ form the difference triangle of the Catalan numbers. The Catalan numbers are accross the top row, and each number in the triangle is the the difference of the number above and the number to the upper left.

variables.

$$R_N(n,k) = \frac{B(n+1,k)}{B(n,k)} \tag{2.21}$$

$$R_{NK}(n,k) = \frac{B(n+1,k+1)}{B(n,k)}. \tag{2.22}$$

Lemma 2.12. *The following are equivalent for $n \geq 2$ and $0 \leq k \leq n$.*

$$R_N(n,k) < 4 \tag{2.23}$$

$$R_K(n+1,k) < 3/4 \tag{2.24}$$

$$R_{NK}(n,k) < 3. \tag{2.25}$$

Also, the following are equivalent for $n \geq 2$ and $0 \leq k \leq n$.

$$R_N(n,k) < R_N(n+1,k) \tag{2.26}$$

$$R_N(n+1,k) < R_N(n+1,k+1) \tag{2.27}$$

$$R_N(n,k) < R_N(n+1,k+1) \tag{2.28}$$

$$R_K(n+1,k) < R_K(n+2,k) \tag{2.29}$$

$$R_K(n+2,k) < R_K(n+2,k+1) \tag{2.30}$$

$$R_K(n+1,k) < R_K(n+2,k+1) \tag{2.31}$$

$$R_{NK}(n,k) < R_{NK}(n+1,k) \tag{2.32}$$

$$R_{NK}(n+1,k) < R_{NK}(n+1,k+1) \tag{2.33}$$

$$R_{NK}(n,k) < R_{NK}(n+1,k+1) \tag{2.34}$$

Proof. We require $n \geq 2$ so that $B(n,k) > 0$, avoiding $B(1,1) = 0$. It is worth emphasizing that the lemma just says the inequalities are equivalent. Later we show that (2.23) through (2.25) are false for $(3,3)$ and true for every other $n \geq 2, 0 \leq k \leq n$. This will establish (2.18).

Suppose $4B(n+1,k+1) < 3B(n+1,k)$. Then $4B(n+1,k) - 4B(n,k) < 3B(n+1,k)$ (using (2.19)), which gives $B(n+1,k) < 4B(n,k)$. These steps are reversible, so (2.23) and (2.24) are equivalent. Now suppose $B(n+1,k+1) < 3B(n,k)$. Then $B(n+1,k) - B(n,k) < 3B(n,k)$ or $B(n+1,k) < 4B(n,k)$, which is again reversible. Hence (2.23) and (2.25) are equivalent.

We show later that (2.26) through (2.34) are false for $(2,1)$, $(3,3)$, and $(5,5)$, and true for every other $n \geq 2, 0 \leq k \leq n$. The equivalence of these nine inequalities will enable us to prove that an increase in any of the ratios implies an increase in every other ratio.

We first show $R_K(n+1,k) = 1 - 1/R_N(n,k)$, using (2.19). Hence, an increase in R_K is equivalent to an increase in R_N. This shows (2.26) and (2.29) are equivalent, as well as (2.27) and (2.30), and also (2.28) and (2.31). Similarly, we show $R_{NK}(n,k) = R_N(n,k) - 1$, which proves the equivalence of (2.26) and (2.32), and also (2.27) and (2.33), and also (2.28) and (2.34).

We show that (2.27) and (2.29) are equivalent by first writing (2.27) in terms of $B(n,k)$. We swap a numerator with a denominator, which gives (2.29). A similar argument shows that (2.33) and (2.31) are equivalent, which completes the proof. □

The equivalence of (2.23), (2.24), and (2.25) will be used in the proof that $N_n < (8+4\sqrt{3})^n$. For now we restrict our attention to the terms $b_n = B(n,n)$ (the diagonal entries of the difference triangle of the Catalan numbers). The numbers b_n are given various names: Riordan numbers, ring numbers, or Motzkin sums (see [2] or [11]).

Lemma 2.13. *The diagonal entries $b_n = B(n,n)$ of the difference triangle of the Catalan numbers (i.e., the Riordan numbers) have the generating function*

$$B(x) = \frac{1 + x - \sqrt{1 - 2x - 3x^2}}{2x(1+x)}$$

and satisfy the recurrence relation (for $n \geq 2$)

$$B(n,n) = \frac{n-1}{n+1}\left(2B(n-1,n-1) + 3B(n-2,n-2)\right).$$

The generating function $B(x)$ can be found by an Euler transformation (see Lemma 2.14), from which follows the recurrence relation. I will provide some details which are not included in [2]. A reader familiar with these results should jump to Lemma 2.16.

Lemma 2.14 (The Euler Transformation)**.** *Let* $C(x) = \sum_{n=0}^{\infty} c_n x^n$ *be the generating function of the sequence* $\{c_n\}_{n=0}^{\infty}$*. Let* $b_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} c_{n-i}$ *be the terms on the diagonal of the difference array of the sequence* $\{c_n\}_{n=0}^{\infty}$*. Let* $B(x) = \sum_{n=0}^{\infty} b_n x^n$ *be the generating function of the sequence* $\{b_n\}_{n=0}^{\infty}$*. Then* $B(x)$ *and* $C(x)$ *are related by the Euler transformation*

$$(1+x)B(x) = C\left(\frac{x}{1+x}\right).$$

Proof. First work with $\left(\frac{x}{1+x}\right)^k$.

$$\begin{aligned}
\left(\frac{x}{1+x}\right)^0 &= 1 \\
\left(\frac{x}{1+x}\right)^1 &= x - x^2 + x^3 - x^4 + x^5 - x^6 + \cdots \\
\left(\frac{x}{1+x}\right)^2 &= x^2 - 2x^3 + 3x^4 - 4x^5 + 5x^6 - 6x^7 + \cdots \\
\left(\frac{x}{1+x}\right)^3 &= x^3 - 3x^4 + 6x^5 - 10x^6 + 15x^7 - 21x^8 + \cdots
\end{aligned}$$

Rewriting all the terms as binomial coefficients, gives the following:

$$\begin{aligned}
\left(\frac{x}{1+x}\right)^0 &= 1 \\
\left(\frac{x}{1+x}\right)^1 &= \binom{0}{0}x^1 - \binom{1}{0}x^2 + \binom{2}{0}x^3 - \binom{3}{0}x^4 + \binom{4}{0}x^5 - \binom{5}{0}x^6 + \cdots \\
\left(\frac{x}{1+x}\right)^2 &= \binom{1}{1}x^2 - \binom{2}{1}x^3 + \binom{3}{1}x^4 - \binom{4}{1}x^5 + \binom{5}{1}x^6 - \binom{6}{1}x^7 + \cdots \\
\left(\frac{x}{1+x}\right)^3 &= \binom{2}{2}x^3 - \binom{3}{2}x^4 + \binom{4}{2}x^5 - \binom{5}{2}x^6 + \binom{6}{2}x^7 - \binom{7}{2}x^8 + \cdots \\
\left(\frac{x}{1+x}\right)^k &= \binom{k-1}{k-1}x^k - \binom{k}{k-1}x^{k+1} + \cdots + (-1)^i\binom{k-1+i}{k-1}x^{k+i} + \cdots
\end{aligned}$$

One way to observe the binomial coefficients is to note that the absolute value of the coefficient of x^{k+i} in $\left(\frac{x}{1+x}\right)^k$ is found by summing the absolute values of the coefficients of x^{k-1+j} in $\left(\frac{x}{1+x}\right)^{k-1}$ for $j \in \{0, \ldots, i\}$. This recurrence relation is a well known recurrence relation of the binomial coefficients. The coefficient of x^n $(n \geq 1)$ in the power series expansion of $\left(\frac{x}{1+x}\right)^{n-k}$ $(0 \leq k \leq n-1)$ is easily seen to be $(-1)^k \binom{n-1}{n-1-k} c_{n-k}$ which shows that

$$\begin{aligned} C\left(\frac{x}{1+x}\right) &= \sum_{n=0}^{\infty} c_n \left(\frac{x}{1+x}\right)^n \\ &= c_0 + \sum_{n=1}^{\infty} x^n \left(\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{n-1-k} c_{n-k}\right) \\ &= c_0 + \sum_{n=1}^{\infty} x^n \left(\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} c_{n-k}\right) \end{aligned}$$

We now examine $(1+x)B(x)$.

$$\begin{aligned} (1+x)B(x) &= (1+x) \sum_{n=0}^{\infty} \sum_{i=0}^{n} (-1)^i \binom{n}{i} c_{n-i} x^n \\ &= (1+x)\left((c_0)x^0 + (c_1 - c_0)x^1 + (c_2 - 2c_1 + c_0)x^2 + \cdots\right) \\ &= (c_0)x^0 + (c_1 - c_0)x^1 + (c_0)x^1 + (c_2 - c_1 + c_0)x^2 + (c_1 - c_0)x^2 + \cdots \\ &\quad + \left(\binom{n}{0} c_n - \binom{n}{1} c_{n-1} + \cdots + (-1)^n \binom{n}{n} c_0\right) x^n \\ &\quad + \left(\binom{n-1}{0} c_{n-1} - \binom{n-1}{1} c_{n-2} + \cdots + (-1)^{n-1} \binom{n-1}{n-1} c_0\right) x^n + \cdots \\ &= c_o + \sum_{n=1}^{\infty} x^n \left(\binom{n}{0} c_n + \sum_{k=1}^{n} (-1)^k c_{n-k} \left(\binom{n}{k} - \binom{n-1}{k-1}\right)\right) \\ &= c_o + \sum_{n=1}^{\infty} x^n \left(\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} c_{n-k}\right) \\ &= C\left(\frac{x}{1+x}\right) \end{aligned}$$

$\square$

We now prove the recurrence relation given for the Riordan numbers in Lemma

2.13. This recurrence follows from the theory of D-finite generating functions. For a sequence $\{c_n\}_{n=0}^{\infty}$ in a field $\mathbb{F}$, it can be shown that the generating function $C(x) = \sum_{n=0}^{\infty} c_n x^n$ is a rational function (the quotient of two polynomials with coefficients in $\mathbb{F}$) if and only if the coefficients satisfy, for large enough n, a linear recurrence relation of the form

$$a_k c_{n+k} + a_{k-1} c_{n+k-1} + \cdots + a_0 c_n = 0,$$

where $a_k, a_0 \neq 0$ and each a_i is a constant. Many generating functions satisfy a polynomial recurrence relation of the form

$$a_k(n) c_{n+k} + a_{k-1}(n) c_{n+k-1} + \cdots + a_0(n) c_n = 0,$$

where this time each $a_i(n)$ is a polynomial in n (with coefficients in $\mathbb{F}$), with $a_k(n), a_0(n)$ not identically zero. Richard Stanley gives a thorough overview of this theory on pages 187-195 in [13]. The key result I will emphasize is the following.

Lemma 2.15. *A generating function which is algebraic (over a field $\mathbb{F}$) satisfies a polynomial recurrence relation, and there is an algorithm to find it.*

We give the details for finding the recurrence relation of the Riordan numbers. Namely, letting $b_n = B(n,n)$, and $B(x) = \sum_{n=0}^{\infty} b_n x^n$, we show that $(n+1)b_n = (n-1)(2b_{n-1} + 3b_{n-2})$. Here our field is $\mathbb{Q}$, the rational numbers. Start by recalling the generating function

$$B(x) = \frac{1 + x - \sqrt{1 - 2x - 3x^2}}{2x(1+x)}.$$

After a little manipulation, this becomes

$$1 - 2xB(x) = \frac{\sqrt{1 - 2x - 3x^2}}{1+x}.$$

Let $u = 1 - 2xB$ and $f = \dfrac{1-2x-3x^2}{(1+x)^2} = \dfrac{1-3x}{1+x}$, so that $u = f^{1/2}$, or $u^2 = f$. Differentiating gives $2uu' = f'$, or $2u^2u' = uf'$, or $2fu' = uf'$. We calculate $f' = -\dfrac{4}{(1+x)^2}$ and $u' = -2xB' - 2B$. The equation $2fu' = uf'$ then gives

$$2\left(\frac{1-2x-3x^2}{(1+x)^2}\right)(-2xB'-2B) = (1-2xB)\left(-\frac{4}{(1+x)^2}\right),$$

which simplifies to

$$3x^3B' + 2x^2B' + 3x^2B - xB' - B + 1 = 0. \tag{2.35}$$

Replace each occurrence of B and B' with $B = \sum_{n=0}^{\infty} b_n x^n$ and $B' = \sum_{n=1}^{\infty} nb_n x^{n-1}$. Collecting the coefficients of x^n gives the recurrence of the Riordan numbers,

$$(n+1)b_n = (n-1)(2b_{n-1} + 3b_{n-2}). \tag{2.36}$$

This same procedure can be carried out using the equivalent equation $1 + x - 2x(1+x)B(x) = \sqrt{1-2x-3x^2}$. Let $u = 1+x-2x(1+x)B(x)$ and $f = 1-2x-3x^2$ and calculate $f' = -2 - 6x$ and $u' = 1 - 2xB'(x) - 2x^2B'(x) - 2B(x) - 4xB(x)$ The equation $2fu' = uf'$ gives $2(1-2x-3x^2)(1-2xB'-2x^2B'-2B-4xB) = (1+x-2x(1+x)B)(-2-6x)$ which simplifies to

$$1 - B - xB' + x^2B' + x + 3x^2B + 5x^3B' + 3x^3B(x) + 3x^4B' - xB(x) = 0. \tag{2.37}$$

After collecting the coefficients of x^n, we get a new recurrence

$$(n+1)b_n = (n-2)b_{n-1} + (5n-7)b_{n-2} + (3n-6)b_{n-3}$$

which does not seem to me to be the same recurrence relation already given. However, if you divide both sides of equation (2.37) by $1 + x$, you get equation (2.35) which lead to the original recurrence. This new recurrence relation is actually implied by (2.36), so we have not introduced any contradictions.

We are now prepared to prove the upper bound for N_n. As mentioned, estimating the terms $B(n,k)$ is the crucial step. The following lemma proves (2.18) for the diagonal terms of the Catalan difference triangle.

Lemma 2.16. *The Riordan numbers ($B(n,n)$ in our notation) satisfy the following inequalities for $n \geq 7$.*

$$\frac{3n}{n+2} < \frac{B(n,n)}{B(n-1,n-1)} < \frac{3n}{n+3/2}$$

Proof. We manually verify the inequalities for $n = 4, 5, 7, 8$, and begin our induction with $n \geq 9$.

$$\begin{aligned}
\frac{B(n+1,n+1)}{B(n,n)} &= \frac{n}{n+2}\left(2+\frac{3}{\dfrac{B(n,n)}{B(n-1,n-1)}}\right) \\
&= \frac{n}{n+2}\left(2+\frac{3}{\dfrac{n-1}{n+1}\left(2+\dfrac{3}{\dfrac{B(n-1,n-1)}{B(n-2,n-2)}}\right)}\right) \\
&< \frac{n}{n+2}\left(2+\frac{3}{\dfrac{n-1}{n+1}\left(2+\dfrac{3}{\dfrac{3(n-1)}{(n-1)+3/2}}\right)}\right) \\
&= \frac{6n^2}{(2n-1)(n+2)} \\
&< \frac{3(n+1)}{(n+1)+3/2}
\end{aligned}$$

Similarly,

$$\frac{B(n+1,n+1)}{B(n,n)} > \frac{n}{n+2}\left(2+\frac{3}{\frac{n-1}{n+1}\left(2+\frac{3}{\frac{3(n-1)}{(n-1)+2}}\right)}\right)$$

$$= \frac{(9n+1)(n)}{(3n-1)(n+2)}$$

$$> \frac{3(n+1)}{(n+1)+2}$$

□

Recalling the notation $R_{NK}(n,k) = \frac{B(n+1,k+1)}{B(n,k)}$, we just showed $R_{NK}(n,n) < 3$ for $n \geq 6$.

Corollary 2.17. *For all $n \geq 6$ and all $0 \leq k \leq n$,*

$$R_N(n,k) < 4, R_K(n+1,k) < \frac{3}{4}, R_{NK}(n,k) < 3$$

Proof. Lemma 2.16 shows that $R_{NK}(n,n) < 3$ for $n \geq 6$. By the equivalences of Lemma 2.12, it is enough to only prove $R_{NK}(n,k) < 3$ for $n \geq 6$ and $0 \leq k \leq n$. We check that $R_{NK}(n,k) < 3$ for all k when $n = 6$. Assume that $R_{NK}(n,k) < 3$ and $R_{NK}(n-1,k-1) < 3$. Using the equivalences of Lemma 2.12, we have $R_N(n,k) < 4$ and $R_K(n,k-1) < \frac{3}{4}$. By the definition of R_N and R_K, we have $B(n+1,k) < 4B(n,k) < 3B(n,k-1)$, which gives $R_{NK}(n,k-1) < 3$. Hence, if $R_{NK}(n,k) < 3$ and $R_{NK}(n-1,k-1) < 3$, then $R_{NK}(n,k-1) < 3$. Using induction (with the infinitely many base cases $n = 6$ and $k = n$) we have $R_{NK}(n,k) < 3$ for all $n \geq 7$ and $0 \leq k \leq n$. □

Theorem 2.18. *The number N_n of elements of Thompson's Group F represented by a reduced pair of n-caret trees satisfies*

$$N_n < (8 + 4\sqrt{3})^n \approx (14.93)^n.$$

Proof. This follows from Corollary 2.17 and the comments following Lemma 2.10.

□

Corollary 2.19. *The probability of randomly choosing a reduced ordered pair of n-caret trees approaches 0 as $n \to \infty$. In other words, most pairs of caret trees are not reduced.*

Proof. We just finished a proof that $\lim_{n\to\infty} \frac{N_n}{(8+4\sqrt{3})^n} < 1$. Since $\lim_{n\to\infty} \frac{(4-\epsilon)^n}{C_n} = 0$ for small $\epsilon > 0$, we pick ϵ so that $(8 + 4\sqrt{3}) < (4 - \epsilon)^2$. Then

$$\lim_{n\to\infty} \frac{N_n}{C_n^2} = \lim_{n\to\infty} \frac{N_n}{(8+4\sqrt{3})^n} \frac{(8+4\sqrt{3})^n}{C_n^2} \leq \lim_{n\to\infty} \frac{N_n}{(8+4\sqrt{3})^n} \frac{(4-\epsilon)^{2n}}{C_n^2} = 0$$

□

2.2.2 Upper and Lower Bounds

We now develop the tools needed to show

$$\lim_{n\to\infty} \frac{N_n}{(8+4\sqrt{3})^n} = 0$$

for small $\epsilon > 0$. This gives an upper estimate of the asymptotic growth for N_n. The $B(n, k)$ terms play a central role in our analysis of Thompson's Group F.

Lemma 2.20. *For large n, the ratios $R_N(n, k)$, $R_K(n + 1, k)$, and $R_{NK}(n, k)$ increase as either n or k increase, which gives the inequalities*

$$\frac{3(n-1)}{2(2n-1)} \leq \frac{B(n,k+1)}{B(n,k)} \leq \frac{3}{4} \tag{2.38}$$

$$\left(\frac{3(n-1)}{2(2n-1)}\right)^k C_n \leq B(n,k) \leq \left(\frac{3}{4}\right)^k C_n. \tag{2.39}$$

Proof. In regards to the ratios increasing, by the equivalences established in Lemma 2.12 it is enough to prove that $R_{NK}(n-1,k-1) < R_{NK}(n,k)$. Using the recurrence of the Riordan numbers (Lemma 2.13), we calculate

$$\begin{aligned} R_{NK}(n,n) &= \frac{B(n+1,n+1)}{B(n,n)} \\ &= \frac{n}{n+2}\left(2\frac{B(n,n)}{B(n,n)} + 3\frac{B(n-1,n-1)}{B(n,n)}\right) \\ &= \frac{n}{n+2}\left(2 + \frac{3}{R_{NK}(n-1,n-1)}\right) \end{aligned}$$

Letting $x = R_{NK}(n-1,n-1)$, we see that proving $R_{NK}(n-1,n-1) < R_{NK}(n,n)$ reduces to solving the inequality

$$x < \frac{n}{n+2}\left(2 + \frac{3}{x}\right). \tag{2.40}$$

The solution set is

$$x \in \left(\frac{n - \sqrt{2n(2n+3)}}{n+2}, \frac{n + \sqrt{2n(2n+3)}}{n+2}\right). \tag{2.41}$$

When $n \geq 7$, the inequality

$$\frac{3n}{n+3/2} < \frac{n + \sqrt{2n(2n+3)}}{n+2}$$

shows (using Lemma 2.16) that $R_{NK}(n-1,k-1)$ is a solution of (2.40), and hence induction proves that $R_{NK}(n-1,n-1) < R_{NK}(n,n)$.

Assume that $R_{NK}(n-1,k-1) < R_{NK}(n,k)$ and $R_{NK}(n,k) < R_{NK}(n+1,k+1)$. Lemma 2.12 shows that $R_K(n,k-1) < R_K(n+1,k)$ and $R_N(n,k) < R_N(n+1,k+1)$. Hence, simplifying $R_K(n,k-1)R_N(n,k) < R_K(n+1,k)R_N(n+1,k+1)$ shows that $R_{NK}(n,k-1) < R_{NK}(n+1,k)$. We check that $R_{NK}(6,k) < R_{NK}(7,k+1)$ for $0 \leq k \leq 6$. Following the proof of Corollary 2.17, we use induction (with the infinitely many base cases $n = 7$ and $k = n$) to show $R_{NK}(n-1,k-1) < R_{NK}(n,k)$ for all $n \geq 7$ and $1 \leq k \leq n$.

Using the monotonicity of R_K and Corollary 2.17, we observe that

$$\frac{3(n-1)}{2(2n-1)} = \frac{B(n,1)}{B(n,0)} \leq \frac{B(n,k+1)}{B(n,k)} \leq \frac{3}{4}.$$

We then calculate (recall $B(n,0) = C_n$)

$$B(n,k) = \frac{B(n,k)}{B(n,k-1)} \frac{B(n,k-1)}{B(n,k-2)} \cdots \frac{B(n,2)}{B(n,1)} \frac{B(n,1)}{B(n,0)} C_n,$$

from which follows (2.39). □

Lemma 2.21. *The number N_n of elements of Thompson's Group F represented by a reduced pair of n-caret trees satisfies*

$$\lim_{n\to\infty} \frac{N_n}{(8+4\sqrt{3})^n} = 0.$$

Proof. The preceding lemma gives the estimate $B(n,k) < \left(\frac{3}{4}\right)^k C_n$. We then calculate

$$\lim_{n\to\infty} \frac{B(n,k)}{4^{n-k}3^k} \leq \lim_{n\to\infty} \frac{\left(\frac{3}{4}\right)^k C_n}{4^{n-k}3^k} = \lim_{n\to\infty} \frac{C_n}{4^n} = 0,$$

and

$$\begin{aligned}
N_n &= \sum_{k=1}^{\lceil n/2 \rceil} T(n,k)B(n,k) \\
&< \sum_{k=1}^{\lceil n/2 \rceil} 2^{n-2k+1} \binom{n-1}{n-2k+1} C_{k-1} C_n \left(\frac{3}{4}\right)^k \\
&< 2^{n+1} C_n \sum_{k=1}^{\lceil n/2 \rceil} \left(\frac{3}{4}\right)^k \binom{n-1}{2(k-1)} \\
&= \frac{2^n \cdot 6 \cdot C_n}{4} \sum_{k=1}^{\lceil n/2 \rceil} \binom{n-1}{2(k-1)} \left(\sqrt{\frac{3}{4}}\right)^{2(k-1)} \\
&< \frac{2^n \cdot 6 \cdot C_n}{4} \sum_{i=0}^{n-1} \binom{n-1}{i} \left(\sqrt{\frac{3}{4}}\right)^{i} \\
&< \frac{2^n \cdot 6 \cdot C_n}{4} \left(1 + \frac{\sqrt{3}}{2}\right)^{n-1}
\end{aligned}$$

We finally compute

$$\lim_{n\to\infty}\frac{N_n}{\left(8+4\sqrt{3}\right)^n} \le \lim_{n\to\infty}\frac{\frac{2^n\cdot 6\cdot C_n}{4}\left(1+\frac{\sqrt{3}}{2}\right)^{n-1}}{\left(8\left(1+\frac{\sqrt{3}}{2}\right)\right)^n} \le \lim_{n\to\infty}\frac{C_n}{4^n} = 0.$$

□

Having found an asymptotic upper bound for N_n, we conjecture that the best asymptotic upper bound of the form $N_n < c^n$ for some constant c occurs when $c = 8 + 4\sqrt{3}$.

Conjecture 2.22. *For every small $\epsilon > 0$ ($\epsilon < 8 + 4\sqrt{3}$),*

$$\lim_{n\to\infty}\frac{N_n}{\left(8+4\sqrt{3}-\epsilon\right)^n} = \infty.$$

Numerically (for at least all $n \le 200$, after which computations become time intensive) it appears that N_n grows at the rate

$$\frac{\left(8+4\sqrt{3}\right)^n}{n^3}.$$

The n^3 appears by replacing the two instances of the Catalan numbers in the sum used to calculate N_n with Sterling's formula estimates. The C_n term contributes an immediate $n^{3/2}$, while it is not clear how C_{k-1} contributes as it is involved in an alternating sum. Further investigation is needed to determine the exact growth rate of N_n.

2.3 Tree Functions

In this section we give an alternative way of viewing caret trees. We use this description to give a proof that the number $B(n, k)$ of n-caret trees that can be paired (resulting in a reduced tree pair) with a given n-caret tree with k exposed carets depends only on n and k. We define a split set to be a set of integers with no pair of consecutive integers. The counts involving $B(n, k)$ depend on the fact that a set of positions of exposed carets is split. The study of split sets led to interesting relationships to continued fractions, due to Jim Cannon.

First, we can view an n-caret tree as an ordered list of n horizontal rays from $-\infty$ to some given height, with the property that any two rays with the same height must have a taller ray in between them. Starting with an n-caret tree, draw all of the leaves down so they intersect the same horizontal line. Think of the white space between lines as channels, and number the channels from left to right (our convention being to start the numbering at 1, see Figure 3). Note that each channel is topped off above by a caret, and the caret number is the same as the channel number. Through the bottom of each channel we draw a vertical ray upwards and stop when the height of the ray is the same as the height of the vertex of the corresponding caret. This gives the vertical rays, with the ordering being left to right. The rays we can think of as functions.

Definition 2.23. A tree function (or n-tree function) is a map $t : \{1, 2, \ldots, n\} \to \mathbb{R}$ with the following property

- If $t(x) = t(y), x \neq y$, then there is a z between x and y with $t(z) > t(x)$.

We say that tree functions s and t are equivalent if there exists a homotopy of tree functions from s to t. This is an equivalence relation, and the corresponding set of equivalence classes is in a bijective correspondence with the set of caret trees.

One reason for using tree functions is the ease of recognizing an exposed caret in the tree function description. For $2 \leq i \leq n-1$, caret i is exposed if and only if $t(i)$ is less than $t(i+1)$ and $t(i-1)$, i.e., $t(i)$ is a local minimum. Similar results hold for carets 0 and n, by just checking one side. This makes spotting exposed carets, inserting exposed carets, and removing exposed carets a visually local problem. In the caret tree representation, consecutive carets may be situated very far from each other. Another nice property of tree functions is that we can slide a local minimum (an exposed caret) downward to be as low as we would like, due to the homotopy equivalence.

Definition 2.24. We say a caret tree (tree function) covers a set $X \subset \{1, 2, \ldots, n\}$ if none of the carets in X are exposed.

Definition 2.25. We say a set $X \subset \{1, 2, \ldots, n\}$ is split if for every $i, j \in X$, $i \neq j$, we have $|i - j| \geq 2$.

Definition 2.26. We define $B(n, X) =$ the number of n-caret trees that cover X.

Definition 2.27. We say X and X' are adjacent if $|X| = |X'|$ and $X \setminus \{x\} \cup x' = X'$, where $|x - x'| = 1$.

In Lemma 2.7, we calculated $B(n, X)$ when X was split. We found that $B(n, X)$ depends only on n and $k = |X|$. We now give an alternative proof of that fact.

Theorem 2.28. *If X and X' are two split sets of the same cardinality, then $B(n, X) = B(n, X')$.*

Proof. We may assume that X and X' are adjacent, since any two split sets can be connected through a finite number of adjacent sets. We now construct a bijection φ from the set of n-caret trees that cover X to the set of n-caret trees which cover X'. Remember that X and X' differ only in elements x and x', and without loss of generality we can assume $x + 1 = x'$. Let t be a tree function which covers X. If t covers X', then let $\varphi(t) = t$. If t does not cover X', then $t(x')$ is a local minimum. Via homotopy, adjust the tree function t so that $t(x') < t(y)$ for all $y \neq x'$. We then create a new tree function t' by saying $t'(x) = t(x'), t'(x') = t(x)$, and $t'(y) = t(y)$

for $y \neq x, x'$. Hence t' has a local minimum at x (meaning t' does not cover X), and t' covers X'. Let $\varphi(t) = t'$. As swapping consecutive carets is an invertible process, φ is a bijection. □

2.4 Counting the number of n-caret trees with a prescribed set of exposed carets

In Theorem 2.7 we calculated $B(n, X)$, when X was split. We now calculate $B(n, X)$ if X is not split. This will tell us precisely how many n-caret trees have no exposed carets with numbers in X. Using this count, we also give a formula for counting the number of n-caret trees that have a prescribed split set of positions of exposed carets. The count is highly dependent on the positions of the carets, which shows that the results of Lemma 2.7 are rather surprising.

We begin by generalizing the binomial coefficients. Let $\binom{X}{i}$ be the number of subsets of X of cardinality i which are split. Note that if X is split, then $\binom{X}{i} = \binom{|X|}{i}$. Let $H(n, T)$ be the number of n-caret trees having at least T as its set of numbers of exposed carets. If T is split, then $H(n, T) = C_{n-|T|}$ (Lemma 2.5). If T is not split, then $H(n, T) = 0$. Using an inclusion-exclusion argument as in the proof of Theorem 2.7, we find that

$$\begin{aligned} B(n, X) &= \sum_{i=0}^{k} \left((-1)^i \sum_{T \subset X; |T| = i} H(n, T) \right) \\ &= \sum_{i=0}^{k} \left((-1)^i \sum_{T \subset X; |T| = i, T \text{ is split}} C_{n-i} \right) \\ &= \sum_{i=0}^{k} (-1)^i \binom{X}{i} C_{n-i} \end{aligned}$$

We are now left with the question, how many subsets of cardinality i of a set $X \subset \{1, 2, \ldots, n\}$ are split? Clearly the answer depends only on the configuration of the clusters of consecutive integers in X. Let $S = (s_1, s_2, s_3, \ldots)$ be a sequence of integers which is eventually all zero. Associated with every set X is a sequence $S_X = (s_1, s_2, s_3, \ldots)$ where s_j denotes the number of clusters of j consecutive integers. For example, if $X = \{1, 2, 4, 6, 7, 8, 9, 13, 14\}$, then $S_X = (1, 2, 0, 1, 0, 0, \ldots)$. When restricted to a cluster of m consecutive integers, we enumerated the number of split subsets of cardinality j when we enumerated, in Lemma 2.4, the number $A(m, j)$ of possible configurations of j exposed carets for m-caret trees. Let $A_m(y)$ be the generating function which counts the number of split subsets of cardinality j of m consecutive integers, namely $A_m(y) = \sum_{j=0}^{m} A(m, j)y^j$. Now given a set $X \subset \{1, 2, \ldots, n\}$, we construct S_X and notice that the number of split subsets of cardinality j in each cluster is independent of other clusters. This proves the following lemma.

Lemma 2.29. *The generating function for the number of split subsets of $X \subset \{1, 2, \ldots, n\}$ of cardinality j is given by*

$$\sum_{j=0}^{\infty} \binom{X}{j} y^j = X(y) = \prod_{j=0}^{\infty} A_j(y)^{s_j},$$

where we assume $A_0(y) = 1$. The sum and product are finite since $\binom{X}{j} = 0$ for large j and S_X is eventually all zero.

We already saw that there are exactly $\binom{n-1}{i-1}$ n-caret trees that have caret i as the only exposed caret (where we number carets from left to right beginning at 1). We now give a formula for calculating the number of n-caret trees having precisely the set I as the positions of its exposed carets.

We begin with an example. Suppose $n = 10$ and $I = \{4, 6, 9\}$. Remove the 3 exposed carets to get a 7-caret tree. To get back the original tree, we would have to add back a caret onto each of the leaves with numbers in the set $\{3, 4, 6\} =$

$\{4-1, 6-2, 9-3\}$. What are the positions of exposed carets of a 7-caret tree T which, after adding a caret to leaves 3,4, and 6, will result in a 10-caret tree with exposed carets numbered by I? Since leaf 3 belongs to either caret 3 or caret 4, either carets 3 or 4 of T can be exposed. Similarly, either caret 4 or 5, and either caret 6 or 7 can be exposed in T. Hence, a seven caret tree with exposed carets having positions in the set $\{3, 4, 5, 6, 7\}$ becomes a 10-caret tree with exposed caret set I after adding the three carets back to the appropriate leaves. Hence, the number of 10 caret trees with exposed carets $\{4, 6, 9\}$ is equal to the number of 7-caret trees that cover (have no exposed carets in) $\{1, 2\}$, i.e., $B(7, \{1, 2\})$. The general case is similar.

Theorem 2.30. *The number of n-caret trees that have the split set $I = \{i_1, i_2, \ldots, i_k\}$ ($i_i < i_2 < \cdots < i_k$) as the positions of its exposed carets is $B(n-k, X_I)$, where X_I is the complement in $\{1, \ldots, n-k\}$ of the set $\{i_1 - 1, i_1, i_2 - 2, i_2 - 1, i_3 - 3, i_3 - 2, \ldots, i_k - k, i_k - k + 1\}$.*

It is interesting to note that since $\binom{X_I}{j}$ depends only on the size of clusters of consecutive integers, then a slightly simpler algorithm (which avoids taking complements) can be developed for finding $X_I(y) = \sum_{j=0}^{n} \binom{X_I}{j} y^j$, which is needed to compute $B(n - |I|, X_I)$. There are at most $k+1$ clusters of consecutive integers in X_I, and the size of each cluster depends on the distance between consecutive elements of I. The sizes of clusters of consecutive integers can be calculated using $i_1 - 2, i_2 - i_1 - 3, i_3 - i_2 - 3, \ldots, i_k - i_{k-1} - 3, n - i_k - 1$, where a -1 means that the size of the cluster is zero. Letting $A_{-1}(y) = 1$, we compute

$$X_I(y) = \sum_{j=0}^{n} \binom{X_I}{j} y^j = A_{i_1-2}(y) A_{n-i_k-1}(y) \prod_{p=2}^{k} A_{i_p - i_{p-1} - 3}(y).$$

2.5 How Multiplication Affects the Number Of Carets

Near the end of our summer research group, Sharleen De Gaston suggested counting the average number of carets that are on the left edge of an n-caret tree, as this count is directly related to multiplication by the generator x_0. Using Fordham's notation and labels for caret types, these carets are labeled $L_\emptyset$ and L_L (a reader unfamiliar with this notation should look ahead in Section 3.1 for a brief description before continuing). Interchanging the label $L_\emptyset$ on the first caret with the label L_L on the root caret, we see that the number of carets on the left edge of the tree (excluding the root) is the same as the number of left-lefts (carets with label L_L). Our work, joint with De Gaston, shows that there are only two left-lefts on average on a random n-caret tree.

If $w \in F$ has no left-lefts in the domain tree of its reduced tree pair, then the number of carets in a reduced tree pair for wx_0 is greater than the the number of carets in the tree pair for w. Using Blake's Theorem (Theorem 3.1), this implies that the word length of wx_0 is greater than the word length of w in the standard two generator presentation. If w has a left-left in the domain tree of its reduced tree pair, then wx_0 has one less left-left in the domain tree of its reduced tree pair. Hence, if the number of left-lefts in the domain tree of the reduced tree pair for w is k, then $|wx_0^k| < |wx_0^{k+1}| < |wx_0^{k+2}| < \cdots$. Hence, counting the average number of left-lefts is related to how many times, on average, we may right multiply by x_0 before increasing the number of carets, and hence increasing the word length.

The generator x_1 acts on the right subtree of the domain tree of a reduced tree pair in an entirely similar way as x_0. The number of left-lefts on the right subtree of an n-caret tree we will call interior left-lefts. The number of interior left-lefts in the reduced tree pair for an element $w \in F$ tells us how many times we may right multiply by x_1 before increasing the number of carets, and hence increasing word length.

Our work explicitly counts the average number of left-lefts and the average number of interior left-lefts in an n-caret tree. We outline our work, using De Gaston's notation.

Theorem 2.31 (De Gaston's Theorem)**.** *The average number of left-lefts in an n-caret tree is*

$$A_{K_n} = \frac{2(n-1)}{n+2}.$$

The average number of interior left-lefts in an n-caret tree is

$$W_n = \frac{5n^2 - 15n + 10}{4n^2 + 6n - 4}.$$

Proof. Let K_n be the total number of left-lefts in all n-caret trees. Show that

$$K_n = \sum_{i-1}^{n-1} C_{n-1-i}(K_i + C_i).$$

Hence the average number of left-lefts is

$$A_{K_n} = \frac{1}{C_n} \sum_{i-1}^{n-1} C_{n-1-i}(K_i + C_i).$$

Numerical evidence suggests the result $A_{K_n} = \frac{2(n-1)}{n+2}$. Verify the identity

$$\frac{2(n-1)}{n+2} C_n = \sum_{i-1}^{n-1} C_{n-1-i}(K_i + C_i)$$

by computing the generating function of both sides, and noticing the generating functions are equal.

For interior left-lefts, let V_n be the total number of interior left-lefts in all n-caret trees. Show that

$$V_n = \sum_{i=1}^{n-1} C_{n-i-1} K_i$$

so that

$$W_n = V_n/C_n = \frac{1}{C_n}\sum_{i=1}^{n-1} C_{n-i-1}K_i.$$

We then use the previous calculations for A_{K_n} to write

$$\frac{2(n-1)}{n+2} = W_n + 1 - \frac{C_{n-1}}{C_n}.$$

Solve for W_n and use the recurrence for C_n to conclude. □

De Gaston's Theorem suggests the following conjectures:

Conjecture 2.32. *1. We can multiply by x_0 on average 2 times before increasing the number of carets in reduced tree pairs.*

2. We can multiply by x_1 on average $\frac{5}{4}$ times before increasing the number of carets in reduced tree pairs.

If each n-caret tree appeared equally often as the domain tree in reduced pairs of n-caret trees, then De Gaston's Theorem would immediately imply both of these conjectures. However, we have already shown that an n-caret tree with k exposed carets appears $B(n, k)$ times (see Theorem 2.7) as the domain tree in reduced n-caret tree pairs, and so more work is needed to prove these conjectures.

We now describe some modifications to De Gaston's work. The numbers $B(n, k)$ seem central to counting the average number of left-lefts. We let $L_L(n, k)$ (resp. $IL_L(n, k)$) represent the total number of left-lefts (resp. interior left-lefts) in the set of all n-caret trees with k-exposed carets. Then the average number of left-lefts in an n-caret tree is given by the sum

$$\frac{1}{N_n}\sum_{k=1}^{\lceil n/2\rceil} L_L(n, k)B(n, k)$$

$L_L(n,k)$	n=1	2	3	4	5	6	7	8	9	10
$k=1$	0	1	3	7	15	31	63	127	255	511
2			1	7	30	103	312	873	2314	5899
3					3	31	188	876	3475	12361
4							9	126	1000	5925
5									28	498

Table 3: The numbers $L_L(n,k)$ count the number of left-lefts in an n-caret tree with k exposed carets.

and the average number of interior left-lefts in an n caret tree is

$$\frac{1}{N_n}\sum_{k=1}^{\lceil n/2\rceil} IL_L(n,k)B(n,k)$$

Numerically (for $n \leq 200$), it appears that these averages do monotonically increase to 2 and $\frac{5}{4}$, but a proof is still unknown. We list some partial results.

We start with a recurrence. Let $L_L(0,0) = -1$, and $L_L(n,0) = L_L(0,k) = 0$ for $n, k > 0$. Using the convention $T(n,k) = 1$, we then get the recurrence

$$L_L(n+1,k) = \sum_{m=0}^{n}\sum_{i=0}^{k}\left((L_L(m,i) + T(m,i)) \cdot T(n-m,k-i)\right).$$

Some values of $L_L(n,k)$ are given in Table 3. We conjecture that $L_L(n,k)$ is given by the formula

$$L_L(n,k) = (-1)^k\binom{n+1}{k-1} - T(n,k)$$
$$+\, 2^{n-2k+2}(2k-1)C_{k-1}\left(\sum_{i=0}^{k-1}(-1)^i\frac{\binom{k-1}{i}}{\binom{2k-1}{2i+1}}\left(\binom{n-(2i+1)}{2(k-i-1)} + \binom{n-(2i+1)-1}{2(k-i-1)-1}\right)\right)$$

This conjecture came from taking each row (fix k) of Table 3 and computing its difference triangle. Then take the diagonal of that difference triangle, and compute

1	7	30	103	312	873	2314
	6	23	73	209	561	1441
		17	50	136	352	880
			33	86	216	528
				53	130	312
					77	182
						105

1	6	17	33	53	77	105	137	173	213
	5	11	16	20	24	28	32	36	40
		6	5	4	4	4	4	4	4
			-1	-1	0	0	0	0	0
				0	1	0	0	0	0
					1	-1	0	0	0
						-2	1	0	0
							3	-1	0
								-4	1
									5

Table 4: The first table is the difference triangle of $L_L(n,2)$. The second table is the difference triangle of the diagonal of the first table. Notice the zeros that begin in the fourth row. Similar tables occur for $L_L(n,k)$ for other k.

its difference triangle (see Table 4). After doing this you find that the second difference triangle consists of mostly zeros. We then deduce (for fixed k) the generating function for $L_L(n,k)$, by using the Euler transformation in Lemma 2.14. We then write out $L_L(n,k)$ in terms of binomial coefficients. The generating function from row k is of the form

$$\frac{p_k(x)}{(2x-1)^{2k-1}(x-1)^k}$$

for some polynomial $p_k(x)$. After converting to a partial fraction decomposition, we notice that the coefficients involving $(2x-1)^{2k-1}$ depend on $C(k-1)$ and the terms involving $(x-1)^k$ simplify to $\binom{n+1}{k-1}$. These results are all experimental. The same technique gives a conjectured formula for $IL_L(n,k)$.

3 Fordham's Table

In his Ph.D. dissertation [8], Blake Fordham revolutionized computing the word length of elements of Thompson's Group F, using the standard two generator presentation given in Section 1.2. He gave an algorithm to compute the word length of $w = (D,R)$ based on the reduced n-caret tree pair (D,R). In addition, his algorithm returns a list of generators, which can be rearranged to multiply together and give a minimal word representing w.

This section is mostly review for a reader unfamiliar with the notation used in Fordham's dissertation. In addition we discuss the following:

- We wrote a computer algorithm (Section 3.2) to generate Fordham's Table (Table 5), with the intent to use this algorithm to find a corresponding table using a different generating set for F. Two features of the standard two generator presentation which make it ideal for study are the existence of nor-

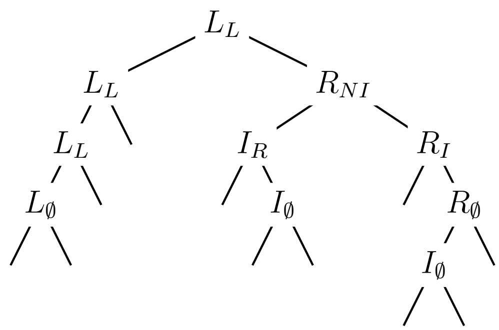

Figure 22: Caret Types. Notice that caret 7 is of type R_{NI}, caret 8 is of type R_I, and caret 9 is an interior caret. If caret i is of type R_I, then caret $i+1$ is always an interior caret.

mal forms and Fordham's Table. The generating set we explored includes an additional generator y_1 which is the mirror image of x_1. We give a set of normal forms using this new generating set. We discuss attempts at finding a corresponding Fordham's Table in 3.3. Using this generating set allows one to immediately see a $\mathbb{Z} \times \mathbb{Z}$ in the Cayley graph of F at every element of F. In addition, the presentation for F becomes shorter with this new generator, and, letting $y_0 = x_0^{-1}$, interchanging x and y is equivalent to mirror imaging the domain and range trees of a reduced tree pair.

In Section 3.1 we provide the basic notation and results of Fordham's work which we will use throughout our paper. In Section 3.2 we describe the computer algorithm we implemented which computes Fordham's Table (Table 5). In Section 3.3 we discuss an alternative, more symmetric, generating set for F, and provide a set of normal forms using these generators. We also discuss finding a corresponding "Fordham's Table" in this new generating set, using our algorithm.

3.1 Notation and Fordham's Theorem

We begin by defining caret types. Let T be a caret tree. We say a caret is a left caret of T if its left edge is part of the leftmost edge of T. We say a caret is a right caret if it is not the root caret (which is a left caret) and its right edge is part of the rightmost edge of T. An interior caret is a caret that is neither a left nor right caret. We further subdivide left, right, and interior carets into the following seven types.

- $L_\emptyset$ — a left caret with no left child.
- L_L — a left caret with a left child.
- $R_\emptyset$ — a right caret with no interior caret to the right.
- R_{NI} — a right caret with an interior caret to the right, but not to the immediate right.
- R_I — a right caret with an interior caret to the immediate right.
- $I_\emptyset$ — a interior caret with no right child.
- I_R — a interior caret with a right child.

Figure 22 gives an example of labeling a caret tree with these caret types.

Let w be an element of F represented by the reduced tree pair (D, R). Let D_i and R_i be the caret types of caret i of D and R, respectively. Associated to each caret type pair (D_i, R_i) is a weight $c(D_i, R_i)$ given in Fordham's Table (Table 5). Fordham's main result in [8] can be restated as follows.

Theorem 3.1 (Fordham's Theorem)**.** *The word length of $w = (D, R)$ in the two generator presentation of Thompson's group F is equal to the sum $\sum_{i=1}^{n} c(D_i, R_i)$,*

	$R_\emptyset$	R_{NI}	R_I	L_L	$I_\emptyset$	I_R
$R_\emptyset$	0	2	2	1	1	3
R_{NI}	2	2	2	1	1	3
R_I	2	2	2	1	3	3
L_L	1	1	1	2	2	2
$I_\emptyset$	1	1	3	2	2	4
I_R	3	3	3	2	4	4

Table 5: Blake's Table. The weight $c(L_\emptyset, L_\emptyset) = 0$. The weight $c(D_i, R_i)$ is the entry in row D_i and column R_i, or column D_i and row R_i, as the table is symmetric.

where (D, R) is a reduced pair of n-caret trees, and $c(D_i, R_i)$ is the weight of the caret type pair (D_i, R_i) from Table 5.

Fordham's Theorem simplifies calculating word length to labeling carets. Remarkably, right multiplication by one of the generators $x_0^{\pm 1}, x_1^{\pm 1}$ alters very few caret type pairs, which allows us to track all possible changes to the word length of w after right multiplication by a generator. Tables 7, 8, 9, and 10 in Section 4 enumerate the effects on word length of multiplication by each generator.

3.2 Discovering Fordham's Table Algebraically

The proof of Fordham's Theorem given in his Ph.D. dissertation, as well as in [8], does not describe the process he used to discover Fordham's Table. We now describe a computer algorithm we wrote to discover Fordham's Table.

Let $CT = \{L_\emptyset, L_L, R_\emptyset, R_{NI}, R_I, I_\emptyset, I_R\}$ be a set of caret types. Let $w \in F$ be represented by a reduced n-caret tree pair (D, R), with D_i and R_i the caret types of caret i in D and R, respectively. Let $f(w) = \sum_{i=1}^{n} c(D_i, R_i)$, where $c(D_i, R_i)$ is an unknown. We then consider the countably infinite linear system of equations $f(w) = |w|$ for all $w \in F$. This system of linear equations has 49 unknowns. Fordham's Theorem says that this linear system of equations is consistent.

We wrote a computer program to solve this linear system of equations. We ordered the elements $w_1, w_2, w_3, \ldots$ of Thompson's Group F so that the word length of w_i is not more than the word length of w_{i+1}. This gives, for each k, a linear system of k equations in 49 unknowns using the equations $f(w_i) = |w_i|$ for $1 \leq i \leq k$. When we considered all the elements with word length no more than 8, a Gaussian elimination scheme, tailored to keep the arithmetic to small integer addition and subtraction, produced Fordham's Table (Table 5). Hence Fordham's Table is easily generated by a computer, once we know what caret types to consider. The genius of Fordham's Table is his choice of caret types, and his faith that a table would appear.

The algorithm we used is completely determined by the set of caret types CT. We can change the set of caret types and carry out the algorithm. For example, the choice to label the root caret as left caret instead of a right caret is not readily obvious. We could use the caret type C to label the root (or center) caret when it is not the first caret, and C_f when it is the first caret. Then our set of caret types would be $CT = \{C, C_f, L_\emptyset, L_L, R_\emptyset, R_{NI}, R_I, I_\emptyset, I_R\}$. After running the algorithm, we find that the columns corresponding to C and L_L are identical, as well as the columns corresponding to $L_\emptyset$ and C_f. Hence the algorithm validates the choice to label the root caret as a left caret instead of a right caret. Alternatively, we could label each root caret with one of the three right caret types. Carrying out the algorithm reveals that such a linear system is inconsistent, and hence our choice of caret types is incorrect. We tried similar experiments with different sets of caret types. Some systems grew too large to efficiently carry out the computations on computer.

3.3 A New Generating Set

Our purpose in using an algorithm to generate Fordham's Table was to prepare ourselves to find a corresponding Fordham's Table for a different generating set. Recall that two standard presentations for Thompson's group F are

$$F_\infty = \left\langle x_0, x_1, x_2, \ldots | x_{k+1} = x_i^{-1} x_k x_i \text{ for } 0 \leq i < k < \infty \right\rangle$$
$$F_2 = \left\langle x_0, x_1 | x_1^{-1} x_2 x_1 = x_3, x_1^{-1} x_3 x_1 = x_4 \right\rangle$$

The element $y_1 = x_0 x_1 x_0^{-2}$ is represented by a reduced tree pair whose domain and range trees are the mirror images of the domain and range trees of the reduced tree pair for x_1 (see Figure 23). Just as applying x_1 can be regarded as a clockwise rotation on the right subtree of the domain tree, applying y_1 can be regarded as a counterclockwise rotation on the left subtree of the domain tree. This suggests that x_1 and y_1 commute, which is easily proved.

We let $y_0 = x_0^{-1}$, and then note that $y_{k+1} = y_0^{-k} y_1 y_0^k$ is represented by a reduced tree pair whose domain and range trees are the mirror images of the domain and range tree of the reduced tree pair for x_{k+1}. Just as applying x_k corresponds to a clockwise rotation, applying y_k can be regarded as a counterclockwise rotation on the kth caret along the left side of the domain tree.

We now consider the following presentation for Thompson's group F

$$F_3 = \left\langle x_0, x_1, y_1 | y_1 = x_0 x_1 x_0^{-2}, x_1^{-1} x_2 x_1 = x_3, x_1^{-1} x_3 x_1 = x_4 \right\rangle.$$

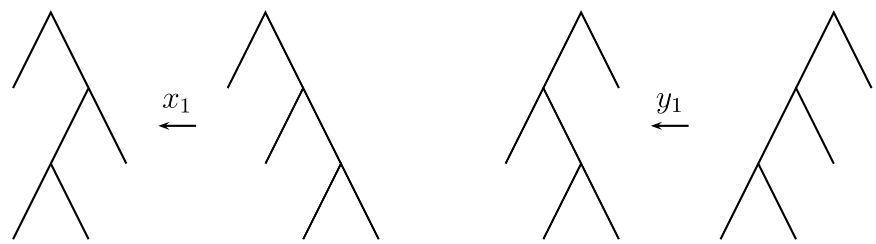

Figure 23: The element y_1 has a reduced tree pair whose domain and range trees are the mirror images of the domain and range trees of the reduced tree pair for x_1.

Note that the relator $x_1^{-1}x_2x_1 = x_3$ can be replaced by the relator $[x_1, y_1]$, as seen in the following calculation.

$$\begin{aligned}
x_1^{-1}x_2x_1x_3^{-1} &= x_1^{-1}x_0^{-1}x_1x_0x_1x_0^{-2}x_1^{-1}x_0^2 \\
&= (x_0^{-2}x_0^2)x_1^{-1}x_0^{-1}x_1x_0x_1x_0^{-2}x_1^{-1}x_0^2 \\
&= x_0^{-2}(x_0^2x_1^{-1}x_0^{-1})x_1(x_0x_1x_0^{-2})x_1^{-1}x_0^2 \\
&= x_0^{-2}y_1^{-1}(x_1y_1)x_1^{-1}x_0^2 \\
&= x_0^{-2}y_1^{-1}y_1x_1x_1^{-1}x_0^2 \\
&= 1
\end{aligned}$$

This allows us to write the presentation F_3 as

$$F_3 = \left\langle x_0, x_1, y_1 | y_1 = x_0x_1x_0^2, [x_1, y_1], x_1^{-1}x_3x_1 = x_4 \right\rangle.$$

We will now discuss advantages and disadvantage of this presentation.

The presentation F_3 contains symmetry that is not available in F_2. For example, interchanging x and y is analogous to mirror imaging domains and ranges of tree pairs. Since x_1 and y_1 commute, the Cayley graph of Thompson's group F using the presentation F_3 contains, at each vertex, a copy of the integer lattice of the plane. We can visualize the Cayley graph as integer lattices, folded together by right multiplication by x_0. One disadvantage of F_3 is that now the presentation contains a relator of odd length. Having an odd length relator complicates measuring the affects of right multiplication by a generator, as now multiplication by a generator may neither increase nor decrease word length.

We found a set of normal forms for F which used the elements y_k. Every element of Thompson's group F has a representation of the form

$$x_{i_1}^{r_1} x_{i_2}^{r_2} \cdots x_{i_m}^{r_m} x_0^a y_{j_1}^{s_1} y_{j_2}^{s_2} \cdots y_{j_n}^{s_n} y_{k_p}^{-t_p} \cdots y_{k_2}^{-t_2} y_{k_1}^{-t_1} x_{l_q}^{-u_q} \cdots x_{l_2}^{-u_2} x_{l_1}^{-u_1},$$

where $r_i, s_i, t_i, u_i > 0$, $i_1 < i_2 < \cdots i_m, j_1 < j_2 < \cdots j_n, k_1 < k_2 < \cdots k_p, l_1 < l_2 < \cdots l_q$, and $a \in \mathbb{Z}$. This normal form is obtained as follows. Given a reduced pair (D, R), let D_L and R_L denote the left subtrees of D and R, respectively. Similarly define D_R and R_R. Compute the exponent of each leaf of D_R and R_R as done in Section 1.2. For each leaf in D_L and R_L, we say that the exponent (in the three generator presentation) is the exponent (in the two generator presentation) computed from a mirror image of D_L and R_L. The number a is equal to the number of carets in D_L minus the number of carets in R_L. The leaves of R_R contribute the positive powers of x_k, while the leaves of D_R contribute the negative powers of x_k. The positive and negative powers of y_k come from the exponents (in the three generator presentation) of the leaves of R_L and D_L, counted from right to left. These normal forms are entirely analogous to the normal form in Section 1.2, we just consider left and right subtrees separately.

Fordham's Theorem relies entirely on the two generator presentation. In Section 3.2 we discussed an algorithm which produced Fordham's Table. The algorithm requires specifying a set CT of caret types, and either says that the choice of caret types results in an inconsistent linear system, or the algorithm runs forever and hopefully completes a table similar to Fordham's Table. We have tried various sets of caret types in our algorithm. To date, every set of caret types we have tried resulted in an inconsistent system. We hope this algorithm can be used to either find a good choice of caret types which gives a Fordham's Table in the three generator presentation, or suggest a proof that no such table exists.

The algorithm of Section 3.2 uses an exhaustive search through every element of Thompson's group based on word length. The same algorithm can be used in any finitely presented group, provided there is an analog to caret types.

4 Dead End Elements

Blake Fordham discovered elements of Thompson's Group F which, when multiplied by any of the four generators $\{x_0^{\pm 1}, x_1^{\pm 1}\}$, resulted in an element of smaller word length [8]. These elements he called dead end elements, or dead end words. If we try to move away from the identity element in the Cayley graph of F, dead end elements are problematic, since once we reach a dead end element we must turn around and move back toward the identity (see Figure 20). Recognizing and avoiding dead elements may be crucial if we hope to prove Thompson's group is nonamenable by modifying the token passing argument used to prove that the free group on two generators is nonamenable (see Figure 18).

In this section we address the following:

- Theorem 4.3 gives a set of conditions on tree pairs which completely characterizes dead end elements in Thompson's Group F. This expounds upon the work of Cleary and Taback in [5], by providing a converse to Theorem 4.1 in their paper.

- The classification of dead end elements allows us to discuss the spread of dead end elements, as well as define equivalence classes and minimal dead end elements (Section 4.2, in particular Lemma 4.4 and Corollary 4.8). In addition, we describe elements we call k-deep roots (Definition 4.11, Section 4.3) which have the geometric property that along a path of length k toward the origin, multiplication by all but one generator reduces word length. Deep roots give an extension of the dead end phenomenon by providing a path of elements where the only way to leave the path is to first reduce word length. We also use the characterization of dead ends to define equivalence classes and minimal deep roots. This is further discussed in Section 4.3.

4.1 A Characterization of Dead End Elements

We begin by stating a result of Sean Cleary and Jennifer Taback which forces a certain structure on dead end elements.

Theorem 4.1 (Cleary and Taback [5]). *All dead end elements in Thompson's group F are given by reduced tree pairs of the form in Figure 24, where the subtrees E, A', and E' are non-empty. Caret b may or may not be the root of the range tree.*

The key step in proving this theorem is to enumerate the conditions under which each generator $\alpha \in \{x_0^{\pm 1}, x_1^{\pm 1}\}$ decreases the word length of an element w (i.e., enumerate when $|w\alpha| < |w|$). The tables listed in [5] at the bottom of page 2836 are incomplete. We will first complete these tables. Then we add an extra condition to the cth caret type pair, from which follows a converse to Theorem 4.1.

Let $w = (D, R)$ be an element of Thompson's group. We will use α to represent one of the generators $\{x_0^{\pm 1}, x_1^{\pm 1}\}$. In his Ph.D. thesis, Blake Fordham gives the following conditions which guarantee that exactly one pair of caret types changes when α is applied to w, provide the tree pair representing $w\alpha$ has the same number of carets as the tree pair (D, R).

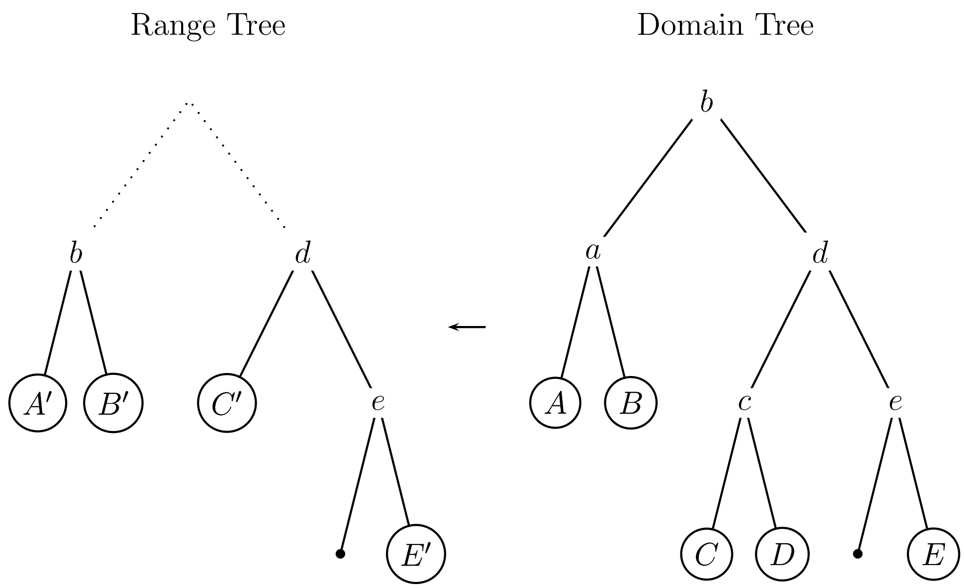

Figure 24: The general tree pair diagram of dead end elements. Capital letters represent (possibly empty) subtrees, while lower case letters label the carets. Caret b in the range tree could possibly be the root. Both E and E' are nonempty. The left subtree of caret e is empty in both the domain and range trees.

1. If $\alpha = x_0$, we require that the left subtree of the root of D is non-empty.

2. If $\alpha = x_0^{-1}$, we require that the right subtree of the root of D is non-empty.

3. If $\alpha = x_1$, we require that the left subtree of the right child of the root of D is nonempty.

4. If $\alpha = x_1^{-1}$, we require that the right subtree of the right child of the root of D is nonempty.

Blake Fordham also showed that if D does not satisfy these conditions with respect to α, then the word length of w increases when α is applied to w. This forces any dead end element to satisfy all four of the conditions listed above.

If $w = (D, R)$ satisfies Fordham's conditions, then there is a unique i for which the caret type pair (C_i, X) changes (or is removed). Table 6 enumerates the possible changes in word length, based on a change from (C_i, X) to (C_i', X).

We now introduce some notation (borrowed from [5]). Let C denote the root caret of the domain tree D. Let C_R denote the right child of C, and C_L the left child of C. Continue this notation inductively, for example C_{RRL} is the left child of C_{RR} which is the right child of C_R. Similarly, let S_R denote the subtree of the domain tree whose root caret is C_R, and S_X denote the subtree with root caret C_X (where X is a string of R's and L's). If the domain tree is missing caret C_X, we say that S_X is empty and write $S_X = \emptyset$.

For each generator $\alpha \in \{x_0^{\pm 1}, x_1^{\pm 1}\}$, we will enumerate the conditions on $w = (D, R)$ which cause a decrease in word length when α is applied to w (i.e., $|w\alpha| < |w|$). We also enumerate the conditions which cause an increase in word length. The following observation aids in enumerating these conditions.

Observation 4.2. The following list restricts the types of certain carets.

C_i	C'_i	Decrease if X is	Increase if X is
L_L	$R_\emptyset$	$R_\emptyset, L_L, I_\emptyset$	R_{NI}, R_I, I_R
L_L	R_{NI}	$L_L, I_\emptyset$	$R_\emptyset, R_{NI}, R_I, I_R$
L_L	R_I	L_L	$R_\emptyset, R_{NI}, R_I, I_\emptyset, I_R$
$I_\emptyset$	$R_\emptyset$	$R_\emptyset, R_I, L_L, I_\emptyset, I_R$	R_{NI}
$I_\emptyset$	R_{NI}	$R_I, L_L, I_\emptyset, I_R$	$R_\emptyset, R_{NI}$
I_R	R_I	$R_\emptyset, R_{NI}, R_I, L_L, I_\emptyset, I_R$	—
$R_\emptyset$	L_L	R_{NI}, R_I, I_R	$R_\emptyset, L_L, I_\emptyset$
R_{NI}	L_L	$R_\emptyset, R_{NI}, R_I, I_R$	$L_L, I_\emptyset$
R_I	L_L	$R_\emptyset, R_{NI}, R_I, I_\emptyset, I_R$	L_L
$R_\emptyset$	$I_\emptyset$	R_{NI}	$R_\emptyset, R_I, L_L, I_\emptyset, I_R$
R_{NI}	$I_\emptyset$	$R_\emptyset, R_{NI}$	$R_I, L_L, I_\emptyset, I_R$
R_I	I_R	—	$R_\emptyset, R_{NI}, R_I, L_L, I_\emptyset, I_R$

Table 6: Effects on word length based on a caret type change. The effects on word length of a caret type pair change from (C_i, X) to (C'_i, X) are listed above for changes resulting from multiplication by a generator.

Conditions on D		Type change of caret C	Decrease if X is	Increase if X is
empty	nonempty			
S_L	—	Extra carets are added		
S_R	S_L	L_L to $R_\emptyset$	$R_\emptyset^*, L_L$	—
S_{RL}, S_{RR}	S_L, S_R	L_L to $R_\emptyset$	$L_L, I_\emptyset$	—
S_{RL}	S_L, S_{RR}	L_L to $R_\emptyset$	$L_L, I_\emptyset$	R_{NI}, R_I, I_R
S_{RL}	S_L, S_{RR}	L_L to R_{NI}	$L_L, I_\emptyset$	$R_\emptyset, R_{NI}, R_I, I_R$
—	S_L, S_{RL}	L_L to R_I	L_L	$R_\emptyset, R_{NI}, R_I, I_\emptyset, I_R$

Table 7: The effects of multiplication by x_0 on word length. When x_0 is applied to $w = (D, R)$, either extra carets are added or the caret type pair (C, X) changes or is removed. *The number of carets may decrease if $X = R_\emptyset$.

- The caret type pair $(R_\emptyset, R_\emptyset)$ can only occur as the last pair.
- The caret type R_{NI} cannot be the type of the last three carets.
- The caret type R_I cannot be the type of the last two carets.
- The caret type I_R cannot be the type of the last two carets.
- The caret type $I_\emptyset$ cannot be the type of the last caret.

Using Table 6 and the previous observation, we list the effects of multiplying w by each generator in Tables 7, 8, 9, and 10. Some pairings are purposely omitted, as they cannot occur. Note that because the length of each relator in the two generator presentation for F is even, every generator either increases word length or reduces word length when applied to w.

Tables 7, 8, 9, and 10 expand the tables provided in [5]. Of particular interest

Conditions on D		Type change	Decrease	Increase
empty	nonempty	of caret C_R	if X is	if X is
S_R	—	Extra carets are added		
S_L, S_{RL}	S_R	$R_\emptyset$ to L_L	R_{NI}, R_I, L_L^*, I_R	$I_\emptyset$
		R_{NI} to L_L	$R_\emptyset, R_{NI}, R_I, L_L^*, I_R$	$I_\emptyset$
		R_I to L_L	$R_\emptyset, R_{NI}, R_I, L_L^*, I_\emptyset, I_R$	—
All others		$R_\emptyset$ to L_L	R_{NI}, R_I, I_R	$R_\emptyset, L_L, I_\emptyset$
		R_{NI} to L_L	$R_\emptyset, R_{NI}, R_I, I_R$	$L_L, I_\emptyset$
		R_I to L_L	$R_\emptyset, R_{NI}, R_I, I_\emptyset, I_R$	L_L

Table 8: The effects of multiplication by x_0^{-1} on word length. When x_0^{-1} is applied to $w = (D, R)$, either extra carets are added or the caret type pair (C_R, X) changes or is removed. *The number of carets may decrease if $X = L_L$.

Conditions on D		Type change	Decrease	Increase
empty	nonempty	of caret C_{RL}	if X is	if X is
S_{RL}	—	Extra carets are added		
S_{RR}, S_{RLR}	S_{RL}	$I_\emptyset$ to $R_\emptyset$	$R_\emptyset^*, L_L, I_\emptyset$	—
S_{RLR}	S_{RL}, S_{RR}	$I_\emptyset$ to $R_\emptyset$	$R_I, L_L, I_\emptyset, I_R$	R_{NI}
S_{RLR}	S_{RL}, S_{RR}	$I_\emptyset$ to R_{NI}	$R_I, L_L, I_\emptyset, I_R$	$R_\emptyset, R_{NI}$
—	S_{RLR}	I_R to R_I	$R_\emptyset, R_{NI}, R_I, L_L, I_\emptyset, I_R$	—

Table 9: The effects of multiplication by x_1 on word length. When x_1 is applied to $w = (D, R)$, either extra carets are added or the caret type pair (C_{RL}, X) changes or is removed. *The number of carets may decrease if $X = R_\emptyset$.

Conditions on D		Type change	Decrease	Increase
empty	nonempty	of caret C_R	if X is	if X is
S_{RR}	—	Extra carets are added		
S_{RL}, S_{RRL}	S_{RR}	$R_\emptyset$ to $I_\emptyset$	$R_{NI}, I_\emptyset^*$	$R_I, L_L, I_\emptyset^*, I_R$
S_{RL}, S_{RRL}	S_{RR}	R_{NI} to $I_\emptyset$	$R_\emptyset, R_{NI}, I_\emptyset^*$	$R_I, L_L, I_\emptyset^*, I_R$
S_{RRL}	S_{RR}, S_{RL}	$R_\emptyset$ to $I_\emptyset$	R_{NI}	$R_\emptyset, R_I, L_L, I_\emptyset, I_R$
S_{RRL}	S_{RR}, S_{RL}	R_{NI} to $I_\emptyset$	$R_\emptyset, R_{NI}$	$R_I, L_L, I_\emptyset, I_R$
—	S_{RRL}	R_I to I_R	—	$R_\emptyset, R_{NI}, R_I, L_L, I_\emptyset, I_R$

Table 10: The effects of multiplication by x_1^{-1} on word length. When x_1^{-1} is applied to $w = (D, R)$, either extra carets are added or the caret type pair (C_R, X) changes or is removed. *The word length decreases if both $X = I_\emptyset$ and the number of carets decreases. Otherwise, the word length increases when $X = I_\emptyset$.)

is the table for x_1^{-1}, where the caret type pair $(C_R, I_\emptyset)$ may result in an increase or decrease, depending on whether or not $I_\emptyset$ is exposed.

The changes made in these tables do not affect Cleary and Taback's proof that a dead end element has the form of Figure 24. We provide the details for completeness. Suppose w is a dead end element. By Fordham's conditions, we have $S_L, S_R, S_{RL}, S_{RR} \neq \emptyset$. Table 7 shows that C is paired with type L_L, hence A' is nonempty and caret b in R is a left caret (possibly the root caret). Table 10 shows that C_R is paired with either $R_\emptyset$ or R_{NI}, and that $S_{RRL} = \emptyset$. Since caret d (in the range tree) is not of type R_I (as the pair (C_R, R_I) means an increase in Table 10), we get an empty subtree on the left of caret e in R. If either E or E' is empty, then so is the other. Since caret e cannot be exposed, this means that both E and E' are nonempty, and that at least one of them contains an interior caret. This completes a proof of Theorem 4.1.

Theorem 4.3. *An element $w = (D, R)$ of Thompson's group F is a dead end element if and only if the reduced tree pair representing w has the form of Figure 24, and caret C_{RL} satisfies either of the following conditions.*

1. *C_{RL} is of type I_R.*

2. *C_{RL} is of type $I_\emptyset$ and is not paired with $R_\emptyset$ or R_{NI}.*

Proof. Suppose w is a dead end element. Then the reduced tree pair of w has the form of Figure 24. Table 9 shows that if C_{RL} is of type I_R, then it can be paired with any caret. In addition, if C_{RL} is of type $I_\emptyset$, then it cannot be paired with $R_\emptyset$ or R_{NI}. Notice that in Table 9, when $S_{RLR} = \emptyset$, $S_{RL}, S_{RR} \neq \emptyset$, and $I_\emptyset$ changes to $R_\emptyset$, the caret type $X = R_\emptyset$ is omitted from both the decrease and increase list. It is omitted precisely because if C_{RL} were originally paired with $R_\emptyset$ (prior to applying

x_1), then caret C (which is of type $R_\emptyset$ since C_{RL} changes to type $R_\emptyset$) would also be paired with $R_\emptyset$. This is impossible since $(R_\emptyset, R_\emptyset)$ can only appear as the last pair of caret types. This small observation is central to the proof.

Now suppose w is of the form provided in Figure 24 and satisfies either condition. We show that w is a dead end element. As C is paired with L_L, $|wx_0| < |w|$ by Table 7. Since the caret type pair associated with the dth carets cannot be $(R_\emptyset, R_\emptyset)$, Table 8 shows that $|wx_0^{-1}| < |w|$ as X is a right caret. The extra conditions added to the lemma force $|wx_1| < |w|$ by Table 9. Since the caret type pair associated with the dth carets cannot be $(R_\emptyset, R_\emptyset)$, and since the left subtree of caret e in R is empty implies that caret d of R is not of type R_I, Table 10 gives $|wx_1^{-1}| < |w|$. □

4.2 The Spread of Dead Ends

In the previous section we classified all the dead end elements in Thompson's group F. We now use this classification to partition dead end elements into equivalence classes. The equivalence relations are described in Corollaries 4.6 and 4.9. We start with the following lemma.

Lemma 4.4. *If w is a dead end element of Thompson's group F, then the element $wx_0^{-2}x_1^kx_0^2 = wx_3^k$ is also a dead end element for all but at most one $k \in \mathbb{Z}$.*

Proof. Since $w = (D, R)$ is a dead end element, subtrees S_L, S_R and S_{RR} are nonempty. This means that the reduced tree pair $(\hat{D}, \hat{R})$ for wx_0^{-2} has the same number of carets as the reduced tree pair for w. The subtrees E and E' from Figure 24 form a reduced tree pair as E and E' have the same number of carets. Let y be the element of F whose reduced tree pair is (E, E'). Applying x_1^k to wx_0^{-2} affects only the right subtree of the root of $\hat{D}$, which is E, and the right subtree of the root of $\hat{R}$, which is E'. Hence, we can think of applying x_1^k to wx_0^{-2} as replacing

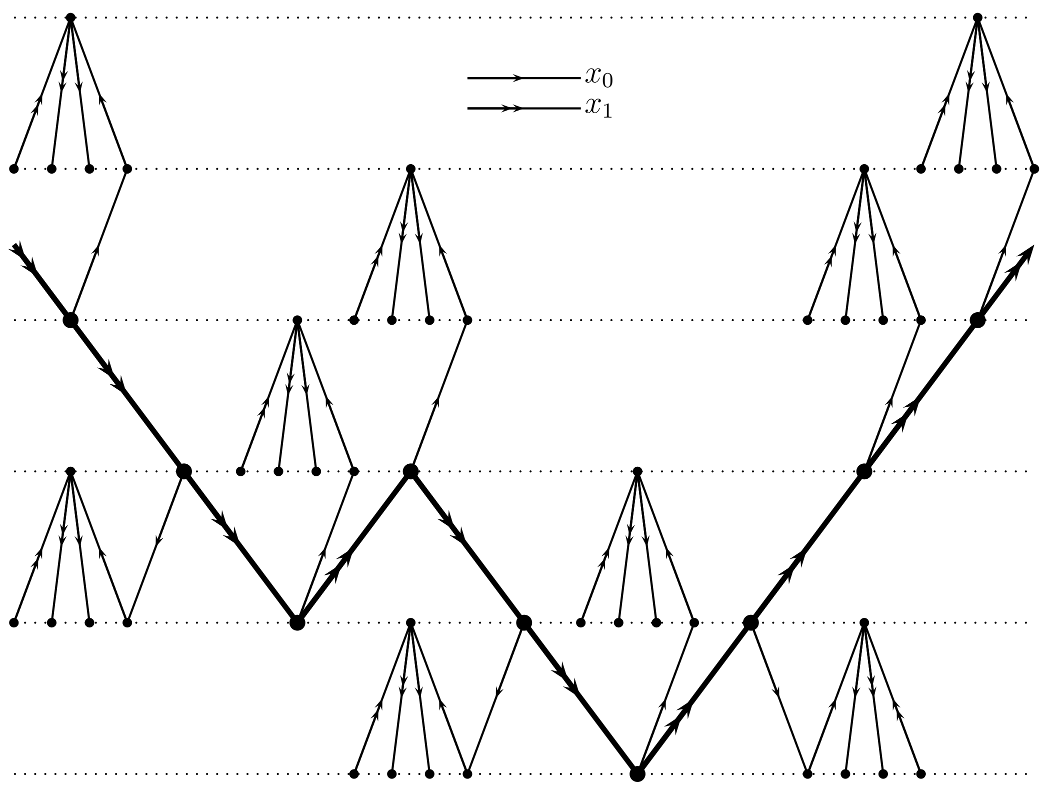

Figure 25: The local spread of dead end elements in the (vertically drawn) Cayley graph of F. A single arrow represents multiplication by x_0. Double arrows represent multiplication by x_1. Inverse multiplication is achieved by following an arrow backwards. From every vertex (except possibly one) on the thickened path, multiplying by x_0^2 results in a dead end element.

the right subtrees of $\hat{D}$ and $\hat{R}$ with the domain and range trees of the reduced tree pair for the product yx_0^k. If yx_0^k is the identity, then we erase the right subtrees of $\hat{D}$ and $\hat{R}$. We then apply x_0^2 to $wx_0^{-2}x_1^k$. If yx_0^k is the identity, then the reduced tree pair diagram for $wx_0^{-2}x_1^kx_0^2$ has an empty subtree S_{RR}, hence is not a dead end word. Otherwise, $wx_0^{-2}x_1^kx_0^2$ is identical to w prior to caret e which implies that it is a dead end element. As yx_0^k can be the identity for at most one k, $wx_0^{-2}x_1^kx_0^2$ is a dead end element for all but at most one k. □

Given a dead end element $w \in F$, each dead end element of the form $wx_0^{-2}x_1^kx_0^2$ is distance two (using the word metric) from a path P formed by connecting $wx_0^{-2}x_1^k$ to $wx_0^{-2}x_1^{k+1}$ for $k \in \mathbb{Z}$. A possible local Cayley graph structure (drawn vertically) of some dead end elements is shown in Figure 25. Notice that the difference in word length between wx_3^k and wx_3^{k+1} is ± 1 or ± 3. As k increase, the path P eventually always moves upwards in the vertically drawn Cayley graph. This can be shown by noticing that eventually the left subtree of the caret C_R of $wx_0^{-2}x_1^k$ will be empty after applying x_1 enough times. Once the left subtree of C_R is empty, every subsequent application of x_1 will require the addition of a new caret, and hence the word length will increase. By a similar argument, the path P moves upwards for large negative k as k decreases. The previous argument validates the following.

Corollary 4.5. *There is an n (perhaps not unique) for which $wx_0^{-2}x_1^n$ has minimal word length. Alternatively, there is a m (perhaps not unique) for which wx_3^m has minimal word length.*

We naturally ask, is n or m unique? An answer to this question is further motivated by the following equivalence relation.

Corollary 4.6. *We introduce an equivalence relation on the set of dead end elements by saying that w and w' are equivalent if there exists a k such that $w' = wx_3^k$.*

If either n or m from Corollary 4.5 is unique, this allows us to associate with each equivalence class of dead end elements a unique dead end element representative. The usefulness of such a representative may help in discovering Følner sets in F.

In the proof of Lemma 4.4, the element x_1 was used because it affects only the right subtrees of $\hat{D}$ and $\hat{R}$.

Observation 4.7. Let $w = (D, R)$ be any element of F. Applying an element $z \in F$ to w affects only the right subtree of D if and only if z can be written as a product of elements $x_j^{\pm 1}$ for $j \geq 1$ (i.e, $z = \prod_{i=1}^{n} x_{j_i}^{\epsilon_i}$ for $j_i \geq 1$ and $\epsilon_i = \pm 1$). In other words, applying z to w affects only the right subtree of D if and only if z can be written as a word in the standard infinite presentation for F without using x_0 or x_0^{-1}.

This observation allows us to strengthen Lemma 4.4, by replacing x_1^k with any z which affects only the right subtree of the domain tree of its reduced pair.

Corollary 4.8. *If w is a dead end element of Thompson's group F, then the element $wx_0^{-2}zx_0^2$ (where z is as given in Observation 4.7) is also a dead end element for all but exactly one such z.*

Proof. First, write $z = \prod_{i=1}^{n} x_{j_i}^{\epsilon_i}$ for $j_i \geq 1$ and $\epsilon_i = \pm 1$. Let $z' = \prod_{i=1}^{n} x_{(j_i - 1)}^{\epsilon_i}$. We then repeat the proof of Lemma 4.4 with one modification. We replace the right subtrees of $\hat{D}$ and $\hat{R}$ with the reduced tree pair for the product yz'. If $z' = y^{-1}$, then $wx_0^{-2}zx_0^2$ is not a dead end element. □

The previous corollary gives us an equivalence relation on the set of dead end elements by saying w and w' are equivalent if there exists a z as in Observation 4.7 such that $w' = wx_0^{-2}zx_0^2$. These equivalence classes also contain minimal word length elements. By counting the number of carets, the choice of z for which $wx_0^{-2}zx_0^2$ is not a dead end satisfies $|wx_0^{-2}zx_0^2| < |wx_0^{-2}z'x_0^2|$ for any other choice of z (where $|x|$ is the word length). This is summed up in the following corollary.

Corollary 4.9. *The relation $w \sim w'$ if $w' = wx_0^{-2}zx_0^2$ for some z as giving in Observation 4.7 is an equivalence relation on the set of dead end elements of Thompson's* *group F. If w is some representative of an equivalence class $[w]$ of dead end elements, then the unique element $\hat{w} = wx_0^{-2}zx_0^2$, which is not a dead end element, has smaller word length than every element in $[w]$.*

Even thought $\hat{w}$ is not a dead end element, we will say that $\hat{w}$ is the unique representative of the equivalence class $[w]$. This allows us to reduce our search for dead end elements to elements whose reduced tree pair diagram is of the form in Figure 24, except that caret e is missing.

4.3 Deep Roots

We now describe some interesting elements we call k-deep roots, which appeared from the classification of dead end elements in Theorem 4.3. The extra conditions on C_{RL} were only needed to guarantee that $|wx_1| < |w|$. The following corollary, whose proof is in the proof of Theorem 4.3 will be used to discover the k-deep roots.

Corollary 4.10. *Suppose the reduced tree pair for $w \in F$ has the form of Figure 24. Then $|w\alpha| > |w|$ for a generator $\alpha \in \{x_0^{\pm 1}, x_1^{\pm 1}\}$ implies $\alpha = x_1$. In other words, applying any generator, except possibly x_1, reduces word length.*

Definition 4.11. Let $G = \langle C|R \rangle$ be a finitely presented group with symmetric generating set C (i.e., $g \in C$ implies $g^{-1} \in C$) and Cayley graph $\Gamma = \Gamma(G, C)$. An element $w \in G$ is a k-deep root (with respect to C) if along a shortest path in Γ to the origin from w, every element x on that path, within distance k of x, satisfies $|x\alpha| < |x|$ for all but at most one generator $\alpha \in C$.

An example of a local Cayley graph structure (drawn vertically) of a 3-deep root is shown in Figure 21 (Section 1.3). A k-deep root is the element at the top of the path, however it is convenient to call the entire path of length k a root. We now show that Thompson's group F has k-deep roots for every $k \geq 1$.

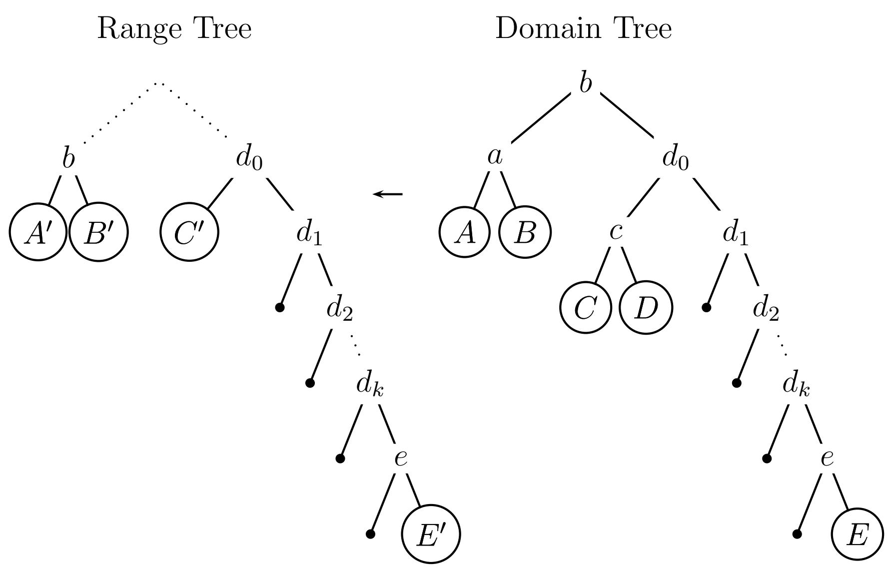

Figure 26: The general tree pair diagram of deep roots. Capital letters represent (possibly empty) subtrees, while lower case letters label the carets. Caret b in the range tree could possibly be the root. Both E and E' are nonempty. The left subtree of carets $d_1, \ldots, d_k$ and e are empty in both the domain and range trees.

Lemma 4.12. *An element $w = (D, R) \in F$ of the form of Figure 26 is a k-deep root. Hence there are k-deep roots for every $k \geq 1$.*

Proof. An element $w = (D, R)$ of the form of Figure 26 has the form of Figure 24. Hence, $|w\alpha| > |w|$ implies $\alpha = x_1$ by Corollary 4.10. Note that wx_1^{-1} is obtained by a counterclockwise rotation on the right subtree of D (see Figure 15). Hence wx_1^{-1} has the form of Figure 24 also, and so $|wx_1^{-1}\alpha| > |wx_1^{-1}|$ again implies $\alpha = x_1$. We repeat this argument to show that $|wx_1^{-i}\alpha| > |wx_1^{-i}|$ implies $\alpha = x_1$ for $1 \leq i \leq k$. This shows that w is a k-deep root, and the path back to the origin is formed by connecting $wx_1^{-(i-1)}$ to wx_1^{-i} for $1 \leq i \leq k$. □

Combining this lemma with the classification of dead end elements in Theorem 4.3, we immediately get the following corollary.

Corollary 4.13. *If w is of the form of Figure 26, and C_{RL} is of type I_R or is paired with a caret of type $R_\emptyset$ or R_{NI}, then w is both a k-deep root and a dead end element.*

Deep roots have a spread very similar to the spread of dead end elements. Compare the following corollary with Corollary 4.8.

Corollary 4.14. *If w is a k-deep root of the form of Figure 26, then $wx_1^{-k}x_0^{-2}zx_0^2x_1^k$ (where z is as given in Observation 4.7) is also a k-deep root for all but exactly one such z.*

This corollary shows, in particular, that along the path containing $wx_1^{-k}x_0^{-2}x_1^i$ for $i \in Z$, every vertex (except possibly one) on the path is distance two from the bottom vertex of a k-deep root. We could hence replace every dead end in Figure 25 with a k-deep root to get a figure depicting the local spread of deep roots.

Both deep roots and dead ends pose a threat to using a token passing argument (see Section 1.3) to prove that Thompson's group is nonamenable. The fundamental concept of a token passing argument is to pass tokens upwards, when possible, along a set S until you reach the border, without creating a stockpile of tokens at any element of the border. If a set S contains a deep root, and we place tokens on every element of S, then every token along the root will be passed to the top of the root, creating a stockpile of tokens that must be dispersed to the border. Dispersing the tokens to the border may require sending some of the tokens back down the root.

Corollary 4.14 shows that deep roots appear in clusters. The k-deep roots we described only followed a single path back to the origin. We could say these roots are 1-wide k-deep roots, and define a j-wide k-deep root to be a k-deep root which satisfies the definition of being a k-deep root for j distinct paths back to the identity. A further analysis of k-deep roots could include discovering for what pairs j, k we have j-wide k-deep roots, which could aid in locating Følner sets in Thompson's group. Cleary and Taback ([5]) showed that there are no 4-wide 1-deep roots, when they showed there are no 2-deep pockets in F. Are there 3^k-wide k-deep roots? Are there 2^k-wide k-deep roots?

Part II

On Random Pseudo Manifolds

As a freshman in college, I learned a get to know you game called "The Human Knot." A group of people stand in a circle and randomly grab each others hands. The goal of the game is to unknot your group without having to break any grips (see Figure 27). Recently I played this game with family, and we had formed a trefoil knot. It took some effort to convince my anti-mathematics family that it was

Figure 27: The Human Knot Game

impossible to unknot our hands. This led me to ask, what is a typical Human Knot game? Do most games end with the unknot, or a more complex knot? Do most games end with a single knot, or a collection of disjoint knots?

In Section 5, we describe how the Human Knot game can be simplified to randomly pairing the endpoints of segments. Using this simplification, we show that most games end in a collection of disjoint knots. We explore some variations of the game, as well as describe some unanswered probabilistic questions. Knuth studied a related problem with random permutations in [10], p. 176.

Section 6 generalizes the counts in Section 5 to higher dimensions. We explore random pairings of facets of n-simplexes. Every endpoint pairing of k segments, as well as every affine edge pairing of k triangles (assuming k is even) results in a quotient space that is a closed manifold. In dimensions greater than three, however, there are pairings where the resulting quotient space is not a manifold. I began this research to show that most face pairings of k tetrahedra (dimension $n = 3$) are not manifolds (conjectured by Jim Cannon). I have succeeded in showing that for dimensions $n \geq 2$, most quotient spaces resulting from an affine facet pairings of k

n-simplexes are connected. These results are related to work by Erdős and Rényi on random graphs [6].

In this portion of our work we prove the following:

- A random one-manifold formed from an endpoint pairing of k segments is connected with probability approaching zero as $k \to \infty$ (Corollary 5.2). This shows that most Human Knot games end with more than one component. If players are not allowed to grab their own hand, then Lemma 5.4 shows that most games still result with more than one component. Our work suggests playing the Human Knot game with fewer than 10 people.

- A random n-manifold formed from a facet pairing of k n-simplexes is connected with probability approaching one as $k \to \infty$ (Theorem). In particular this shows that most random 2-manifolds are connected. The links of the vertices of a random pseudo 3-manifold is a 2 manifold, which is connected if and only if all the vertices of the random pseudo 3-manifold are identified together. Since a random 2-manifold is connected, we conjecture that most pseudo 3-manifolds have only one vertex after identification. We leave as an open problem finding the exact relationship between the link of a random pseudo 3-manifold and a random 2-manifold.

5 Random One-Manifolds

In Section 5.1 we simplify the Human Knot game to a combinatorial pairing of endpoints of segments. We show that most Human Knot games end with more than one component. We also discuss the average number of components, and describe a scaling of probability functions which could help in future analysis. In

Section 5.2 we introduce an extra rule to the Human Knot game, and show that with this extra rule, most games still end with more than one component.

5.1 The Human Knot

We begin by simplifying the Human Knot game to a one dimensional combinatorial problem. Each person playing the game has two hands, so we replace each person with a segment. We then randomly pair the endpoints of the segments, where every endpoint is paired with exactly one other endpoint. The quotient space obtained by identifying paired endpoints is a closed 1-manifold.

How many connected components are there in the quotient space formed by a random endpoint pairing of k segments? To answer this problem, we will ignore crossings. For $k = 2$ segments, there are 3 possible pairings, 2 of which result in a connected quotient. For five segments, there are $1 \cdot 3 \cdot 5 \cdot 7 \cdot 9$ pairings, $2 \cdot 4 \cdot 6 \cdot 8$ of which result in a connected quotient.

Lemma 5.1. *Given k segments, the number of endpoint pairings is given by*

$$\frac{\binom{2k}{2}\binom{2k-2}{2}\cdots\binom{2}{2}}{k!} = \frac{(2k)!}{2^k k!} = 1 \cdot 3 \cdots (2k-1).$$

Of those pairings,

$$2^{k-1}(k-1)! = 2 \cdot 4 \cdots (2k-2)$$

result in a connected quotient.

Proof. The number of endpoint pairings is found by choosing 2 endpoints at a time from the remaining segments. The order in which we make the choices is irrelevant, so we have over counted $k!$ times. This gives the first formula. To count the number of connected pairings, fix a starting endpoint e_1 and label the segment to which it belongs s_1. There are $2(k-1)$ choices of endpoints which can be paired with e_1.

k	2	3	4	5	10	15	20	30	50	100
$P(k)$	0.667	0.533	0.457	0.406	0.284	0.231	0.199	0.162	0.126	0.089

Table 11: The Human Knot Probabilities: With k players, $P(k)$ represents the probability of ending a Human Knot game as a single connected knot.

Whichever endpoint is paired with e_1 we call e_2, and the segment containing e_2 we call s_2. The other endpoint of s_2 can be paired with $2(k-2)$ other endpoints, since it cannot be paired with the unpaired endpoint of s_1, unless $k = 2$. We continue in this fashion until we have made $(2k-2)(2k-4)\cdots 4 \cdot 2$ choices. The remaining endpoint of s_k is paired with the remaining endpoint of s_1. □

Corollary 5.2. *The probability $P(k)$ that a random endpoint pairing of k segments results in a connected quotient tends to 0 as $k \to \infty$. For easy reference, the first few values of $P(k)$ are given in Table 11.*

Proof. The previous lemma shows that $P(k) = \dfrac{1 \cdot 2 \cdot 4 \cdots (2k-2)}{1 \cdot 3 \cdot 5 \cdots (2k-1)}$, the product of the evens divided by the product of the odds up to $(2k-1)$. After rearranging terms and multiplying and dividing by $2k$, we get $P(k) = \dfrac{2^{2k} k! k!}{2k(2k)!}$. Then Stirling's formula $k! \sim \left(\dfrac{k}{e}\right)^k \sqrt{2\pi k}$ gives the estimate

$$P(k) \sim \frac{2^{2k}\left(\left(\frac{k}{e}\right)^k \sqrt{2\pi k}\right)^2}{2k\left(\frac{2k}{e}\right)^{2k}\sqrt{4\pi k}} = \frac{\sqrt{\pi}}{2\sqrt{k}}$$

which tends to zero as $k \to \infty$. □

The preceding corollary shows that most Human Knot games will end with multiple connected groups of people. The next question is how many connected components are there on average (compare this to the average number of cycles of a random permutation, p. 176 of [10])? Consider the discrete random variable X_k which represents the number of connected components of a random endpoint pairing of k segments. Let $f_k(x)$ be the probability of having exactly $\lceil x \rceil$ connected components after pairing k segments. We see that $f_k(x)$ represents the probability density function of X_k, and $\int_{-\infty}^{\infty} f_k(x)\,dx = 1$. Experimentally, we observe that the pointwise limit $\lim\limits_{k\to\infty} f_k(x) = 0$. The limit flattens out, in the sense that the integral over any compact set can be made as small as possible by increasing k. More technically, consider the cumulative distribution function $F_k(x) = \int_{-\infty}^{x} f_k(t)\,dt$. Then pointwise we also observe experimentally that $\lim\limits_{k\to\infty} F_k(x) = 0$. It is unfortunate that these limits of density and distribution functions are neither density nor distribution functions.

It is desirable to scale $f_k(x)$ in such a way that the limit is a density function. Our first attempt is to let $g_k(x) = kf_k(kx)$. Hence

$$\int_{-\infty}^{\infty} g_k(x)\,dx = \int_{-\infty}^{\infty} kf_k(kx)\,dx = \int_{-\infty}^{\infty} f_k(u)\,du = 1$$

so that $g_k(x)$ is a density function. We think of $g_k(x)$ as the probability density function of a "continuous" random variable X_k/k. Again we observe experimentally that the pointwise limit $\lim\limits_{k\to\infty} g_k(x) = 0$, however this time the limit spikes at 0. More technically, the cumulative distribution function $G_k(x) = \displaystyle\int_{-\infty}^{x} g_k(t)\,dt$ limits to the step function, i.e., for $x \le 0$ we have $\lim_{k\to\infty} G_k(x) = 0$, and for $x > 0$ we have $\lim_{k\to\infty} G_k(x) = 1$. Hence, the limiting probability distribution can be thought of as Dirac's Delta distribution, centered at 0. This shows that the proportion of the number of components to the number of segments is very small. So we see that in a typical Human Knot game there are likely to be multiple components, but the number of components will be very small in comparison with the number of people playing.

The last scaling of $f_k(x)$ resulted in a limiting distribution which was completely concentrated at 0. The scaling was too much, in that we squashed all the possibly interesting trends to zero. Does there exists a scaling of $f_k(x)$ whose limit is an elementary probability density function? We seek a function $\phi(k)$, with $h_k(x) = \phi(k) f_k(\phi(k)x)$ and $H_k(x) = \int_{-\infty}^{x} h_k(t)\,dt$, which satisfies

1. $\displaystyle\int_{-\infty}^{\infty} h_k(x)\,dx = 1,$
2. $\displaystyle\lim_{n\to\infty} h_k(x) \neq 0$ pointwise,
3. $H(x) = \displaystyle\lim_{k\to\infty} H_k(x)$ is an elementary function,
4. and $\displaystyle\int_{-\infty}^{\infty} \lim_{k\to 0} h_k(x)\,dx = 1.$

Such a scaling could give valuable information about the random variable X_k. Similar considerations can be considered on any sequence of probability density functions, where the limit of the distributions is zero under one scaling, and Dirac's Delta distribution under another scaling. Is there a general method of finding a "nice" scaling, or is this a case specific problem?

5.2 Modifications

When playing the Human Knot game, there are factors which complicate the game beyond the simple endpoint pairing of segments which we counted in the previous section. For one, there are over and under crossing issues. In addition, some people unconsciously think it is a rule not to grab your own hand. This can be added as a specific rule to the game, although I have participated in a game where one person could not find any free hands, and was forced to grasp his own hands since his were the only two hands left unpaired. This latter situation I will address in this section, and show that adding a rule which prevents someone from grabbing their own hand does not change the fact that most games will end with more than one component. I will leave it as an open problem to explore what happens when you consider over and under crossing issues. A simple, probably impossible to answer question is, what is the expected minimal crossing number of a random Human Knot game? Other simple questions arise, which are probably just as hard to answer.

We now focus on counting the total number of possible ways of pairing up the endpoints of k segments, with the additional rule that the endpoints of a single segment cannot be paired together. The total number of connected pairings is the same as in the previous section, provided $k > 1$. When $k = 1$, there are no pairings. When $k = 2$, there are 2 total pairings. When $k = 3$, there are 8 total pairings.

Lemma 5.3. *The number a_k of endpoint pairings of k segments where the endpoints of a single segment cannot be paired together satisfies the recurrence $a_{k+1} = 2k(a_k + a_{k-1})$ for $k \geq 2$, with $a_1 = 0, a_2 = 2$. (see [11] A053871)*

Proof. Suppose there are $k+1$ segments. Pick a starting segment s_1 and a starting endpoint e_1. The other endpoint of s_1 we label e_2. There are $2k$ choices of endpoints to pair with e_1. Whichever choice is made at this stage is independent of future choices. We label the chosen endpoint e_3, the segment containing e_3 we label s_2, and the other endpoint of s_2 we label e_4. If we pair e_4 and e_2, then there are a_{k-1} possible ways to pair the remaining $k-1$ segments. Otherwise, we consider s_1 and s_2 as a single segment with endpoints e_2 and e_4. Since we are not connecting e_2 to e_4, we seek the total number of ways of pairing the endpoints of k segments, where the endpoints of a single segment cannot be paired together. Hence there are a_k total pairings. This gives the desired recurrence $a_{k+1} = 2k(a_k + a_{k-1})$. Note that by stating $a_0 = 1$, the recurrence is valid for $k \geq 1$. □

Lemma 5.4. *Let $Q(k)$ (we use Q because it comes after P) denote the probability that a random endpoint pairing of k segments, where the endpoints of a single segment cannot be paired together, results in a connected quotient. Then $Q(k) \to 0$ as $k \to \infty$.*

Proof. Using the notation a_k from the previous lemma, we calculate

$$\begin{aligned} Q(k) &= \frac{2^{k-1}(k-1)!}{a_k} \\ &= \frac{(2k-2)(2k-4)\cdots 4\cdot 2}{(2k-2)(a_{k-1}+a_{k-2})} \\ &= \frac{(2k-2)(2k-4)\cdots 4\cdot 2}{(2k-2)((2k-4)(a_{k-2}+a_{k-3})+a_{k-2})} \\ &\leq \frac{(2k-4)\cdots 4\cdot 2}{(2k-3)a_{k-2}} \\ &= \frac{(2k-4)}{(2k-3)}Q(k-2) \end{aligned}$$

The above calculation gives, for even k, $Q(k) \leq \frac{2k-4}{2k-3}\frac{2k-8}{2k-7}\cdots Q(2)$. For odd k, the calculation the same, it just ends in $Q(3)$. Since $Q(2) = Q(3) = 1$, it remains to show that the product of every other even, divided by the product of every other odd tends to zero as $k \to \infty$. My attempts at a Stirling's formula proof were unfruitful, especially after noting experimentally that $P(k)/Q(k) \to 0$ as $k \to \infty$, where $P(k)$ is the probability computed in Corollary 5.2, i.e., the probability that a random endpoint pairing of k segments results in a connected quotient, where segments are allowed to self pair. Let $R(k) = \frac{2k-4}{2k-3}\frac{2k-8}{2k-7}\cdots 1$ (R stands for ratio), so that $Q(k) \leq R(k)$ for $k \geq 4$. Notice that $R(k)R(k-1) = P(k-1)$, hence $R(k)R(k-1) \to 0$ as $k \to \infty$. Now, $\{R(2k)\}_{k=2}^{\infty}$ is a decreasing sequence, bounded below by 0, so it converges to $\alpha \geq 0$. Similarly, $\{R(2k+1)\}_{k=2}^{\infty}$ is a decreasing bounded sequence which converges to $\beta \geq 0$. We have $0 \leq \alpha\beta \leq R(k)R(k-1) = P(k-1)$ for all k. Taking limits shows that $\alpha\beta = 0$. Since $R(2k-1) \leq R(2k)$, we see that $\beta \leq \alpha$ which

k	2	3	4	5	9	10	15	20	30	50	100
$Q(k)$	1	1	.8	0.706	0.508	0.480	0.387	0.333	0.270	0.208	0.147

Table 12: The Human Knot Modified Probabilities: With k players, $Q(k)$ represents the probability of ending a Human Knot game as a single connected knot, where players are not allowed to clasp their own hands.

implies that $\beta = 0$. Since $R(2k) \leq \frac{3}{2}R(2k-1)$, we have $\alpha \leq \frac{3}{2}\beta = 0$, which gives $\alpha = 0$. Since $Q(k) \leq R(k)$ for large k, we conclude that $Q(k) \to 0$ as $k \to \infty$. □

Under the assumptions that no one is permitted to clasp their own hands, we see that if 10 or more people play the Human Knot game, the chance of unknotting into a single knot is less than half. Table 12 gives more values of $Q(k)$. I would suggest using less than 10 people per group when playing The Human Knot.

6 Pseudo Manifolds and Brooms

In the previous section, we considered a random pairing of the endpoints of a collection of segments. The resulting quotient space was a topological 1-manifold. We now increase the dimension of our problem to consider a random pairing of the facets (faces of dimension $(n-1)$) of a collection of n-simplexes. As there is more than one way to affinely identify a pair of facets, we must specify an affine homeomorphism between each facet pair in order to create a quotient space from a facet pairing. For example, when $n = 2$, we consider random edge pairings of triangles, i.e., the facets are the edges. A pair of edges can be identified affinely in exactly two ways, so our selection of an affine homeomorphism just tells us which endpoints of the edges to pair together. Notice that we need an even number of triangles to successfully pair every edge. In even dimensions, we will always require that the number of simplexes is even.

In the two dimensional case, a facet pairing always results in a quotient space that is a manifold. However, for $n \geq 3$, a facet pairing need not result in a quotient space that is a manifold. Thurston gives an example of a facet pairing of two tetrahedra where the resulting quotient space is not a 3-manifold (see [14]). Our work began as an attempt to show that most facet pairings of 3-simplexes do not result in manifolds. In higher dimensions, similar difficulties arrive.

Definition 6.1. A pseudo n-manifold is a space obtained as the quotient of finitely many n-simplexes by affine homeomorphisms on each facet pairing.

A natural question follows, with what probability will a random face pairing of k n-simplexes result in a quotient space that is a genuine n-manifold? From his work on twisted face pairings, my advisor, Jim Cannon, conjectured that, when $n = 3$, this probability approaches 0 as $k \to \infty$. Attempting to proving this conjecture began this portion of our work. To avoid trivialities, we will assume that all the facet pairings are orientation reversing, so that the manifolds obtained are orientable.

Let $P_n(k)$ be the probability that a facet pairing of k n-simplexes results in a connected pseudo n-manifold. Our main result is that as $k \to \infty$, $P_n(k) \to 1$, if $n \geq 2$. Informally speaking, most pseudo n-manifolds are connected. When $n = 1$, Section 5 showed that $P_1(k) \to 0$. Hence, there is a stark contrast between dimensions one and two with regards to connectedness. This stark contrast is most likely the key which will show that most random pseudo n-manifolds ($n \geq 3$) have only one vertex, and that the link of that vertex is not a sphere.

Section 6.1 introduces our notation, as well as restates the facet pairing questions in terms of multi-graphs, using what we call n-brooms. In Section 6.2, we first show that $P_3(k) \to 0$ as $k \to \infty$, and then generalize to other dimensions. In Section 6.3, we address using polyhedra, instead of simplexes, as the building blocks of random pseudo manifolds. In particular, in dimension two we discuss using polygons instead of triangles as the building blocks of 2-manifolds. We show that as long as each polygon has at least 3 edges (we don't allow degenerate 1 or 2-gons), then as the number k of polygons increases, the probability that a random edge pairing of the k polygons results in a connected 2-manifold with probability approaching 1 as $k \to \infty$.

6.1 Brooms and Notation

We now show how the counts involved in computing $P_n(k)$ can be replaced by a graph theory question regarding $(n+1)$-regular multi-graphs. Suppose we have k n-simplexes. we can ignore the choice of homeomorphism when computing the probability $P_n(k)$ that a random facet pairing results in a connected pseudo manifold, because the number of choices of homeomorphisms factors out of the total number of connected facet pairings, and also factors out of the total number of facet pairings.

Definition 6.2. We define an n-broom to be a star on $n+1$ vertices, i.e., the complete bipartite graph $K_{1,n}$.

An $(n+1)$-broom corresponds to placing a vertex in the barycenter of an n-simplex and connecting to it $n+1$ vertices placed in the barycenter of each $(n-1)$ dimensional face. A facet pairing of k n-simplexes corresponds to a vertex pairing on the pendant (valence 1) vertices of k $(n+1)$-brooms. A graph theory equivalent of the the main result is, informally, that most n-regular multi-graphs are connected

$(n \geq 3)$. Similar results hold for multi-graphs where the degree of each vertex is at least 3.

The number of pairings of $2n$ elements is (Lemma 5.1)

$$\frac{1}{n!}\binom{2n}{2}\binom{2n-2}{2}\cdots\binom{2}{2} = \frac{(2n)!}{2^n n!} = 1 \cdot 3 \cdots (2n-3) \cdot (2n-1).$$

Given k n-brooms, there are kn vertices to pair, where kn should be even for a pairing to exist. We define

$$L_n(k) = 1 \cdot 3 \cdots (kn-3) \cdot (kn-1), \tag{6.1}$$

$$L_n(j, k-j) = \binom{k}{j} L_n(j) L_n(k-j). \tag{6.2}$$

Line (6.1) gives the total number of pairings of the pendant vertices of k n-brooms. Line (6.2) represents an overestimate on the number of ways of pairing the pendant vertices of k n-brooms, where the vertices of j n-brooms are paired among themselves, and the vertices of the remaining $k-j$ n-brooms are paired among themselves. Line (6.2) is an exact count unless $j = k/2$, in which case the exact count includes dividing by 2. As an example,

$$L_4(2, k-2) = \binom{k}{2}(1 \cdot 3 \cdot 5 \cdot 7)[(4k-9)(4k-11)\cdots 3 \cdot 1].$$

6.2 Most Pseudo Manifolds are Connected

We now show that most pseudo manifolds are connected. We first work with just tetrahedra in dimension three.

Theorem 6.3. *Most pseudo 3-manifolds are connected, i.e., $P_3(k) \to 1$ as $k \to \infty$.*

Proof. In dimension three, there are four facets for each simplex, so we replace the k tetrahedra with k 4-brooms. The proof requires three steps.

We first prove, for $2 < 2j \le k$, the inequality

$$L_4(j-1, k-(j-1)) \ge L_4(j, k-j). \tag{6.3}$$

Writing out both sides and canceling common terms, (6.3) is equivalent to showing

$$f(j,k) = j(4k-4j+3)(4k-4j+1) - (k-j+1)(4j-3)(4j-1) \ge 0 \tag{6.4}$$

for $2 < 2j \le k$. To prove this, fix $k > 2$, and consider the cubic $g(x) = f(x,k)$. Then $g(1) = 16k^2 - 19k + 3$ and $g(\frac{k}{2}) = 4k^2 + 8k - 3$ are positive since $k > 1$. Also, $g(x)$ has a point of inflection at $x = \frac{k+1}{2}$, and $g''(x) < 0$ for $x < \frac{k+1}{2}$. If $f(j,k) = 0$ for some j with $2 < 2j \le k$, then $g(1) > g(j)$ and $g(\frac{k}{2}) > g(j)$ would imply (using the Mean Value Theorem twice) that $g''(x) > 0$ for some $x \in (1, \frac{k}{2})$, which is impossible. This shows that $f(j,k)$ is positive in the desired domain.

Second, we show that there is a constant $c_4 > 0$ so that

$$L_4(1, k-1) \ge c_4 k L_4(2, k-2)$$

for k large. This is equivalent to showing

$$2(4n-5)(4n-7) \ge c_4 n(n-1)(5)(7).$$

Here we can pick $c_4 = \frac{1}{35}$, and then

$$L_4(1, k-1) \ge c_4 k L_4(2, k-2) \ge c_4 \sum_{j=2}^{\lfloor k/2 \rfloor} L_4(2, k-2) \ge c_4 \sum_{j=2}^{\lfloor k/2 \rfloor} L_4(j, k-j).$$

Finally, we overestimate the probability that a facet pairing results in a quotient manifold with at least 2 components, and show this probability approaches zero. Assuming k is large enough, we overestimate the number of facet pairings resulting in a quotient with at least 2 components by

$$\begin{aligned}\sum_{j=1}^{\lfloor k/2 \rfloor} L_4(j, k-j) &= L_4(1, k-1) + \sum_{j=2}^{\lfloor k/2 \rfloor} L_4(j, k-j) \\ &\leq L_4(1, k-1) + \frac{1}{c_4} L_4(1, k-1) \\ &= \frac{c_4+1}{c_4} L_4(1, k-1).\end{aligned}$$

We then compute

$$\frac{\frac{c_4+1}{c_4} L_4(1, k-1)}{L_4(k)} = \frac{(c_4+1)\cdot k \cdot 1 \cdot 3}{c_4(4k-1)(4k-3)} \approx \frac{k}{(4k-1)(4k-3)}$$

which approaches zero as $k \to \infty$. □

In the first part of the proof, (6.3) is equivalent to showing

$$\frac{k-j+1}{j} \leq \frac{(4k-4j+3)}{(4j-3)} \frac{(4k-4j+1)}{(4j-1)} \tag{6.5}$$

Writing the inequality (6.3) in the form (6.5) is the key to generalizing the proof of Theorem 6.3 to other dimensions. We need the following inequality, whose proof is similar, when the number of facets is odd (i.e., the dimension of an n-simplex is even)

$$\frac{(2k-2j+1)}{(2j-1)} \frac{(k-j+1)}{(j)} \leq \frac{(4k-4j+3)}{(4j-3)} \frac{(4k-4j+1)}{(4j-1)}. \tag{6.6}$$

We now generalize Theorem 6.3 to dimensions $n \geq 2$. We begin with odd dimensions, since pairing even dimensional simplexes requires the extra hypothesis that there are an even number of simplexes.

Theorem 6.4. *Most pseudo* $(n-1)$*-manifolds* $(n \geq 3)$ *are connected, i.e.,* $P_{n-1}(k) \to 1$ *as* $k \to \infty$.

Proof. An $(n-1)$ simplex has n facets to pair, so we replace the k $(n-1)$-simplexes with k n-brooms. Again the proof requires three steps. First, for $n = 6$ and $2 < 2j \leq k$, we prove the inequality

$$L_n(j-1, k-(j-1)) \geq L_n(j, k-j). \tag{6.7}$$

After canceling common terms and rearranging, (6.7) is equivalent to

$$\frac{k-j+1}{j} \leq \frac{(6k-6j+5)(6k-6j+3)(6k-6j+1)}{(6j-5)(6j-3)(6j-1)}.$$

Using (6.5), and multiplying and dividing by $\frac{6}{4}$ gives

$$\begin{aligned}\frac{k-j+1}{j} &\leq \frac{(4k-4j+3)(4k-4j+1)}{(4j-3)(4j-1)}\\ &= \frac{(6k-6j+4.5)(6k-6j+1.5)}{(6j-4.5)(6j-1.5)}\\ &\leq \frac{(6k-6j+5)(6k-6j+3)}{(6j-5)(6j-3)}\\ &\leq \frac{(6k-6j+5)(6k-6j+3)(6k-6j+1)}{(6j-5)(6j-3)(6j-1)}\end{aligned}$$

which proves (6.7) for $n = 6$. We can now prove (6.7) for even $n \geq 6$. Multiply and divide the right side of (6.5) by $n/4$, overestimate the remaining fractions, and then add extra terms to the right side to obtain the needed inequality, as shown below.

$$\begin{aligned}\frac{k-j+1}{j} &\leq \frac{(nk-nj+3n/4)(nk-nj+n/4)}{(nj-3n/4)(nj-n/4)}\\ &\leq \frac{(nk-nj+n-1)(nk-nj+n-3)}{(nj-(n-1))(nj-(n-3))}\\ &\leq \frac{(nk-nj+n-1)(nk-nj+n-3)(nk-nj+n-5)\cdots(nk-nj+1)}{(nj-(n-1))(nj-(n-3))(nj-(n-5))\cdots(nj-1)}\end{aligned}$$

The first and last terms of the previous inequality are precisely inequality (6.7). Note that $1 \leq \frac{nk-nj+1}{nj-1}$ because $2j \leq k$.

For n odd, $L_n(j, k-j)$ is defined only when both kn and jn are even, so we use $2k$ n-brooms. The following is proved similarly, using (6.6) as the starting inequality.

For odd $n \geq 3$ and $2 < 2j \leq k$,

$$L_3(2(j-1), 2(k-(j-1))) \geq L_3(2j, 2(k-j)). \tag{6.8}$$

Second, for each $n \geq 4$ (n even), pick a constant c_n such that

$$L_n(1, k-1) \geq c_n k L_n(2, k-2)$$

for k large, which gives

$$L_n(1, k-1) \geq c_n \sum_{j=2}^{\lfloor k/2 \rfloor} L_n(j, k-j).$$

Similarly, for $n \geq 3$ (n odd), pick a constant c_n such that

$$L_n(2, 2k-2) \geq c_n k L_n(4, 2k-4)$$

for large k, which gives

$$L_n(2, 2k-2) \geq c_n \sum_{j=2}^{\lfloor k/2 \rfloor} L_n(2j, 2k-2j).$$

For $n \geq 5$, $c_n = 1$ works.

Lastly, we repeat the final argument in the proof of Theorem 6.3. We compute for even n

$$\frac{\frac{c_n+1}{c_n} L_n(1, k-1)}{L_n(k)} = \frac{(c_n+1) \cdot k \cdot 1 \cdot 3 \cdots (n-3)(n-1)}{c_n(nk-1)(nk-3)\cdots(nk-(n-3))(nk-(n-1))}$$

and we compute for odd n

$$\frac{\frac{c_n+1}{c_n} L_n(2, 2k-2)}{L_n(2k)} = \frac{(c_n+1) \cdot (k)(2k-1) \cdot 1 \cdot 3 \cdots (2n-3)(2n-1)}{c_n(2nk-1)(2nk-3)\cdots(2nk-(2n-3))(2nk-(2n-1))}$$

all of which approach zero as $k \to \infty$ when $n \geq 3$. $\square$

6.3 Adding Straws to the Brooms

Until now, we have required that all k of the brooms have the same number of straws, or that each pseudo n-manifold be created from a facet pairing of n-simplexes. What happens if the number of straws in each broom is allowed to vary from broom to broom, or, equivalently, if the building blocks of a pseudo manifold are polyhedra instead of simplexes? For example, instead of using triangles to create a 2-manifold, we could use squares, pentagons, or a mixture of different polygons? Intuitively, increasing the number of edges to pair should only increase the probability of getting a connected pseudo manifold, however, that is not always the case. For example, every pendant vertex (edge) pairing of two 3-brooms (triangles) results in a connected multi-graph (manifold). After adding two straws, one to each broom, it is possible to get a disconnected multi-graph (manifold) by pairing the pendant vertices (edges) of each 4-broom (square) among itself.

We now consider k brooms, with the ith broom being an n_i-broom, i.e., it has n_i straws. We require that $n_i \geq 3$ for each i. Note that $\sum n_i$ must be even for a pendant vertex pairing to exists. The number of ways of paring $2n+1$ vertices is zero, however we will over count the number of pairings of $2n+1$ vertices with a "fake" estimate, namely the product of the evens up to $2n+1$. We then extend the definition of $L_n(k)$ to be the product of the numbers up to kn of opposite parity. Again, $L_n(j, k-j) = \binom{k}{j} L_n(j) L_n(k-j)$ represents an overestimate of the number of ways of pairing the pendant vertices of k n-brooms, where the vertices of j of the brooms are paired among themselves, and the vertices of the remaining $k-j$ are paired among themselves. When the subscript n is omitted, we will assume $n = 1$, i.e., $L(k) = L_1(k)$. Using the "fake" estimate, when kn is odd, makes $L_n(k)$ a strictly increasing function of k.

The following facts are proved in a similar way as done in Section 6.2.

Lemma 6.5. *For $4 < 2j < k$,*

$$L_3(j-2, k-(j-2)) \geq L_3(j, k-j).$$

For large k there are constants e_3 (even), and o_3 (odd) so that

$$\begin{aligned}
L_3(2, k-2) &\geq e_3 k L_3(4, k-4), \\
L_3(2, k-2) &\geq e_3 \sum_{j=2}^{\lfloor k/4 \rfloor} L_3(2j, k-2j), \\
L_3(3, k-3) &\geq o_3 k L_3(5, k-5), \\
L_3(3, k-3) &\geq o_3 \sum_{j=3}^{\lfloor (k+2)/4 \rfloor} L_3(2j-1, k-2j+1)
\end{aligned}$$

Let X represent a vertex pairing of a collection of k brooms, with the ith broom having n_i straws. Let $P(X)$ represent an overestimate of the probability that X results in a disconnected multi-graph, given by

$$\begin{aligned}
L\left(\sum_{i=1}^{k} n_i\right) P(X) = &\sum_{j_1} \left(L(n_{j_1}) \cdot L\left(\sum_{i \neq j_1} n_i\right)\right) \\
&+ \sum_{j_1 < j_2} \left(L(n_{j_1} + n_{j_2}) \cdot L\left(\sum_{i \neq j_1, j_2} n_i\right)\right) \\
&+ \sum_{j_1 < j_2 < j_3} \left(L\left(\sum_{i=1}^{3} n_{j_i}\right) \cdot L\left(\sum_{i \neq j_1, j_2, j_3} n_i\right)\right) \\
&\vdots \\
&+ \sum_{j_1 < j_2 < \cdots < j_{\lfloor k/2 \rfloor}} \left(L\left(\sum_{i=1}^{\lfloor k/2 \rfloor} n_{j_i}\right) \cdot L\left(\sum_{i \neq j_1, j_2, \ldots, \lfloor k/2 \rfloor} n_i\right)\right).
\end{aligned}$$

Theorem 6.6. *Fix a sequence $\{n_i\}_{i=1}^{\infty}$. Let X_k be a pairing of k brooms, where the ith broom has n_i straws ($3 \leq n_i$). Then $P(X_k) \to 0$, as $k \to \infty$. In other words, most pairings of k brooms, regardless of the number of straws on each broom, result in a connected multi-graph, as long as each broom has at least 3 straws.*

Proof. First fix $n_i = 3$ for each i. Overestimate the number of disconnected pairings by

$$L_3(k)P(X_k) = \sum_{j=1}^{\lfloor k/2 \rfloor} L_3(j, k-j) \leq L_3(1, k-1) + \frac{2}{e_3} L_3(2, k-2) + \frac{2}{o_3} L_3(3, k-3)$$

Divide by $L_3(k)$ and take limits as $k \to \infty$. The last two summands approach 0 by simply comparing $L_3(k)$ and $L_3(j, k-j)$ for $j = 2, 3$. Stirling's formula shows the first term approaches zero. This is the calculation when $k = 2m$ is even.

$$\begin{aligned}
\frac{L_3(1, k-1)}{L_3(k)} &= \frac{2kL_3(2m-1)}{1} \frac{1}{L_3(2m)} \\
&= \frac{4m(3m)!\, 2^{3m}}{(6m)(6m-2)} \frac{(3m)!\, 2^{3m}}{(6m)!} \\
&= \frac{((3m)!\, 2^{3m})^2}{3(3m-1)(6m)!} \\
&\approx \frac{\left(\left(\frac{3m}{e}\right)^{3m} \sqrt{6\pi m}\, 2^{3m}\right)^2}{3(3m-1)\left(\frac{6m}{e}\right)^{6m} \sqrt{12\pi m}} \\
&= \frac{\sqrt{3\pi m}}{3(3m-1)}
\end{aligned}$$

which approaches 0 as $2m = k \to \infty$. This establishes the theorem when $n_i = 3$ for each i.

Now consider the addition of a single 2-broom to a collection of k 3-brooms. Let X_k be a pairing of k 3-brooms. Let X_k' be a pairing of k 3-brooms and a single 2-broom. We now show $P(X_k') \to 0$ as $k \to \infty$. For now we assume k is even, and calculate

$$L(3k)P(X_k) = \sum_{j=1}^{k/2} \binom{k}{j} L(3j)L(3k-3j) \tag{6.9}$$

$$L(3k+2)P(X_k') = \sum_{j=0}^{k/2-1} \binom{k}{j} L(3j+2)L(3k-3j) + \sum_{j=1}^{k/2} \binom{k}{j} L(3j)L(3k-3j+2). \tag{6.10}$$

Note that $L(3j+2)L(3k-3j) \leq L(3j)L(3k-3j+2)$, and so after dividing by $L(3k+2) = (3k+1)L(3k)$ we have (since $\frac{3k-3j+1}{3k+1} < 1$)

$$\begin{aligned} P(X'_k) &< \binom{k}{0}\frac{L(2)L(3k)}{L(3k+2)} + \frac{2}{(3k+1)L(3k)}\sum_{j=1}^{k/2}(3k-3j+1)\binom{k}{j}L(3j)L(3k-3j) \\ &< \frac{L(2)L(3k)}{L(3k+2)} + 2P(X_k) \end{aligned}$$

which approaches zero as $k \to \infty$. If k is odd, then we change line (6.10) so that it reads

$$L(3k+2)P(X'_k) = \sum_{j=0}^{\lfloor k/2 \rfloor}\binom{k}{j}L(3j+2)L(3k-3j) + \sum_{j=1}^{\lfloor k/2 \rfloor}\binom{k}{j}L(3j)L(3k-3j+2).$$

The rest of the proof that $P(X'_k) \to 0$ is equivalent when k is odd, if we replace each $k/2$ with $\lfloor k/2 \rfloor$.

Claim 6.7. *Increasing the number of straws decreases the overestimate $P(X)$ used in computing the probability of X resulting in a disconnected graph. Formally, let X and X' be collections of k brooms (the ith broom having n_i, resp. n'_i, straws). If $n_i \leq n'_i$ for each i and $2 + \sum n_i = \sum n'_i$, then $P(X) \geq P(X')$.*

To prove this claim, it is enough to show, for $1 \leq m \leq \lfloor k/2 \rfloor$ and $1 \leq j_1 < j_2 < \cdots < j_m \leq k$, that

$$\frac{L\left(\sum_{i=1}^m n_{j_i}\right) \cdot L\left(\sum_{i \neq j_1, j_2, \ldots, j_m} n_i\right)}{L\left(\sum_{i=1}^k n_i\right)} > \frac{L\left(\sum_{i=1}^m n'_{j_i}\right) \cdot L\left(\sum_{i \neq j_1, j_2, \ldots, j_m} n'_i\right)}{L\left(\sum_{i=1}^k n'_i\right)}$$

This follows at once from the following inequalities.

$$\frac{L(a)L(b)}{L(a+b)} \geq \frac{L(a)L(b+2)}{L(a+b+2)}, \tag{6.11}$$

$$\frac{L(a)L(b)}{L(a+b)} \geq \frac{L(a+2)L(b)}{L(a+b+2)}, \tag{6.12}$$

$$\frac{L(a)L(b)}{L(a+b)} \geq \frac{L(a+1)L(b+1)}{L(a+b+2)}. \tag{6.13}$$

The proofs of (6.11) and (6.12) are trivial term by term comparison. We now prove (6.13). After a suitable rearranging, we find that (6.13) is equivalent to

$$\frac{L(a+b+2)}{L(a+b)} \geq \frac{L(a+1)}{L(a)} \frac{L(b+1)}{L(b)}$$

The following motivates our next step. Stirling's formula gives

$$\frac{L(a+1)}{L(a)} \sim \begin{cases} \sqrt{\frac{\pi}{2}a} & a \text{ even} \\ \sqrt{\frac{2}{\pi}a} & a \text{ odd} \end{cases}$$

and so we find asymptotically that (6.13) is similar to showing

$$a+b+1 \geq \sqrt{\frac{\pi}{2}a}\sqrt{\frac{\pi}{2}b} \qquad\qquad a+b+1 \geq \sqrt{\frac{\pi}{2}a}\sqrt{\frac{2}{\pi}b}$$
$$a+b+1 \geq \sqrt{\frac{2}{\pi}a}\sqrt{\frac{\pi}{2}b} \qquad\qquad a+b+1 \geq \sqrt{\frac{2}{\pi}a}\sqrt{\frac{2}{\pi}b}$$

The above four inequalities hold as long as

$$a+b \geq \sqrt{2a}\sqrt{2b} = 2\sqrt{ab}$$

which is just the geometric-arithmetic mean. The above should motivate this next step. For $n \geq 1$ we prove

$$\sqrt{2n} \geq \frac{L(n+1)}{L(n)}. \tag{6.14}$$

When $n = 1$ the inequality says $\sqrt{2} \geq 1$ and when $n = 2$ it gives $2 \geq 2$. Proceed by induction. Suppose (6.14) is true for $n-2$. We show (6.14) is true for n.

$$\begin{aligned} \frac{L(n+1)}{L(n)} &= \frac{nL(n-1)}{(n-1)L(n-2)} \\ &\leq \frac{n}{n-1}\sqrt{2(n-2)} \\ &\leq \sqrt{2n}. \end{aligned}$$

We can now prove (6.13).

$$\frac{L(a+b+2)}{L(a+b)} = a+b+1 > a+b \geq 2\sqrt{ab} = \sqrt{2a}\sqrt{2b} \geq \frac{L(a+1)}{L(a)}\frac{L(b+1)}{L(b)}$$

This concludes the proof of (6.13), and consequently the proof of Claim 6.7.

To conclude a proof of the theorem, let X_k be a random pairing of of k brooms, with the ith broom having n_i straws. If $\sum_{i=1}^{k} n_i$ and $3k$ have the same parity, then let Y_k be a random pairing of k 3-brooms. Otherwise let Y_k be a random pairing of $k-1$ 3-brooms and one 2-broom. Since $n_i \geq 3$ for each i, we add straws, two at a time, to Y_k, until the number of straws is n_i in each broom. We then have $P(Y_K) > P(X_k)$ from the claim. Since $P(Y_k) \to 0$ as $k \to \infty$, we see that $P(X_k) \to 0$. □

The proof of the last theorem showed that one 2-broom did not affect the high probability of getting a connected graph. I'll end with the question, how many two brooms can be added to arrive at a similar conclusion? I conjecture that as $k \to \infty$, the percentage of two brooms must approach zero.

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