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Classifying Homotopy Types of One-Dimensional Peano Continua

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CLASSIFYING HOMOTOPY TYPES OF ONE-DIMENSIONAL PEANO CONTINUA

by

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ABSTRACT

CLASSIFYING HOMOTOPY TYPES OF
ONE-DIMENSIONAL PEANO CONTINUA

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Determining the homotopy type of one-dimensional Peano continua has been an open question of some interest. We give a complete invariant of the homotopy type of such continua, which consists of a pair of subspaces together with a relative homology group. Along the way, we describe reduced forms for one-dimensional Peano continua.

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1 Introduction

This paper addresses the problem of classifying one-dimensional Peano continua, up to homotopy type. This is a step towards solving an interesting conjecture from the mid-90's, namely that the homotopy type of a one-dimensional Peano continuum is determined by its fundamental group. Cannon, Conner and Zastrow proved that one-dimensional spaces are aspherical [2], thus the conjecture can be proven for CW-complexes.

To see how this paper addresses the conjecture, we first state a few definitions. We also present our main results, and then mention how the main theorem may be useful in proving the conjecture.

1.1 Definitions

Throughout this paper, all neighborhoods will be open, and dimension will be covering dimension. We will denote the identity map on X by Id_X. In a one-dimensional Peano continuum X, a point x is *bad* if it has no simply connected neighborhood. We denote the set of all bad points of X by *B(X)*, and its complement by $G(X) = X \setminus B(X)$. A one-dimensional Peano continuum X is a *core* continuum if it admits no proper strong deformation retract. A *graph* is a one dimensional CW-complex. A graph is *locally finite* if each 0-cell intersects the closure of only finitely many 1-cells. A loop (or path) is *reduced* if the only nulhomotopic intervals in the domain (either S^1 or $[0,1]$) are constant.

1.2 Results

We prove the following theorems about one-dimensional Peano continua. Some of these theorems prove the existence of certain reduced forms for homotopy types of such continua, while others discuss three homotopy invariants which together completely determine the homotopy type of the space.

Theorem 2.4. *Every non-contractible one-dimensional Peano continuum is homotopy equivalent to a unique minimal deformation retract, which we call a* core.

Theorem 2.7. *If X is a core one-dimensional Peano continuum then G(X) is a locally finite graph.*

Theorem 2.9 (Arc-reduced). *Let X be a one-dimensional Peano continuum with $B(X) \neq \emptyset$. Then there is a continuum Y which is homotopy equivalent to X, such that $G(Y)$ is a disjoint union of a null sequence of (open) arcs. We call such a continuum Y* arc-reduced.

Definition 2.12. *The subset $Q(X) \subset B(X)$ is the set of points of X such that every neighborhood contains an essential reduced loop intersecting $G(X)$.*

Theorem 2.11 and Theorem 2.13. *A point in a one-dimensional Peano continuum X is bad if and only if it is fixed by every self-homotopy of X. Moreover, every homotopy equivalence from X to Y restricts to a homeomorphism of pairs from $(B(X), Q(X))$ onto $(B(Y), Q(Y))$.*

Theorem 2.14. *Two one-dimensional Peano continua X and Y are homotopy equivalent if and only if one of the following hold*

1. *$Q(X) \neq \emptyset$ and there exists a homeomorphism of pairs*

$$h : (B(X), Q(X)) \to (B(Y), Q(Y))$$

2. *$Q(X) = \emptyset$, there exists a homeomorphism $h : B(X) \to B(Y)$ and*

$$H_1(X, B(X)) \cong H_1(Y, B(Y)).$$

Theorem 2.14B. *The triple $(B(\cdot), Q(\cdot), H_1(\cdot, B(\cdot)))$ is a complete invariant of the homotopy type of one-dimensional Peano continua.*

The main theorem of this paper is Theorem 2.14 (or Theorem 2.14B), and as stated earlier is aimed toward proving that the fundamental group actually determines the homotopy type of one-dimensional Peano continua. Conner and Eda have shown that the set $B(X)$ can be recovered from the fundamental group of a one-dimensional Peano continuum [3, 4]. Given Theorem 2.14, it only remains to identify the set $Q(X)$ from the fundamental group in order to prove the conjecture for all one-dimensional Peano continua.

2 Proofs

In order to facilitate later arguments, we state a few lemmas. Many of our arguments will use reduced loops. Recall that a loop is reduced if the only intervals in the domain where the loop is nulhomotopic are in fact constant. For a proof of Lemma 2.1 and more information on reduced loops see the work of Cannon and Conner [1].

Lemma 2.1. In a one-dimensional Peano continuum, every non-nulhomotopic loop f is homotopy equivalent to a reduced loop $\widetilde{f}$, which is unique up to reparameterization. Furthermore,the image of $\widetilde{f}$ is contained in the image of f. We call $\widetilde{f}$ a *reduced representative* for f.

The next lemma proves all of the continuity statements in the following theorems. This lemma is one form of an infinite pasting lemma, stating that you can paste together a null sequence of continuous moves. When using this lemma, we will either take Y to be a singleton or an interval.

Lemma 2.2. Let H be a function from the first-countable space $X \times Y$ into Z. Let $\{C_i\}$ be a null sequence of closed sets whose union is X. Suppose that $\{D_i = H(C_i \times Y)\}$ is a null sequence of sets in Z and that H is continuous on each $C_i \times Y$. If for every subsequence $C_{i_k} \to x_0$ there exists a $z_0 \in Z$ such that $D_{i_k} \to z_0$ and $H(\{x_0\} \times Y) = \{z_0\}$, then H is continuous on $X \times Y$.

Proof. Consider a sequence $(x_n, y_n) \to (x_0, y_0)$. For each n, choose an i_n such that $x_n \in C_{i_n}$. If $\{C_{i_n}\}$ is finite, then by restricting H to $\bigcup_n C_{i_n} \times Y$ we have $H(x_n, y_n) \to H(x_0, y_0)$ by a finite application of the pasting lemma. If $\{C_{i_n}\}$ is infinite, then since $\{C_i\}$ is a null sequence and $x_n \in C_{i_n}$, we have $C_{i_n} \to x_0$ and thus $H(x_n, y_n) \in D_{i_n} \to z_0 = H(x_0, y_0)$. Thus H is continuous on all of $X \times Y$. $\square$

We will use the following lemma to give an alternate definition of a locally finite graph, which we will use in proving Theorem 2.7.

Lemma 2.3. If X is a second countable metric space such that each $x \in X$ has a deleted neighborhood that is a 1-manifold with finitely many components, then X is homeomorphic to a locally finite graph.

Proof. For $x \in X$, we can choose a small deleted neighborhood of x that is a 1-manifold with finitely many components, each component limiting on x 'exactly once.' For example, if any component together with x forms a circle, then delete one point of the circle, as well as deleting any components that do not limit on x. Define the valence of a point, $v(x)$, to be the number of components of such a deleted neighborhood. Let $V = \{x \in X \mid v(x) \neq 2\}$, which will be a subset of our set of 0-cells. Then $X \setminus V$ is just a disjoint union of open arcs. Suppose there is one of these arcs, a, without compact closure. Let a be parameterized by $(0, 1)$, and consider the sequences $\{a(1/n)\}$ and $\{a(1 - 1/n)\}$ $(n \geq 2)$. If either sequence does not converge in X, then include that sequence in V as well. It can be seen that V is discrete and that $X \setminus V$ is a collection of open arcs with boundary in V, each having compact closure. Also, each 0-cell intersects only finitely many closed 1-cells, thus X is a locally finite graph. □

2.1 Theorems

Theorem 2.4. *Every non-contractible one-dimensional Peano continuum is homotopy equivalent to a unique core (a subcontinuum which admits no proper deformation retract).*

To prove this theorem, we will need the following technical lemma, which will also be used in the proof of Theorem 2.7.

Lemma 2.5. Let C be a closed subset of a simply connected neighborhood W of x in a one-dimensional Peano continuum. Given a collection of paths $\{p_i\}$ in W starting at x and intersecting C, let y_i be the first intersection of $\mathrm{im}(p_i)$ and C. Then $\{y_i\}$ is finite.

Proof. If $x \in C$, then $\{y_i\} = \{x\}$. Suppose then that $x \notin C$ and that $\{y_i\}$ is infinite. Then there exists a limit point $q \in C$ of $\{y_i\}$. In a path connected neighborhood of q in $W \setminus \{x\}$, there are distinct y_j and y_k that are joined by a short path, as well as a path contained in $\mathrm{im}(p_j) \cup \mathrm{im}(p_k)$. This loop cannot be nullhomotopic since the y_i's where chosen as the first point of intersection, contradicting the fact that W is simply connected. Thus $\{y_i\}$ must be finite. □

Proof of Theorem 2.4. Given such a continuum X, let Y be the set of points $x \in X$ such that there exists an essential reduced loop passing through x. In other words, Y is the union of all essential reduced loops in X. Since X is not contractible, it is not simply connected and thus there must be an essential reduced loop, so that Y is nonempty. We claim that Y is the unique minimal deformation retract of X. Since any loop homotopic to a reduced loop ℓ must contain ℓ by Lemma 2.1, we see that Y admits no proper deformation retract. We now show that Y is a deformation retract of X.

First, we show that Y is closed. Let y be a limit point of Y:

If $y \in B(X)$, then take a null sequence of essential reduced loops converging to y. Since X is locally path connected, we can conjugate each of these loops by a short path connecting them to y. Concatenating all of these conjugated loops gives an essential loop ℓ. The reduced representative $\widetilde{\ell}$ must contain each of the original reduced loops, and must therefore contain y, being closed. Thus $y \in Y$.

If $y \in G(X)$, suppose that $y \notin Y$. Consider simply connected neighborhoods $U \subset \overline{U} \subset W$ of y. For each point $z \in Y \cap U$ there is an essential reduced loop ℓ going through it, which must miss y since $y \notin Y$. Choose a sequence of points $\{z_i\}$ in $Y \cap U$, and for each z_i an essential reduced loop ℓ_i passing through z_i, such that z_k is chosen from a path connected neighborhood in U that misses all ℓ_j for $j < k$. Since W is simply connected, each ℓ_i must intersect ∂U at one (or two) points distinct from the previous ℓ_j's. Since we can connect the paths ℓ_i to y, Lemma 2.5 tells us that there can only be finitely many points in $(\bigcup \ell_i) \cap \partial U$, which is a contradiction.

Thus Y is closed.

Since X is path connected, for every $x \in X \setminus Y$ and for any $y_0 \in Y$ there is a path from x to y_0. Since Y is closed, there is a first point on this path that is in Y. Call this point y. We claim that given x, this point y is independent of the point y_0 or path chosen. Suppose not. Then there are distinct paths $p_1,\ p_2$ from x to $y_1,\ y_2$, each of which is the only point on the path contained in Y. Then there are $\ell_1,\ \ell_2$ essential reduced loops passing through $y_1,\ y_2$. If the paths $p_1,\ p_2$ are disjoint, then $p_1\ell_1\overline{p}_1p_2\ell_2\overline{p}_2$ is an essential reduced loop which passes through x. If the paths agree for some time and then become disjoint, the last point on which they agree must be a point of Y as before. If the paths are disjoint at first and then intersect, this also creates an essential reduced loop which passes through x. Thus for any path from x to any point of Y, there is a unique first point of Y on the path, depending solely on x.

For each such y, we claim that at least one complementary component of y is contractible, namely the one containing x. If it were not contractible, then it would contain an essential reduced loop, and a path from x to that loop contained in the component would intersect Y at some point other than y. We will call all such contractible components unioned with the point y *attached dendrites*. More precisely, an attached dendrite is a dendrite C such that for some $y \in C$, there is a strong deformation retract $r : X \to (X \setminus C) \cup \{y\}$. Denote all such attached dendrites $\{C_i\}$, each with a strong deformation retraction $r_i : C_i \to y_i$, for some $y_i \in Y$. Since any limit point of the collection $\{C_i\}$ is contained in Y (by local path connectivity), then by Lemma 2.2 the maps r_i paste together to give a strong deformation retraction from X to Y.

So we see that $Y \subset X$ is the image of X, hence Y is compact, connected and metric. Y is locally path connected as well, since the sets C_i did not connect to more than one point of Y. Thus Y is a core Peano continuum, and is unique since any core continuum must contain all of the points of Y. (Note that if X is contractible, the core is not uniquely defined, as X can be contracted to any point.) $\square$

We now state a few convenient equivalent characterizations of core continua.

Corollary 2.6. If X is a one-dimensional Peano continuum then the following are equivalent:

1. X is a core continuum.
2. X admits no proper strong deformation retract.
3. X has no attached dendrites.
4. $\forall\, x \in X$ and $\forall$ path component p of $X \setminus \{x\}$, $p \cup \{x\}$ is not simply connected.
5. Every point of X is on an essential loop that cannot be homotoped off it.

Proof. (1)⇔(2) by definition. (2)⇒(3) since an attached dendrite can be contracted, giving a proper strong deformation retract. (3)⇒(4) since a path component p of $X \setminus \{x\}$ such that $p \cup \{x\}$ is simply connected is an attached dendrite. (4)⇒(5) Let x be a point such that every essential loop in X can be homotoped off of x. Then as in the proof of the theorem, either x or a point on the path from x to the deformation retract Y has a complementary path component that is contractible. (5)⇒(2) If X admits a proper strong deformation retract Y, then any loop through a point in $X \setminus Y$ can be homotoped into Y (by the deformation retract). $\square$

Theorem 2.7. *If X is a core one-dimensional Peano continuum then $G(X)$ is a locally finite graph.*

In proving this theorem we will need another technical lemma.

Lemma 2.8. Let K be a simply connected closed set in a non-degenerate core one-dimensional Peano continuum X. Let $y \in K$ and p be an arc emanating from y contained in a component C of $K \setminus \{y\}$. Then p can be extended in C to an arc from y to ∂K.

Proof. First note that $\partial K \neq \emptyset$, since otherwise K would be both open and closed, hence $K = X$ would be a simply connected core continuum, and must be degenerate, a contradiction.

If $C \cap \partial K = \emptyset$, then C is also a component of $\operatorname{int}(K) \setminus \{y\}$ hence of $X \setminus \{y\}$, but $C \cup \{y\} \subset K$ is simply connected, contradicting the fact that X is a core continuum. So $C \cap \partial K \neq \emptyset$, and since K is arc connected $C \cup \{y\}$ is also arc connected, and must contain an arc p' from y to ∂K.

To see that p' can be chosen to extend p, consider the last point z of p that can be used in an arc from y to the boundary. If p and p' are disjoint after y, then since they are both in the path component $C \subset K \setminus \{y\}$, they must be connected by an arc at some point $z' \neq x$ of p. Similarly, if p and p' are not disjoint, $z \neq y$. Thus by repeating the above argument with y replaced by z, we see that z must actually be the last point of p, so that p' can be chosen to extend p in C. $\square$

Proof of Theorem 2.7. Assume that $G(X) \neq \emptyset$. By Lemma 2.3, it suffices to show that each point in $G(X)$ has a deleted neighborhood that is a finite disjoint union of arcs.

Fix $x \in G(X)$, and let W be a simply connected neighborhood of x. Let U be a path connected neighborhood of x such that $\overline{U} \subset W$. Then for any $w \in U$, there is a unique arc $p(w)$ from x to w. For points $w_1, w_2 \in U$, if neither w_1 nor w_2 lies on the arc from x to the other, we will call the last point in $p(w_1) \cap p(w_2)$ a *y-point* (relative to x). So each y-point y is the endpoint of (at least) three arcs. By Lemma 2.8, we see that these arcs can be extended to join y to ∂U, and intersect only at y since $\overline{W}$ is simply connected. For a set $A \subset U$ define $Y(A)$ to be the set of all y-points in A. Note that if A is finite, then so is $Y(A)$.

We claim that $Y(U)$ is finite. If not, then we may choose a sequence $\{y_n\}$ in $Y(U) \setminus \{x\}$ such that $y_i \notin Y(\{y_1, \ldots, y_{i-1}\})$. Choose the unique arc p from x to y_1 in U. Since y_1 is a y-point in U, there are three arcs emanating from y_1. Choose one of these arcs that is disjoint from p, and call it $\widetilde{p}$. By Lemma 2.8, $\widetilde{p}$ can be extended to p' which joins y_1 to ∂U. Let p_1 be $p\,p'$. Continuing by induction, since y_i is a y-point, but not for any two of the previous y_k's, there are three arcs emanating from y_i, of which at most two intersect the previous p_k's. Thus there is an arc p_i joining x to y_i to ∂U, with the segment of p_i from y_i to ∂U not intersecting the previous p_k's. Let r_i be the first intersection of p_i with ∂U. Then $\{r_i\}$ is infinite, which contradicts Lemma 2.5, so $Y(U)$ must be finite.

Then there is a path connected neighborhood $A \subset W$ of x with no y-points (other than possibly x). Each point in A lies on an arc with x as one endpoint, and which extends to ∂A by Lemma 2.8.

Then a path connected neighborhood of each point contained in $A \setminus \{x\}$ is exactly a portion of that arc, since any other point would need to be connected by a path to that arc, but there are no y-points in $A \setminus \{x\}$. Again by Lemma 2.5, there can only be finitely many of these arcs from x to ∂A. So $A \setminus \{x\}$ is a disjoint union of finitely many arcs, hence $G(X)$ is a locally finite graph. $\square$

Theorem 2.9. *Let X be a one-dimensional Peano continuum with $B(X) \neq \emptyset$. Then there is a continuum Y homotopy equivalent to X, with $G(Y)$ a disjoint union of a null sequence of (open) arcs. We call such a continuum Y* arc-reduced.

Proof. We may assume that X is a core continuum by Theorem 2.4. The theorem is obviously true if $X = B(X)$, so we assume that $G(X) \neq \emptyset$. Then $G(X)$ is a locally finite graph by Theorem 2.7.

To define the continuum Y and the homotopy equivalence between X and Y, we will choose rays in $G(X)$ along which we will retract the vertices of $G(X)$ into $B(X)$. In order to do this, we will define a sequence of covers $\mathcal{U}_i$ each refining a smaller cover $\mathcal{O}_i$ of $B(X)$, with the following properties:

1. $\mathcal{U}_i$ is a finite open cover of $B(X)$,
2. $\forall\, U \in \mathcal{U}_i$, $\forall\, x \in U$, $\exists$ a path from x to $B(X)$ contained in U, and
3. $\mathcal{U}_i = \mathcal{V}_i \cup \mathcal{W}_i$ with each collection $\mathcal{V}_i, \mathcal{W}_i$ pairwise disjoint.

Set $\mathcal{U}_0 = \{X\}$. Define $\mathcal{U}_i$ iteratively as follows. Let $\epsilon_i = \lambda_{i-1}/2$, where λ_{i-1} is a Lebesgue number for the cover $\mathcal{U}_{i-1} \cup \{G(X)\}$. Let $\mathcal{O}_i = \{O_1, \ldots, O_n\}$ be a finite, order 2 cover of $B(X)$ by open sets in X of diameter $\leq \epsilon_i$. For $j \neq k$, define the set V_{jk} to be the union of all path connected neighborhoods contained in $O_j \cap O_k$ about points $b \in O_j \cap O_k \cap B(X)$. Let $O'_j = O_j \setminus \bigcup_{k \neq j} \overline{O_k}$. Define V'_j to be the union of all path connected neighborhoods contained in O'_j about points $b \in O'_j \cap B(X)$. All such sets V_{jk} and V'_j will form the finite subcover $\mathcal{V}_i \subset \mathcal{U}_i$.

The only points $b \in B(X)$ that are not covered by $\mathcal{V}_i$ are those $b \in \partial O_m$ for some m. Note that each such point b is in exactly one set $O_k \in \mathcal{O}_i$: if it were in $O_j \cap O_k$, then that would be a neighborhood which must intersect O_m since $b \in \partial O_m$, but our cover has order 2. Define W_k to be the union of all path connected neighborhoods C of points $b \in B(X) \cap O_k \cap \partial O_m$, for any m, such that $C \subset B_{\delta/2}(b) \subset B_\delta(b) \subset O_k$ for some $\delta > 0$. Denote the collection of all such sets W_k by $\mathcal{W}_i$.

Then $\mathcal{U}_i = \mathcal{V}_i \cup \mathcal{W}_i$ refines the cover $\mathcal{O}_i$, and it is clear that 1 and 2 hold, and that the collection $\mathcal{V}_i$ is pairwise disjoint. To see that $\mathcal{W}_i$ is pairwise disjoint, consider $p \in W_j \cap W_k$. Then $p \in C_j \subset B_{\delta_j/2}(b_j) \subset O_j$, and also $p \in C_k \subset B_{\delta_k/2}(b_k) \subset O_k$. Then $d(b_j, b_k) \leq d(b_j, p) + d(p, b_k) < \delta_j/2 + \delta_k/2 \leq \max\{\delta_j, \delta_k\}$. But $B_{\delta_m}(b_m) \subset O_m$ for each m, so both $b_j, b_k \in O_j \cap O_k$, hence j must equal k by the discussion in the previous paragraph.

We now begin to choose the rays by which we will retract the vertices of $G(X)$ to $B(X)$. Set A_i to be the union of all $U \in \mathcal{U}_i$. By our choice of ϵ_i, we have the following inclusion: $A_i \supset A_{i+1} \supset B(X)$. Consider the vertices in $A_i \cap G(X)$. First consider those vertices $x \in A_i$ that are in some $V \in \mathcal{V}_i$. There is a path from x to $B(X)$ contained in V. Let $p(x)$ be the initial segment of that path, until it hits a vertex in A_{i+1} (this will be constant if $x \in A_{i+1}$). Denote the end vertex of $p(x)$ by $v(x)$. Continue choosing such paths with the added condition that for $x, y \in A_i$, the paths $p(x)$ and $p(y)$ are either disjoint, or they coincide after their first intersection. After choosing such paths for all vertices in A_i that are in some $V \in \mathcal{V}_i$, choose paths with the same criterion (chosen paths are either disjoint or coincide) for vertices covered by $\mathcal{W}_i$. Because of disjointness of the collections $\mathcal{V}_i$ and $\mathcal{W}_i$, the path for a vertex in some $V \in \mathcal{V}_i$ will remain in V, while the path for a vertex in some $W \in \mathcal{W}_i$ may go into a neighboring $V \in \mathcal{V}_i$. This, together with the fact that each neighborhood has diameter $\leq \epsilon_i$, shows for all vertices $x \in A_i, \forall\, y \in p(x)$, that $d(x, y) \leq 2\epsilon_i$. (Note that it can be shown that $d(x, y) \leq \epsilon_i$, since $\mathcal{U}_i$ is 'nearly' a star refinement of the cover $\mathcal{O}_i$ – at least for the sets $W \in \mathcal{W}_i$.)

After all the paths $p(x)$ have been chosen, set $r(x)$ to be the ray defined by the paths $p(x), p(v(x)), p(v^2(x)), \ldots$ (i.e. $r(x)$ follows the path $p(x)$ from x, then the path from the endpoint of that path, and so on). The sequence $\{v^k(x)\}$ is Cauchy since $\epsilon_k \leq \lambda_{k-1}/2 \leq \epsilon_{k-1}/2$, and thus limits on a unique point $b(x) \in B(X)$. It will be important to note that if $x \in U \in \mathcal{U}_i$, then the ray $r(x)$ has diameter at most $\sum_{k=i}^{\infty} 2\epsilon_k \leq 2\sum_{k=0}^{\infty} \epsilon_i 2^{-k} = 4\epsilon_i$, and thus the rays form a null sequence. This follows since each segment of the ray is contained in two elements of $\mathcal{U}_k$, each having diameter $\leq \epsilon_k$, together with the inequality $\epsilon_k \leq \lambda_{k-1}/2 \leq \epsilon_{k-1}/2$.

We now define the space Y and the maps $f : X \to Y$ and $g : Y \to X$, and show that they are homotopy inverses. The space Y is a quotient space of X, where each ray $r(x)$ is identified with the point $b(x) \in B(X)$, and f is the corresponding quotient map. This is well defined since if any of the paths $p(x)$ intersect, they coincide the rest of the way to $B(X)$. Thus Y is a one-dimensional Peano continuum with $G(Y)$ a disjoint union of arcs, corresponding to those edges in $G(X)$ that are not part of any ray $r(x)$.

To define the map g, first label all of the edges in $G(X)$. Then subdivide each arc a in $G(Y)$ into a bi-infinite sequence of subarcs. Label the middle third of a with the label for $b = f^{-1}(a)$, the corresponding arc in X. Then for an end third of a, let x be the corresponding endpoint of b. Label the infinite sequence of subarcs with the labels of the edges in $r(x)$, noting orientation (i.e. which direction is going to the bad set). The map g is also a quotient, identifying all (closed) subarcs in $G(Y)$ that have the same (oriented) label.

The composition $f \circ g$ simply takes the arcs of $G(Y)$ and slides the end thirds to their endpoints in $B(Y)$, and stretches the middle third over the whole arc. This is clearly homotopic to the identity map on each arc. Since the arcs in $G(Y)$ form a null sequence, and any limit point of the arcs is in $B(Y)$ which is fixed, we see that $f \circ g$ is homotopic to Id_Y by Lemma 2.2.

We now show that $g \circ f$ is homotopic to Id_X. The union of all the rays $r(x)$ and their limit points $b(x)$ is a forest F. Let $h : F \times I \to F$ be a strong deformation of F onto $\{b(x)\}$. For each arc a in $G(X) \setminus F$ with endpoints $a(0)$ and $a(1)$, define $C(a)$ to be the concatenation of paths $\overline{r}(a(0))\ a\ r(a(1))$. The homotopy $H : X \times I \to X$ fixes $B(X)$ for all times, retracts F by h, and stretches each arc a in $G(X) \setminus F$ over the arc $C(a)$. Explicitly, for $x \in C(a)$ the homotopy looks like

$$H(x,t) = \begin{cases} h(x,t) & \text{if } x \in r(a(0)), \\ h(a(0), t - s(2t+1)) & \text{if } x = a(s) \text{ for } s \in [0, t/(2t+1)], \\ a(s(2t+1) - t) & \text{if } x = a(s) \text{ for } s \in [t/(2t+1), 1 - t/(2t+1)], \\ h(a(1), t - (1-s)(2t+1)) & \text{if } x = a(s) \text{ for } s \in [1 - t/(2t+1), 1], \\ h(x,t) & \text{if } x \in r(a(1)), \end{cases}$$

Again we use Lemma 2.2 to paste the homotopies on each $C(a)$ together to get a homotopy on all of X. This follows since the collection $\{C(a)\}$ forms a null sequence, with each $C(a)$ mapping into itself continuously for all time, and as before, any limit point x_0 of the $C(a)$'s will be in $B(X)$, which is fixed for all time. Thus $g \circ f$ is homotopic to Id_X.

Thus $f : X \to Y$ is a homotopy equivalence of one-dimensional Peano continua X and Y, where $G(Y)$ is a disjoint union of arcs and f maps $B(X)$ homeomorphically onto $B(Y)$. $\square$

Corollary 2.10. If X is a one-dimensional Peano continuum with $B(X) = \emptyset$ then X is homotopy equivalent to a bouquet of finitely many (possibly zero) circles.

Proof. If $B(X) = \emptyset$, then X is a locally finite graph. Pick any vertex of X to play the role of $B(X)$ above. The rest of the proof follows that of the theorem. The bouquet must be finite since X is compact, and has no bad points by hypothesis. $\square$

Theorem 2.11. *In a one-dimensional Peano continuum X, a point is bad if and only if it is fixed by every self-homotopy $f : X \to X$. Moreover, if $h : X \to Y$ is a homotopy equivalence, then $h|_{B(X)}$ is a homeomorphism onto $B(Y)$.*

Proof. Suppose $x \notin B(X)$. If x is not in the core of X, then the deformation retract to the core of X does not fix x. Otherwise we may assume that X is core by Theorem 2.4, so that by Theorem 2.7 x has a neighborhood that is a locally finite graph, and can clearly be moved by a self-homotopy of X. If $x \in B(X)$, then suppose $f : X \to X$ is homotopic to Id_X, and $f(x) \neq x$. Choose a neighborhood U of x such that $U \cap f(U) = \emptyset$. Then any non-nulhomotopic loop ℓ in U is disjoint from its image $f(\ell) \subset f(U)$. But ℓ and $f(\ell)$ are freely homotopic, and thus must both contain the same reduced representative by Lemma 2.1, which is a contradiction. Thus f fixes x for every $x \in B(X)$.

Now let $h : X \to Y$ and $g : Y \to X$ be homotopy inverses. For $b \in B(X)$, if $h(b) \notin B(Y)$, then there is a simply connected neighborhood of $h(b)$, and thus h_* is not injective since h maps small loops to small loops. So h maps $B(X)$ into $B(Y)$, and similarly g maps $B(Y)$ into $B(X)$.

Then $g \circ h \cong Id_X$, so as above $g \circ h|_{B(X)} = Id_{B(X)}$, and $h|_{B(X)}$ is injective. Also, $h \circ g \cong Id_Y$, so $h \circ g|_{B(Y)} = Id_{B(Y)}$, which implies that $h|_{B(X)}$ is surjective (onto $B(Y)$) since g maps $B(Y)$ into $B(X)$. Since both h and g are continuous, it follows that h maps $B(X)$ homeomorphically onto $B(Y)$. $\square$

Definition 2.12. The subset $Q(X) \subset B(X)$ is the set of points of X such that every neighborhood contains an essential reduced loop intersecting $G(X)$.

In an arc-reduced continuum X, $Q(X)$ is equivalently the set of limit points of the set consisting of one point from each open arc in $G(X)$. The pair $(Q(X), B(X))$ will give a complete invariant of the continuum X, up to homotopy type. We first prove the analog of Theorem 2.11 for $Q(X)$.

Theorem 2.13. *If $f : X \to Y$ is a homotopy equivalence of one-dimensional Peano continua, then $f|_{Q(X)}$ is a homeomorphism onto $Q(Y)$. Furthermore, if f is a self-homotopy of X, then f fixes $Q(X)$ pointwise.*

Proof. Let $x \in Q(X)$. We claim that any essential reduced loop ℓ near x that intersects $G(X)$ cannot map into $B(Y)$. If it did, then $g \circ f(\ell)$ would be contained in $B(X)$ by Theorem 2.11. This is not homotopic to Id_X because $g \circ f(\ell)$ does not contain its reduced representative ℓ (see Lemma 2.1). So f maps $Q(X)$ into $Q(Y)$.

Then since $g \circ f|_{Q(X)} = Id_{Q(X)}$, $f|_{Q(X)}$ must be injective, and since $f \circ g|_{Q(Y)} = Id_{Q(Y)}$, $f|_{Q(X)}$ must be surjective because g maps $Q(Y)$ into $Q(X)$. Since f and g are continuous, we see that f maps $Q(X)$ homeomorphically onto $Q(Y)$.

If f is a self-homotopy of X, by Theorem 2.11 f must fix $Q(X)$ pointwise since $Q(X) \subset B(X)$. □

By Theorem 2.11 we see that the relative homology group $H_1(X, B(X))$ is a homotopy invariant of one-dimensional Peano continua, and consequently so is its rank. If X is arc reduced, then the rank of $H_1(X, B(X))$ is just the number of arcs in $G(X)$. We note that the rank is finite if and only if $Q(X) = \emptyset$. This dichotomy (whether or not the rank is finite) separates our characterization of one-dimensional Peano continua into two cases.

Theorem 2.14. *If X and Y are one-dimensional Peano continua, then X and Y are homotopy equivalent if and only if one of the following hold*

1. *$Q(X) \neq \emptyset$ and there exists a homeomorphism of pairs*

$$h : (B(X), Q(X)) \to (B(Y), Q(Y))$$

2. *$Q(X) = \emptyset$, there exists a homeomorphism $h : B(X) \to B(Y)$ and*

$$H_1(X, B(X)) \cong H_1(Y, B(Y)).$$

Theorem 2.14B. *The triple $(B(\cdot), Q(\cdot), H_1(\cdot, B(\cdot)))$ is a complete invariant of the homotopy type of one-dimensional Peano continua.*

Proof. By Theorem 2.11 and Theorem 2.13, if X and Y are homotopy equivalent, then there is a homeomorphism of pairs $h : (B(X), Q(X)) \to (B(Y), Q(Y))$, and h induces an isomorphism $H_1(X, B(X)) \cong H_1(Y, B(Y))$.

First we prove the converse when $Q(X) \neq \emptyset$. Suppose there is a homeomorphism of pairs $h : (B(X), Q(X)) \to (B(Y), Q(Y))$. By Theorem 2.9 we may assume that X and Y are both arc-reduced. We will extend the homeomorphism $h : B(X) \to B(Y)$ to a homotopy equivalence of X and Y. It suffices to prove that X is homotopy equivalent to the adjunction space $X \cup_h Y$. Due to the homeomorphism h, we will refer to the sets $B(X)$ and $Q(X)$ as subsets of the spaces X, Y, and $X \cup_h Y$.

We will construct our homotopy equivalence by choosing for each arc in $G(Y)$ a sequence of arcs in $G(X)$ converging to a nearby point in $Q(X)$. The map will send the arc in Y to the first arc in the sequence, which will be mapped to the next arc, and so on. In order to ensure continuity, we must be careful in our choices of sequences of arcs, which will be made explicit in the following paragraphs.

Label the null sequence of arcs in $G(X)$ as the collection $\{x_k \mid k \in \mathbb{N}\}$, and those in $G(Y)$ as $\{y_i \mid i \in \mathbb{N}\}$. For any path p and any point q, we define $\tilde{d}(p, q) = \operatorname{diam}(p \cup q)$. Since $Q(X)$ is closed hence compact, for each y_i we can find a point $q_i \in Q(X)$ such that $\tilde{d}(y_i, q_i)$ is minimal

Let X^0 denote the collection of all the arcs in $G(X)$. When X^i has been defined, choose a sequence of arcs $\{w^i_j \mid j \in \mathbb{N}\} \subset X^i$ that converge to q_i such that $\tilde{d}(w^i_j, q_i) < 2^{-(i+j)}$. Now for this i we define the sequence of arcs $Z^i = \{z^i_j \mid j \in \mathbb{N}\}$ by $z^i_0 = y_i$ and $z^i_j = w^i_{2j}$. Also set $X^{i+1} = X^i \setminus Z^i$. Since we have only included those arcs with even index into the set Z^i, we have that $Q(B(X) \cup X^{i+1}) = Q(B(X) \cup X^i) = Q(X)$, so that we will always be able to find such a sequence $\{w^i_j \mid j \in \mathbb{N}\}$ for all i.

We now describe a homotopy equivalence $f : X \cup_h Y \to X$ and its homotopy inverse g. The set $B(X)$ will be fixed, as will each arc in $X^0 \setminus \bigcup Z^i$. It remains to specify where each arc in $\bigcup Z^i$ is sent.

Since our space is locally path connected and compact, we know that two points are close if and only if there is a small diameter path between them. For each i, j, find a pair of small diameter paths p^i_j, r^i_j in X from the endpoints of z^i_j to the endpoints of z^i_{j+1}. Note that these paths will have arbitrarily small diameter as i, j increase, due to the choice of the z^i_j. Then f maps the arc z^i_j to $p^i_j z^i_{j+1} \overline{r}^i_j$. The map g is defined similarly: the arc z^i_{j+1} (in X) is mapped to $\overline{p}^i_j z^i_j r^i_j$. Note that f maps each $y_i = z^i_0$ into $G(X)$, and that both f and g are surjective.

It remains to show that f and g are continuous, and that they are homotopy inverses. To see that f is continuous, we use Lemma 2.2. In the lemma we take Y to be a singleton, take X to be the adjunction space $X \cup_h Y$, and set $\{C_i \mid i \in \mathbb{N}\}$ to be the collection consisting of the set $B(X)$, and also the null sequence of arcs in $G(X \cup_h Y) = G(X) \cup G(Y)$. Any point x_0 in the lemma will be in $B(X)$ and thus fixed, so the lemma applies. Thus f is continuous, and similarly so is g.

The composition $g \circ f$ fixes $B(X)$ as well as each arc in $X^0 \setminus \bigcup Z^i$. Also, $g \circ f$ maps z^i_j onto the concatenation of paths $p^i_j \overline{p}^i_j z^i_j \overline{r}^i_j r^i_j$. Let C^i_j be (the image of) the closed arc z^i_j, and D^i_j the union of (the images of) the paths p^i_j, z^i_j, and r^i_j. Clearly, on any one of the z^i_j, there is a homotopy $H^i_j : C^i_j \times I \to D^i_j$ between $g \circ f$ and the identity. Again, by Lemma 2.2 we can paste these homotopies together to get a homotopy on the entire space. Thus $g \circ f$ is homotopic to the identity, and similarly so is $f \circ g$.

Therefore, when $Q(X) \neq \emptyset$, X and Y are homotopy equivalent if and only if there is a homeomorphism of pairs $h : (B(X), Q(X)) \to (B(Y), Q(Y))$.

Now suppose that $Q(X) = \emptyset$ and that there is a homeomorphism $h : B(X) \to B(Y)$, where X and Y are arc-reduced and that $H_1(X, B(X)) \cong H_1(Y, B(Y))$. Since $Q(X) = \emptyset$, there must only be finitely many arcs in $G(X)$, namely the rank of $H_1(X, B(X))$. So there are only finitely many path components of $B(X)$, each of which will be a Peano continuum. Clearly we can homotop any non-separating arc in $G(X)$ to a loop based at any point of $B(X)$. Thus it suffices to prove that if each arc separates, the space is homotopy equivalent to one with the same path components of $B(X)$ connected in any given configuration. It is clear that given any two path components A and B of $B(X)$, there is a homotopy equivalence that connects A and B by one arc in $G(X)$. This arc can have any given endpoints in A and B, since they are path connected. Thus we can construct a homotopy equivalence from X to Y.

Thus the triple $(B(\cdot), Q(\cdot), H_1(\cdot, B(\cdot)))$ is a complete invariant of one-dimensional Peano continua, where we consider the triples for X and Y equivalent if there is a homeomorphism of pairs $(B(X), Q(X)) \cong (B(Y), Q(Y))$, and an isomorphism $H_1(X, B(X)) \cong H_1(Y, B(Y))$. $\square$

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