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Maximal Surfaces in Complexes

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MAXIMAL SURFACES IN COMPLEXES

by

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of a thesis submitted by

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This thesis has been read by each member of the following graduate committee and by majority vote has been found to be satisfactory.

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BRIGHAM YOUNG UNIVERSITY

As chair of the candidate's graduate committee, I have read the thesis of Allen Dickson in its final form and have found that (1) its format, citations, and bibliographical style are consistent and acceptable and fulfill university and department style requirements; (2) its illustrative materials including figures, tables, and charts are in place; and (3) the final manuscript is satisfactory to the graduate committee and is ready for submission to the university library.

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ABSTRACT

MAXIMAL SURFACES IN COMPLEXES

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Master of Science

Cubical complexes are defined in a manner analogous to that for simplicial complexes, the chief difference being that cubical complexes are unions of cubes rather than of simplices. A very natural cubical complex to consider is the complex $C(k_1, \ldots, k_n)$ where $k_1, \ldots, k_n$ are nonnegative integers. This complex has as its underlying space $[0, k_1] \times \cdots \times [0, k_n] \subset \mathbb{R}^n$ with vertices at all points having integer coordinates and higher dimensional cubes formed by the vertices in the natural way.

The genus of a cubical complex is defined to be the maximum genus of all surfaces that are subcomplexes of the cubical complex. A formula is given for determining the genus of the cubical complex $C(k_1, \ldots, k_n)$ when at least three of the k_i are odd integers. For the remaining cases a general solution is not known.

When $k_1 = \cdots = k_n = 1$ the genus of $C(k_1, \ldots, k_n)$ is shown to be $(n-4)2^{n-3}+1$ which is equivalent to the genus of the graph of the n-cube. Indeed, the genus of the complex and the genus of the graph of the 1-skeleton of the complex, are shown to be equal when at least three of the k_i are odd, but not equal in general.

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1 Introduction

The objective of this paper is to provide a solution to the problem of determining the maximal genus of a surface which is a subcomplex of a given cubical complex. A cubical complex is similar to a simplicial complex. The only real difference is that instead of constructing the complex from simplices, one constructs the complex from cubes. Typically we refer to 0-cubes as vertices, 1-cubes as lines or edges, and 2-cubes as squares. We will present some cubical complexes that are of particular interest for this problem of determining the maximal genus of surfaces in the complex.

For example, let us consider the cubical complex which has as its underlying space $[0,3]\times[0,3]\times[0,1] \subset \mathbb{R}^3$, having vertices at all points with integer coordinates, having edges between all vertices of distance 1 apart, and having squares and 3-cubes that are determined by the edges in the obvious way. You can think of this complex as a $3 \times 3 \times 1$ stack of cubes. This is the complex which we will call $C(3,3,1)$. If we take the union of all squares on the boundary of this complex, we will have a sphere of genus 0. It is easy to see that the maximal genus of any surface which is a subcomplex of this complex is 1, since we cannot do better than the torus shown in Figure 1.

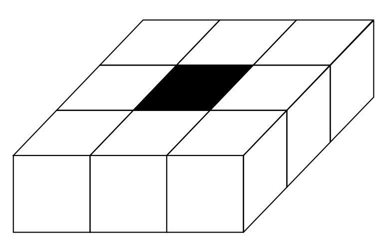


Figure 1: A surface of genus 1 in $C(3,3,1)$

In general, we will prove that the maximal genus of a surface which is a subcomplex of $C(k_1, \ldots, k_n)$ is given by a certain symmetric equation when at least three of the k_i are odd positive integers. In addition, there is a standard way of constructing a surface which realizes this maximal genus. When less than three of the k_i are odd, the situation is much more complex and a general solution has not been found.

When at least three of the k_i are odd, it can be shown that the maximal genus of a surface in the cubical complex $C(k_1, \ldots, k_n)$ is equal to the genus of the graph that is the 1-skeleton of $C(k_1, \ldots, k_n)$. Therefore the solution to this problem generalizes a result by Beineke and Harary, which gives the genus of the n-cube Q_n as

$$\gamma(Q_n) = (n-4)2^{n-3} + 1.$$

The dependence of both of these results upon the condition that at least three of the k_i are odd, reflects the fact that this condition ensures that we can construct a surface of maximal genus that contains the entire 1-skeleton of the cubical complex. If this condition is not met, there are many examples where constructing such a surface is not possible.

2 Constructing Cubical Complexes

The construction of cubical complexes is analogous to the construction of simplicial complexes. First we will define cubical complexes in general, then we will give some examples of cubical complexes. Mostly we will work with a simple type of cubical complex in which the cubes are all of a uniform size and are arranged nicely in a rectangular stack.

We begin by defining an n-cube C^n, which is analogous to an n-simplex of a simplicial complex.

Definition 2.1. An n-cube C^n is a subset of $\mathbb{R}^N$, $n \leq N$, determined by two vectors in $\mathbb{R}^N$, $r = (r_1, r_2, \ldots, r_N)$, $s = (s_1, s_2, \ldots, s_N)$, such that

$$C^n = \left\{(x_1, x_2, ..., x_N) \in \mathbb{R}^N | r_i \leq x_i \leq s_i, \text{ for } 1 \leq i \leq N\right\}$$

where $r_i < s_i$ for n of the i, and $r_i = s_i$ for the remaining i.

We will commonly refer to 0-cubes as vertices, 1-cubes as edges, and 2-cubes as squares. When the value n need not be specified, we refer to n-cubes simply as cubes.

Definition 2.2. A subcube of an n-cube determined by vectors $r, s \in \mathbb{R}^N$, is an m-cube, $m \leq n$, determined by $r' = (r'_1, r'_2, \ldots, r'_N)$, $s' = (s'_1, s'_2, \ldots, s'_N)$, such that for each i, one of the following is true:

1. $r'_i = r_i$ and $s'_i = s_i$
2. $r'_i = s'_i = r_i$
3. $r'_i = s'_i = s_i$.

For example, the proper subcubes of a square are its four edges and four vertices. We are now ready to define a cubical complex.

Definition 2.3. A cubical complex $\mathcal{C}$, is a union of cubes with the following properties:

1. Every subcube of a cube in $\mathcal{C}$, is in $\mathcal{C}$.
2. If $C, C' \in \mathcal{C}$, then $C \cap C'$ is either the empty set, or a subcube of both C and C'.

It will be useful to give a definition analogous to that of a polytope of a simplicial complex for cubical complexes.

Definition 2.4. Let $|\mathcal{C}|$ be the subset of $\mathbb{R}^N$ that is the union of the cubes of $\mathcal{C}$, where every cube of $\mathcal{C}$ is of dimension less than or equal to N. Give each cube of $\mathcal{C}$ its natural topology as a subset of $\mathbb{R}^N$ then define the topology of $|\mathcal{C}|$ by declaring $A \subset \mathcal{C}$ to be closed if and only if $A \cap C$ is closed in C for each C in $|\mathcal{C}|$.

The topology of $|\mathcal{C}|$ is finer in general than the topology it inherits as a subspace of $\mathbb{R}^N$. However, we will only work with finite cubical complexes. In this case it is elementary to show that these two topologies are the same. The proof is identical to that for simplicial complexes [3, p.9].

We can now introduce the cubical complex $C(k_1, k_2, ..., k_n)$. This is the rectangular stack of cubes to which we referred earlier. We will work with surfaces that are subcomplexes of this cubical complex.

Definition 2.5. Let $k_1, k_2, ..., k_n \in \mathbb{Z}_+ \cup \{0\}$. Define $C(k_1, k_2, ..., k_n)$ to be the cubical complex that is the union of all possible n-cubes determined by the vectors

$$r = (z_1 - 1, z_2 - 1, \ldots, z_n - 1),$$

$$s = (z_1, z_2, \ldots, z_n),$$

where $z_i \in \mathbb{Z}$ such that $0 < z_i \leq k_i$ for $k_i > 0$, and $r_i = s_i = 0$ for $k_i = 0$; together with all subcubes of these n-cubes.

In order to clarify the construction of some of the surfaces in these complexes, we will label each n-cube of $C(k_1, k_2, ..., k_n)$ with an n-tuple of integers $(z_1, z_2, \ldots z_n)$, where the integers z_i are exactly the s_i from Definition 2.5.

Figure 2 shows the complex $C(2, 3, 4)$ with some of the 3-cubes labeled.

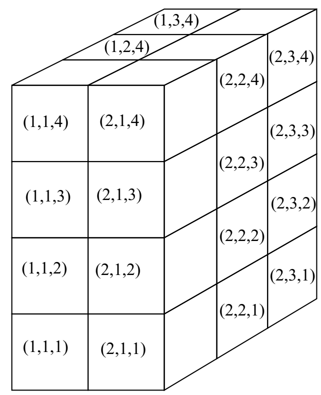


Figure 2: $C(2, 3, 4)$ with 3-cubes labeled

3 Maximal Surfaces

Let us consider the complex $C(3,3,3)$. There are various ways of constructing subcomplexes of $C(3,3,3)$ which are compact orientable surfaces. These surfaces will consist of some union of squares in the cubical complex. For example, one might take the union of all squares which are on the boundary of the $3\times3\times3$ stack of cubes. This surface is a 2-sphere having genus 0. Alternatively, one could punch a hole through the middle of the stack of cubes by removing the cubes $(2,2,1),(2,2,2)$, and $(2,2,3)$. The boundary squares of the closure of this punched stack form a torus of genus 1. By repeating the previous construction and removing the four additional cubes $(2,1,2),(2,3,2),(1,2,2),(3,2,2)$, we obtain the 5 holed torus of genus 5. It may be easier to visualize this by stretching the top hole of the surface so that the surface is laid out flat. One can verify by exhausting all possibilities, or by using an argument with Euler's characteristic formula, that 5 is the maximum possible genus of a surface that is a subcomplex of $C(3,3,3)$. We call 5 the *maximal genus* of $C(3,3,3)$, and the surface with genus 5, the *maximal surface* for $C(3,3,3)$. Figures 3-6 illustrate these surfaces.

Throughout this paper, the term surface will always refer to a connected compact orientable 2-manifold. All such surfaces are classified up to homeomorphism by their genus, which corresponds to the number of handles on the surface [2, p.29,119-128].

Definition 3.1. Let $\gamma(S)$ denote the genus of a surface S. A maximal surface M in a complex $\mathcal{C}$, is a surface such that if S is any other surface in $\mathcal{C}$, then $\gamma(S) \leq \gamma(M)$.

Definition 3.2. The maximal genus of a complex $\mathcal{C}$, denoted $\gamma(\mathcal{C})$, is equal to $\gamma(M)$, where M is a maximal surface in $\mathcal{C}$.

The example above with $C(3,3,3)$ leads us to the following question. What is

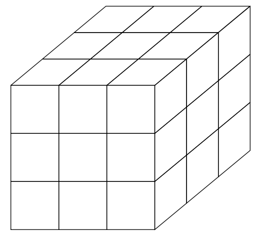


Figure 3: A surface of genus 0 in $C(3,3,3)$

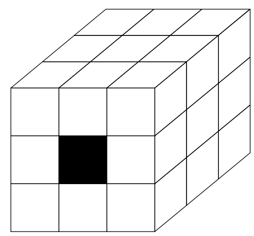


Figure 4: A surface of genus 1 in $C(3,3,3)$

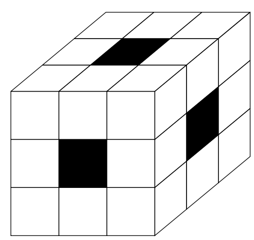


Figure 5: The maximal surface of genus 5 in $C(3,3,3)$

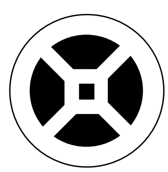


Figure 6: The maximal surface of genus 5 stretched flat

the maximal genus of a surface that is a subcomplex of $C(k_1, ..., k_n)$, and what does the maximal surface look like?

For many cases the answer to this question is intuitively clear. Consider the complex $C(k, 1, 1)$. It is clear that the maximal genus for any value of k is zero. We will not provide proofs for such cases, but it should be noted that a rigorous proof of this claim is not easy to express clearly in words. A rough sketch of such a proof might be first to show that any surface in this complex must contain one square, then show that there are certain squares bordering the first square which must be added so that the surface is a 2-manifold. This leaves us with the only possible surface being a sphere of some varying integer length along the 1st dimension $[0, k]$.

Here is an example that illustrates how things become different when dealing in dimensions greater than 3. For $C(1, 1, 1)$, the only possible surface is of genus 0. This surface is formed from the six squares on the boundary of the 3-cube. However, for $C(1, 1, 1, 1)$ the maximal genus is 1, and the maximal surface is the torus as shown in Figure 7 below. It may be helpful to think of $C(1, 1, 1, 1)$ as $[0, 1]^4 = [0, 1]^2 \times [0, 1]^2$, where $[0, 1]^2$ is a square. The torus is formed by taking the product of the boundaries of the two copies of $[0, 1]^2$. This is homeomorphic to $S^1 \times S^1$, which is the torus.

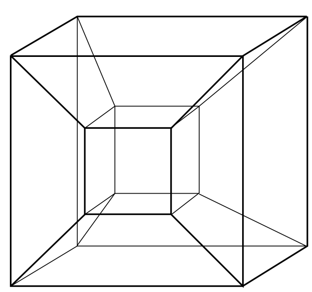


Figure 7: The maximal surface of genus 1 in $C(1, 1, 1, 1)$

The question above will be answered by the main theorem of this paper, so we will restate the question more precisely. Is there a standard construction for a maximal surface in $C(k_1, k_2, ..., k_n)$; is it unique; and what is its genus?

4 A Bound for the Genus of $C(k_1, \ldots, k_n)$

To answer the question of the previous section we will find an upper bound for the genus of an arbitrary surface in the complex $C(k_1, \ldots, k_n)$, then we will show how to construct a surface whose genus is equal to this upper bound when all of the k_i are odd. It will be helpful to have the following definitions.

Definition 4.1. Define $Q(k_1, \ldots, k_n)$ to be the 1-skeleton of the cubical complex $C(k_1, \ldots, k_n)$.

$Q(k_1, \ldots, k_n)$ is a graph in the traditional sense, and the 1-skeleton of any subcomplex of $C(k_1, \ldots, k_n)$ is a subgraph of $Q(k_1, \ldots, k_n)$.

Definition 4.2. For any subgraph Γ of $Q(k_1, \ldots, k_n)$, define the complement

$$\Gamma' = Q(k_1, \ldots, k_n) - \Gamma.$$

Γ' is a union of cubes, though not necessarily a graph or a cubical complex since it may contain edges (1-cubes) which do not contain both of their vertices, thereby violating the properties of Definition 2.3. In particular, suppose that e is an edge in $Q(k_1, \ldots, k_n)$ that is not contained in Γ, and that e contains a vertex v that is contained in Γ. Then e is an edge in Γ' but v is not a vertex of Γ'. Note that $|\Gamma'| \cap |\Gamma| \neq \emptyset$ since the intersection contains at least the vertex v.

We want to count the difference between the number of edges and vertices in $Q(k_1, \ldots, k_n)$, and the number of edges and vertices in Γ. This is equivalent to counting the number of edges and vertices in Γ'. It will sometimes be necessary to make a distinction between edges of Γ' which contain both of their vertices and those which do not. We will refer to edges that are missing one or both of their vertices as *incomplete* edges when it is necessary to make such a distinction.

Now suppose that Γ is a subgraph of $Q(k_1, \ldots, k_n)$ and that v is a vertex of Γ. There are between n and $2n$ edges of $Q(k_1, \ldots, k_n)$ which have v as an endpoint (up to two in each dimension), i.e. the vertex v has valence between n and $2n$. If every edge in $Q(k_1, \ldots, k_n)$ which has v as an endpoint is also an edge in Γ, then we will say that v is a *complete* vertex of Γ.

Definition 4.3. For any cellular complex C, define V_C, E_C, F_C to be the number of vertices, edges, and faces respectively of C. If the complex C is clear from the context, these will be denoted simply by V, E, F.

Proposition 4.4. *For any surface S in $C(k_1, \ldots, k_n)$,*

$$\gamma(S) = -\frac{V}{2} + \frac{E}{4} + 1.$$

Proof. Suppose that S is any surface in $C(k_1, \ldots, k_n)$. The 1-skeleton of S, denoted $S^{(1)}$, embeds cellularly into S by inclusion, where the 2-cells of the embedding are the squares in the complex S. Thus we may use Euler's characteristic formula to calculate the genus of S. Recall that Euler's characteristic formula is

$$\chi(S) = V - E + F$$

and that the relation

$$2 - 2\gamma(S) = \chi(S)$$

holds for any cellular embedding in the surface S [2, p.121]. Because S is a 2-manifold, every edge in the surface S must intersect exactly two faces. Conversely, every face must have exactly four edges because every face is a square. By counting each edge twice we obtain the relation $2E = 4F$ giving us the essential equality

$$E = 2F.$$

It follows that the genus of S is given by

$$\gamma(S) = \frac{-V + E - F + 2}{2} = \frac{-V + E - \frac{E}{2} + 2}{2} = -\frac{V}{2} + \frac{E}{4} + 1.$$

■

In particular, the genus of S depends only on the number of vertices and edges in $S^{(1)}$.

Definition 4.5. For any graph Γ, define the function

$$g(\Gamma) = -\frac{V}{2} + \frac{E}{4} + 1.$$

The function g does not necessarily have any relation to the genus of a surface, but for any surface S in $C(k_1, \ldots, k_n)$, we have $\gamma(S) = g(S^{(1)})$. Our goal is to maximize the function g over all graphs $\Gamma \subset Q(k_1, \ldots, k_n)$. By so doing, we will obtain an upper bound for the function g, hence for the genus of a surface in $C(k_1, \ldots, k_n)$.

Proposition 4.6. *For $n \geq 4$, if Γ is a subgraph of $Q(k_1, \ldots, k_n)$, then*

$$g(\Gamma) \leq g(Q(k_1, \ldots, k_n)).$$

Proof. Suppose that Γ is a subgraph of $Q(k_1, \ldots, k_n)$. We will assume that Γ contains all possible edges between its vertices since if it does not, then we may simply add the missing edges to Γ, which will increase the value of g. Now consider the complement Γ'. By counting the number of vertices and edges of Γ' (some of the edges may be incomplete), we will be able to determine that $g(\Gamma) \leq g(Q(k_1, \ldots, k_n))$. In particular, if we can show that

$$E_{\Gamma'} \geq 2V_{\Gamma'}$$

then this will imply

$$-\frac{V_{\Gamma'}}{2} + \frac{E_{\Gamma'}}{4} \geq 0$$

so that

$$\begin{aligned} g(Q(k_1, \ldots, k_n)) &= \left(-\frac{V_\Gamma}{2} + \frac{E_\Gamma}{4}\right) + \left(-\frac{V_{\Gamma'}}{2} + \frac{E_{\Gamma'}}{4}\right) + 1 \\ &= g(\Gamma) + \left(-\frac{V_{\Gamma'}}{2} + \frac{E_{\Gamma'}}{4}\right) \\ &\geq g(\Gamma). \end{aligned}$$

The key property of Γ' is that every vertex in Γ' is complete. This is because Γ' is the complement of a graph. If there were a vertex v in Γ' that were missing an edge, then the missing edge would be an incomplete edge in Γ. Since Γ is a graph, every edge in Γ must contain both of its vertices, a contradiction.

Because the complex $Q(k_1, \ldots, k_n)$ is of dimension n, every vertex of Γ' will have between n and $2n$ edges extending from it. On the other hand, every edge will contain either 1 vertex if it is an incomplete edge, or 2 vertices if it is not. Note that because we are assuming Γ contains all possible edges between vertices, that Γ' does not contain any edges which are missing both endpoints. By counting the number of edges extending from each vertex of Γ' we obtain $cV = kE$ where $n \leq c \leq 2n$ and $1 \leq k \leq 2$. This gives us the inequality $nV \leq 2E$ which implies

$$E \geq \frac{n}{2}V.$$

When $n \geq 4$, this yields $E \geq 2V$, so that $g(\Gamma) \leq g(Q(k_1, \ldots, k_n))$ as desired. ■

When $n = 3$ the inequality used in the proof of Proposition 4.6 yields

$$E \geq \frac{3}{2}V.$$

In general, this inequality cannot be improved since if Γ' is equal to $Q(1, 1, 1)$, then $E = 12$ and $V = 8$. However, if k_1, k_2, k_3 are not too small then we can show that $E \geq 2V$.

Proposition 4.7. *If* Γ *is a subgraph of* $Q(k_1, k_2, k_3)$*, and if* k_1, k_2, k_3 *satisfy the inequality*

$$k_1k_2k_3 \geq k_1 + k_2 + k_3 + 2,$$

then

$$g(\Gamma) \leq g(Q(k_1, k_2, k_3)).$$

Proof. Let Γ and Γ' be as in the proof of the previous proposition. Again we wish to show that $E_{\Gamma'} \geq 2V_{\Gamma'}$, from which it follows that $g(\Gamma) \leq g(Q(k_1, k_2, k_3))$ as in the previous proposition. If we are to have any hope of the inequality $E \geq 2V$ holding for Γ', then it must hold when $\Gamma' = Q(k_1, k_2, k_3)$, in which case

$$V = k_1k_2k_3 + k_1k_2 + k_1k_3 + k_2k_3 + k_1 + k_2 + k_3 + 1$$

and

$$E = 3k_1k_2k_3 + 2k_1k_2 + 2k_1k_3 + 2k_2k_3 + k_1 + k_2 + k_3.$$

Thus the inequality $E \geq 2V$ implies that

$$k_1k_2k_3 \geq k_1 + k_2 + k_3 + 2.$$

This condition reduces to

$$k_1 \geq 1, k_2 \geq 3, k_3 \geq 3$$

or

$$k_1, k_2, k_3 \geq 2$$

up to reordering of the indices. We will prove that the inequality $E \geq 2V$ holds for Γ' when $k_1 \geq 1, k_2 \geq 3, k_3 \geq 3$, and when $k_1 = 2, k_2 = 2, k_3 \geq 2$.

Let $k_1 \geq 1, k_2 \geq 3, k_3 \geq 3$. Recall that all of the vertices of Γ' are complete. We will guarantee that there are at least twice as many edges as vertices in Γ' by associating two edges with every vertex. The argument can be simplified by imagining a token placed on every edge of Γ' and then sliding the tokens to nearby vertices in such a way that we end up with at least two tokens on every vertex. We will refer to the three dimensions as:

dimension 1: front to back (corresponding to $k_1 \geq 1$)

dimension 2: left to right (corresponding to $k_2 \geq 3$)

dimension 3: top to bottom (corresponding to $k_3 \geq 3$).

Begin by sliding the tokens lying on edges in dimension 2 to the nearest vertex on the left. Next slide tokens lying on edges in dimension 3 to the nearest vertex below. If the token lies on an incomplete edge, do not slide the token if the vertex you are supposed to slide to is missing. After the leftward and downward slides, many of the vertices of Γ' will already have 2 tokens. We can classify the types of vertices of Γ' according to the number and direction of edges extending from each vertex. Figure 8 shows the types of vertices that will already have 2 tokens.

Now slide all of the tokens in the topmost layer ($[0, k_1] \times [0, k_2] \times \{k_3\}$), lying along edges in dimension 1, backward to the nearest vertex. Then slide all of the tokens on the rightmost layer ($[0, k_1] \times \{k_2\} \times [0, k_3]$),lying along edges in dimension 1, backward to the nearest vertex. Figure 9 shows the additional types of vertices that now have 2 tokens.

Figure 10 shows how to slide tokens for vertices lying along the top front and right front edges, except the top right front corner vertex and the nearest vertex to this corner along either edge. From the diagram you should notice that for each vertex, one of the slides depends upon the existence of another vertex in Γ'. If this other vertex does not exist, then we simply use the previously unused token lying on the incomplete edge leading to the missing vertex.

Figure 11 shows how to slide the tokens for any vertex along the top right edge except the top front right corner vertex.

Finally, Figure 12 shows how to slide tokens for the remaining 3 types of vertices. These include the top front right corner vertex and the nearest vertices to it along either the top front edge or the right front edge.

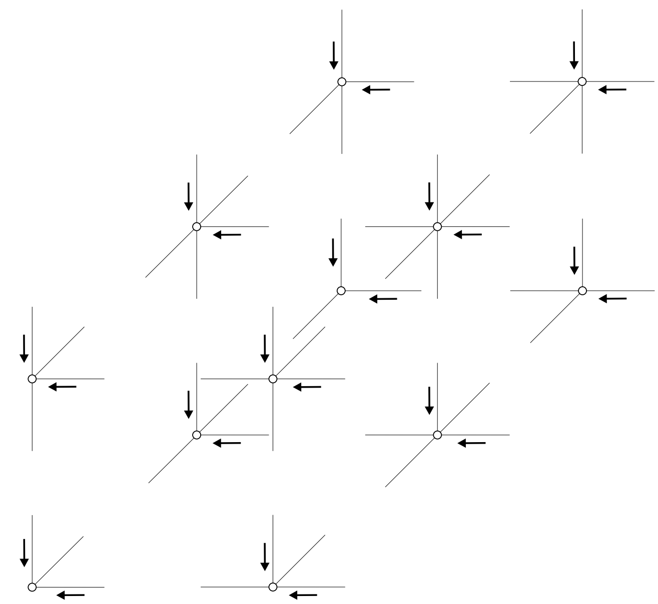


Figure 8: Down and left (12 types)

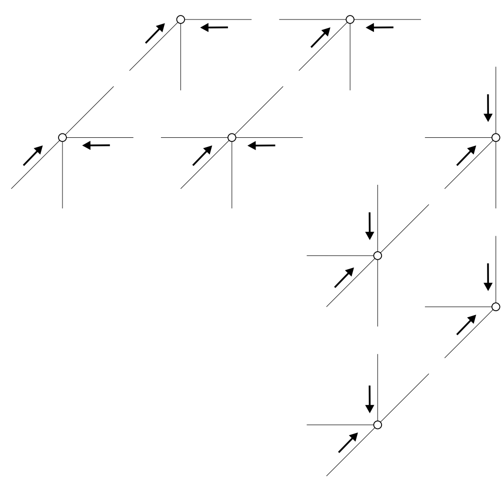


Figure 9: Backward and left or backward and down (8 types)

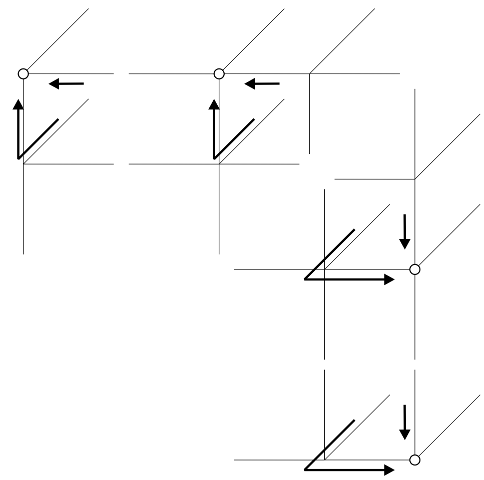


Figure 10: Top front and right front edges (4 types)

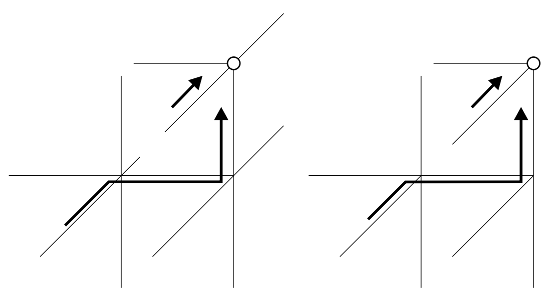


Figure 11: Top right edge (2 types)

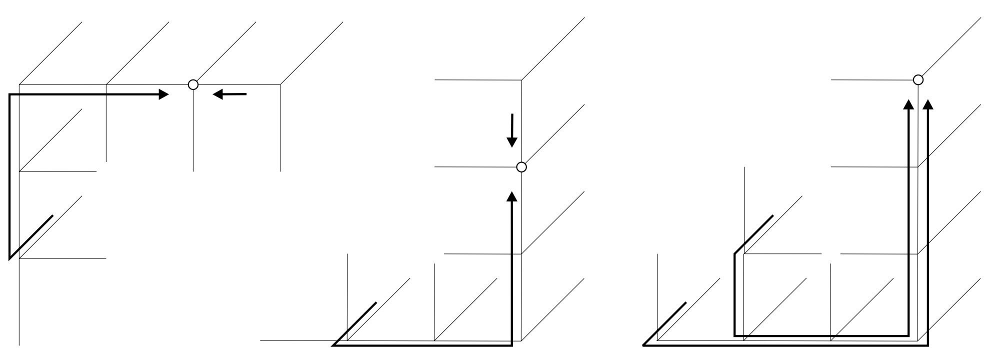


Figure 12: 3 remaining types of vertices

Since we are able to place two tokens on every vertex, there must be at least twice as many edges as vertices. Therefore for Γ' we have $E \geq 2V$ which implies that $g(\Gamma) \leq g(Q(k_1, k_2, k_3))$.

For the case $k_1 = 2, k_2 = 2, k_3 \geq 2$, slide tokens in the same manner as above for all of the types of vertices except the last three types. For these last three types of vertices we cannot assume the existence of 3 edges in a row along dimensions 2 and 3, but since now $k_1 \geq 2$, we can assume the existence of 2 edges in a row along dimension 1. Figure 13 shows alternate patterns for sliding tokens to the remaining 3 types of vertices for the case $k_1 = 2, k_2 = 2, k_3 \geq 2$. With this, the proof of the proposition is complete. ■

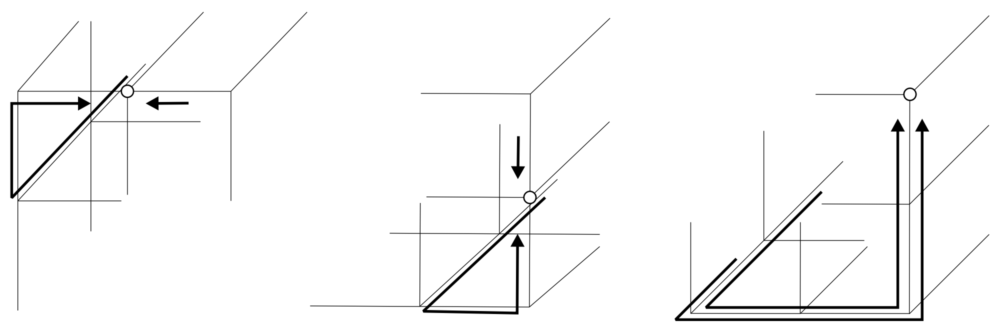


Figure 13: 3 remaining types for the case $k_1 = 2, k_2 = 2, k_3 \geq 2$

Corollary 4.8. *For any i, the function $g(Q(k_1, \ldots, k_n))$ is increasing with respect to the variable k_i.*

Proof. $Q(k_1, \ldots, k_i, \ldots, k_n)$ is a subgraph of $Q(k_1, \ldots, k_i + 1, \ldots, k_n)$ for any i. By the previous proposition, $g(Q(k_1, \ldots, k_i, \ldots, k_n)) \leq g(Q(k_1, \ldots, k_i + 1, \ldots, k_n))$.

■

We now calculate $g(Q(k_1, \ldots, k_n))$ in terms of the k_i. This gives an upper bound on the genus of a surface in $C(k_1, \ldots, k_n)$.

Proposition 4.9. *For any surface S in $C(k_1, \ldots, k_n)$,*

$$\gamma(S) \leq g(Q(k_1, \ldots, k_n)) = \frac{1}{4}\sum_{r=1}^{n}(r-2)F(r, n) + \frac{1}{2},$$

where

$$F(r,n) = \sum_{i_1<i_2<\cdots<i_r} k_{i_1}k_{i_2}\cdots k_{i_r},$$

for $i_1, i_2, \ldots, i_r \in \{1, \ldots, n\}$ *.*

Proof. For the graph $Q(k_1, \ldots, k_n)$, we have that

$$V = (k_1+1)(k_2+1)\cdots(k_n+1)$$

and

$$E = [k_1(k_2+1)\cdots(k_n+1)] + [k_2(k_1+1)(k_3+1)\cdots(k_n+1)] + \cdots$$
$$+[k_n(k_1+1)\cdots(k_{n-1}+1)].$$

Therefore

$$\begin{aligned} g(Q(k_1,\ldots,k_n)) &= -\frac{V}{2} + \frac{E}{4} + 1 \\ &= \frac{1}{4}\sum_{r=1}^{n}(r-2)F(r,n) + \frac{1}{2} \end{aligned}$$

by a straightforward algebraic simplification. Now let S be a surface in the complex $C(k_1, \ldots, k_n)$. By Propositions 4.6 and 4.7,

$$\gamma(S) = g(S^{(1)}) \leq g(Q(k_1, \ldots, k_n))$$

whenever $n \geq 4$ or $n = 3$ and k_1, k_2, k_3 satisfy the inequality

$$k_1k_2k_3 \geq k_1 + k_2 + k_3 + 2.$$

The only remaining cases are $n = 3$ with

$$k_1 = 1, k_2 = 1, k_3 \geq 1$$

or

$$k_1 = 1, k_2 = 2, k_3 \geq 1$$

up to reordering of the indices. It is clear that for either case, any surface S in $C(k_1, k_2, k_3)$ is of genus 0. On the other hand, $g(Q(k_1, k_2, k_3)) = 0$ in the first case and

$$g(Q(k_1, k_2, k_3)) = \frac{1}{4}(k_3 - 3) + \frac{1}{2} \geq 0$$

in the second case, so $\gamma(S) \leq g(Q(k_1, \ldots, k_n))$. ■

5 Constructing Maximal Surfaces in $C(k_1, \ldots, k_n)$

In this section we will learn how to construct maximal surfaces that will realize the bound on the genus given in the previous section. This will prove the main theorems of the paper.

5.1 Constructing $M(k_1, k_2, k_3)$

We will begin by showing how to construct a surface $M(k_1, k_2, k_3)$ in $C(k_1, k_2, k_3)$ for k_1, k_2, k_3 odd integers. The construction is much easier to understand by looking at the picture below. We proceed as follows. Begin with the complex $C(k_1, k_2, k_3)$. Remove the 3-cubes labeled

$$(2i, 2j, x),$$

$$1 \leq i \leq \frac{k_1 - 1}{2}, 1 \leq j \leq \frac{k_2 - 1}{2}, 1 \leq x \leq k_3,$$

then remove the 3-cubes labeled

$$(2i, x, 2j),$$

$$1 \leq i \leq \frac{k_1 - 1}{2}, 1 \leq j \leq \frac{k_3 - 1}{2}, 1 \leq x \leq k_2,$$

and finally remove the 3-cubes labeled

$$(x, 2i, 2j),$$

$$1 \leq i \leq \frac{k_2 - 1}{2}, 1 \leq j \leq \frac{k_3 - 1}{2}, 1 \leq x \leq k_1.$$

Now take the closure of the resulting subcomplex and let $M(k_1, k_2, k_3)$ be the surface consisting of the squares on the boundary of this subcomplex. Note that the maximal surface described above for $C(3, 3, 3)$ is simply $M(3, 3, 3)$.

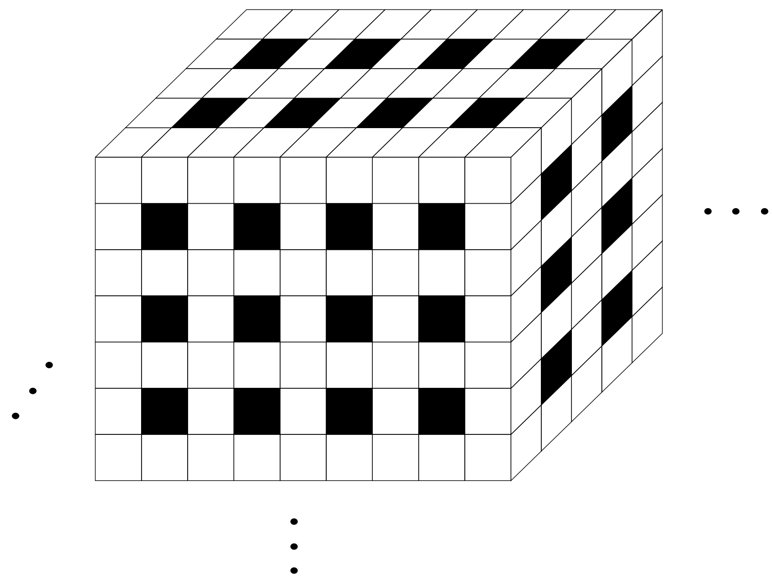


Figure 14: $M(k_1, k_2, k_3)$

Definition 5.1. For k_1, k_2, k_3 odd positive integers, define $M(k_1, k_2, k_3)$ to be the surface described above in the cubical complex $C(k_1, k_2, k_3)$.

We have already calculated the genus of $M(k_1, k_2, k_3)$ because the 1-skeleton of $M(k_1, k_2, k_3)$ is in fact the graph $Q(k_1, k_2, k_3)$. Therefore the genus of $M(k_1, k_2, k_3)$ is given by the formula in Proposition 4.9, and $M(k_1, k_2, k_3)$ is a maximal surface. This proves the following theorem.

Theorem 5.2. *The surface $M(k_1, k_2, k_3)$ is a maximal surface for the cubical complex $C(k_1, k_2, k_3)$ and has genus given by*

$$\gamma(M(k_1, k_2, k_3)) = \frac{1}{4}\left(k_1k_2k_3 - k_1 - k_2 - k_3\right) + \frac{1}{2}.$$

From this point on we will refer to $M(k_1, k_2, k_3)$ more succinctly as M_3, the maximal surface in three dimensions, when the values k_1, k_2, k_3 are clear from the context. This surface will serve as a building block for constructing maximal surfaces in higher dimensions, which will be similarly denoted $M(k_1, \ldots, k_n)$ or M_n for short.

To construct the maximal surfaces in higher dimensions, we will join two surfaces together via *tubes*. Each surface will reside in a copy of $C(k_1, \ldots, k_n)$, and these copies of $C(k_1, \ldots, k_n)$ along with the tubes will together reside in the product complex $C(k_1, \ldots, k_n) \times C(k_{n+1})$.

A tube is an annulus formed from 4 squares by taking the squares of the 3-cube and removing two end squares. To connect two surfaces with a tube, remove a square from each surface. Then glue one end of a tube along the boundary of the first surface, and the other end of the tube along the boundary of the second surface. We want to connect the two surfaces by as many tubes as possible in order to increase the genus of the resulting surface by as much as possible. To connect an additional tube, we must remove an additional square from each surface that is disjoint from the first square that was removed. For this reason, determining the number of disjoint squares on the surface $M(k_1, \ldots, k_n)$ will be important. Figure 15 shows how a tube connects two surfaces. Edges with the same labels are identified.

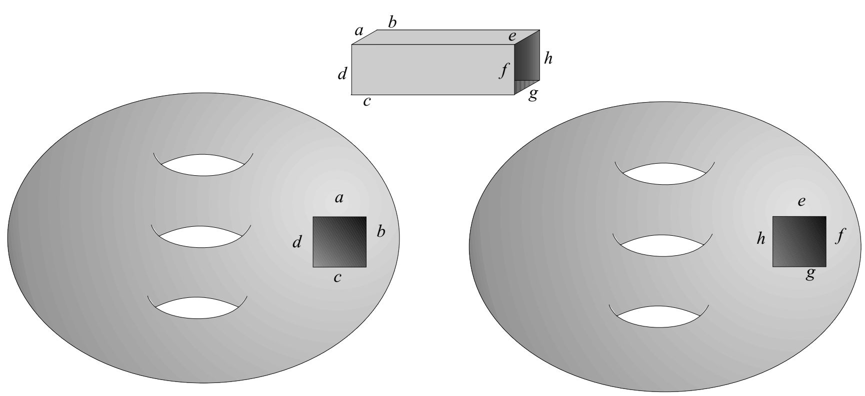


Figure 15: Connecting two surfaces with a tube

Definition 5.3. Let $m(k_1, \ldots, k_n)$ denote the maximum number of disjoint squares on the surface $M(k_1, \ldots, k_n)$. When the values k_i are known, we will use the abbreviated notation m_n.

Proposition 5.4. *Let k_1, k_2, k_3 be odd positive integers. Then*

$$m_3 = \frac{1}{4}(k_1+1)(k_2+1)(k_3+1)$$

and M_3 has at least three sets of m_3 disjoint squares.

Proof. Note that $C(k_1, k_2, k_3)$ has $(k_1+1)(k_2+1)(k_3+1)$ vertices. Disjoint squares must contain distinct vertices, so $4m_3 \leq (k_1+1)(k_2+1)(k_3+1)$ Thus

$$m_3 \leq \frac{1}{4}(k_1+1)(k_2+1)(k_3+1).$$

It is not difficult to choose 3 sets of $\frac{1}{4}(k_1+1)(k_2+1)(k_3+1)$ disjoint squares from M_3 in some standard way. There are in fact many configurations of disjoint squares that are possible. It is more trouble than it is worth to describe a pattern for choosing disjoint squares with an algebraic notation. Figure 16 shows one possible pattern for choosing a set of m_3 disjoint squares on the surface $M(k_1, k_2, k_3)$. To find two additional sets of disjoint squares, simply act by symmetry. Figure 17 shows 2 additional sets of m_3 disjoint squares on the surface $M(k_1, k_2, k_3)$ by using symmetry (use the same pattern as before, but permute the dimensions). ■

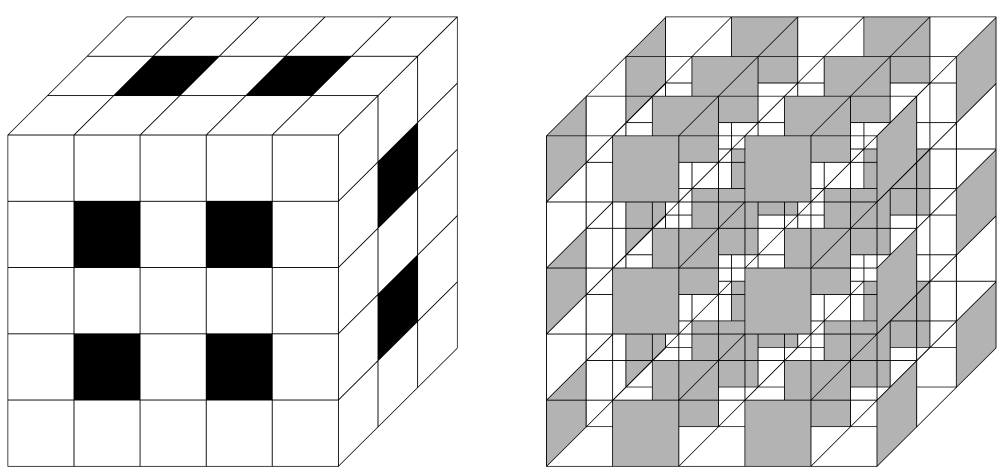


Figure 16: m_3 disjoint squares on $M(k_1, k_2, k_3)$

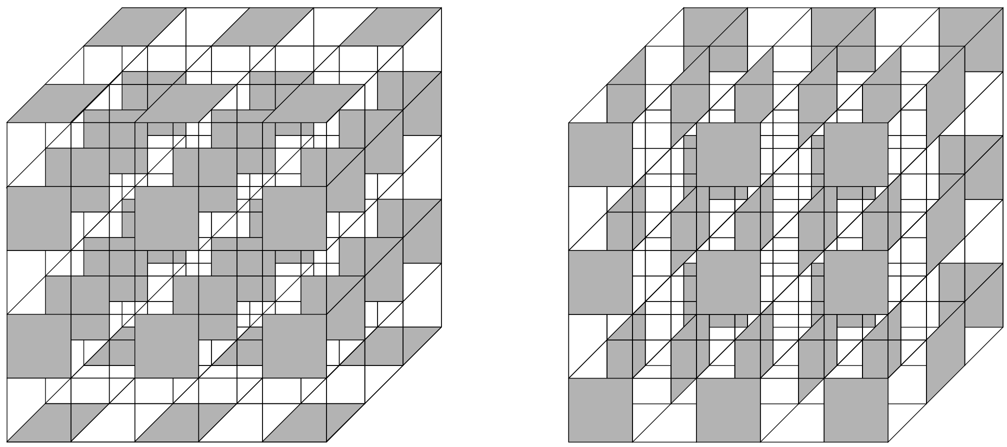


Figure 17: 2 additional sets of disjoint squares on $M(k_1, k_2, k_3)$

A useful consequence of this proposition will be that a set of m_3 disjoint squares on the surface M_3 contains all of the vertices of $C(k_1, k_2, k_3)$.

5.2 Constructing $M(k_1, \ldots, k_n)$

We will now show how to construct a maximal surface in $C(k_1, k_2, k_3, k_4)$ for k_1, k_2, k_3 odd positive integers and k_4 any positive integer. The complex $C(k_4)$ is simply a line with $k_4 + 1$ vertices and k_4 edges whose underlying space is $[0, k_4] \subset \mathbb{R}$. The complex $C(k_1, k_2, k_3, k_4)$ is equal to the complex $C(k_1, k_2, k_3) \times C(k_4)$ so we may think of it as $k_4 + 1$ copies of the complex $C(k_1, k_2, k_3)$ located at the integral values of $[0, k_4]$. These copies of $C(k_1, k_2, k_3)$ are connected by edges, squares and cubes extending over the intervals between integer values of $[0, k_4]$. We will refer to $[0, k_1] \times [0, k_2] \times [0, k_3] \times \{i\}$ as the ith level of the complex.

Begin with one copy of the maximal surface M_3 at the 0th level and one copy at the 1st level. Choose a set of m_3 disjoint squares on the 0th copy of M_3 and an identical set of disjoint squares on the 1st copy of M_3. Given a pair of squares for which one square is from the ith level and one square is from the jth level, we will call the squares identical if they share the same coordinates in $[0, k_1] \times [0, k_2] \times [0, k_3]$, i.e. the squares are the same when projected onto $[0, k_1] \times [0, k_2] \times [0, k_3]$. Take a pair of identical squares A_0, A_1 where A_0 is from the set of disjoint squares at the 0th level and A_1 is from the set of disjoint squares at the 1st level. Let the coordinates of the vertices of A_0 be given by

$$(x, y, z, 0), (x, y, z + 1, 0), (x + 1, y, z + 1, 0), (x + 1, y, z, 0)$$

and the coordinates of the vertices of A_1 given by

$$(x, y, z, 1), (x, y, z + 1, 1), (x + 1, y, z + 1, 1), (x + 1, y, z, 1).$$

Since the two squares are identical, their vertices are joined by edges along the dimension k_4, and their edges are joined by squares along the dimension k_4. We will join the 0th and 1st copies of the surface M_3 by a tube formed from the four squares

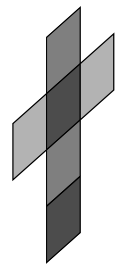
$$\begin{aligned} [x, x + 1] \times \{y\} \times \{z\} \times [0, 1], \\ [x, x + 1] \times \{y\} \times \{z + 1\} \times [0, 1], \\ \{x\} \times \{y\} \times [z, z + 1] \times [0, 1], \\ \{x + 1\} \times \{y\} \times [z, z + 1] \times [0, 1]. \end{aligned}$$

Simply remove the interior of the square A_0 and the interior of the square A_1, then add the four squares that form the tube. We have just joined the two copies of the surface M_3 via a tube at a pair of identical disjoint squares. Repeat this procedure to connect the two copies of M_3 via tubes at every pair of identical disjoint squares. Altogether, we will have connected the surfaces with m_3 tubes. Thus the genus of the resulting surface is $2\gamma(M_3) + m_3 - 1$. Note that because the identical sets of m_3 disjoint squares at the 0th and 1st levels contain all of the vertices at the 0th and

1st levels, the connecting tubes contain all of the edges between the 0th and 1st levels of the complex. Since the copies of M_3 at the 0th and 1st levels contain all of the edges in $C(k_1, k_2, k_3) \times \{0\}$ and $C(k_1, k_2, k_3) \times \{1\}$ respectively, and since the process of connecting with tubes does not remove any edges, the surface we have constructed contains the graph $Q(k_1, k_2, k_3, 1)$. Therefore the surface is maximal in the complex $C(k_1, k_2, k_3, 1)$ and the genus of the resulting surface is given by $g(Q(k_1, k_2, k_3, 1)) = 2\gamma(M_3)+m_3-1$. Now extend this construction by taking copies of the surface M_3 at every level of the complex $C(k_1, k_2, k_3, k_4)$. Connect every level to its adjacent levels by means of tubes attached at disjoint squares. For the ith level with $i \neq 0, k_4$, we must connect to the $(i-1)$th level and the $(i+1)$th level. Thus we need 2 sets of disjoint squares on the surface M_3 at the ith level. One set is used to connect tubes to the $(i-1)$th level and the other to connect tubes to the $(i+1)$th level. Thus we will alternate the locations of the tubes between levels. The resulting surface is defined to be the surface $M(k_1, \ldots, k_4)$ or M_4 for short. This surface contains the graph $Q(k_1, k_2, k_3, k_4)$ because again, the tubes between levels contain all of the edges between the levels, and the copies of M_3 contain all of the edges at every level. Thus M_4 is a maximal surface of genus $\gamma(M_4) = g(Q(k_1, k_2, k_3, k_4))$, which by the way is equivalent to $(k_4 + 1)\gamma(M_3) + k_4(m_3 - 1)$ corresponding to the $k_4 + 1$ copies of M_3 and the m_3 tubes between each level that contribute $m_3 - 1$ to the genus. $M(k_1, \ldots, k_4)$ is shown in Figure 18. The surface M_3 in the figure is actually a closed surface, but is shown flat to make the drawing more clear. You can think of the six squares of M_3 in the figure as folding up to make a cube.

Let us count the maximum number of disjoint squares on the surface M_4. As in the 3 dimensional case,

$$m_4 \leq \frac{1}{4}(k_1 + 1) \cdots (k_4 + 1) = m_3(k_4 + 1)$$



M_3 with 3 sets of disjoint squares

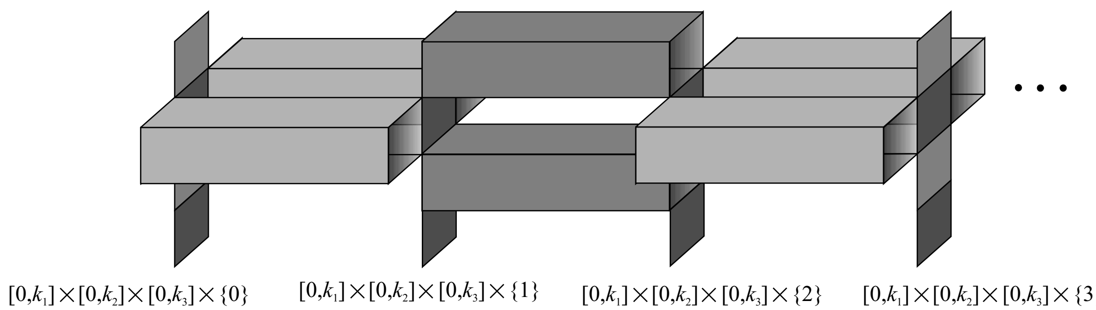


Figure 18: $M(k_1, \ldots, k_4)$

and we can realize this bound by finding a set of $m_3(k_4+1)$ disjoint squares. When connecting copies of M_3 via tubes, we only removed at most two sets of disjoint squares on each copy of M_3. There remains a third set of m_3 disjoint squares for each of the k_4+1 copies of M_3 in the surface M_4. Because the copies of M_3 are disjoint, we have $m_3(k_4+1)$ disjoint squares on the surface M_4. Thus

$$m_4 = \frac{1}{4}(k_1+1)\cdots(k_4+1)$$

and this set of disjoint squares contains all of the vertices of $C(k_1, \ldots, k_4)$.

We need to find two additional sets of m_4 disjoint squares on the surface M_4. If k_4 is odd, take the top and bottom squares of every tube between alternating levels in M_4. This gives us a set of

$$2m_3\frac{k_4+1}{2} = m_4$$

disjoint squares. By taking the two side squares of every tube between alternating levels we obtain a third set of m_4 disjoint squares. When k_4 is even, take the top and bottom squares of all tubes between alternating levels in addition to a set of m_3 disjoint squares at the k_4th level. Note that we have a set of disjoint squares to spare at this level since we removed only one set of disjoint squares to construct the tubes at this last level. This gives us

$$2m_3\frac{k_4}{2} + m_3 = m_4$$

disjoint squares. For the third set of m_4 disjoint squares, take the top and bottom squares of tubes between the other set of alternating levels, and use m_3 disjoint squares at the 0th level.

We have shown that the surface M_4 contains the graph $Q(k_1, \ldots, k_4)$ and thus is maximal. We have also shown that M_4 contains at least 3 sets of

$$m_4 = \frac{1}{4}(k_1 + 1)\cdots(k_4 + 1)$$

disjoint squares. This means that a set of disjoint squares on M_4 contains every vertex of $C(k_1, \ldots, k_4)$. These conditions are sufficient to construct the maximal surface $M(k_1, \ldots, k_5)$ for k_1, k_2, k_3 odd integers and k_4, k_5 any integers, from copies of M_4. The construction is identical to that of M_4. By induction, we may construct the surface $M(k_1, \ldots, k_n)$ for any dimension n, where k_1, k_2, k_3 are odd and the rest of the k_i are any positive integers.

In general, to construct $M(k_1, \ldots, k_{n+1})$, we connect $k_{n+1} + 1$ copies of M_n via tubes at disjoint squares. Since a set of disjoint squares on M_n contains all of the vertices of $C(k_1, \ldots, k_n)$, the tubes connecting the adjacent levels of M_{n+1} will contain all of the edges between levels of $C(k_1, \ldots, k_{n+1})$. Additionally, the ith copy of M_n contains all of the edges of $C(k_1, \ldots, k_n) \times \{i\}$ so that the surface M_{n+1}

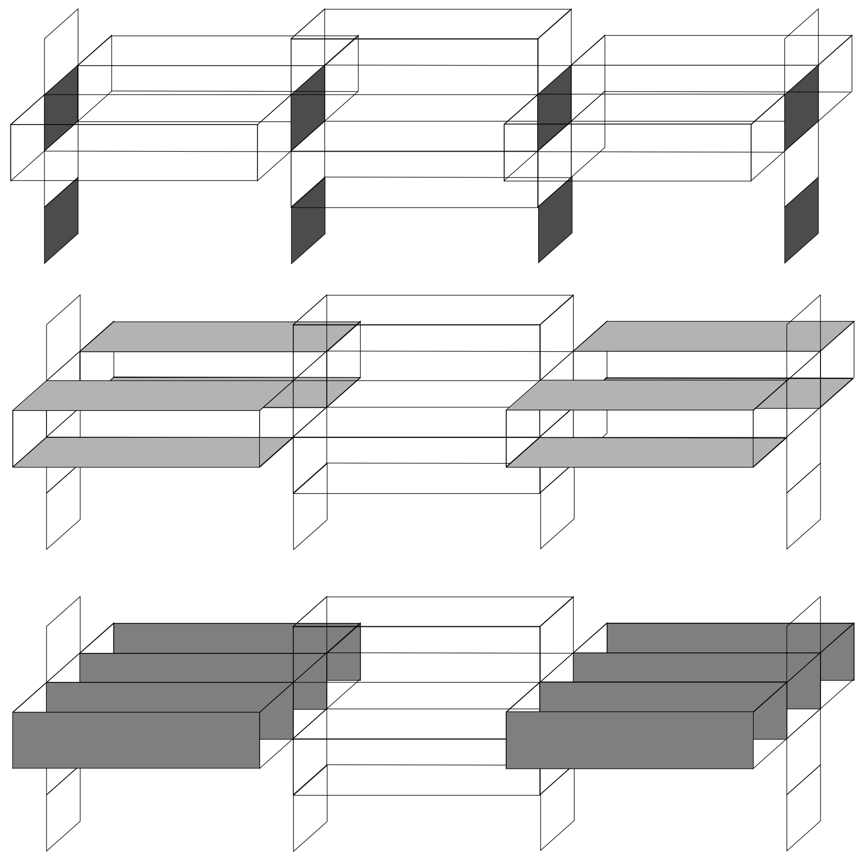


Figure 19: 3 sets of disjoint squares on M_4 for k_4 odd

contains the entire graph $Q(k_1, \ldots, k_{n+1})$. Thus the surface M_{n+1} is maximal with genus given by the formula in Proposition 4.9. Then we choose 3 sets of

$$m_{n+1} = \frac{1}{4}(k_1 + 1) \cdots (k_{n+1} + 1)$$

disjoint squares on the surface M_{n+1} just as we did for M_4.

Definition 5.5. For k_1, k_2, k_3 odd, define $M(k_1, \ldots, k_n)$ to be the maximal surface in $C(k_1, \ldots, k_n)$ as described above.

Think of $k_1, \ldots, k_n$ as being an ordered sequence of numbers and let $P(k_1, \ldots, k_n)$ denote some permutation of the k_i. For example, k_3, k_1, k_2 is a permutation of k_1, k_2, k_3. The cubical complex $C(P(k_1, \ldots, k_n))$ is isomorphic to the complex

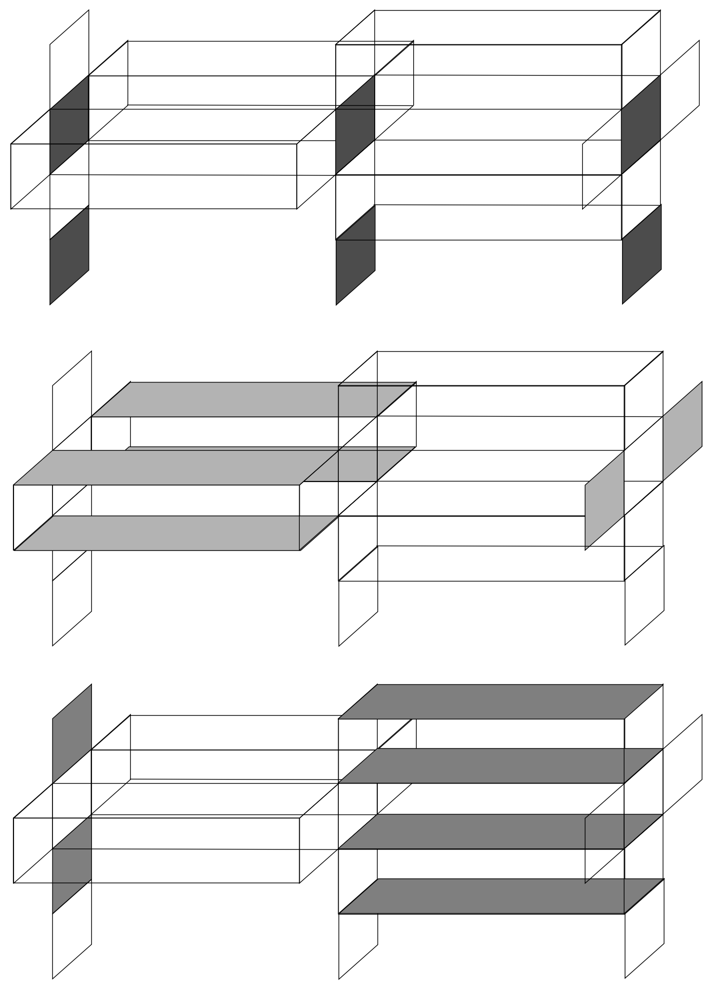


Figure 20: 3 sets of disjoint squares on M_4 for k_4 even

$C(k_1, \ldots, k_n)$ (the two complexes are equivalent by rigid motions). Thus a surface S in $C(P(k_1, \ldots, k_n))$ can be thought of as a surface in $C(k_1, \ldots, k_n)$ under the isomorphism. For example, $M(1, 3, 5, 6)$ is a surface in $C(6, 3, 5, 1)$ under the isomorphism which interchanges dimensions 1 and 4.

Definition 5.6. Let $k_1, \ldots, k_n$ be positive integers such that at least three of the k_i are odd. Let $k_{i_1}, k_{i_2}, k_{i_3}$ be the first three odd integers in the sequence $\{k_1, \ldots, k_n\}$ and let $P(k_1, \ldots, k_n)$ be the permutation which moves the values $k_{i_1}, k_{i_2}, k_{i_3}$ to the first three positions in the sequence $\{k_1, \ldots, k_n\}$. Define $M(k_1, \ldots, k_n)$ to be the surface $M(P(k_1,\ldots,k_n))$ under the isomorphism given by the permutation P.

Note that when k_1, k_2, k_3 are odd, this definition of $M(k_1,\ldots,k_n)$ agrees with the original definition. Also, because the genus of a surface and the function g are both invariant under isomorphisms,

$$\begin{aligned}\gamma(M(k_1,\ldots,k_n)) &= \gamma(M(P(k_1,\ldots,k_n))) \\ &= g(Q(P(k_1,\ldots,k_n))) \\ &= g(Q(k_1,\ldots,k_n)).\end{aligned}$$

In particular, $M(k_1,\ldots,k_n)$ is a maximal surface. We have proved the main theorem of the paper.

Theorem 5.7. *For $k_1,...,k_n$ positive integers and at least three of the k_i odd, $M(k_1,\ldots,k_n)$ is a maximal surface in $C(k_1,...,k_n)$ having genus*

$$\gamma(M(k_1,...,k_n)) = \frac{1}{4}\sum_{r=1}^{n}(r-2)F(r,n) + \frac{1}{2},$$

where

$$F(r,n) = \sum_{i_1<i_2<\cdots<i_r} k_{i_1}k_{i_2}\cdots k_{i_r}\ ,\ \textit{for}\ i_1,i_2,...,i_r \in \{1,2,...,n\}\,.$$

Note that whereas the surface M_3 is the unique maximal surface for $C(k_1,k_2,k_3)$, M_n for $n \geq 4$ is simply *a* maximal surface. Different maximal surfaces could be obtained if for example, a different pattern of choosing disjoint squares was used, or the surface was constructed with the k_i in a different ordering.

We now prove a corollary for the special case of the n-cube $C^n = C(1,\ldots,1)$.

Corollary 5.8. *The maximal genus of the n-cube is $\gamma(C^n) = (n-4)2^{n-3}+1$.*

Proof. For the cubical complex C^n, we have $k_1 = k_2 = \cdots = k_n = 1$ so that $F(r,n) = \binom{n}{r}$ for all r. Thus Theorem 5.7 gives us

$$\gamma(C^n) = \frac{1}{4}\sum_{r=1}^{n}(r-2)\binom{n}{r} + \frac{1}{2}.$$

We use the identities $2^n = \sum_{i=0}^{n} \binom{n}{i}$ and $\binom{n-1}{i-1} = \frac{i}{n}\binom{n}{i}$ for the calculation below.

$$\begin{aligned}
\gamma(C^n) &= \frac{1}{4}\sum_{i=1}^{n}(i-2)\binom{n}{i} + \frac{1}{2} \\
&= \frac{1}{4}\left[\sum_{i=1}^{n} i\binom{n}{i} - 2\sum_{i=1}^{n}\binom{n}{i}\right] + \frac{1}{2} \\
&= \frac{1}{4}\left[n\sum_{i=1}^{n} \frac{i}{n}\binom{n}{i} - 2\sum_{i=1}^{n}\binom{n}{i}\right] + \frac{1}{2} \\
&= \frac{1}{4}\left[n\sum_{i=1}^{n} \binom{n-1}{i-1} - 2\sum_{i=1}^{n}\binom{n}{i}\right] + \frac{1}{2} \\
&= \frac{1}{4}\left[n\sum_{i=0}^{n-1} \binom{n-1}{i} - 2(2^n-1)\right] + \frac{1}{2} \\
&= \frac{1}{4}\left[n2^{n-1} - 2^{n+1} + 2\right] + \frac{1}{2} \\
&= \frac{1}{4}\left[n2^{n-1} - 2^{n+1}\right] + 1 \\
&= n2^{n-3} - 2^{n-1} + 1 \\
&= (n-4)2^{n-3} + 1
\end{aligned}$$

and the proof is complete. ■

6 Relation to the Genus of a Graph

Recall that the genus of a graph is the minimum genus among all surfaces for which the graph can be drawn on the surface without crossing edges. The result from Corollary 5.8 may look familiar to those acquainted with the genus of the n-cube graph Q_n. Indeed, Beineke and Harary proved that the genus of the n-cube graph Q_n is

$$\gamma(Q_n) = (n-4)2^{n-3} + 1$$

[1]. Thus the two notions: finding the maximal genus of a surface in the cubical complex C^n; and finding the genus of the graph Q_n, which is the 1-skeleton of C^n, seem to be equivalent. Is this true in general for the cubical complex $C(k_1, ..., k_n)$?

The answer is no and we need not look very far for an example. Consider the cubical complex $C(2,2,1)$. It is clear by inspection that $\gamma(C(2,2,1)) = 0$. However, the 1-skeleton of this cubical complex, $Q(2,2,1)$, contains the bipartite graph $K_{3,3}$. Therefore by Kuratowski's Theorem [2, p.28], it cannot be drawn on a sphere without edges crossing. Figure 22 shows that $Q(2,2,1)$ can be drawn on a torus, so the genus of $Q(2,2,1)$ is equal to one.

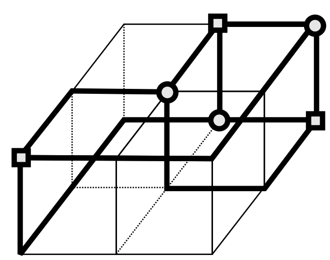


Figure 21: $K_{3,3}$ as a subgraph of $Q(2,2,1)$

We will prove that the notions of maximal genus of $C(k_1, ..., k_n)$ and the genus of the graph $Q(k_1, ..., k_n)$ are equivalent when at least three of the integers k_i are

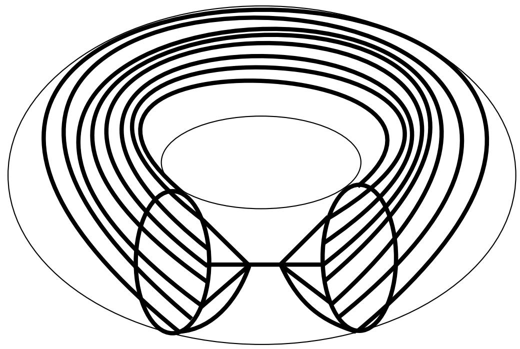


Figure 22: $Q(2,2,1)$ on the torus

odd.

Theorem 6.1. *When at least three of the k_i are odd, finding the maximal genus of the cubical complex $C(k_1,...,k_n)$ is equivalent to finding the genus of the graph $Q(k_1,...,k_n)$, which is the 1-skeleton of the cubical complex $C(k_1,...,k_n)$.*

Proof. The inequality $\gamma(Q(k_1,...,k_n)) \geq \gamma(C(k_1,...,k_n))$ holds for k_i any positive integer value. Let M be a maximal surface in $C(k_1,...,k_n)$, and let $M^{(1)}$ denote the 1-skeleton of M. It is a standard result of topological graph theory that there is a cellular embedding of $M^{(1)}$ into a surface of genus $\gamma(M^{(1)})$ [2, p.132]. Therefore we may calculate

$$\gamma(M^{(1)}) = \frac{-V+E-F+2}{2}$$

where V, E are the number of vertices and edges of the graph $M^{(1)}$ and F is the number of faces for the corresponding cellular embedding. However, $M^{(1)}$ also embeds cellularly by inclusion into the surface M so that

$$\gamma(M) = \frac{-V+E-F_M+2}{2}$$

where F_M is the number of faces in M. Now since the graph $M^{(1)}$ is a subcomplex of $C(k_1,...,k_n)$, the edges of the graph cannot form a triangle anywhere. Thus all of the 2-cells of the embedding into the surface of genus $\gamma(M^{(1)})$, must have number of sides greater than or equal to four. The example above shows how the number of sides of a 2-cell may be strictly greater than 4. On the other hand, for the embedding into the surface M, each 2-cell has exactly four edges, so it must be that $F \leq F_M$ because for both embeddings, the number of edges is the same. Thus $-F \geq -F_M$ so that

$$\gamma(M^{(1)}) \geq \gamma(M) = \gamma(C(k_1, ..., k_n)).$$

Now since the graph $Q(k_1, ..., k_n)$ contains the graph $M^{(1)}$, it is clear that

$$\gamma(Q(k_1, ..., k_n)) \geq \gamma(M^{(1)}) \geq \gamma(C(k_1, ..., k_n)).$$

For the reverse inequality, we must require that at least three of the k_i are odd. Since the maximal surface in $C(k_1, ..., k_n)$ for at least three of the k_i odd contains the graph $Q(k_1, ..., k_n)$, we have that $Q(k_1, ..., k_n)$ can be drawn without intersection on a surface of genus $\gamma(C(k_1, ..., k_n))$. Thus $\gamma(Q(k_1, ..., k_n)) \leq \gamma(C(k_1, ..., k_n))$. ■

As a result of this theorem, the calculation of maximal genus of surfaces in $C(k_1, ..., k_n)$ generalizes the calculation of the genus of the n-cube graph to the much larger class of graphs $Q(k_1, ..., k_n)$.

7 Cases Where the Genus of $C(k_1, \ldots, k_n)$ is Unknown

We have determined the genus of $C(k_1, \ldots, k_n)$ when at least three of the k_i are odd. For the remaining cases, we have upper and lower bounds on the genus. We will investigate some examples that illustrate the complexities of determining the genus in these cases.

Definition 7.1. Define $\langle k_i \rangle$ to be the greatest odd positive integer less than or equal to k_i when $k_i \neq 0$ and $\langle 0 \rangle = 0$.

Proposition 7.2. *For k_i any positive integers,*

$$\gamma(C(\langle k_1 \rangle, \ldots, \langle k_n \rangle)) \leq \gamma(C(k_1, \ldots, k_n)) \leq \lfloor g(Q(k_1, \ldots, k_n)) \rfloor.$$

Proof. The first inequality follows from the fact that $C(\langle k_1 \rangle, \ldots, \langle k_n \rangle) \subset C(k_1, \ldots, k_n)$ and therefore any surface S in $C(\langle k_1 \rangle, \ldots, \langle k_n \rangle)$ is a surface in the complex $C(k_1, \ldots, k_n)$. The second inequality was proved without the floor function in Proposition 4.9. The floor function is added because $\gamma(C(k_1, \ldots, k_n))$ must be an integer. ■

For the 3 dimensional case, the bound of Proposition 7.2 gives us

$$\begin{aligned} \frac{1}{4}\left(\langle k_1 \rangle\langle k_2 \rangle\langle k_3 \rangle - \langle k_1 \rangle - \langle k_2 \rangle - \langle k_3 \rangle\right) + \frac{1}{2} &\leq \gamma(C(k_1, k_2, k_3)) \\ &\leq \lfloor \frac{1}{4}\left(k_1 k_2 k_3 - k_1 - k_2 - k_3\right) + \frac{1}{2} \rfloor. \end{aligned}$$

This bound on the genus becomes increasingly worse (the range of possible genus values becomes larger) as the values of the k_i increase. The 3 dimensional case is very interesting however, because after considering a few examples it becomes intuitively obvious that

$$\gamma(C(k_1, k_2, k_3)) = \gamma(C(\langle k_1 \rangle, \langle k_2 \rangle, \langle k_3 \rangle)).$$

In other words, if the value of k_i is odd, then there is nothing to be gained in terms of genus, by adding one to k_i. Although there are heuristic arguments of why this must be true, there is no general proof. For simple cases, this can be proved by exhaustion or even by the bound from Proposition 7.2. For example, for $C(2,2,1)$, the bound above yields $0 \le \gamma(C(2,2,1)) \le \lfloor \frac{1}{4} \rfloor = 0$. This result will be stated as a conjecture.

Conjecture 7.3. For the cubical complex $C(k_1, k_2, k_3)$ where the values k_1, k_2, k_3 are any positive integers,

$$\gamma(C(k_1, k_2, k_3)) = \gamma(C(\langle k_1 \rangle, \langle k_2 \rangle, \langle k_3 \rangle)).$$

So it seems that for 3 dimensions, when the k_i are even, we get no increase in the genus. This is not the case for dimensions greater than or equal to 4.

Consider the cubical complex $C(1,1,2,2)$. Then $\gamma(C(1,1,1,1)) = 1$ and we have that $\lfloor g(Q(1,1,2,2)) \rfloor = 4$ so that

$$1 \le \gamma(C(1,1,2,2)) \le 4.$$

It is possible to construct a surface in this complex of genus 4 by taking 3 copies of a $1 \times 1 \times 2$ sphere of genus 0 in $C(1,1,2)$, each of which has at least 2 sets of 3 disjoint squares, and connecting adjacent copies with tubes, resulting in a surface of genus $3(0) + 2(2) = 4$. Thus the upper bound on genus is achieved by a surface in the complex.

Now consider the cubical complex $C(2,2,2,2)$. Then $\gamma(C(1,1,1,1)) = 1$ and $\lfloor g(Q(2,2,2,2)) \rfloor = 14$ so that

$$1 \le \gamma(C(2,2,2,2)) \le 14.$$

It is possible to construct a surface in this complex of genus 10 by taking 3 copies of a $2 \times 2 \times 2$ sphere of genus 0 in $C(2,2,2)$, each of which has at least 2 sets of 6 disjoint squares, and connecting adjacent copies with tubes, resulting in a surface of genus $3(0)+2(5)=10$. This complex is small enough that by exhausting all possibilities of surfaces in the complex, we can show that the maximal genus is indeed 10. Therefore there exist complexes which we know have genus strictly between the bounds of Proposition 7.2.

Finally, consider the complex $C(3,3,2,4)$. Then $\gamma(C(3,3,1,3)) = 25$ and we have that $\lfloor g(Q(3,3,2,4)) \rfloor = 59$ so that

$$25 \leq \gamma(C(3,3,2,4)) \leq 59.$$

It is possible to construct a surface in this complex of genus 53 by taking 3 copies of a surface of genus 5 in $C(3,3,4)$, each of which has at least 2 sets of 20 disjoint squares, and connecting adjacent copies with tubes resulting in a surface of genus $3(5)+2(19) = 53$. In this case there may exist a surface of genus greater than 53, but it is unlikely that there exists a surface of genus 59. This is a relatively simple example, but giving a proof of the maximal genus would probably require exhausting all possible surfaces in the complex by computer.

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