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# An Exposition of the Deterministic Polynomial-Time Primality Testing Algorithm of Agrawal-Kayal-Saxena

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AN EXPOSITION OF THE DETERMINISTIC POLYNOMIAL-TIME PRIMALITY  
TESTING ALGORITHM OF AGRAWAL-KAYAL-SAXENA

by  
Robert Lawrence Anderson

A thesis submitted to the faculty of  
Brigham Young University  
in partial fulfillment of the requirements for the degree of

Master of Science

Department of Mathematics  
Brigham Young University  
August 2005

BRIGHAM YOUNG UNIVERSITY

GRADUATE COMMITTEE APPROVAL

of a thesis submitted by  
Robert Lawrence Anderson

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As chair of the candidate's graduate committee, I have read the thesis of Robert Lawrence Anderson in its final form and have found that (1) its format, citations, and bibliographical style are consistent and acceptable and fulfill university and department style requirements, (2) its illustrative materials including figures, tables, and charts are in place; and (3) the final manuscript is satisfactory to the graduate committee and is ready for submission to the university library.

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## **Abstract**

I present a thorough examination of the unconditional deterministic polynomial-time algorithm for determining whether an input number is prime or composite proposed by Agrawal, Kayal and Saxena in their paper [1].

All proofs cited have been reworked with full details for the sake of completeness and readability.

# Contents

Abstract.....	4
Contents .....	5
Introduction.....	6
An Overview of Computational Complexity Theory .....	9
Notation.....	11
Part 1 – Conceptual Foundation.....	12
<u>Lemma 1</u> .....	12
<u>Definition 0.1</u> .....	15
<u>Definition 0.2</u> .....	15
<u>Definition 0.3</u> .....	15
<u>Theorem 0.4</u> .....	15
<u>Corollary 0.5</u> .....	16
Part 2 – The AKS Algorithm .....	17
Part 3 – The Correctness of the AKS Algorithm .....	23
<u>Lemma 2</u> .....	23
<u>Definition 1</u> .....	26
<u>Lemma 3</u> .....	26
<u>Lemma 4</u> .....	27
<u>Proposition 1</u> .....	28
<u>Proposition 2</u> .....	29
<u>Proposition 3</u> .....	30
<u>Lemma 5</u> .....	31
<u>Lemma 6</u> .....	32
<u>Lemma 7</u> .....	33
<u>Theorem 1 (Correctness Theorem)</u> .....	34
Appendix.....	35
<u>Theorem A1</u> .....	35
<u>Theorem A2</u> .....	36
References.....	39

# Introduction

This work is intended to provide enough detail that an advanced undergraduate with background in Abstract Algebra and Number Theory should be able to comprehend the construction of this important result on primality without consulting other sources.

The references in the paper have been verified and all proofs have been checked and we have added references to standard literature in algebra, number theory, and computational complexity theory.

Computational complexity theory is part of the theory of computation dealing with the resources required during computation to solve a given problem. The most common resources are time (how many steps does it take to solve a problem) and space (how much memory does it take to solve a problem).

In this theory, the class **P** consists of all those decision problems that can be solved on a deterministic sequential machine in an amount of time that is polynomial in the size of the input; the class **NP** consists of all those decision problems whose positive solutions can be verified in polynomial time given the right information, or equivalently, whose solution can be found in polynomial time on a non-deterministic machine.

The class **RP** (Randomized Polynomial-Time) consists of all those decision problems solvable by an **NP** machine such that:

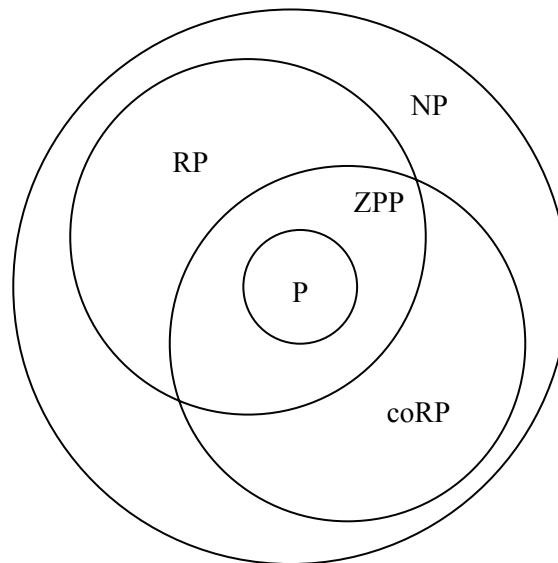
- If the answer is 'yes,' at least  $1/2$  of computation paths accept.
- If the answer is 'no,' all computation paths reject.

The class **coRP** is the complement of **RP** and consists of all those decision problems solvable by an **NP** machine such that:

- If the answer is 'yes,' all computation paths accept.

- If the answer is 'no,' at least 1/2 of computation paths reject.

The class **ZPP** (Zero-Error Probabilistic Polynomial-Time) is defined to be the intersection of **RP** and **coRP**. It is the class of problems solvable by randomized algorithms that always return the correct answer, and whose expected running time (on any input) is polynomial. The relationship of these classes is given by the following diagram:



Arguably, the biggest open question in theoretical computer science concerns the relationship between those two classes:

Is **P** equal to **NP**?

A \$1,000,000 USD prize has been offered for a correct solution. Garey[2] is an excellent reference on the theory of **NP**-completeness for those interested in pursuing this prize.

The historical classification of the problem of testing whether an input integer is prime roughly follows this timeline:



<u>Year</u>	<u>Class</u>	<u>By Whom</u>
pre-1975	$coNP$	Trivial. A single factor of $n$ is a short certificate that can be used to verify that $n$ is not prime. This can be done in linear time using the Euclidean algorithm.
1975	$NP$	Pratt [3]
1975	$NP \cap coNP$	
1977	$RP$	Solovay-Strassen [4]
1987	$coRP$	Adleman-Huang [5]
1987	$ZPP$	$= RP \cap coRP$
2004	$P$	Agrawal-Kayal-Saxena [1]

The classes  $RP$ ,  $coRP$ , and  $ZPP$  were defined by Gill [6] in 1977.

Given a problem such as determining whether or not a number is prime, the question arises, “What is the best algorithm for doing this?”. The difficulty in such a question is in deciding what is meant by “best”.

One decision that often comes into play is as to the necessity of a deterministic algorithm. For instance, if you are testing to see if a certain number is prime for the purpose of using it to construct a cryptographic system, the fastest known algorithms are probabilistic ones (e.g., the Miller-Rabin test [7]). Although they provide no guarantee of theoretical correctness, the error bound can be made arbitrarily small so that for all practical considerations they are reliable algorithms. However, this sort of primality testing will not be of class of  $P$  no matter how fast it is.

The AKS algorithm is the first known deterministic algorithm for primality testing that can be proved to run unconditionally in polynomial-time on all inputs.

# An Overview of Computational Complexity Theory

A *deterministic one-tape Turing machine* (DTM) is computational model consisting of a finite state control, a read-write head, and a tape made up of a two-way infinite sequence of tape squares, labeled ... -2, -1, 0, 1, 2, ...

Let  $\Sigma$  be a finite set of symbols. Let  $\Sigma^*$  denote the set consisting of the empty string  $\varepsilon$  and all finite concatenations of elements of  $\Sigma$ . Then  $L$  is a **language** over  $\Sigma^*$  if  $L \subset \Sigma^*$ . (e.g., the set of binary representations of integers is a language over  $\{0,1\}$ ).

A **program** for a DTM consists of

- a finite set  $\Gamma$  of tape symbols, including a subset  $\Sigma \subset \Gamma$  of input symbols and a distinguished blank symbol  $b \in \Gamma - \Sigma$ ;
- a finite set  $Q$  of states, including a distinguished start state  $q_0$  and two distinguished halt-states  $q_Y$  and  $q_N$ ;
- a transition function  $\delta : (Q - \{q_Y, q_N\}) \times \Gamma \rightarrow Q \times \Gamma \times \{-1, +1\}$ .

A DTM program  $M$  with input alphabet  $\Sigma$  **accepts**  $x \in \Sigma^*$  if and only if  $M$  halts in state  $q_Y$  when applied to input  $n$ .

The language  $L_M$  **recognized** by the program  $M$  is given by  $L_M = \{n \in \Sigma^* : M \text{ accepts } n\}$ .

For a DTM program which halts on all inputs  $n \in \Sigma^*$ , its **time complexity function**  $T_M : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$  is given by:

$$T_M(x) = \max \left\{ m : \begin{array}{l} \text{there is an } n \in \Sigma^* \text{ with } x = \lfloor \log_2 n \rfloor + 1, \text{ such that} \\ \text{the computation of } M \text{ on input } n \text{ takes time } m \end{array} \right\}.$$

Any function  $f(x)$  is considered to be  $O(g(x))$  (pronounced “big-oh of  $g$ ”) if given some function  $g(x)$  there exists a constant  $c$  such that  $|f(x)| \leq c \cdot |g(x)|$  for all values of  $x \geq 0$ , where  $x$  is defined to be the input length of  $n$ .

A DTM program which halts on all inputs  $n \in \Sigma^*$  is called a **polynomial time** DTM program if there exists a polynomial  $p(x)$  such that, for all  $x \in \mathbb{Z}^+$ ,  $T_M(x) \leq p(x)$ .

Alternately, a DTM program which halts on all inputs  $x \in \Sigma^*$ , is called a **polynomial time** DTM program if its time complexity function is  $O(p(x))$ . Again,  $x$  is defined to be the input length of  $n$ .

The class **P** is the class of languages defined as follows:

$$\mathbf{P} = \{L : \text{there is a polynomial time DTM program } M \text{ for which } L = L_M\}$$

Within this framework, the purpose of the paper is to show that with  $M =$  “the AKS algorithm” and  $L_M =$  “binary representation of positive integers”, then  $M$  is a polynomial time DTM program with input set  $\Sigma = \mathbb{N} - \{1\}$  and halt states  $q_Y =$  “PRIME” and  $q_N =$  “COMPOSITE”, and time complexity function  $T_M(x)$  which is  $O(x^{25/2})$ .

## Notation

We will use  $\log n$  to denote the base 2 logarithm instead of  $\log_2 n$  or  $\lg n$ . Base 10 logarithms and natural logarithms will be denoted as  $\log_{10} n$  and  $\ln n$  respectively.

The notation  $\text{ord}_r(a)$  represents the *order of a modulo r*, which is the smallest positive integer  $k$ , such that  $a^k \equiv 1 \pmod{r}$ .

The notation  $\phi(r)$  will be used to represent *Euler's totient function*, which is defined as the number of positive integers less than or equal to  $r$  that are relatively prime to  $r$ .

The notation  $f(x) \equiv g(x) \pmod{h(x), p}$  is used throughout to mean that  $f(x) = g(x)$  in the ring  $\mathbb{Z}_p[x] / (h(x))$ . In some cases,  $p$  will be prime and  $h(x)$  will have degree  $d$  and be irreducible in  $\mathbb{Z}_p[x]$ , so that  $\mathbb{Z}_p[x] / (h(x))$  will be a finite field of order  $p^d$ .

Time complexity functions will be written in “big-O” notation. A function  $f(x)$  is considered to be  $O(g(x))$  (pronounced “big-oh of g”) if given some function  $g(x)$  there exists a constant  $c$  such that  $|f(x)| \leq c \cdot |g(x)|$  for all values of  $x \geq 0$ , where  $x$  is defined to be the binary input length of  $n$ .

The function  $M(n)$  will be used to represent the time complexity function for multiplication. The fastest known algorithm for multiplying two  $n$  digit numbers is due to Schönhage and Strassen [17] and has time complexity  $O(n \log n \log \log n)$ . Ordinary multiplication (the kind you learn in school) has time complexity  $O(n^2)$ .

## Part 1 – Conceptual Foundation

The algorithm is based upon a generalization of Fermat’s Little theorem given in the following lemma:

### **Lemma 1**

Let  $a \in \mathbb{Z}$ ,  $n \in \mathbb{N}$ ,  $n \geq 2$ , and  $(a, n) = 1$ . Then  $n$  is prime if and only if

$$(x + a)^n \equiv x^n + a \pmod{n}$$

This says that if after we reduce all of the coefficients of the binomial expansion of  $(x + a)^n$  modulo  $n$  we are left with  $x^n + a$ , then  $n$  must be prime, otherwise  $n$  is not prime.

### **Proof**

We want  $n \in \mathbb{N}$ ,  $n \geq 2$  since we are only interested in showing whether or not  $n$  is a prime. If  $n < 2$ , then  $n$  will not be prime by definition. The  $\gcd(a, n) = 1$  condition is necessary to use Fermat’s Little Theorem which this generalizes.

Instead of proving “a implies b” and then “b implies a”, we prove the contrapositive of the second direction (i.e., instead of proving “b implies a” we prove “not a implies not b”, which is logically equivalent).

Recall that the binomial expansion of  $(x + a)^n$  is given by

$$\sum_{i=0}^n \binom{n}{i} x^i a^{n-i} \quad \text{where} \quad \binom{n}{i} = \frac{n!}{i!(n-i)!}.$$

First we will prove that if  $n$  is prime then  $(x + a)^n \equiv x^n + a \pmod{n}$ .

Suppose  $n$  is prime. If  $0 < i < n$ , then  $\gcd(n, i!) = 1$  and  $\gcd(n, (n-i)!) = 1$  since both  $i$  and  $n-i$  are less than  $n$  and  $n$  is prime. Thus  $\frac{n!}{i!(n-i)!} = n \cdot c$ , where  $c = \frac{(n-1)!}{i!(n-i)!}$  and

$c \in \mathbb{Z}$ , since  $\frac{n!}{i!(n-i)!} \in \mathbb{Z}$  and  $i!$  and  $(n-i)!$  contain no divisors of  $n$ . This gives us that

$$\binom{n}{i} = \frac{n!}{i!(n-i)!} = n \cdot c \equiv 0 \pmod{n} \text{ for } 0 < i < n, \text{ and hence } \binom{n}{i} x^i a^{n-i} \equiv 0 \pmod{n} \text{ for}$$

$0 < i < n$ .

Thus we see that when  $n$  is prime, the middle terms of the binomial expansion are all equivalent to zero mod  $n$ . The only thing left to show is that  $a^n \equiv a \pmod{n}$ . By Fermat's Little Theorem, we know that if  $n$  is prime, then  $a^{(n-1)} \equiv 1 \pmod{n}$  whenever  $\gcd(a, n) = 1$ .

Multiplying both sides by  $a$ , we get  $a^n \equiv a \pmod{n}$  as desired.

Now we will prove that if  $(x+a)^n \equiv x^n + a \pmod{n}$  then  $n$  is prime by proving its contrapositive, "if  $n$  is not prime, then  $(x+a)^n \not\equiv x^n + a \pmod{n}$ ".

Suppose  $n$  is not prime (i.e., composite since  $n > 1$ ). Let  $q$  be a prime factor of  $n$  and let  $k$  be the largest integer such that  $q^k$  divides  $n$ . We can then write  $n = q^k m$  where  $q$  does not divide  $m$ . Then

$$\binom{n}{q} = \frac{n!}{q!(n-q)!} = \frac{q^k m(n-1)\dots(n-q+1)}{q(q-1)!} = \frac{q^{k-1} m(n-1)\dots(n-q+1)}{(q-1)!}.$$

Notice that  $q$  does not divide the product  $m(n-1)\dots(n-q+1)$  on the far right, since  $q$  does not divide  $m$  and the first multiple of  $q$  less than  $n$  is  $n-q$ . Thus  $q^k$  does not divide  $\binom{n}{q}$  since we cannot factor out another  $q$  on top to make  $q^k$ . Also note that  $\gcd(q^k, a^{n-q}) = 1$

since  $\gcd(a, n) = 1$  and  $q^k$  divides  $n$ . So the coefficient of  $x^q$  in the binomial expansion of  $(x-a)^n$ , given by  $\binom{n}{q} a^{n-q}$ , cannot be equivalent to zero mod  $n$  since it is not even divisible by  $q^k$ , let alone by  $n$ .

Thus  $((x+a)^n - (x^n + a)) \not\equiv 0 \pmod{n}$ , since even if all other middle coefficients go to zero, we still have

$$((x+a)^n - (x^n + a)) \equiv \left( x^n + \binom{n}{q} a^{n-q} x^q + a^n - x^n - a \right) \pmod{n} \equiv \binom{n}{q} a^{n-q} x^q \pmod{n}.$$

Therefore,  $(x+a)^n \not\equiv x^n + a \pmod{n}$ . □

In order to reduce the number of coefficients that need to be examined, the authors decided to consider Lemma 1 after reducing modulo the ideal generated by  $x^r - 1$  for a sufficiently small  $r$ . This way at most  $r-1$  coefficients need to be examined.

There is one small problem. Lemma 1 is not generally true modulo  $(x^r - 1)$ . It turns out, however, that if  $r$  is chosen to be the smallest positive integer such that the order of  $n$  in  $(\mathbb{Z}/r\mathbb{Z})^\times$  is greater than  $\log^2 n$ , then  $n$  can be shown to be a prime power if

$(x+a)^n \equiv x^n + a \pmod{x^r - 1, n}$  for all integer values of  $a$  ranging from 1 to  $\lfloor 2\sqrt{\phi(r)} \log n \rfloor$ . This will be shown in Lemma 7.

Next we will review some definitions and theorems from finite field theory that will be necessary to show the correctness of the algorithm.

### **Definition 0.1**

Let  $n$  be a positive integer. The splitting field of  $x^n - 1$  over a field  $K$  is called the  $n^{\text{th}}$  cyclotomic field over  $K$  and is denoted by  $K^{(n)}$ . The roots of  $x^n - 1$  in  $K^{(n)}$  are called the  $n^{\text{th}}$  roots of unity over  $K$  and the set of all these roots is denoted by  $E^{(n)}$ .

### **Definition 0.2**

Let  $K$  be a field of characteristic  $p$  and  $n$  a positive integer not divisible by  $p$ . Then a generator of the cyclic group  $E^{(n)}$  is called a *primitive  $n^{\text{th}}$  root of unity over  $K$* .

### **Definition 0.3**

Let  $K$  be a field of characteristic  $p$ ,  $n$  a positive integer not divisible by  $p$ , and  $\zeta$  a primitive  $n^{\text{th}}$  root of unity over  $K$ . Then the polynomial

$$\mathcal{Q}_n(x) = \sum_{\substack{s=1 \\ \gcd(s,n)=1}}^n (x - \zeta^s)$$

is called the  $n^{\text{th}}$  cyclotomic polynomial over  $K$ .

### **Theorem 0.4**

Let  $K = \mathbb{F}_q$  with  $\gcd(q, n) = 1$ . Then  $\mathcal{Q}_n$  factors into  $\phi(n)/d$  distinct monic irreducible polynomials in  $K[x]$  of the same degree  $d$ ,  $K^{(n)}$  is the splitting field of any such irreducible factor over  $K$ , and  $[K^{(n)} : K] = d$ , where  $d$  is the least positive integer such that  $q^d \equiv 1 \pmod{n}$ .



**Proof**

Let  $\eta$  be a primitive  $n^{\text{th}}$  root of unity over  $K$ . Then  $\eta \in \mathbb{F}_{q^k}$  if and only if  $\eta^{q^k} = \eta$  and the later identity is equivalent to  $q^k \equiv 1 \pmod{n}$ . The smallest positive integer for which this holds is  $k = d$ , and so  $\eta$  is in  $\mathbb{F}_{q^d}$  but in no proper subfield thereof. Thus the minimal polynomial of  $\eta$  over  $K$  has degree  $d$ , and since  $\eta$  is an arbitrary root of  $Q_n$ , the desired result follows. □

**Corollary 0.5**

$\text{ord}_r(n)$  divides  $\phi(r)$ .

**Proof**

In Theorem 0.4,  $d$  is by definition  $\text{ord}_n(q)$ , which divides  $\phi(n)$ . □

## Part 2 – The AKS Algorithm

The Agrawal-Kayal-Saxena (AKS) algorithm consists of six steps:

1. **Test  $n$  to see if it is a perfect power.**

**Output COMPOSITE if it is.**

This computationally inexpensive step is necessary to allow us to generalize Lemma

1 to the quotient ring  $\mathbb{Z}_n[x]/(x^r - 1)$ . Bernstein [8] has devised an algorithm to perform such

a test in time  $O\left((\log n)^{\left(1 + \frac{\sqrt{\log \log n \log \log \log n}}{\log n}\right)}\right)$ , which is essentially linear on the binary length of  $n$ .

The algorithm is essentially a binary search to find a root  $z$  of  $z^k y - 1$ . The midpoint of an interval  $R$  containing a root is evaluated. Then depending on the sign of  $z^k y - 1$ ,  $R$  is replaced with the left or right half  $R$  and the process is repeated. The *middle half* of the interval is taken if the approximation is too close to zero to be sure of the sign. To save time, the calculation of  $z^k y - 1$  at the midpoint is approximated. This method effectively determines the binary representation of  $z$  one bit at a time. The algorithm outputs  $(x, p)$  where  $n = x^p$  if  $n$  is a perfect power, and  $(n, 1)$  otherwise.

The proofs of the correctness and time complexity of this algorithm are beyond the scope of this paper and will not be given here. The interested reader may consult Bernstein's paper [8] for a more complete treatment.

The algorithm is as follows:

1. Set  $f = \lfloor \log 2n \rfloor$

2. Compute  $y \leftarrow \text{nroot}_{3+\lceil f/2 \rceil}(n, 1)$ . The algorithm for computing  $\text{nroot}_j(y, k)$  immediately follows this main algorithm.
3. For each prime number  $p < f$  :
  - a. Set  $b = 3 + \lceil f / p \rceil$ .
  - b. Compute  $r \leftarrow \text{nroot}_b(y, p)$ .
  - c. Find an integer  $x$  with  $|r - x| \leq \frac{5}{8}$ .
  - d. If  $x = 0$  or  $|r - x| \geq \frac{1}{4}$ , start over with the next prime.
  - e. Compute the sign of  $n - x^p$  :
    - i. Set  $b \leftarrow 1$ .
    - ii. Compute  $r \leftarrow \text{pow}_{b+\lceil \lg 8p \rceil}(x, p)$ . The algorithm for computing  $\text{pow}_b(r, k)$  is given on the next page.
    - iii. If  $n < r$ , output  $-1$  and stop.
    - iv. If  $r(1 + 2^{-b}) \leq n$ , output  $1$  and stop.
    - v. If  $b \geq f$ , output  $0$  and stop.
    - vi. Set  $b \leftarrow \min\{2b, f\}$ . Go back to step ii.
  - f. If  $n = x^p$ , output  $(x, p)$  and stop.
  - g. Start over with the next prime.
4. Output  $(n, 1)$ .

To compute  $\text{nroot}_b(y, k)$  (a floating point number):

1. Find the exponent  $g$  satisfying  $2^{g-1} < y \leq 2^g$ .
2. Set  $a = \lfloor -g/k \rfloor$ , so that  $2^a \leq y^{-1/k} < 2^{a+1}$ .
3. Set  $B = \lceil \log(66(2k+1)) \rceil$ .
4. Set  $z \leftarrow 2^a + 2^{a-1}$ ,  $j \leftarrow 1$ .
5. Now  $\text{nroot}_j(y, k) = z$ . If  $j = b$ , stop.
6. Compute  $r \leftarrow \text{trunc}_B(\text{pow}_B(z, k) \text{trunc}_B y)$ , where the function  $\text{trunc}_b r$  is the truncation of  $r$  to  $b$  bits.
7. If  $r \leq 993/1024$ , set  $z \leftarrow z + 2^{a-j-1}$ .
8. If  $r > 1$ , set  $z \leftarrow z - 2^{a-j-1}$ .
9. Set  $j \leftarrow j+1$ . Go back to step 5.

To compute  $\text{pow}_b(r, k)$  (the  $b$ -bit approximate  $k^{\text{th}}$  power of  $r$ ):

1. If  $k = 1$ , output  $\text{trunc}_b r$  and stop.
2. If  $k$  is even, compute  $\text{pow}_b(r, k/2)$ , output  $\text{trunc}_b(\text{pow}_b(r, k/2)^2)$  and stop.
3. Compute  $\text{pow}_b(r, k-1)$  and output  $\text{trunc}_b(\text{pow}_b(r, k-1) \text{trunc}_b r)$ .

Examples:

$$\text{pow}_b(r, 7) = \text{trunc}_b \left( \text{trunc}_b \left( \text{trunc}_b \left( \text{trunc}_b(r)^2 \right) \text{trunc}_b(r) \right)^2 \text{trunc}_b(r) \right).$$

$$\text{pow}_b(r, 8) = \text{trunc}_b \left( \text{trunc}_b \left( \text{trunc}_b \left( \text{trunc}_b(r)^2 \right)^2 \right)^2 \right).$$

**2. Find the smallest  $r$  such that  $\text{ord}_r(n) > \log^2 n$ .**

Lemma 2 guarantees the existence of such an  $r \leq \lceil \log^5 n \rceil$ . Since each  $r$  can be tested using at most  $O(\log^2 n)$  multiplications, this step has time complexity

$O(r \log^2 n)M(\log n) = O(\log^7 n)M(\log n)$ , where  $M(x)$  is the time complexity function for multiplication. So using fast Fourier multiplication, we get a time complexity of  $O(\log^8 n \log \log n \log \log \log n)$ .

Improving this bound for  $r$  is the key to improving the expected run time of the algorithm. It is believed that  $r$  is bounded by  $O(\log^2 n)$  – a far better result than the bound of  $O(\log^5 n)$  used in this proof. Agrawal [1] claims that this result has actually been proven by Hendrik Lenstra, Jr. and Carl Pomerance in a private communication. I have not been able to verify this since this source is not available to me.

**3. Compute  $\text{gcd}(n, a)$  for each  $2 \leq a \leq r$ .**

**Output COMPOSITE if  $1 < \text{gcd}(n, a) < n$  for some  $a \leq r$ .**

This is just some basic trial division. Computing the GCD of  $r$  numbers takes  $O(r \log n)$ , so using the bound in step 2, this step can be computed in time  $O(\log^6 n)$ .

**4. Output PRIME if  $n \leq r$ .**

This is a result of the trial division in step 3. This step takes no time of any consequence. Note that since  $r \leq \lceil \log^5 n \rceil$ , this step is only relevant when  $n \leq 5,690,034$ . In practice, one would generally look up such a small number in a table of primes.

Furthermore, reducing the bound on  $r$  decreases the likelihood of this step occurring. For instance, with  $r \leq \lceil \log^3 n \rceil$ , this step is only relevant when  $n \leq 982$ . There are only 165 primes less than 982, with 983 being the 166<sup>th</sup> prime.

**5. Compute  $(x+a)^n \bmod (x^r - 1, n)$  for each  $1 \leq a \leq \lfloor \sqrt{\phi(r)} \log(n) \rfloor$ .**

**Output COMPOSITE if  $(x+a)^n \not\equiv x^n + a \pmod{x^r - 1, n}$  for some  $a$ .**

Lemma 1 guarantees that if the test fails for any  $a$ , then  $n$  is composite. On the other hand, if step 5 is satisfied for each  $a$ , a group can be defined with a lower bound on its order. Lemma 6 will show that  $n$  must be some power of a prime  $p$  by showing that on the contrary that the order of this same group would be too small. Step 1 of the algorithm will guarantee that that power is 1, thus making  $n$  a prime.

In this step there are  $\lfloor \sqrt{\phi(r)} \log n \rfloor$  equations which require  $O(\log n)$  multiplications of degree  $r$  polynomials with coefficients of size  $O(\log n)$  to verify. Each equation then

takes time  $O(r \log^2 n)M(\log n)$  to verify. This gives a time complexity of

$O(r \sqrt{\phi(r)} \log^3 n)M(\log n) = O(r^{3/2} \log^3 n)M(\log n) = O(\log^{3/2} n)M(\log n)$ . This step then

represents the time complexity of the algorithm, since it dominates all others. Thus using

even classical multiplication, we get  $M(\log n) = O(\log^2 n)$ , so that the overall time complexity of the algorithm is  $O(\log^{25/2} n)$ , which is polynomial on the binary length of  $n$ . This allows the categorization of this algorithm in  $\mathbf{P}$ , the collection of deterministic polynomial-time algorithms after the algorithm is proved to be correct in Theorem 1. Fast Fourier multiplication, gives a time complexity of  $O(\log^{23/2} n \log \log n \log \log \log n)$ .

## 6. Output PRIME

We will show in Theorem 1 that the algorithm outputs prime if and only if  $n$  is prime.

## Part 3 – The Correctness of the AKS Algorithm

The following lemma validates step 2 by setting an upper bound for finding a suitable  $r$ . It is the principal theorem used to show that the AKS algorithm is polynomial-time.

### **Lemma 2**

There exists an  $r \leq \max \{ 3, \lceil \log^5 n \rceil \}$  such that  $\text{ord}_r(n) > \log^2 n$ .

The proof uses a Chebyshev type bound on the lcm of the first  $n$  numbers by Nair [9]. This bound is given with proof in Theorem A2 in the Appendix. The authors opted to use this result instead of the sieve theory result of Fouvry [10] in order to keep the proof as elementary as possible.

### **Proof**

For  $n = 2, r = 3$  satisfies the lemma, so we may let  $n > 2$ . Let  $r_1, r_2, \dots, r_t$  be all integers such that either  $\text{ord}_{r_i}(n) \leq \log^2 n$  or  $r_i$  divides  $n$ . The first condition implies that for each  $r_i$  there exists  $k_i \leq \log^2 n$  such that  $n^{k_i} \equiv 1 \pmod{r_i}$ , or in other words,

$n^{k_i} - 1 = r_i q$  for some integer  $q$ , so that  $r_i$  divides  $(n^{k_i} - 1)$ . Each of the  $r_i$  must divide the

product  $n \cdot \prod_{i=1}^{\lfloor \log^2 n \rfloor} (n^i - 1)$ , since the fact that each  $k_i$  is less than or equal to  $\log^2 n$  ensures

that  $n^{k_i} - 1$  is one of the terms of this product. This gives us that

$$\text{lcm}(r_1, r_2, \dots, r_t) \leq n \cdot \prod_{i=1}^{\lfloor \log^2 n \rfloor} (n^i - 1).$$



We also know that

$$n \cdot \prod_{i=1}^{\lfloor \log^2 n \rfloor} (n^i - 1) < n^{\log^4 n}.$$

To prove this, first notice that for  $k > 0$  and  $x > 1$ ,  $x^{\left(\sum_{i=1}^k i\right)+1} > \prod_{i=1}^k (x^i - 1)$ . And since

$n > 2$ , we have

$$n \cdot \prod_{i=1}^{\lfloor \log^2 n \rfloor} (n^i - 1) < n^{\left(\sum_{i=1}^{\lfloor \log^2 n \rfloor} i\right)+1} = n^{\frac{(\lfloor \log^2 n \rfloor)(\lfloor \log^2 n \rfloor + 1)}{2} + 1} < n^{\log^4 n}.$$

Thus since  $n = 2^{\log n}$ , we have that

$$\text{lcm}(r_1, r_2, \dots, r_t) \leq n \cdot \prod_{i=1}^{\lfloor \log^2 n \rfloor} (n^i - 1) < 2^{\log^5 n}.$$

Since  $n > 2$ , we have  $\lceil \log^5 n \rceil > 10$  and we may apply Theorem A2 to get

$$\text{lcm}(1, 2, \dots, \lceil \log^5 n \rceil) \geq 2^{\lceil \log^5 n \rceil}.$$

Therefore there must exist an  $s < \lceil \log^5 n \rceil$  such that  $s \notin \{r_1, r_2, \dots, r_t\}$ . If  $(s, n) = 1$

then  $\text{ord}_s(n) > \log^2 n$  and we are done. If  $(s, n) > 1$ , then since  $s$  does not divide  $n$  and

$$(s, n) \in \{r_1, r_2, \dots, r_t\}, r = \frac{s}{(s, n)} \notin \{r_1, r_2, \dots, r_t\} \text{ and so } \text{ord}_r(n) > \log^2 n. \quad \square$$

For the remainder of this paper we will assume that the algorithm returns PRIME and show that  $n$  must be prime. This result will be restated later as Lemma 7.

Let  $p$  be prime and suppose that  $p \mid n$ . In order to pass through step 1 without getting an output of COMPOSITE, we must have that  $n \neq a^b$  for any positive integers  $a$  and  $b$ .

In step 2, no output is given.

In order to pass through step 3 without getting an output of COMPOSITE, we need  $\gcd(a, n) = 1$  or  $n$  for all  $a \leq r$ . Since  $p \mid n$ , this means that  $p$  must be greater than  $r$ , otherwise step 3 would have output COMPOSITE. Since  $p$  is prime, we have  $\gcd(r, p) = 1$ .

If  $n \leq r$ , then step 3 amounts to trial division and is a complete test on the primality of  $n$ . So we can pass over step 4 by assuming that  $n > r$  without any loss of generality.

Recall that since  $r \leq \lceil \log^5 n \rceil$ , this step is only relevant when  $n \leq 5,690,034$ .

Next we assume that  $n$  and  $p$  are fixed and set  $\ell = \lfloor \sqrt{\phi(r)} \log n \rfloor$ . Step 5 of the algorithm verifies  $\ell$  equations. Since the algorithm does not output COMPOSITE in this step, we have

$$(x+a)^n \equiv x^n + a \pmod{x^r - 1, n} \text{ for every } a, 0 \leq a \leq \ell.$$

This implies that

$$(x+a)^n \equiv x^n + a \pmod{x^r - 1, p} \text{ for every } a, 0 \leq a \leq \ell. \quad (1)$$

By Lemma 1, we have

$$(x+a)^p \equiv x^p + a \pmod{x^r - 1, p} \text{ for every } a, 0 \leq a \leq \ell. \quad (2)$$

Since  $p$  divides  $n$ , from (2) we get

$$(x+a)^n \equiv \left( (x+a)^p \right)^{n/p} \equiv (x^p + a)^{n/p} \pmod{x^r - 1, p} \text{ for every } a, 0 \leq a \leq \ell, \quad (3)$$

and from (1) we get

$$(x+a)^n \equiv x^n + a = (x^p)^{n/p} + a \pmod{x^r - 1, p} \text{ for every } a, 0 \leq a \leq \ell. \quad (4)$$

(3) and (4) imply that

$$(x^p + a)^{\not/p} \equiv (x^p)^{\not/p} + a \pmod{x^r - 1, p} \text{ for every } a, 0 \leq a \leq \ell.$$

Replacing  $x^p$  with  $x$ , we get

$$(x + a)^{\not/p} \equiv x^{\not/p} + a \pmod{x^r - 1, p} \text{ for every } a, 0 \leq a \leq \ell.$$

Thus both  $n$  and  $\frac{n}{p}$  behave like prime  $p$  in the above equation. At this point we give

the following definition:

### **Definition 1**

For polynomial  $f(x)$  and number  $m \in \mathbb{N}$ , we say that  $m$  is *introspective* for  $f(x)$  if

$$[f(x)]^m \equiv f(x^m) \pmod{x^r - 1, p}. \quad \square$$

The next two lemmas show that the set of introspective numbers for a given polynomial is closed under multiplication, and that the set of polynomials for which  $m$  is introspective is also closed under multiplication.

### **Lemma 3**

If  $m$  and  $m'$  are introspective for  $f(x)$ , then so is  $m \cdot m'$ .

### **Proof**

By definition,  $[f(x)]^m \equiv f(x^m) \pmod{x^r - 1, p}$ . Thus

$$[f(x)]^{mm'} \equiv [f(x^m)]^{m'} \pmod{x^r - 1, p}. \quad (1)$$

Since  $m'$  is also introspective, replacing  $x$  with  $x^m$  gives:

$$\left[ f(x^m) \right]^{m'} \equiv f(x^{mm'}) \pmod{x^{mr} - 1, p}.$$

Notice that there exists an injective map

$$\varphi: \mathbb{Z}_p[x] / (x^r - 1) \rightarrow \mathbb{Z}_p[x] / (x^{mr} - 1).$$

Or in other words, since

$$x^{mr} - 1 = (x^r - 1) \left( \sum_{i=1}^m x^{(m-i)r} \right),$$

we can see that  $x^r - 1$  divides  $x^{mr} - 1$ , and so the ideal generated by  $x^r - 1$  contains the ideal generated by  $x^{mr} - 1$ . Thus any congruence modulo  $(x^{mr} - 1)$  must also hold modulo the larger ideal  $(x^r - 1)$  in precisely the same way that any number divisible by 6 must also be divisible by 2 or 3. Thus we see that

$$\left[ f(x^m) \right]^{m'} \equiv f(x^{mm'}) \pmod{x^r - 1, p}, \quad (2)$$

and from (1) and (2) we get

$$\left[ f(x) \right]^{mm'} \equiv f(x^{mm'}) \pmod{x^r - 1, p}. \quad \square$$

#### **Lemma 4**

If  $m$  is introspective for  $f(x)$  and  $g(x)$ , then  $m$  is introspective for  $f(x) \cdot g(x)$ .

### **Proof**

First we have that  $[f(x)g(x)]^m = [f(x)]^m [g(x)]^m$ . And since  $m$  is introspective for both  $f(x)$  and  $g(x)$ , we get  $[f(x)g(x)]^m = f(x^m)g(x^m) \pmod{x^r - 1, p}$ .  $\square$

We next define the following sets:

$$I = \{ n^i p^j \mid i, j \geq 0 \}$$

$$P = \left\{ \prod_{a=1}^{\ell} (x+a)^{e_a} \mid e_a \geq 0 \right\}$$

Notice that since Step 5 guarantees that  $(x+a)^n \equiv x^n + a \pmod{x^r - 1, n}$  for  $1 \leq a \leq \ell$ , it follows immediately from Lemma 4 and Lemma 5 that every  $m \in I$  is introspective for every  $f(x) \in P$ .

### **Proposition 1**

$G = \{ i \pmod r \mid i \in I \}$  is a multiplicative subgroup of  $(\mathbb{Z}/r\mathbb{Z})^\times$ .

### **Proof**

Since  $G$  is a finite set, it is sufficient to show that  $G$  is a multiplicatively closed subset of the multiplicative group  $(\mathbb{Z}/r\mathbb{Z})^\times$ . We can see that  $G \subset (\mathbb{Z}/r\mathbb{Z})^\times$  since  $n, p \in \mathbb{Z}^+$ . Now let  $g = n^i p^j \pmod r$ ,  $g' = n^k p^l \pmod r$ . Then  $gg' = n^{i+k} p^{j+l} \pmod r \in G$ .  $\square$

## **Proposition 2**

Let  $Q_r(x)$  be the  $r^{\text{th}}$  cyclotomic polynomial over  $\mathbb{Z}_p$ . Let  $h(x)$  be any irreducible factor of  $Q_r(x)$  of degree  $\text{ord}_r(p)$ . Theorem 0.4 guarantees the existence of such an  $h(x)$ .

Then

$$\mathcal{G} = \left\{ f(x) \bmod (h(x), p) \neq 0 \mid f(x) \in P \right\} \text{ is a subgroup of } \left( \mathbb{Z}_p[x] / (h(x)) \right)^\times.$$

### **Proof**

Again it is sufficient to show closure under multiplication since it is a finite set. It is clear that  $\mathcal{G}$  is a subset of the multiplicative group since we are taking non-zero residues of polynomials in  $P$ . Let  $f, g \in \mathcal{G}$ . Recall that

$$P = \left\{ \prod_{a=1}^{\ell} (x+a)^{e_a} \mid e_a \geq 0 \right\}.$$

Then

$$f = \prod_{a=1}^{\ell} (x+a)^{e_a} \pmod{h(x), p}, \quad g = \prod_{a=1}^{\ell} (x+a)^{e'_a} \pmod{h(x), p}$$

$$fg = \prod_{a=1}^{\ell} (x+a)^{e_a} \pmod{h(x), p} \prod_{a=1}^{\ell} (x+a)^{e'_a} \pmod{h(x), p}$$

$$= \left( \prod_{a=1}^{\ell} (x+a)^{e_a+e'_a} \right) \pmod{h(x), p} \in \mathcal{G}. \quad \square$$

### **Proposition 3**

Let  $F = \mathbb{Z}_p[x]/(h(x))$ . If  $f(x), g(x)$  are distinct elements of  $P$ , then they are

distinct in  $F$ .

### **Proof**

We show by way of contradiction that if  $f$  and  $g$  are not distinct in  $F$ , then a polynomial  $Q(y)$  with degree less than  $t$  has  $t$  distinct roots in  $F$ , which is impossible.

Let  $f(x), g(x)$  be distinct elements of  $P$ . Suppose  $f(x) = g(x)$  in  $F$ . Let  $m \in I$ . Then  $[f(x)]^m = [g(x)]^m$  in  $F$ . Recall that every  $m \in I$  is introspective for every  $f(x) \in P$ . Since  $m$  is introspective for  $f$  and  $g$ , we have

$$\begin{aligned} [f(x)]^m &= f(x^m) \quad \text{in } \mathbb{Z}_p[x]/(x^r - 1), \\ [g(x)]^m &= g(x^m) \quad \text{in } \mathbb{Z}_p[x]/(x^r - 1). \end{aligned}$$

And since  $Q_r(x) \mid x^r - 1$ , then  $h(x) \mid x^r - 1$ .

Thus the equalities above also hold in  $F$ , and we have that  $f(x^m) = g(x^m)$  in  $F$ .

This implies that  $x^m$  is a root of the polynomial  $Q(y) = f(y) - g(y)$  for every  $y \in G$ . By proposition 1, we have  $G \leq (\mathbb{Z}/r\mathbb{Z})^\times$ . Thus  $\gcd(m, r) = 1$ , and each such  $x^m$  is a primitive  $r^{\text{th}}$  root of unity. Thus there will be  $t = |G|$  distinct roots of  $Q(y)$  in  $F$ . But the degree of  $Q(y)$  must be less than  $t$ , since  $\deg f, \deg g < t$  in  $F$ .  $\square$

**Lemma 5**

Let  $\mathcal{G}$  be defined as in Proposition 2. Then  $|\mathcal{G}| \geq \binom{t+\ell}{t-1}$ .

**Proof**

Let  $F = \mathbb{Z}_p[x] / (h(x))$ . Since  $h(x)$  is a factor of  $\mathcal{Q}_r(x)$ , then any root of  $h(x)$  is a primitive  $r^{\text{th}}$  root of unity in  $F$ . By Lemma 2,  $r$  was chosen such that  $\text{ord}_r(n) > \log^2 n$ . By Corollary 0.5,  $\text{ord}_r(n) \mid \phi(r)$ . Thus we have  $\text{ord}_r(n) < \phi(r) < r$ . Thus  $r \log^2 n < r^2$ , and  $\sqrt{r} \log n < r$ .

Recall that  $\ell = \lfloor \sqrt{\phi(r)} \log n \rfloor$ . This gives  $\ell < r$  and we have already chosen  $p > r$ .

Thus  $i \neq j$  in  $\mathbb{Z}_p$  for  $1 \leq i \neq j \leq \ell$ . Thus  $x, x+1, x+2, \dots, x+\ell$  are all distinct in  $P$ . So by Proposition 3,  $x, x+1, x+2, \dots, x+\ell$  are all distinct in  $F$ .

Selecting with repetition  $t-1$  of  $\ell+1$  distinct objects gives us the number of distinct polynomials of degree  $t-1$ , and is given by the formula [16]:

$$\frac{((\ell+1)+(t-1)-1)!}{(t-1)!(\ell)!} = \binom{t+\ell-1}{t-1}.$$

Similarly counting polynomials of degree  $d$ ,  $1 \leq d \leq t-1$ , we get

$$\sum_{a=1}^{t-1} \binom{t+\ell-1}{t-a} \geq \binom{t+\ell}{t-1}, \text{ since}$$

$$\binom{t+\ell}{t-1} = \binom{t+\ell-1}{t-1} + \binom{t+\ell-1}{t-2}.$$

Thus there are at least  $\binom{t+\ell}{t-1}$  distinct polynomials of degree  $< t$  in  $\mathcal{G}$ . □



**Lemma 6**

If  $n$  is not a power of  $p$  then  $|\mathcal{G}| \leq n^{\sqrt{t}}$ .

**Proof**

Consider  $\hat{I} \subset I$  where

$$\hat{I} = \left\{ \left( \frac{n}{p} \right)^i p^j \mid 0 \leq i, j \leq \lfloor \sqrt{t} \rfloor \right\}.$$

If  $n$  is not a power of  $p$ , then by taking  $0 \leq i \leq \lfloor \sqrt{t} \rfloor$ , we see that  $\hat{I}$  has at least

$(\lfloor \sqrt{t} \rfloor + 1)^2 > t$  distinct elements, namely

$$\begin{array}{ccccccc} 1 & , & p & , & p^2 & , & \dots , p^{\lfloor \sqrt{t} \rfloor} \\ \frac{n}{p} & , & \frac{n}{p} \cdot p & , & \frac{n}{p} \cdot p^2 & , & \dots , \frac{n}{p} \cdot p^{\lfloor \sqrt{t} \rfloor} \\ \vdots & & & & & & \vdots \\ \left( \frac{n}{p} \right)^{\lfloor \sqrt{t} \rfloor} & , & \left( \frac{n}{p} \right)^{\lfloor \sqrt{t} \rfloor} \cdot p & , & \left( \frac{n}{p} \right)^{\lfloor \sqrt{t} \rfloor} \cdot p^2 & , & \dots , \left( \frac{n}{p} \right)^{\lfloor \sqrt{t} \rfloor} \cdot p^{\lfloor \sqrt{t} \rfloor}. \end{array}$$

Since  $|G| = t$ , at least two elements of  $\hat{I}$  must be equal mod  $r$ . Let these be

$m_1$  and  $m_2$ ,  $m_1 > m_2$ . And we have,

$$x^{m_1} \equiv x^{m_2} \pmod{x^r - 1} \tag{1}$$

Let  $f(x) \in P$ . Since  $m_1$  is introspective,

$$[f(x)]^{m_1} \equiv f(x^{m_1}) \pmod{x^r - 1, p}.$$

From (1), we get

$$[f(x)]^{m_1} \equiv f(x^{m_2}) \pmod{x^r - 1, p},$$

and since  $m_2$  is introspective,

$$[f(x)]^{m_1} \equiv [f(x)]^{m_2} \pmod{x^r - 1, p}.$$

Thus  $[f(x)]^{m_1} = [f(x)]^{m_2}$  in the field  $F$ . Therefore,  $f(x) \in \mathcal{G}$  is a root of the polynomial  $Q'(Y) = Y^{m_1} - Y^{m_2}$  in the field  $F$ . And since  $f(x)$  is an arbitrary element of  $\mathcal{G}$ , the polynomial  $Q'(Y)$  has at least  $|\mathcal{G}|$  distinct roots in the field  $F$ . The degree of  $Q'(Y)$  is

$$m_1 \leq \left(\frac{n}{p} \cdot p\right)^{\lfloor \sqrt{t} \rfloor} \leq n^{\sqrt{t}}. \text{ Thus } |\mathcal{G}| \leq n^{\sqrt{t}}. \quad \square$$

### **Lemma 7**

If the algorithm returns PRIME, then  $n$  is prime.

### **Proof**

Suppose that the algorithm returns PRIME. Lemma 5 implies that for  $t = |G|$  and

$$\ell = \lfloor \sqrt{\phi(r)} \log n \rfloor:$$

$$|\mathcal{G}| \geq \binom{t + \ell}{t - 1}$$

Since  $G$  is generated by  $n$  and  $p$ , and  $o_r(n) > \log^2 n$ , we have:

$$\begin{aligned} t &> o_r(n) > \log^2 n \\ t^2 &> t \log^2 n \\ t &> \sqrt{t} \log n \geq \lfloor \sqrt{t} \log n \rfloor. \end{aligned}$$

Thus

$$|\mathcal{G}| \geq \binom{\ell+1 + \lfloor \sqrt{t} \log n \rfloor}{\lfloor \sqrt{t} \log n \rfloor}.$$

Since  $G$  is a subgroup of  $\mathbb{Z}_r^*$ , which has order  $\phi(r)$ ,

$$\phi(r) \geq t; \ell = \lfloor \sqrt{\phi(r)} \log n \rfloor \geq \lfloor \sqrt{t} \log n \rfloor.$$

Thus

$$|\mathcal{G}| \geq \binom{2 \lfloor \sqrt{t} \log n \rfloor + 1}{\lfloor \sqrt{t} \log n \rfloor}.$$

Now  $\lfloor \sqrt{t} \log n \rfloor > \lfloor \log^2 n \rfloor \geq 1$  since  $t > \log^2 n$ , so by Theorem A1 in the Appendix,

$$|\mathcal{G}| > 2^{\lfloor \sqrt{t} \log n \rfloor + 1} > 2^{\sqrt{t} \log n} = n^{\sqrt{t}}.$$

By Lemma 6,  $|\mathcal{G}| \leq n^{\sqrt{t}}$  if  $n$  is not a power of  $p$ . Therefore  $n = p^k$  for some  $k > 0$ . If  $k > 1$ ,

then the algorithm will output COMPOSITE in step 1. Therefore,  $k = 1$  and  $n = p$ .  $\square$

### **Theorem 1 (Correctness Theorem)**

The algorithm returns PRIME if and only if  $n$  is prime.

#### **Proof**

Suppose that  $n$  is prime. Step 1 cannot output COMPOSITE since  $n$  is not a perfect power. Step 3 cannot output COMPOSITE since  $\gcd(a, n) = 1$  or  $n$  for all  $a < r$ . Step 5 cannot output COMPOSITE by Lemma 1. Thus the algorithm correctly outputs PRIME in either step 4 or step 6.

The converse was proved in Lemma 7.  $\square$

## Appendix

### Theorem A1

If  $n \geq 1$ , then  $\binom{2n+1}{n} > 2^{n+1}$ .

#### Proof

Expanding the notation, we get:

$$\binom{2n+1}{n} = \frac{(2n+1)!}{n!(n+1)!} = \frac{(2n+1)(2n)\cdots(n+2)}{n!}.$$

There are exactly  $n$  terms in the product in the numerator, so we can rewrite this as:

$$\binom{2n+1}{n} = \prod_{i=1}^n \frac{n+i+1}{i} > \prod_{i=1}^n \frac{2i+1}{i},$$

since  $n \geq 1$ . Since each term in this last product is greater than 2, we have:

$$\binom{2n+1}{n} > 2^n.$$

Noticing that  $\prod_{i=1}^3 \frac{2i+1}{i} = \frac{3 \cdot 5 \cdot 7}{1 \cdot 2 \cdot 3} = \frac{35}{2} > 16 = 2^4$ , we see that indeed

$$\binom{2n+1}{n} > 2^{n+1}.$$

□

## **Theorem A2**

Let  $d_n = \text{lcm}_{1 \leq m \leq n}(m)$  denote the least common multiple of the first  $n$  positive integers.

Then for any integer  $n \geq 7$ ,  $d_n \geq 2^n$ .

### **Proof**

For  $1 \leq m \leq n$ , consider the integral,  $I = \int_0^1 x^{m-1} (1-x)^{n-m}$ . Using the binomial

expansion for  $(1-x)^{n-m}$ , we get

$$I = \int_0^1 x^{m-1} \sum_{r=0}^{n-m} (-1)^r \binom{n-m}{r} x^r = \sum_{r=0}^{n-m} (-1)^r \binom{n-m}{r} \int_0^1 x^{m+r-1}$$

$$I = \sum_{r=0}^{n-m} (-1)^r \binom{n-m}{r} \frac{1}{m+r}$$

Since  $0 \leq r \leq n-m$ ,  $m+r \mid d_n$  and we see that  $I d_n \in \mathbb{N}$ .

On the other hand, integration by parts gives us:

$$I = \int_0^1 x^{m-1} (1-x)^{n-m}, \quad dv = x^{m-1}, \quad u = (1-x)^{n-m}$$

$$v = \frac{x^m}{m}, \quad du = -(n-m)(1-x)^{n-m-1}$$

$$I = \underbrace{\frac{x^m (1-x)^{n-m}}{m}}_0 \Big|_0^1 + \frac{n-m}{m} \int_0^1 x^m (1-x)^{n-m-1},$$

and repeated integration by parts after this manner then yields,

$$I = \frac{(n-m)!}{n \cdot (n-1) \cdot \dots \cdot m} = \frac{(n-m)!(m-1)!}{n!} = \frac{1}{m \binom{n}{m}}.$$

Alternately, recognizing that  $I$  is an Eulerian integral of the 1<sup>st</sup> kind, we could use the well-known (and more general) identity,

$$B(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

And since for positive integers,  $\Gamma(z) = (z-1)!$ , we get that

$$I = B(m, n-m+1) = \frac{\Gamma(m)\Gamma(n-m+1)}{\Gamma(n+1)} = \frac{(m-1)!(n-m)!}{n!} = \frac{1}{m \binom{n}{m}}$$

as before.

But we have already shown that  $Id_n \in \mathbb{N}$ ; therefore,  $m \binom{n}{m}$  divides  $d_n$  for all  $1 \leq m \leq n$ . In particular,  $n \binom{2n}{n}$  divides  $d_{2n}$ .

Notice, however, that

$$\begin{aligned} (2n+1) \binom{2n}{n} &= (2n+1) \frac{(2n)!}{n! \cdot n!} = \frac{(2n+1)!}{n! \cdot n!} \\ &= (n+1) \frac{(2n+1)!}{(n+1)! \cdot n!} = (n+1) \binom{2n+1}{n+1} \end{aligned}$$

so that  $(2n+1) \binom{2n}{n}$  divides  $d_{2n+1}$ . We also know that  $n \binom{2n}{n}$  divides  $d_{2n+1}$  since, in

general,  $d_{n-1}$  divides  $d_n$ . And since  $\gcd(n, 2n+1) = 1$ , we deduce that

$n(2n+1) \binom{2n}{n}$  divides  $d_{2n+1}$ . Also, since  $\binom{2n}{n}$  is the largest of the  $2n+1$  terms in the

binomial expansion of  $(1+1)^{2n}$ , we have that

$$d_{2n+1} \geq n(2n+1) \binom{2n}{n} \geq n4^n.$$

Thus, if  $n \geq 2$ ,  $d_{2n+1} \geq 2^{2n+1}$  and if  $n \geq 4$ ,  $d_{2n+2} \geq d_{2n+1} \geq 2^{2n+2}$ . And since,  $\{2n+1, 2n+2 \mid n \in \mathbb{Z}, n > 0\} = \mathbb{N}$ , we have that  $d_N \geq 2^N$ ,  $N \geq 9$ .

It can be verified by direct computation that this also holds for  $N \geq 7$ . □

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