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Abstract: The hydrostatic pressure assumption has been widely applied in hydrodynamic numerical simulation in rivers and lakes, but it has been found inappropriate in various cases where the vertical acceleration is significant. To this end, this paper proposes a new three-dimensional, non-hydrostatic hydrodynamic model. Based on Navier-Stokes equations with the σ-coordinate transformation, rectangular grid is adopted as the approximation of the study region. For numerical solution, the process includes two steps, the estimated step and the modified step. The FDM is adopted to derive the discretization of the governing equations, and the θ method is employed for equations solution. A numerical test is presented to prove the validity of the proposed model and numerical methods, the results show that the model has the capability and superiority to provide precise prediction for flow field distribution of natural water movements compared to the hydrostatic model, the calculation efficiency of the model is discussed in this paper.

***Keywords**:* non-hydrostatic; three-dimensional model; hydrodynamic; θ method

1 INTRODUCTION

The hydrostatic pressure assumption has been used successfully in studying rivers, lakes and reservoirs, and it has been adopted by many existing hydrodynamic models. However, it is not appropriate in cases where the vertical acceleration is significant, and can make serious deviation to the simulation results. For this reason, more and more researchers have been studying the non-hydrostatic pressure assumption in the past decade and have made great progress. The non-hydrostatic model can be classified into three types: two-dimensional vertical model, depth-integrated model and three-dimensional model, Yamazaki developed a depth-integrated model, which used the Keller-box scheme and solved the governing equations in two steps: the hydrostatic step and the non-hydrostatic step, Kang and Guo built a similar model and adopted a new alternating direction implicit scheme to solve it; the three dimensional model is the most realistic one and the most widely used now, Kocyigit developed a three dimensional model of free surface flows with non-hydrostatic pressure and verify the model with several wave experiments. Based on the above, this paper develops a three-dimensional model in the non-hydrostatic pressure assumption, and get the dynamic pressure using the finite difference method, yield the water level and velocity. This research provides a scientific computing tool for studying the temporal water movements in natural water.

2 NUMERICAL MODELLING

2.1 Three-dimensional Governing Equations

On the basis of the assumption of non-hydrostatic pressure, the three dimensional governing equations for unsteady, incompressible flow and conservation form are:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad (1)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=fv-\frac{1}{\rho}\frac{\partial P}{\partial x}+v_h\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial}{\partial z}\left(v_v\frac{\partial u}{\partial z}\right) \quad (2)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-fu-\frac{1}{\rho}\frac{\partial P}{\partial y}+v_h\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\frac{\partial}{\partial z}\left(\mathrm{v}_v\frac{\partial v}{\partial z}\right) \quad (3)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-fu-\frac{1}{\rho}\frac{\partial P}{\partial z}+v_h\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\frac{\partial}{\partial z}\left(\mathrm{v}_v\frac{\partial v}{\partial z}\right)-g \quad (4)$$

Where *t* is time; *u*, *v* and *w* are the velocity in the *x*-, *y*- and *z*-directions respectively; *f* is the Coriolis parameter; *ρ* is water density; *P* is pressure; v_h and v_v are horizontal and vertical kinematic viscosity coefficient respectively; *g* is gravitational acceleration.

The model adopts the non-hydrostatic assumption, so the pressure *P* can be decomposed into the hydrostatic and dynamic components, respectively, giving:

$$\mathrm{P}=\rho\mathrm{g}(\eta-\mathrm{z})+\mathrm{q} \quad (5)$$

Where *η* is the surface elevation at the still-water level; *q* is the dynamic pressure.

To fit the water surface and the boundary better, the vertical direction adopts the σ coordinate system, in which the calculation area is normalized to -1 to 0 in the vertical direction, the transform equations are:

$$\sigma=\frac{z-\eta(x,\ y,\ \mathrm{t})}{h(\mathrm{x,y})+\eta(x,\ y,\ t)}=\frac{z-\eta(x,\ y,\ \mathrm{t})}{H(x,y,t)} \quad (6)$$

Where *σ* is the transformed vertical coordinate; *z* is the arbitrary distance along the vertical axis; *η(x, y, t)* is the water surface elevation above datum; *h(x, y)* is the elevation of the bed below datum; and *H* is the total depth of the water column.

Thus, the governing equations (1)-(3) in *σ* coordinate system may be written in a conservative form as:

The continuity equation

$$\frac{\partial H\mathrm{u}}{\partial x}+\frac{\partial Hv}{\partial y}+\frac{\partial \omega}{\partial \sigma}+\frac{\partial \eta}{\partial t}=0 \quad (7)$$

The momentum equations

$$\frac{\partial H\mathrm{u}}{\partial t}+\frac{\partial(H\mathrm{uu})}{\partial x}+\frac{\partial(Hv\mathrm{u})}{\partial y}+\frac{\partial(\omega\mathrm{u})}{\partial \sigma}=fHv-Hg\frac{\partial \eta}{\partial x}-\frac{H}{\rho}\frac{\partial q}{\partial x}+v_hH\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial}{\partial \sigma}\left(\frac{v_v}{H}\frac{\partial u}{\partial \sigma}\right) \quad (8)$$

$$\frac{\partial H\mathrm{v}}{\partial t}+\frac{\partial(H\mathrm{vu})}{\partial x}+\frac{\partial(Hv\mathrm{v})}{\partial y}+\frac{\partial(\omega\mathrm{v})}{\partial \sigma}=-fHu-Hg\frac{\partial \eta}{\partial y}-\frac{H}{\rho}\frac{\partial q}{\partial y}+v_hH\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\frac{\partial}{\partial \sigma}\left(\frac{v_v}{H}\frac{\partial v}{\partial \sigma}\right) \quad (9)$$

$$\frac{\partial H\mathrm{w}}{\partial t}+\frac{\partial(H\mathrm{wu})}{\partial x}+\frac{\partial(H\mathrm{wv})}{\partial y}+\frac{\partial(\mathrm{w}\omega)}{\partial \sigma}=-\frac{1}{\rho}\frac{\partial q}{\partial \sigma}+Hv_h\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+\frac{\partial}{\partial \sigma}\left(\frac{v_v}{H}\frac{\partial w}{\partial \sigma}\right) \quad (10)$$

Where a new vertical velocity *ω*, defined as $\omega=H(d\sigma/dt)$, is related to *w* by the following relationship:

$$\omega=\mathrm{w}-\mathrm{u}\left(\sigma\frac{\partial H}{\partial x}+\frac{\partial \eta}{\partial x}\right)-\mathrm{v}\left(\sigma\frac{\partial H}{\partial y}+\frac{\partial \eta}{\partial y}\right)-\left(\sigma\frac{\partial H}{\partial t}+\frac{\partial \eta}{\partial t}\right) \quad (11)$$

By integrating the continuity equation (7) over the depth and applying the kinematic boundary condition at the surface, the well-known free surface equation can be written as:

$$\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x}\left(H\int_{-1}^{0}u d\sigma\right)+\frac{\partial}{\partial y}\left(H\int_{-1}^{0}v d\sigma\right)=0 \quad (12)$$

2.2 Numerical Solution

Figure 1 shows the layout of variables. *i*, *j* and *k* are the cell center indexes in the *x* , *y* and *σ* directions with grid space step of *Δx*, *Δy* and *Δσ* respectively; *u* , *v* and *ω* are defined at the center of the cell faces (i±1/2, j, k) ,(i, j±1/2,k) and (i, j, k±1/2) respectively; *η*, *h* and *H* are located at the center of the cell; *n+1* and *n* are the next time step and the current time step with computing time step of *Δt*.

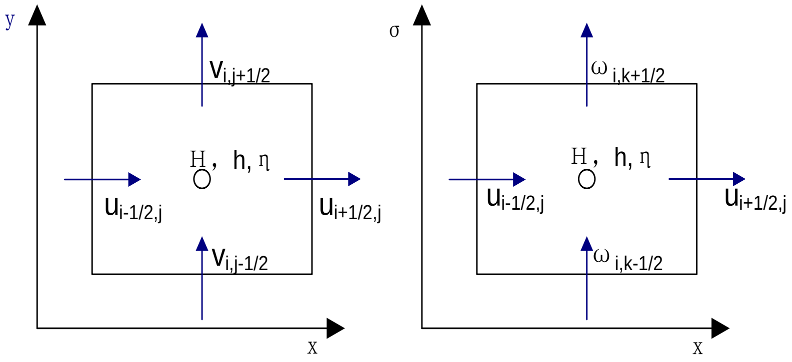


Figure 1 Layout of the variables

A new θ method scheme is applied to solve the equations above, which contains the estimated step and the modified step.

The estimated step processes the convection term, the Coriolis term, the water level gradient term and the viscosity term in explicit scheme, the central difference scheme is applied in space; the free surface equation is discretized in explicit scheme to get water level in the next time step, and the continuity equation is discretized to get the velocity at last.

Based on the results in the estimated step, the modified step processes the dynamic pressure gradient term in implicit scheme using the θ method, yield the equation of the velocity component on the dynamic pressure. Substitute the equations to the discretization scheme of the continuity equation, yield a linear equation system related to the dynamic pressure, the conjugate gradient method is adopted to solve it, update the velocity component and water level in the non-hydrostatic assumption at last.

The estimated step

For equations (8) and (9), discretize all the terms in explicit scheme, use the θ method to process the dynamic pressure gradient term:

$$\tilde{u}^{n+1}_{i+\frac{1}{2},j,k} = Fu^{n}_{i+\frac{1}{2},j,k} - g\frac{\eta^{n}_{i+1,j}-\eta^{n}_{i,j}}{\Delta x}\Delta t - \frac{(1-\theta)}{\rho}\frac{q^{n}_{i+1,j,k}-q^{n}_{i,j,k}}{\Delta x} + \frac{\nu_v}{\left(H^{n}_{i+\frac{1}{2},j}\right)^2}\frac{1}{\Delta\sigma_k}\left[\frac{u^{n}_{i+\frac{1}{2},j,k+1}-u^{n}_{i+\frac{1}{2},j,k}}{0.5(\Delta\sigma_{k+1}+\Delta\sigma_k)} - \frac{u^{n}_{i+\frac{1}{2},j,k}-u^{n}_{i+\frac{1}{2},j,k-1}}{0.5(\Delta\sigma_k+\Delta\sigma_{k-1})}\right] \quad (13)$$

$$\tilde{v}^{n+1}_{i,j+\frac{1}{2},k} = Fv^{n}_{i,j+\frac{1}{2},k} - g\frac{\eta^{n}_{i,j+1}-\eta^{n}_{i,j}}{\Delta y}\Delta t - \frac{(1-\theta)}{\rho}\frac{q^{n}_{i,j+1,k}-q^{n}_{i,j,k}}{\Delta y} + \frac{\nu_v}{\left(H^{n}_{i,j+\frac{1}{2}}\right)^2}\frac{1}{\Delta\sigma_k}\left[\frac{v^{n}_{i,j+\frac{1}{2},k+1}-v^{n}_{i,j+\frac{1}{2},k}}{0.5(\Delta\sigma_{k+1}+\Delta\sigma_k)} - \frac{v^{n}_{i,j+\frac{1}{2},k}-v^{n}_{i,j+\frac{1}{2},k-1}}{0.5(\Delta\sigma_k+\Delta\sigma_{k-1})}\right] \quad (14)$$

Where *F* is a finite difference operator that includes the explicit discretization of the convection terms, the horizontal viscosity and the Coriolis acceleration; θ is the weighting factor, the value of θ is 0.68 in this model.

The discrete form of equation (12) is written:

$$\mathrm{H}^{n+1}_{i,j} = \eta^{n}_{i,j} - \frac{\Delta t}{\Delta x}\left(H^{n}_{I+\frac{1}{2},J}\sum_{k=1}^{KC} u^{n}_{i+\frac{1}{2},j,k}\,\Delta\sigma_k - H^{n}_{i-\frac{1}{2},j}\sum_{k=1}^{KC} u^{n}_{i-\frac{1}{2},j,k}\,\Delta\sigma_k\right) - \frac{\Delta t}{\Delta y}\left(\mathrm{H}^{n}_{I,j+\frac{1}{2}}\sum_{k=1}^{KC} v^{n}_{I,j+\frac{1}{2},k}\,\Delta\sigma_k - \mathrm{H}^{n}_{I,j-\frac{1}{2}}\sum_{k=1}^{KC} v^{n}_{I,j-\frac{1}{2},k}\,\Delta\sigma_k\right) \quad (15)$$

The discrete form of equation (7) is written:

$$\frac{\eta^{n+1}_{i,j}-\eta^{n}_{i,j}}{\Delta t} + \frac{H^{n+1}_{i+\frac{1}{2},j}u^{n+1}_{i+\frac{1}{2},j,k} - H^{n+1}_{i-\frac{1}{2},j}u^{n+1}_{i-\frac{1}{2},j,k}}{\Delta x} + \frac{H^{n+1}_{i,j+\frac{1}{2}}v^{n+1}_{i,j+\frac{1}{2},k} - H^{n+1}_{i,j-\frac{1}{2}}u^{n+1}_{i,j-\frac{1}{2},k}}{\Delta y} + \frac{\omega^{n+1}_{i,j,k+\frac{1}{2}}-\omega^{n+1}_{i,j,k-\frac{1}{2}}}{\Delta\sigma_k} = 0 \quad (16)$$

$\tilde{\omega}^{n+1}_{i,j,k+\frac{1}{2}}$ is determined by substituting equations (13), (14) and (15) to equation (16).

The modified step

Based on the results of the estimated step, discretize the dynamic pressure gradient term in implicit scheme using the θ method:

$$u_{i+\frac{1}{2},j,k}^{n+1} = \tilde{u}_{i+\frac{1}{2},j,k}^{n+1} - \theta\Delta t \frac{q_{i+1,j,k}^{n+1} - q_{i,j,k}^{n+1}}{\rho\Delta x} \tag{17}$$

$$v_{i,j+1/2,k}^{n+1} = \tilde{v}_{i,j+1/2,k}^{n+1} - \theta\Delta t \frac{q_{i,j+1,k}^{n+1} - q_{i,j,k}^{n+1}}{\rho\Delta y} \tag{18}$$

$$\omega_{i,j,k}^{n+1} = \widetilde{\omega}_{i,j,k}^{n+1} - \theta\Delta t \frac{q_{i,j,k+1}^{n+1} - q_{i,j,k}^{n+1}}{0.5\rho(\Delta\sigma_{k+1} + \Delta\sigma_k)} \tag{19}$$

Substituting equation (17), (18) and (19) to equation (16) yields:

$$\mathrm{GC}q_{i,j,k}^{n+1} + \mathrm{GE}q_{i+1,j,k}^{n+1} + \mathrm{GW}q_{i-1,j,k}^{n+1} + \mathrm{GN}q_{i,j+1,k}^{n+1} + \mathrm{GS}q_{i,j-1,k}^{n+1} + \mathrm{GF}q_{i,j,k+1}^{n+1} + \mathrm{GA}q_{i,j,k-1}^{n+1} = \mathrm{GB} \tag{20}$$

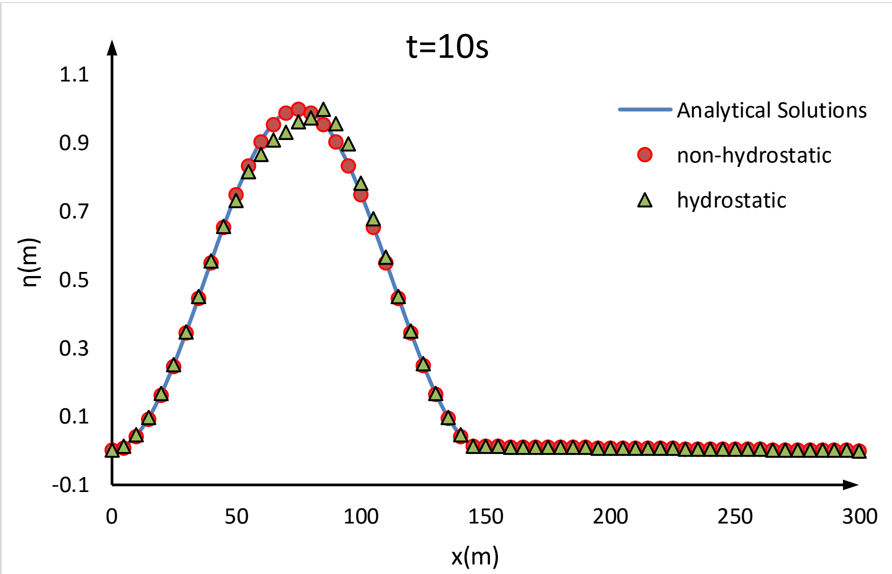
Where *GC*, *GE*, *GW*, *GN*, *GS*, *GF*, *GA* and *GB* are the simplified expressions of known variables. Calculate equation (20) column by column, the matrix of the system is seven-diagonal, the conjugate gradient method is applied to solve it.

Substituting the obtained velocity to equation (15) to update water level in the dynamic pressure situation.

3 MODEL VALIDATION

Solitary wave propagation

The solitary wave maintains its shape and velocity when propagating over a flat bottom, and the change of dynamic pressure is distinct without considering the bottom friction and viscous flow. To validate the model, a solitary wave is exerted in a pool with the size of 300m*1m*10m, the initial crest of which is located at the point x=-20m. The boundary conditions are set according to the analytical solutions of water level and velocity, the Sommerfeld's radiation condition is imposed. Δx=0.5m，Δy=0.25m，Δt=0.01s, the simulation duration is 35s.



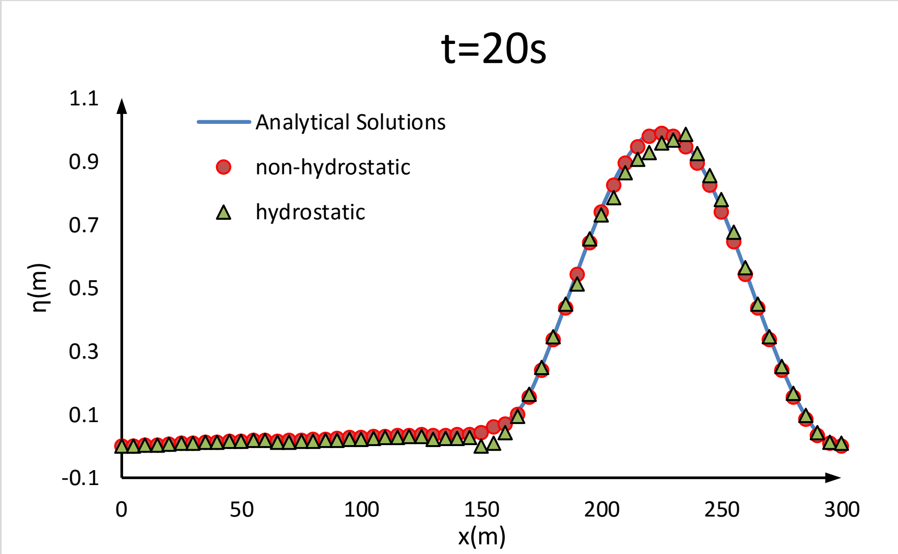


Figure 2 Simulation of water lever in analytical solutions, non-hydrostatic model and hydrostatic model

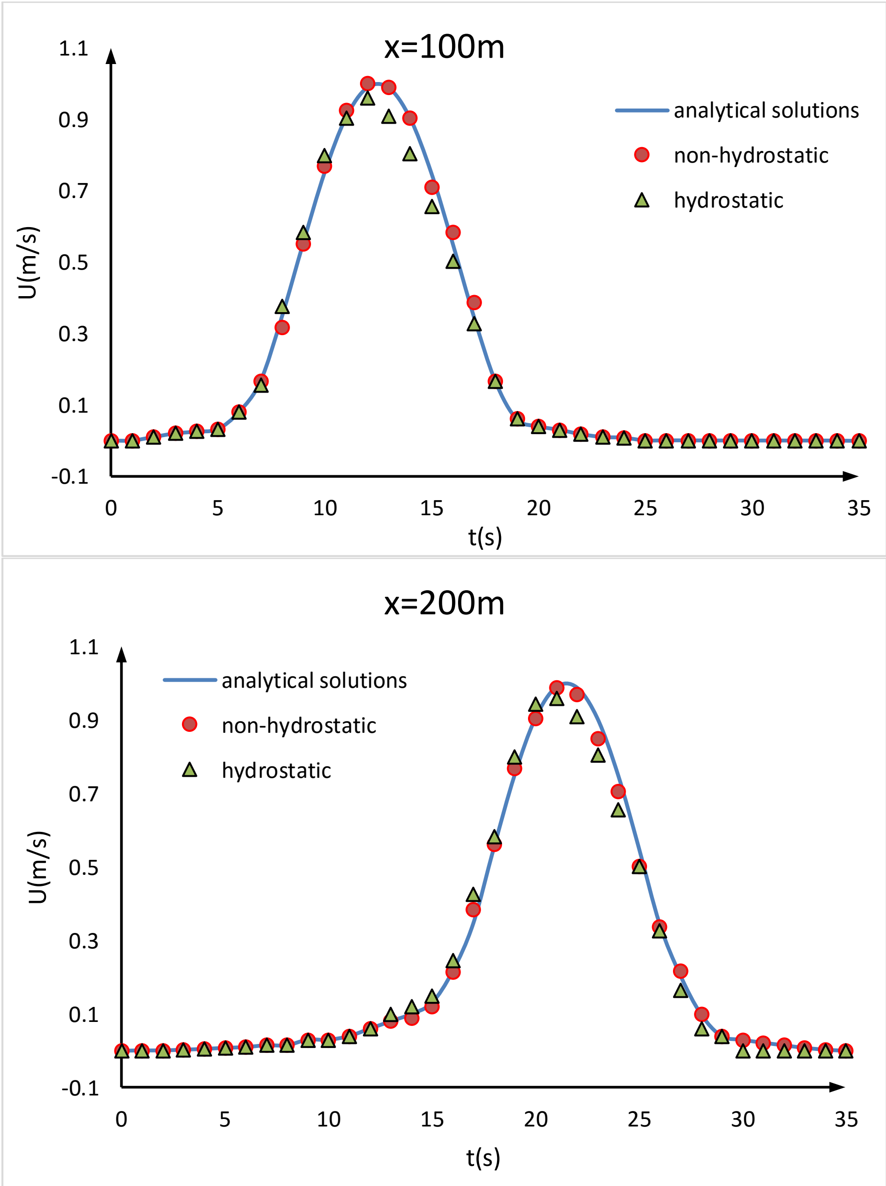


Figure 3 Simulation of velocity in analytical solutions, non-hydrostatic model and hydrostatic model

Figure 2 shows the relationship between water level and location in t=10s and t=20s of analytical solutions, non-hydrodynamic and hydrodynamic respectively, and figure 3 shows the relationship between horizontal velocity and time at x=100m and x=200m, both of them show that the model can simulate actual conditions of water movements well, and the obtained water level and velocity fit well with the analytical solutions. In addition, the non-hydrostatic results show obvious advantages compared to the hydrostatic results, which is an adequate verification for this model.

4 CONCLUSION AND DISCUSSION

This paper presents a new θ method to solve the three-dimensional non-hydrostatic model in the σ-coordinate system using the finite difference method. The model divides the dynamic pressure gradient term using the θ method, and discretize it alternately in explicit and implicit scheme, the conjugate gradient method is applied to solve the linear systems to get the dynamic pressure. The numerical format is concise, the using of explicit scheme can simplify the calculation process, and improve computational efficiency. A wave experiment is made to validate the model, the results show that the model has sufficient accuracy, meanwhile, the non-hydrostatic model shows advantage over the hydrostatic model. The paper provides a reliable and stable method to numerical simulation of water environment of lakes, and also provides technical support to further researches.

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