2003-06-18

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Original Publication Citation
Proceedings of the American Mathematical Society, Vol 132, no 2, pp. 491-498.

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 2, Pages 491–498
S 0002-9939(03)07109-0
Article electronically published on June 18, 2003

MINIMAL GRAPHS IN $\mathbb{R}^3$ OVER CONVEX DOMAINS

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(Communicated by Bennett Chow)

ABSTRACT. Krust established that all conjugate and associate surfaces of a minimal graph over a convex domain are also graphs. Using a convolution theorem from the theory of harmonic univalent mappings, we generalize Krust's theorem to include the family of convolution surfaces which are generated by taking the Hadamard product or convolution of mappings. Since this convolution involves convex univalent analytic mappings, this family of convolution surfaces is much larger than just the family of associated surfaces. Also, this generalization guarantees that all the resulting surfaces are over close-to-convex domains. In particular, all the associate surfaces and certain Goursat transformation surfaces of a minimal graph over a convex domain are over close-to-convex domains.

1. INTRODUCTION

The Weierstrass representation of a minimal surface in $\mathbb{R}^3$ provides a formula connecting minimal surfaces and harmonic mappings. Recently, several papers in complex analysis have investigated the properties of complex-valued harmonic univalent mappings from a classical viewpoint. We demonstrate that theorems in this latter area can be used to prove results about minimal surfaces. In particular, Krust established that all associate surfaces of a minimal graph over a convex domain are also graphs and hence embedded. We generalize Krust's theorem by using a theorem concerning the Hadamard product or convolution of planar harmonic mappings. This establishes that the resulting *convolution surfaces* of a minimal graph over a convex domain are graphs over close-to-convex domains. By restricting one of the harmonic mappings, we derive all the associate minimal surfaces just as in Krust's theorem. However, by allowing this harmonic mapping to vary, we can generate collections of minimal surfaces that are guaranteed to be embedded.

2. BACKGROUND

2.1. **Minimal surfaces.** References for this material include [3], [7], [10], and [11]. We will use the following Weierstrass representation form.

Theorem 2.1. *Let p be an analytic function and q a meromorphic function in some domain $\Omega \in \mathbb{C}$, having the property that at each point where q has a pole of*

Received by the editors December 19, 2000 and, in revised form, October 14, 2002.
2000 *Mathematics Subject Classification.* Primary 49Q05, 53A10, 30C45.
This work was supported in part by a grant from the University of Missouri Research Board.
The author thanks the referee for his suggestions.

order m, p has a zero of order at least 2m. Then every regular minimal surface has a local isothermal parametric representation of the form

$$\begin{aligned} X &= (x_1(z), x_2(z), x_3(z)) \\ &= \left(\mathrm{Re}\left\{ \int p(1+q^2)dw \right\}, \mathrm{Re}\left\{ \int -ip(1-q^2)dw \right\}, \right. \\ &\qquad \left. \mathrm{Re}\left\{ \int -2ipqdw \right\} \right). \end{aligned} \tag{1}$$

Related to a minimal surface is its conjugate or adjoint surface.

Definition 2.2. If a minimal surface $X(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$ is defined on a simply connected domain $\Omega \in \mathbb{C}$, then we define the *conjugate surface* or *adjoint surface*, $X^*(u,v) = (x_1^*(u,v), x_2^*(u,v), x_3^*(u,v))$ to $Z(u,v)$ on Ω as a solution of the Cauchy–Riemann equations

$$X_u = X_v^*, \qquad X_v = -X_u^* \tag{2}$$

in Ω.

Example 2.3. Using $p = 1/(1-z^4)$ and $q = iz$, the Weierstrass representation in eq. (1) yields

$$X = \left(\mathrm{Re}\left\{ \frac{i}{2} \log\left(\frac{z+i}{z-i} \right) \right\}, \mathrm{Re}\left\{ -\frac{i}{2} \log\left(\frac{1+z}{1-z} \right) \right\}, \mathrm{Re}\left\{ \frac{1}{2} \log\left(\frac{1+z^2}{1-z^2} \right) \right\} \right)$$

and generates Scherk's 1st Surface. With $p^* = -ip = -i/(1-z^4)$ and $q^* = q = iz$, we have

$$X^* = \left(\mathrm{Re}\left\{ \frac{1}{2} \log\left(\frac{z+i}{z-i} \right) \right\}, \mathrm{Re}\left\{ -\frac{1}{2} \log\left(\frac{1+z}{1-z} \right) \right\}, \mathrm{Re}\left\{ -\frac{i}{2} \log\left(\frac{1+z^2}{1-z^2} \right) \right\} \right)$$

which forms Scherk's Saddle Tower Surface. These surfaces are conjugate. Note that Scherk's Saddle Tower Surface can be generated with $p^* = 1/(1-z^4)$ and $q^* = z$, although the resulting Weierstrass representation X^* and the original X do not satisfy eq. (2).

A conjugate surface is a minimal surface. Thus, we can construct a one–parameter family of minimal surfaces.

Definition 2.4. For $\theta \in \mathbb{R}$, the surfaces $Z(z,\theta)$ are called *associated minimal surfaces* to the surface $X(z)$, where

$$Z(z,\theta) := X(z)\cos\theta + X^*(z)\sin\theta.$$

The Weierstrass representation gives us a formula for minimal surfaces, but these surfaces may have self-intersections. We are interested in embedded surfaces (i.e., ones with no self-intersections). In a personal correspondence to Karcher (see [7] or [3]), Krust established the following result concerning minimal graphs, which are embedded by definition.

Theorem 2.5 (Krust)**.** *If an embedded minimal surface $X : \mathbb{D} \to \mathbb{R}^3$ can be written as a graph over a convex domain in $\mathbb{C}$, then all associated minimal surfaces $Z : \mathbb{D} \to \mathbb{R}^3$ are graphs.*

Krust's theorem can be used to prove embeddedness of surfaces in general (see [3]) and, in particular, for proving the embeddedness of the fundamental domain of a periodic minimal surface which provides the basis for the conjugate surface method of constructing periodic minimal surfaces with a proposed shape. The conjugate surface method uses properties of conjugate surfaces and their associate family to construct specific minimal surfaces related to the free boundary problem by first considering the conjugate surface, which is more easily constructed, and then relating it back to the desired surface (see [7] and [9]). Some other papers that have used this approach to obtain results include [1], [8], [12], [13], and [15].

2.2. **Harmonic univalent mappings.** References for this material include [2], [4], and [5].

Definition 2.6. A *complex-valued, harmonic function* is a continuous function $f = u + iv$, for which both u and v are real harmonic.

Throughout this paper we will discuss harmonic functions that are univalent (i.e., one-to-one) and sense-preserving on $\mathbb{D} = \{z : |z| < 1\}$. In such a case, Clunie and Sheil-Small [2] established the following.

Theorem 2.7. *If $f = u + iv$ is harmonic in D, then $f = h + \overline{g}$, where h and g are analytic and $|h'(z)| > |g'(z)|$. Hence $f(z) = \sum_{k=0}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k \overline{z}^k$.*

Let S_H be the class of such functions for which $a_1 = 1$ and $a_0 = 0$, and let S_H^O be the subset of S_H in which $b_1 = 0$. The classical family S of analytic univalent functions is the subclass of S_H in which $b_k = 0$ for all k. Also, let K_H, S_H^*, and C_H be the subclasses of S_H mapping $\mathbb{D}$ onto convex, starlike, and close-to-convex domains, similar to K, S^*, and C, which are the subclasses of S mapping $\mathbb{D}$ onto convex, starlike, and close-to-convex domains, respectively. A close-to-convex domain is a domain in which its complement can be written as a union of noncrossing half-lines. We will let the second complex dilatation be denoted by $\omega = \overline{f_{\overline{z}}}/f_z = g'(z)/h'(z)$.

For analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$, their Hadamard product or convolution is defined as $f * F = z + \sum_{n=2}^{\infty} a_n A_n z^n$. In the harmonic case, with

$$f = h + \overline{g} = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \, \overline{z}^n \text{ and}$$

$$F = H + \overline{G} = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B}_n \, \overline{z}^n,$$

define the harmonic convolution as

$$f \,\widetilde{*}\, F = h * H + \overline{g * G} = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n} \, \overline{z}^n. \tag{3}$$

In this paper we will use some theorems about harmonic univalent mappings.

Theorem 2.8 (Clunie and Sheil-Small [2])**.** *If $f = h + \overline{g} \in K_H$ and $\varphi \in K$, then the functions*

$$f \,\widetilde{*}\, (\varphi + \alpha\overline{\varphi}) \in C_H$$

where $|\alpha| \le 1$ and $\widetilde{*}$ denotes harmonic convolution as described in (3).

Theorem 2.9 (Sheil-Small [14]). *If $f = h + \overline{g} \in S_H^*$, then after normalization*

$$\tilde{f} = \int_0^z \frac{h(\zeta)}{\zeta} d\zeta - \overline{\int_0^z \frac{g(\zeta)}{\zeta} d\zeta} \in K_H.$$

There is a nice relationship between minimal surfaces and harmonic univalent mappings. Using the Weierstrass representation, we can construct harmonic univalent maps by projecting embedded minimal surfaces onto their base plane, or form embedded minimal surfaces by lifting harmonic univalent mappings (see [5]). This relationship between conformally parametrized minimal surfaces and harmonic functions becomes clear if one notes that the restriction of linear maps from $\mathbb{R}^3$ to $\mathbb{R}$ to a minimal surface gives a harmonic function on the minimal surface. For example, the coordinate functions lead to harmonic maps this way, and if the surface is a graph, then the two coordinate functions make up a univalent harmonic map.

Theorem 2.10. *If a minimal graph $\{(u, v, F(u, v)) : u + iv \in \Omega\}$ is parametrized by sense-preserving isothermal parameters $z = x + iy \in \mathbb{D}$, the projection onto its base plane defines a harmonic mapping $w = u + iv = f(z)$ of $\mathbb{D}$ onto Ω whose dilatation is the square of an analytic function. Conversely, if $f = h + \overline{g}$ is a sense-preserving harmonic mapping of $\mathbb{D}$ onto some domain Ω with dilatation $\omega = q^2$ for some function q analytic in $\mathbb{D}$, then the formulas*

$$\begin{aligned} u &= \operatorname{Re}\{h(z) + g(z)\}, \\ v &= \operatorname{Im}\{h(z) - g(z)\}, \\ t &= 2 \operatorname{Im}\left\{ \int_0^z q(\zeta) h'(\zeta) d\zeta \right\} \end{aligned} \tag{4}$$

define by isothermal parameters a minimal graph whose projection is f.

Example 2.11. Projecting Scherk's 1st Surface with the formula given in Example 2.3 onto the plane yields the harmonic univalent mapping $f = h + \overline{g} \in K_H$ described in [4], where

$$\begin{aligned} h(z) &= \frac{i}{4} \log\left(\frac{z+i}{z-i}\right) + \frac{1}{4} \log\left(\frac{1+z}{1-z}\right), \\ g(z) &= \frac{i}{4} \log\left(\frac{z+i}{z-i}\right) - \frac{1}{4} \log\left(\frac{1+z}{1-z}\right). \end{aligned}$$

Similarly, using the formula for Scherk's Saddle Tower Surface, we have the harmonic univalent mapping $f = h + \overline{g} \in C_H$ described in [6], where

$$\begin{aligned} h^*(z) &= \frac{1}{4} \log\left(\frac{z+i}{z-i}\right) - \frac{i}{4} \log\left(\frac{1+z}{1-z}\right), \\ g^*(z) &= \frac{1}{4} \log\left(\frac{z+i}{z-i}\right) + \frac{i}{4} \log\left(\frac{1+z}{1-z}\right). \end{aligned}$$

3. Results

We can use theorems from the theory of harmonic univalent functions to prove results about minimal surfaces. In particular, we can prove a generalization of Krust's theorem.

Theorem 3.1. *Let $X : \mathbb{D} \to \mathbb{R}^3$ be an embedded minimal surface which can be written as a graph over a convex domain and which has the Weierstrass representation*

$$X = \left(\operatorname{Re}\left\{ \int p\,(1+q^2)\,dw \right\}, \operatorname{Re}\left\{ \int -i\,p\,(1-q^2)\,dw \right\}, \operatorname{Re}\left\{ \int -2i\,p\,q\,dw \right\} \right),$$

where $p(0) = 1$. Then for

$$\widetilde{p}(z) = p(z) * \frac{\varphi(z)}{z} \quad \text{and} \quad \widetilde{q}(z) = \sqrt{\widetilde{\omega}(z)} = \sqrt{\frac{\alpha\, p(z)q^2(z) * \varphi(z)/z}{p(z) * \varphi(z)/z}},$$

where $\varphi \in K$ and $|\alpha| \le 1$, the convolution minimal surface

$$\widetilde{X} = \left(\operatorname{Re}\left\{ \int \widetilde{p}\,(1+\widetilde{q}^2)\,dw \right\}, \operatorname{Re}\left\{ \int -i\,\widetilde{p}\,(1-\widetilde{q}^2)\,dw \right\}, \operatorname{Re}\left\{ \int -2i\,\widetilde{p}\,\widetilde{q}\,dw \right\} \right)$$

is a graph over a close-to-convex domain and hence embedded, whenever $\widetilde{\omega}$ is a perfect square.

Proof. By Theorem 2.10 and since X is embedded, the projection of X onto the $uv-$plane $= \mathbb{C}$ is $f \in K_H$, where

$$\begin{aligned} f(z) &= \operatorname{Re}\left\{ \int_0^z p(\zeta)(1+q^2(\zeta))d\zeta \right\} + i\operatorname{Im}\left\{ \int_0^z p(\zeta)(1-q^2(\zeta))d\zeta \right\} \\ &= \operatorname{Re}\{h(z)+g(z)\} + i\operatorname{Im}\{h(z)-g(z)\}. \end{aligned}$$

Applying Theorem 2.8, yields

$$\begin{aligned} \widetilde{f}(z) &= \operatorname{Re}\{\widetilde{h}(z)+\widetilde{g}(z)\} + i\operatorname{Im}\{\widetilde{h}(z)-\widetilde{g}(z)\} \\ &= \operatorname{Re}\{\varphi * (h(z)+ag(z))\} + i\operatorname{Im}\{\varphi * (h(z)-ag(z))\} \in S_H. \end{aligned}$$

Since $\widetilde{\omega} = \widetilde{g}'/\widetilde{h}' = (\varphi(z)/z * p(z)q^2(z))/(\varphi(z)/z * p(z))$ is the square of an analytic function, Theorem 2.10 allows us to lift $\widetilde{f}$ to derive the minimal graph

$$\begin{aligned} \widetilde{X} = &\left(\operatorname{Re}\left\{\varphi(z) * (h(z)+ag(z))\right\}, \operatorname{Im}\left\{\varphi(z) * (h(z)-ag(z))\right\}, \right. \\ &\quad \left. 2\operatorname{Im}\left\{ \int_0^z \sqrt{\left[h'(\zeta) * \frac{\varphi(\zeta)}{\zeta}\right]\left[ag'(\zeta) * \frac{\varphi(\zeta)}{\zeta}\right]}d\zeta \right\} \right) \\ = &\left(\operatorname{Re}\left\{\varphi(z) * \int_0^z p(\zeta)(1+aq^2(\zeta))d\zeta\right\}, \right. \\ &\quad \operatorname{Re}\left\{\varphi(z) * \int_0^z -ip(\zeta)(1-aq^2(\zeta))d\zeta\right\}, \\ &\quad \left. \operatorname{Re}\left\{ \int_0^z -2i\sqrt{\left(p(\zeta) * \frac{\varphi(\zeta)}{\zeta}\right)\left(aq(\zeta) * \frac{\varphi(\zeta)}{\zeta}\right)}d\zeta \right\} \right) \\ = &\left(\operatorname{Re}\left\{ \int_0^z \widetilde{p}(\zeta)(1+\widetilde{q}^2(\zeta))d\zeta \right\}, \operatorname{Re}\left\{ \int_0^z -i\widetilde{p}(\zeta)(1-\widetilde{q}^2(\zeta))d\zeta \right\}, \right. \\ &\quad \left. \operatorname{Re}\left\{ \int_0^z -2i\widetilde{p}(\zeta)\widetilde{q}(\zeta)d\zeta \right\} \right). \end{aligned}$$

□

Remark 3.2. Since φ is any analytic univalent mapping of the unit disk onto a convex domain and normalized so that $\varphi(0) = 0$, $\varphi'(0) = 1$, this is not a reparametrization of the Weierstrass representation formula. The coefficients of the power series of p are multiplied by the coefficients of the power series of $\varphi(z)/z$. Similarly, the coefficients of the power series of q are altered. If $\varphi(z) = z/(1-z)$, the convolution identity, then the functions p and q are unaltered and we have the special case given by Krust's theorem. In this case, however, this theorem tells us more than Krust's theorem since it establishes that the resulting minimal graph is over a close-to-convex domain.

The minimal surfaces generated by this theorem are not new, but the fact that these minimal surfaces are automatically embedded is new.

Example 3.3. The minimal graph known as Scherk's 1st Surface can be formed by using the Weierstrass representation with

$$p(z) = \frac{1}{1-z^4} \quad \text{and} \quad q(z) = iz.$$

(1) If $\varphi(z) = \dfrac{z}{1-z}$, then

$$\widetilde{p}(z) = \frac{1}{1-z^4} \quad \text{and} \quad \widetilde{q}(z) = e^{i\theta} z \ (\theta \in \mathbb{R}),$$

and we get the minimal surfaces known as Scherk's 1st Surface, Scherk's Saddle Tower Surface, and all of their associated surfaces.

(2) If $\varphi(z) = z$, then

$$\widetilde{p}(z) = 1 \quad \text{and} \quad \widetilde{q}(z) = 0,$$

and we get the plane.

(3) If $\varphi(z) = z + \frac{1}{9}z^3$, then

$$\widetilde{p}(z) = 1 \quad \text{and} \quad \widetilde{q}(z) = \frac{e^{i\theta}}{3} z \ (\theta \in \mathbb{R}),$$

and we get Enneper-like minimal surfaces.

(4) If we have the 1-parameter family $\varphi_\alpha(z) = \dfrac{1}{2i\sin(\alpha)} \log\Big(\dfrac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\Big)$, where $\pi/2 \le \alpha < \pi$ mapping $\mathbb{D}$ onto vertical strips, then

$$\widetilde{p_\alpha}(z) = \frac{1}{8z\sin(\alpha)}\left[\log\frac{(z-ie^{i\alpha})(z+ie^{-i\alpha})}{(z+ie^{i\alpha})(z-ie^{-i\alpha})} + i\log\frac{(z+e^{i\alpha})(z-e^{-i\alpha})}{(z-e^{i\alpha})(z+e^{-i\alpha})}\right],$$

$$\widetilde{q_\alpha}(z) = \sqrt{\frac{a}{\widetilde{p_\alpha}(z)}}\sqrt{\frac{1}{8z\sin(\alpha)}\left[\log\frac{(z-ie^{i\alpha})(z+ie^{-i\alpha})}{(z+ie^{i\alpha})(z-ie^{-i\alpha})} - i\log\frac{(z+e^{i\alpha})(z-e^{-i\alpha})}{(z-e^{i\alpha})(z+e^{-i\alpha})}\right]}.$$

and we have a 2-parameter family of embedded minimal surfaces where a new *convolution* minimal graph is generated as α varies and all the associated minimal graphs are formed as θ varies.

Corollary 3.4. *If X is a minimal graph over a convex domain, then all of its associated minimal surfaces are over close-to-convex domains.*

Goursat showed how to develop a different one-parameter family of minimal surfaces from a specific minimal surface and its conjugate (see [3], pp. 115-116).

Let $X = (x, y, z)$ and $X^* = (x^*, y^*, z^*)$ be conjugate minimal surfaces. Then for $\kappa \in \mathbb{R}$, $Y = (\xi, \eta, \zeta)$ is a one-parameter family of minimal surfaces, where

$$\begin{aligned} \xi &= \frac{1+\kappa^2}{2\kappa}x + \frac{1-\kappa^2}{2\kappa}y^*, \\ \eta &= \frac{1+\kappa^2}{2\kappa}y + \frac{\kappa^2-1}{2\kappa}x^*, \\ \zeta &= z. \end{aligned}$$

We can use the results above to prove the following.

Corollary 3.5. *If X is a minimal graph over a convex domain, then the Goursat transformation Y with $\kappa \geq 1$ is a minimal graph over a close-to-convex domain.*

Proof. Let X have the Weierstrass representation given in Theorem 2.1. Then its projection is $f = h + \overline{g} \in K_H$. By Theorem 3.1 with $\varphi = \dfrac{z}{1-z}$, we get the convolution minimal graph $\widetilde{X}$ with $\widetilde{p} = p$ and $\widetilde{q} = \sqrt{a}q$ over a close-to-convex domain, where

$$\widetilde{X} = \left(\mathrm{Re}\{h(z) + \overline{a}g(z)\}, \mathrm{Im}\{h(z) - \overline{a}g(z)\}, \mathrm{Re}\left\{ \sqrt{a} \int_0^z 2ipqdw \right\} \right). \tag{5}$$

On the other hand, using the fact that $X = (\mathrm{Re}(h+g), \mathrm{Im}(h-g), z)$ and its conjugate surface is $X^* = ((\mathrm{Im}(h+g), -\mathrm{Re}(h-g), z^*)$, the Goursat transformation $Y = (\xi, \eta, \zeta)$ has

$$\begin{aligned} \xi &= \mathrm{Re}\left(\kappa h + \frac{1}{\kappa}g\right), \\ \eta &= \mathrm{Im}\left(\kappa h - \frac{1}{\kappa}g\right). \end{aligned}$$

Hence, letting $a = 1/\kappa^2 \in \mathbb{R}$ in eq. (5), we see that $Y = 1/\sqrt{a}\widetilde{X}$. $\square$

Unlike the previous situation, if we have a minimal graph over a nonconvex domain, its convolution minimal surfaces need not be embedded. However, there is one particular case in which embeddedness is guaranteed. If the minimal graph is over a starlike domain, then the convolution surface formed by $\varphi(z) = -\log(1-z)$ will be a minimal graph.

Theorem 3.6. *Let $X : \mathbb{D} \to \mathbb{R}^3$ be an embedded minimal surface which can be written as a graph over a starlike domain and which has the Weierstrass representation given in* (1)*. Then the convolution minimal surface*

$$Z = \left(\mathrm{Re}\left\{ \int \widetilde{p}\,(1+\widetilde{q}^2)\,dw \right\}, \mathrm{Re}\left\{ \int -i\,\widetilde{p}\,(1-\widetilde{q}^2)\,dw \right\}, \mathrm{Re}\left\{ \int -2i\,\widetilde{p}\,\widetilde{q}\,dw \right\} \right)$$

is a graph over a convex domain, where

$$\begin{aligned} \widetilde{p}(z) &= \frac{1}{z}\int p(z)\,dz = p(z) * \frac{-\log(1-z)}{z}, \\ \widetilde{q}(z) &= i\sqrt{\frac{\int p(z)q^2(z)\,dz}{\int p(z)\,dz}} = i\sqrt{\frac{p(z)q^2(z) * \log(1-z)/z}{p(z) * \log(1-z)/z}}, \end{aligned}$$

whenever $\widetilde{q}$ is a perfect square.

Proof. From (1) and (4), we have that

$$h(z) = \int_0^z p(\zeta)\, d\zeta \quad \text{and} \quad g(z) = \int_0^z p(\zeta) q^2(\zeta)\, d\zeta.$$

Hence

$$\widetilde{p}(z) = \frac{d}{dz}[\widetilde{h}(z)] = \frac{d}{dz}\Big[\int_0^z \frac{h(\zeta)}{\zeta} d\zeta\Big] = \frac{h(z)}{z} = \frac{1}{z}\int p(z)\, dz = p(z) * \frac{-\log(1-z)}{z}$$

and

$$\widetilde{q}(z) = \sqrt{\frac{\frac{d}{dz}[\widetilde{g}(z)]}{\frac{d}{dz}[\widetilde{h}(z)]}} = \sqrt{\frac{-g(z)}{h(z)}} = i\sqrt{\frac{\int p(z) q^2(z)\, dz}{\int p(z)\, dz}} = i\sqrt{\frac{p(z) q^2(z) * \log(1-z)/z}{p(z) * \log(1-z)/z}}.$$

Now follow the same approach as in the proof of Theorem 3.1 replacing the use of Theorem 2.8 with Theorem 2.9. □

Remark 3.7. By applying Theorem 3.1 to Z, we can construct convolution surfaces (including the conjugate and associated surfaces) of Z which will be minimal graphs over close-to-convex domains.

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