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Topics on the Spectral Theory of Automorphic Forms

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TOPICS ON THE SPECTRAL THEORY

OF AUTOMORPHIC FORMS

by

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ABSTRACT

TOPICS ON THE SPECTRAL THEORY

OF AUTOMORPHIC FORMS

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We study the analytic properties of the Eisenstein Series of $\frac{1}{2}$-integral weight associated with the Hecke congruence subgroup $\Gamma_0(4)$. Using these properties we obtain asymptotics for sums of certain Dirichlet L-series. We also obtain a formula reducing the study of Selberg's Eigenvalue Conjecture to the study of the nonvanishing of the Eisenstein Series $E(z,s)$ for Hecke congruence subgroups $\Gamma_0(N)$ at $s = \frac{1+i}{2}$.

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1 Introduction

Let N be a positive integer. Denote by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,\, b,\, c,\, d \in \mathbb{Z},\ ad - bc = 1,\ N|c \right\}$$

the Hecke congruence subgroup of level N. The non-Euclidean Laplacian Δ on the upper half-plane $\mathcal{H}$ is given by

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Let D be the fundamental domain of $\Gamma_0(N)$. Eigenfunctions of the discrete spectrum of Δ are nonzero real-analytic solutions of the equation $\Delta\psi = -\lambda\psi$ such that $\psi(\gamma z) = \psi(z)$ for all γ in $\Gamma_0(N)$ and such that ψ is square integrable on D with respect to the Poincaré measure dz of the upper half-plane.

The Hecke operators T_n, $n = 1, 2, \cdots$, $(n, N) = 1$, which act in the space of automorphic functions with respect to $\Gamma_0(N)$, are defined by

$$(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n,\, 0 \le b < d} f\left(\frac{az + b}{d} \right).$$

It is well-known (see Iwaniec [7]) that there exists an orthonormal system of eigenfunctions of Δ such that each of them is an eigenfunction of all the Hecke operators. Let λ_j, $j = 1, 2, \cdots$, be an enumeration in increasing order of all positive discrete eigenvalues of Δ for $\Gamma_0(N)$ with an eigenvalue of multiplicity m appearing m times, and let $\kappa_j = \sqrt{\lambda_j - 1/4}$ with $\mathrm{Im}(\kappa_j) > 0$ if $\lambda_j < 1/4$.

In 1965, A. Selberg [12] made the following fundamental conjecture.

Conjecture 1.1 (Selberg's eigenvalue conjecture)**.** *If λ is a nonzero discrete eigenvalue of the non-Euclidean Laplacian for any congruence subgroup, then $\lambda \ge 1/4$.*

A. Selberg [12] proved that $\lambda \geqslant 3/16$. The Selberg eigenvalue conjecture is also true for some non-congruence groups. P. Sarnak [10] has proved it for all Hecke triangle groups.

The best available lower bound for the smallest positive discrete eigenvalue of the Laplacian for congruence subgroups is due to Kim and Sarnak [8] and reads $\lambda \geq 975/4096$. Selberg's eigenvalue conjecture is of fundamental importance in number theory; see for example Iwaniec [7], Sarnak [11], and Takhtajan-Vinogradov [16].

The following paper is composed of 4 sections. In Sections 2 and 3, the author explores the analytic properties of the Eisenstein series of $\frac{1}{2}$-integral weight for $\Gamma_0(4)$ as applied to Dirichlet L-Series to obtain certain asymptotic formulas. The methods used follow those presented in a paper by D. Goldfeld and J. Hoffstein [5], with special attention given to the details and computations used to obtain the results in that paper. The asymptotics obtained in Section 3 are not the best known for series of this type. Our interest is not in the results themselves but in the methods used.

In the last section of this paper, the author, with the help of X.-J. Li, obtains a formula which reduces the study of Selberg's Eigenvalue Conjecture to the non-vanishing of the Eisenstein series $E(z, s)$ for $\Gamma_0(N)$ at $s = \frac{1+i}{2}$.

2 Eisenstein Series of $\frac{1}{2}$-integral Weight for $\Gamma_0(4)$

2.1 Quadratic Gauss Sums

We begin by establishing a few facts about Quadratic Gauss Sums. We will use the following properties in writing out certain series representations of our Eisenstein series. In the following, $\left(\frac{m}{n}\right)$ is the Legendre symbol.

Consider the sum

$$g(m;n) = \sum_{a \,(\mathrm{mod}\, n)} e^{\frac{2\pi i a^2 m}{n}}$$

where $m, n \in \mathbb{Z}, n \neq 0$ (not necessarily relatively prime). Let

$$G(m;n) = g(m;n) / \left| g(m;n) \right| .$$

Notice that the value of $G(m;n)$ depends only on the congruence class of $m \,(\mathrm{mod}\, n)$.

We have some lemmas.

Lemma 2.1.

$$g(m;p) = g'(m;p)$$

where

$$g'(m;p) = \sum_{r \,(mod\, p)} \left(\frac{r}{p}\right) e^{\frac{2\pi i r m}{p}}$$

whenever p is a prime and $p \nmid m$.

Proof. Since p is prime, $\left(\frac{a}{p}\right)$ is 1 if a is a square modulo p and -1 otherwise. So

$$g'(m;p) = \sum_{r \text{ is a square mod } p} e^{\frac{2\pi i r m}{p}} - \sum_{r \text{ not a square mod } p} e^{\frac{2\pi i r m}{p}} .$$

Consider the difference

$$g(m;p) - g'(m;p).$$

As a runs through a complete residue system modulo p, a^2 runs through all of the non-zero squares modulo p twice, plus a term with $a^2 \equiv 0 \pmod{p}$. Thus

$$g(m;p) - g'(m;p) = \sum_{a \pmod{p}} e^{\frac{2\pi i a m}{p}} = 0$$

since $p \nmid m$. □

Notice that $g'(m;p)$ is a separable Gauss sum.

Lemma 2.2. *We have*

$$g(k;mn) = g(km;n)g(kn;m)$$

for m,n relatively prime and when $p \nmid k$.

Proof. This is a trivial exercise in playing with indices in finite sums. See [2, pp. 177]. □

Lemma 2.3. *We have*

$$g(k;p^a) = \begin{cases} p^{a/2}, & \text{if } a \text{ is even;} \\ p^{(a-1)/2} g(k;p), & \text{if } a \text{ is odd.} \end{cases}$$

Proof. This follows from first showing the relation $g(k;p^a) = pg(k;p^{a-2})$ for $a \geq 2$, which is again a trivial exercise in playing with the indices of summation. See [2, pp. 177]. □

Lemma 2.4. *We have the identity*

$$\left(\frac{m}{n}\right) G(a;n) = G(am;n) \tag{2.1}$$

for $(am,n) = 1$.

Proof. We write $n = n_0 s^2$ where n_0 is square free, and appeal to the prime factorization of n. Successive uses Lemmas 2.2, 2.3, and 2.1 together with the separability of g' give us

$$g(ma;n) = \left(\frac{m}{n_0}\right) g(a;n) = \left(\frac{m}{n}\right) g(a;n).$$

Normalizing proves the lemma. □

More importantly, we have Hecke's reciprocity law.

Lemma 2.5.

$$G(m;n) = e^{\frac{\pi i}{4} sgn\left(\frac{m}{n}\right)} G(-n;4m) \tag{2.2}$$

where

$$sgn(a) = \begin{cases} +1 & \textit{for } a > 0 \\ -1 & \textit{for } a < 0. \end{cases}$$

Note that m and n do not have to be relatively prime.

Proof. This follows from a remarkable identity involving the following related sum

$$S(a,m) = \sum_{r \,(\mathrm{mod}\ m)} e^{\pi i a r^2/m}.$$

The identity is: If am is even, we have

$$S(a,m) = \sqrt{\frac{m}{a}} \left(\frac{1+i}{\sqrt{2}}\right) \overline{S(m,a)}. \tag{2.3}$$

For a proof of this, the reader is referred to Apostol [2, pp. 196–199]. The proof involves residue calculus, but is not hard. Using this identity we can prove the reciprocity law.

Notice $g(m;n) = S(2m,n)$. So we have

$$\begin{aligned} g(m;n) &= \sqrt{\frac{n}{2m}} \left(\frac{1+i}{\sqrt{2}}\right) \overline{S(n,2m)} \\ &= \sqrt{\frac{n}{m}} \left(\frac{1+i}{2}\right) \sum_{r (\text{mod } 2m)} e^{\frac{-2\pi i n r^2}{4m}} \\ &= \sqrt{\frac{n}{m}} \left(\frac{1+i}{4}\right) \sum_{r (\text{mod } 4m)} e^{\frac{-2\pi i n r^2}{4m}}. \end{aligned}$$

Here we doubled the sum by letting r run through a complete residue system modulo $4m$. We can do this since $(r+2m)^s = r^2 \,(\text{mod } 4m)$. We now have

$$g(m;n) = \sqrt{\frac{n}{m}} \left(\frac{1+i}{4}\right) g(-n;4m).$$

Normalizing, we obtain the identity we needed. $\square$

Also following from the identity for $S(a,m)$, taking $a=2$ and noticing that $\overline{S(m,2)} = 1 + e^{-\pi i m/2}$, we have Gauss's formula:

$$g(1;m) = \frac{1}{2}\sqrt{m}(1+i)(1+e^{-\pi i m/2}) = \begin{cases} \sqrt{m} & \text{if } m \equiv 1 (\text{mod } 4) \\ 0 & \text{if } m \equiv 2 (\text{mod } 4) \\ i\sqrt{m} & \text{if } m \equiv 3 (\text{mod } 4) \\ (1+i)\sqrt{m} & \text{if } m \equiv 0 (\text{mod } 4) \end{cases}$$

for every $m \geq 1$. Now, normalizing we have the useful identity

$$G(1;d) = \epsilon_d \tag{2.4}$$

for odd $d > 0$, where

$$\epsilon_d = \begin{cases} 1 \text{ if } d \equiv 1 \,(\text{mod } 4) \\ i \text{ if } d \equiv 3 \,(\text{mod } 4), \end{cases} \quad (d > 0).$$

This will be helpful in the following sections.

2.2 Theta Series

Next, we develop the transformation formula for the theta series

$$\theta(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i z n^2}.$$

Let

$$\Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1, 4|c \right\}.$$

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ act on the upper half plane in the usual manner, i.e.

$$\gamma z = \frac{az + b}{cz + d}.$$

As a notational convention we take $|\arg(z)^{\frac{1}{2}}| < \frac{\pi}{2}$. What this guarantees is that for z in the upper half plane, we have $(-z)^{\frac{1}{2}} = -i(z)^{\frac{1}{2}}$ and for z in the lower half plane $(-z)^{\frac{1}{2}} = i(z)^{\frac{1}{2}}$. The form in which we will use this is the case where z is in the upper half plane and c and d are real numbers, $d > 0$, so that

$$(-cz - d)^{\frac{1}{2}} = -\mathrm{sgn}(c) i (cz + d)^{\frac{1}{2}}$$

where $\mathrm{sgn}(c)$ is 1 or -1 according as $c \geq 0$ or $c < 0$.

We have the following transformation for $\theta(z)$.

Lemma 2.6. *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$ *with* $d > 0$. *Then we have*

$$\theta(\gamma z) = \left(\frac{c}{d}\right) \epsilon_d^{-1} (cz + d)^{\frac{1}{2}} \theta(z)$$

where

$$\epsilon_d = \begin{cases} 1 \text{ if } d \equiv 1 \,(mod\ 4) \\ i \text{ if } d \equiv 3 \,(mod\ 4), \end{cases} \qquad (d > 0).$$

Proof. The proof follows those methods used in [13] for a more general result.

For z in the upper half plane, N a positive integer, and h an integer, let

$$\theta(z;h,N) = \sum_{n\equiv h(\text{mod } N)} e^{\frac{\pi i z n^2}{N}}.$$

We have the following transformation formulas:

$$\theta(z+2;h,N) = e^{\frac{2\pi i h^2}{N}}\theta(z;h,N) \tag{2.5}$$

$$\theta(z;h,N) = \sum_{\substack{g\equiv h(\text{mod } N)\\ g \text{ mod } cN}} \theta(cz;g,cN) \tag{2.6}$$

$$\theta\left(-\frac{1}{z};h,N\right) = N^{-1/2}(-iz)^{1/2}\sum_{k \text{ mod } N} e^{\frac{2\pi i k h}{N}}\theta(z;k,N). \tag{2.7}$$

The first two identities are trivial. The third is more subtle and makes use of the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{k=-\infty}^{\infty} e^{2\pi i k x}\int_{-\infty}^{\infty} f(y)e^{-2\pi i k y}dy$$

together with the identity

$$\int_{-\infty}^{\infty} e^{-tx^2+2xy}dx = \sqrt{\frac{\pi}{t}}e^{\frac{y^2}{t}} \tag{2.8}$$

where t and y are complex numbers with $\text{Re}(t) > 0$.

To prove (2.8), we first let t, y be real with $t > 0$, and let I denote the integral in (2.8). We make the change of variable $\sqrt{t}x \to x$ and get

$$I = \int_{-\infty}^{\infty} e^{-tx^2+2xy}dx = \frac{1}{\sqrt{t}}\int_{-\infty}^{\infty} e^{-x^2+\frac{2xy}{\sqrt{t}}}dx.$$

Completing the square in the exponent and making another change of variables we have

$$\begin{aligned} I &= \frac{1}{\sqrt{t}}e^{\frac{y^2}{t}}\int_{-\infty}^{\infty} e^{-x^2}dx \\ &= \sqrt{\frac{\pi}{t}}e^{\frac{y^2}{t}}. \end{aligned}$$

Now we appeal to analytic continuation and the identity holds for complex t and y with $\mathrm{Re}(t) > 0$ (it is easy to show that the expression I is an analytic function in both variables).

To prove (2.7) we apply the Poisson summation formula to the function $f(x) = e^{\pi i z (x+\frac{h}{N})^2 N}$. This gives

$$\begin{aligned}\theta(z; h, N) &= \sum_{n=-\infty}^{\infty} e^{\pi i z (n+\frac{h}{N})^2 N} \\ &= \sum_{k=-\infty}^{\infty} e^{2\pi i k \frac{h}{N}} \int_{-\infty}^{\infty} e^{\pi i z x^2 N - 2\pi i k x} dx \\ &= \sqrt{\frac{1}{-izN}} \sum_{k=-\infty}^{\infty} e^{2\pi i k \frac{h}{N}} e^{\frac{\pi i k^2}{N(-z)}}.\end{aligned}$$

The last equality is obtained from (2.8) taking $t = -\pi i z N$ and $y = -\pi i k$. Notice that since $\mathrm{Im}(z) > 0$ and $N > 0$, $\mathrm{Re}(-\pi i z N) > 0$. Substituting $-\frac{1}{z}$ for z, we have

$$\begin{aligned}\theta\left(-\frac{1}{z}; h, N\right) &= N^{-1/2}(-iz)^{1/2} \sum_{k=-\infty}^{\infty} e^{2\pi i k \frac{h}{N}} e^{\frac{\pi i k^2 z}{N}} \\ &= N^{-1/2}(-iz)^{1/2} \sum_{k \bmod N} e^{2\pi i k \frac{h}{N}} \theta(z; k, N).\end{aligned}$$

Now let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Assume that a and d are both even and that $c < 0$. Notice that $-c\gamma z = \frac{1}{cz+d} - a$. Using (2.6), (2.5), (2.7) and then

(2.5) again, we have

$$\begin{aligned}
\theta(\gamma z; h, N) &= \sum_{\substack{g\equiv h (\text{mod } N)\\ g \text{ mod } -cN}} \theta(-c\gamma z; g, -cN)\\
&= \sum_{\substack{g\equiv h (\text{mod } N)\\ g \text{ mod } -cN}} \theta\left(\frac{1}{cz+d} - a; g, -cN\right)\\
&= \sum_{\substack{g\equiv h (\text{mod } N)\\ g \text{ mod } -cN}} e^{\frac{\pi i g^2 a}{Nc}} \theta\left(\frac{1}{cz+d}; g, -cN\right)\\
&= \sum_{\substack{g\equiv h (\text{mod } N)\\ g \text{ mod } -cN}} e^{\frac{\pi i g^2 a}{Nc}} \theta\left(-\frac{1}{-cz-d}; g, -cN\right)\\
&= \sum_{\substack{g\equiv h (\text{mod } N)\\ g \text{ mod } -cN}} e^{\frac{\pi i g^2 a}{Nc}} (-cN)^{-1/2}(i(cz+d))^{1/2}\\
&\quad\times \sum_{k \text{ mod } -cN} e^{\frac{2\pi i k g}{-cN}} \theta(-cz-d; k, -cN)\\
&= \sum_{\substack{g\equiv h (\text{mod } N)\\ g \text{ mod } -cN}} e^{\frac{\pi i g^2 a}{Nc}} (-cN)^{-1/2}(i(cz+d))^{1/2}\\
&\quad\times \sum_{k \text{ mod } -cN} e^{\frac{2\pi i k g}{-cN}} e^{\frac{\pi i k^2 d}{Nc}} \theta(-cz; k, -cN).
\end{aligned}$$

We have used $\text{Im}(-cz-d) > 0$ when $\text{Im}(z) > 0$ and $c < 0$.

Let

$$\Phi(h,k) = \sum_{\substack{g\equiv h (\text{mod } N)\\ g \text{ mod } -cN}} e^{\frac{\pi i (g^2 a - 2kg + k^2 d)}{cN}}.$$

Then we have

$$\theta(\gamma z; h, N) = (-cN)^{-1/2}(i(cz+d))^{1/2} \sum_{k \text{ mod } -cN} \Phi(h,k)\theta(-cz; k, -cN). \quad (2.9)$$

We examine more closely the properties of $\Phi(h,k)$. It clearly depends only on h modulo N. Now

$$\begin{aligned}\Phi(h,k) &= \sum_{n=0}^{-cN-1} e^{\pi i \frac{(h+nN)^2 a - 2k(h+nN)+dk^2}{cN}} \\ &= \sum_{n=0}^{-cN-1} e^{\pi i \frac{h^2 a + 2hnNa + n^2N^2a - 2kh(ad-bc) - 2nN(ad-bc) + dk^2(ad-bc)}{cN}}\end{aligned}$$

(Recall that $ad - bc = 1$)

$$\begin{aligned} &= e^{\pi i \frac{b(2kh-dk^2)}{N}} \sum_{n=0}^{-cN-1} e^{\pi i \frac{a(h+nN-dk)^2}{cN}} e^{2piinb} \\ &= e^{\pi i \frac{b(2kh-dk^2)}{N}} \sum_{n=0}^{-cN-1} e^{\pi i \frac{a(h-dk+nN)^2}{cN}} \\ &= e^{\pi i \frac{b(2kh-dk^2)}{N}} \Phi(h-dk,0).\end{aligned}$$

This gives us that $\Phi(h,k)$ also only depends on k modulo N. Thus (2.9) becomes

$$\theta(\gamma z; h, N) = (-cN)^{-1/2}(i(cz+d))^{1/2} \sum_{k \bmod N} \Phi(h,k) \sum_{\substack{l \equiv k (\bmod N) \\ l \bmod -cN}} \theta(-cz; k, -cN).$$

So by (2.6) we have

$$\theta(\gamma z; h, N) = (-cN)^{-1/2}(i(cz+d))^{1/2} \sum_{k \bmod N} \Phi(h,k)\theta(z;k,N).$$

Substituting $-\frac{1}{z}$ for z, and using (2.7) we get

$$\begin{aligned}\theta\left(\frac{bz-a}{dz-c}; h, N\right) =&(-cN)^{-1/2}\left(i\left(c\left(-\frac{1}{z}\right)+d\right)\right)^{1/2} \\ &\times \sum_{k \bmod N} \Phi(h,k) \sum_{l \bmod N} N^{-1/2}(-iz)^{1/2}\theta(z;k,N) \\ =&(-c)^{-1/2}N^{-1}(dz-c)^{1/2} \\ &\times \sum_{l \bmod N}\sum_{k \bmod N} \Phi(h,k)e^{2\pi i \frac{lk}{N}}\theta(z;l,N).\end{aligned}$$

Suppose $d \equiv 0 \pmod{2N}$. Then we have

$$\Phi(h,k) = e^{\pi i \frac{b(2kh-dk^2)}{N}}\Phi(h-dk,0) = e^{2\pi i \frac{2kh}{N}}\Phi(h,0).$$

So, $\theta\left(\frac{bz-a}{dz-c}; h, N\right)$

$$= (-c)^{-1/2} N^{-1} (dz-c)^{1/2} \Phi(h,0) \sum_{l \bmod N} \sum_{k \bmod N} e^{2\pi i \frac{(l+bh)k}{N}} \theta(z; l, N).$$

Since,

$$\sum_{k \bmod N} e^{2\pi i \frac{(l+bh)k}{N}} = \begin{cases} 0 & \text{if } N \nmid (l+bh) \\ N & \text{if } N | (l+bh), \end{cases}$$

we have

$$\theta\left(\frac{bz-a}{dz-c}; h, N\right) = (-c)^{-1/2} (dz-c)^{1/2} \Phi(h,0) \theta(z; -bh, N).$$

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ with b even, $c \equiv 0 \pmod{2N}$ and $d > 0$. Then

$$\theta(\gamma z; h, N) = d^{-1/2} (cz+d)^{1/2} \left\{ \sum_{\substack{g \equiv h \pmod N \\ g \bmod N}} e^{\pi i \frac{g^2 b}{Nd}} \right\} \theta(z; -ah, N). \tag{2.10}$$

We are interested in the sum that occurs in the brackets. Since $c \equiv 0 \pmod{2N}$, $ad \equiv 1 \pmod N$, as u runs through a complete residue system modulo d, $g = adh + Nu$ runs through all those integers congruent to h in a complete residue system modulo dN. So the sum is the same as

$$\sum_{u=1}^{d} e^{\pi i \frac{b(adh+Nu)^2}{dN}} = e^{\pi i \frac{ba^2dh^2}{N}} \sum_{u=1}^{d} e^{\pi i \frac{bNu^2}{d}} = e^{\pi i \frac{abh^2}{N}} \sum_{u=1}^{d} e^{\pi i \frac{bNu^2}{d}}.$$

Let $w(b,d) = d^{-1/2} \sum_{u=1}^{d} e^{\pi i \frac{bNu^2}{d}}$. (2.10) becomes

$$\theta(\gamma z; h, N) = e^{\pi i \frac{abh^2}{N}} w(b,d) (cz+d)^{1/2} \theta(z; -ah, N).$$

Notice that $w(b,d) = d^{-1/2} g((b/2)N; d)$, where $g(m;n)$ denotes the quadratic Gauss sum from the previous section. Since $ad - bc = 1$ and $N|c$, N and d are coprime. By Lemma 2.4, we have

$$w(b,d) = d^{-1/2} \left(\frac{(b/2)N}{d}\right) g(1;d) = d^{-1/2} \left(\frac{b/2}{d}\right) \left(\frac{N}{d}\right) g(1;d).$$

Using Gauss's Formula (see (2.4)), this implies that

$$w(b,d) = \left(\frac{b/2}{d}\right)\left(\frac{N}{d}\right)\epsilon_d.$$

So (2.10) is

$$\theta(\gamma z; h, N) = e^{\pi i \frac{abh^2}{N}}\left(\frac{b/2}{d}\right)\left(\frac{N}{d}\right)\epsilon_d (cz+s)^{1/2}\theta(z; -ah, N).$$

Since $ad - bc = 1$, we have

$$\left(\frac{b/2}{d}\right)\left(\frac{2c}{d}\right) = \left(\frac{bc}{d}\right) = \left(\frac{bc-ad}{d}\right) = \left(\frac{-1}{d}\right)$$

and

$$\left(\frac{b/2}{d}\right) = \left(\frac{-1}{d}\right)\left(\frac{2c}{d}\right).$$

It is not hard to see that

$$\left(\frac{-1}{d}\right)\epsilon_d = \epsilon_d^{-1}$$

so (2.10) becomes

$$\theta(\gamma z; h, N) = e^{\pi i \frac{abh^2}{N}}\left(\frac{2c}{d}\right)\left(\frac{N}{d}\right)\epsilon_d^{-1}(cz+d)^{1/2}\theta(z; -ah, N).$$

Taking $N = 1$, we have, for $\mathrm{Im}(z) > 0$,

$$\theta'(\gamma z) = \left(\frac{2c}{d}\right)\epsilon_d^{-1}(cz+d)^{1/2}\theta'(z) \tag{2.11}$$

where $\theta'(z) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 z}$.

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$ with $d > 0$. Then $2\gamma z = \gamma' 2z$ where $\gamma' = \begin{pmatrix} a & 2b \\ \frac{c}{2} & d \end{pmatrix}$ and here $2b \equiv \frac{c}{2} \equiv 0 \pmod 2$. So γ' satisfies our conditions. Hence, by (2.11) we have

$$\theta(\gamma z) = \theta'(\gamma' 2z) = \left(\frac{c}{d}\right)\epsilon_d^{-1}(cz+d)^{1/2}\theta'(2z) = \left(\frac{c}{d}\right)\epsilon_d^{-1}(cz+d)^{1/2}\theta(z).$$

This proves the lemma. □

2.3 Eisenstein series for $\Gamma_0(4)$

We now define the Eisenstein Series associated with $\Gamma_0(4)$.

Let $\Gamma_0(N)$ be the Hecke Congruence subgroup of level N, i.e.

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1, N|c \right\}.$$

By [15, pp. 24-5], we have the following theorem.

Theorem 2.7. *Let $\nu(N)$ denote the number of inequivalent cusps of $\Gamma_0(N)$. We have*

$$\nu(N) = \sum_{d|4, d>0} \phi\left(\left(d, \frac{4}{d}\right)\right),$$

where ϕ is Euler's function.

Hence

$$\nu(4) = \phi(1) + \phi(2) + \phi(1) = 1 + 1 + 1 = 3.$$

Every cusp of $\Gamma_0(4)$ is equivalent to one of the following:

$$\frac{u}{w} \text{ with } u, w > 0, (u, w) = 1, w|4.$$

Two such cusps u/w and u_1/w_1 are equivalent under $\Gamma_0(4)$ if and only if $w = w_1$ and $u \equiv u_1$ modulo $(w, 4/w)$. For $w = 1$ and 4, every such cusp is equivalent to 0. For $w = 2$, every such cusp is equivalent to 1/2. ∞ is not equivalent to either of these since, for $\sigma \in \Gamma_0(4)$, $\sigma\infty = 0$ forces $a = 0$, which means $ad - bc$ cannot be 1 since $4|c$. $\sigma\infty = 1/2$ forces a to be even, so $ad - bc$ is divisible by 2, and therefore cannot be 1. Thus, $\Gamma_0(4)$ has three inequivalent cusps at ∞, 0, and 1/2.

Notice, if $\sigma \in \Gamma_0(4)$, and $\sigma\infty = \infty$, then $c = 0$ and $a = d = 1$. Hence

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z} \right\}$$

is the stabilizer of ∞.

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, we define

$$j_\gamma(z) = \left(\frac{c}{d}\right) \epsilon_d^{-1} (cz+d)^{\frac{1}{2}}$$

where

$$\epsilon_d = \begin{cases} 1 \text{ if } d \equiv 1 \pmod 4 \\ i \text{ if } d \equiv 3 \pmod 4, \end{cases} \qquad (d > 0)$$

and $\left(\frac{c}{d}\right)$ is the Legendre symbol, and $(cz+d)^{\frac{1}{2}}$ is taken so that $|\arg(cz+d)^{\frac{1}{2}}| < \frac{\pi}{2}$. If $d < 0$, then $j_\gamma(z)$ is defined by the relation $j_\gamma(z) = j_{-\gamma}(z)$.

By Lemma 2.6, we have

$$j_\gamma(z) = \frac{\theta(\gamma z)}{\theta(z)}.$$

This makes it easy to see that $j_\gamma(z)$ is a multiplier system of weight $\frac{1}{2}$ for $\Gamma_0(4)$, which means that $j_\gamma(z)$ satisfies the relation

$$j_{\gamma\gamma'}(z) = j_\gamma(\gamma' z) j_{\gamma'}(z) \quad \gamma, \gamma' \in \Gamma_0(4).$$

From this point on, let k be any odd rational integer. We define the Eisenstein series of weight $k/2$ for the group $\Gamma_0(4)$ in the following manner. Let Γ_∞ be as above, let Γ_0 and $\Gamma_{\frac{1}{2}}$ be the stabilizers of the cusps 0 and $\frac{1}{2}$, respectively. Then

$$\Gamma_0 = \left\{ \begin{pmatrix} 1 & 0 \\ -4m & 1 \end{pmatrix} : m \in \mathbb{Z} \right\}$$

and

$$\Gamma_{\frac{1}{2}} = \left\{ \begin{pmatrix} 1+2m & m \\ -4m & 1-2m \end{pmatrix} : m \in \mathbb{Z} \right\}.$$

The Eisenstein series for the cusp ∞ is

$$E_\infty(z,s,k) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)} (\mathrm{Im}(\gamma z))^s j_\gamma(z)^{-k}. \tag{2.12}$$

Then (2.12) converges absolutely and uniformly if Re$(s)> 1-\frac{k}{4}$. Notice that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $z = x+iy$,

$$\mathrm{Im}(\gamma z) = \frac{y}{|cz+d|^2}.$$

We define the Eisenstein series at the cusps 0 and $\frac{1}{2}$ as follows

$$E_0(z,s,k) = z^{-k/2} E_\infty\left(\frac{-1}{4z}, s, k\right)$$

$$E_{\frac{1}{2}}(z,s,k) = (2z+1)^{-k/2} E_0\left(\frac{z}{2z+1}, s, k\right).$$

Lemma 2.8 (Automorphic relations)**.** *The Eisenstein series E_∞, E_0 and $E_{\frac{1}{2}}$ satisfy the following relations.*

$$E_c(\gamma z, s) = j_\gamma(z)^k E_c(z,s) \ \ (\textit{for } c = 0,\infty)$$

$$E_0\left(\frac{-1}{4z}, s\right) = (4z)^{k/2} i^{-k} E_\infty(z,s)$$

$$E_{\frac{1}{2}}\left(z+\frac{1}{2}, s\right) = 2^{k/2}(2z+1)^{-k/2} E_\infty\left(\frac{z}{2z+1}, s\right)$$

$$E_{\frac{1}{2}}(z,s) = i^{-k} E_{\frac{1}{2}}(z+1,s)$$

Proof. First we consider the first identity with $c=\infty$. As γ runs over a complete set of cosets in $\Gamma_\infty/\Gamma_0(4)$, so does $\gamma\sigma$ for any $\sigma\in\Gamma_0(4)$. Using this, the sum in (2.12) becomes

$$\begin{aligned} E_\infty(\sigma z, s) &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)} (\mathrm{Im}(\gamma\sigma z))^s j_\gamma(\sigma z)^{-k} \\ &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)} (\mathrm{Im}(\gamma z))^s j_{\gamma\sigma^{-1}}(\sigma z)^{-k}. \end{aligned}$$

Using the multiplier properties of j_γ we see that $j_{\gamma\sigma^{-1}}(\sigma z)^{-k} = j_\gamma(z) j_{\sigma^{-1}}(\sigma z)$. Also, $1 = j_{id}(\sigma z) = j_\sigma(z) j_{\sigma^{-1}}(\sigma z)$, so that $j_{\sigma^{-1}}(\sigma z) = j_\sigma(z)^{-1}$. The sum then becomes

$$\begin{aligned} E_\infty(\sigma z, s) &= j_\sigma(z)^k \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} (\mathrm{Im}(\gamma z))^s j_\gamma(z)^{-k} \\ &= j_\sigma(z)^k E_\infty(z, s) \end{aligned}$$

and hence the first property holds.

For $c \equiv 0 \pmod 4$, we notice that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $(-1/4\gamma z) = \gamma'(-1/4z)$, where $\gamma' = \begin{pmatrix} d & -\frac{c}{4} \\ -4b & a \end{pmatrix} \in \Gamma_0(4)$ ($\gamma' = \alpha\gamma\alpha^{-1}$ with $\alpha = \begin{pmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{pmatrix}$). So, by definition we have

$$\begin{aligned} E_0(\gamma z, s) &= (\gamma z)^{-k/2} E_\infty\left(\frac{-1}{4\gamma z}, s\right) \\ &= (\gamma z)^{-k/2} E_\infty\left(\gamma'\left(\frac{-1}{4z}\right), s\right) \\ &= j_{\gamma'}\left(\frac{-1}{4z}\right)^k (\gamma z)^{-k/2} E_\infty\left(\frac{-1}{4z}, s\right). \end{aligned}$$

If a and d have the same sign, we can assume both are positive since $-\gamma$ induces the same fractional linear transformation. Thus, we can use the explicit definition for $j_{\gamma'}(-1/4z)$.

$$\begin{aligned} j_{\gamma'}(-1/4z) &= \left(\frac{-4b}{a}\right) \epsilon_a^{-1} \left(-4b\left(\frac{-1}{4z}\right) + a\right)^{1/2} \\ &= \left(\frac{-4b}{a}\right) \epsilon_a^{-1} (az+b)^{1/2} (z)^{-1/2}. \end{aligned}$$

Multiplying by $(\gamma z)^{-1/2}$, we get

$$\left(\frac{-4b}{a}\right) \epsilon_a^{-1} (cz+d)^{1/2} (z)^{-1/2}.$$

Since $ad - bc = 1$, we know that $ad \equiv 1 \pmod 4$, which is only possible if a and d are congruent modulo 4. So $\epsilon_a = \epsilon_d$.

Now we claim that $\left(\frac{-4b}{a}\right) = \left(\frac{c}{d}\right)$. To see this we appeal to the multiplier system property of the j_γ functions. Applying this to γ and γ^{-1}, direct computation gives

$$1 = \left(\frac{-c}{a}\right)\left(\frac{c}{d}\right)\epsilon_a^{-1}\epsilon_d^{-1}(-c(az+b)+a(cz+d))^{1/2} = \left(\frac{-c}{a}\right)\left(\frac{c}{d}\right)\epsilon_a^{-1}\epsilon_d^{-1}.$$

Since $\epsilon_a = \epsilon_d$, $\epsilon_a^{-1}\epsilon_d^{-1} = (-1)^{\frac{a-1}{2}} = (-1)^{\frac{d-1}{2}}$. Hence

$$\left(\frac{-c}{a}\right)\left(\frac{c}{d}\right) = (-1)^{\frac{a-1}{2}}.$$

However,

$$\left(\frac{-c}{a}\right) = \left(\frac{-1}{a}\right)\left(\frac{c}{a}\right) = (-1)^{\frac{a-1}{2}}\left(\frac{c}{a}\right).$$

So

$$\left(\frac{c}{a}\right)\left(\frac{c}{d}\right) = 1.$$

Also, since $ad - bc = 1$, $-bc$ is congruent to 1 modulo a, and so

$$\left(\frac{c}{d}\right) = \left(\frac{-bc}{a}\right)\left(\frac{c}{d}\right) = \left(\frac{-b}{a}\right).$$

Therefore $\left(\frac{-4b}{a}\right) = \left(\frac{c}{d}\right)$ and the identity follows.

If a and d have opposite signs we can again assume that a is positive by making the necessary sign change. Recall that in the case where $d < 0$, $j_\gamma(z)$ is defined to be $j_{-\gamma}(z)$. This means that, in this case,

$$j_\gamma(z) = \left(\frac{-c}{|d|}\right)\epsilon_{-d}^{-1}(-cz-d)^{1/2}.$$

The calculations are similar to those above, so that

$$j_{\gamma'}(-1/4z)(\gamma z)^{-1/2} = \left(\frac{-4b}{a}\right)\epsilon_a^{-1}(cz+d)^{1/2}(z)^{1/2}.$$

However, since a and d are opposite in sign, the fact that they are congruent modulo 4 tells us that $\epsilon_a^{-1} = \epsilon_{-d}^{-1}(-1)^{\frac{a-1}{2}}i$, and the i will conveniently factor through the square root as a -1, giving us

$$j_{\gamma'}(-1/4z)(\gamma z)^{-1/2} = \left(\frac{-4b}{a}\right) \epsilon_{-d}^{-1}(-1)^{\frac{a-1}{2}}(-cz-d)^{1/2}(z)^{1/2}.$$

Finally, we can again use the fact that $1 = j_\gamma(\gamma^{-1}z)j_{\gamma^{-1}}(z)$ to see that

$$\left(\frac{-c}{|d|}\right)\left(\frac{-c}{a}\right) = \epsilon_a\epsilon_{-d}i^{-1} = 1.$$

Recall that $-bc \equiv 1 \pmod{a}$. Now we have

$$\begin{aligned}
\left(\frac{-4b}{a}\right) &= \left(\frac{-b}{a}\right) \\
&= \left(\frac{-b}{a}\right)\left(\frac{-c}{|d|}\right)\left(\frac{-c}{a}\right) \\
&= \left(\frac{bc}{a}\right)\left(\frac{-c}{|d|}\right) \\
&= (-1)^{\frac{a-1}{2}}\left(\frac{-c}{|d|}\right).
\end{aligned}$$

The identity then follows.

For the second property, we appeal directly to the definition of E_0 and notice that $(-1)^{-k/2} = i^{-k}$.

The third identity is also a direct consequence of the definitions of $E_{\frac{1}{2}}$ and E_0. Direct computation gives

$$E_{\frac{1}{2}}\left(z + \frac{1}{2}, s\right) = 2^{k/2}(2z+1)^{-k/2}E_\infty\left(-\frac{z+1}{2z+1}, s\right).$$

Now notice that the first identity, using $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, gives $E_\infty(z+1,s) = E_\infty(z,s)$, since $j_\gamma(z) = 1$. The identity then follows.

Finally, for the last identity, take $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, so $\gamma z = z+1$. Notice that $\frac{\gamma z}{2\gamma z+1} = \gamma'\left(\frac{z}{2z+1}\right)$ where $\gamma' = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}$. Using the first property with

$c = 0$, we see that

$$\begin{aligned}
E_{\frac{1}{2}}(z+1,s) &= (2z+3)^{-k/2} E_0\left(\frac{z+1}{2z+3}, s\right) \\
&= (2z+3)^{-k/2} E_0\left(\gamma'\left(\frac{z}{2z+1}\right), s\right) \\
&= j_{\gamma'}\left(\frac{z}{2z+1}\right)^k (2z+3)^{-k/2} E_0\left(\frac{z}{2z+1}, s\right).
\end{aligned}$$

But $j_{\gamma'}(z) = \left(\frac{-4}{3}\right)\epsilon_3^{-1}(-4z+3)^{1/2} = (-1)(-i)(-4z+3)^{1/2} = i(-4z+3)^{1/2}$. So we have

$$\begin{aligned}
E_{\frac{1}{2}}(z+1,s) &= i^k\left(-4\frac{z}{2z+1}+3\right)^{k/2} (2z+3)^{-k/2} E_0\left(\frac{z}{2z+1}, s\right) \\
&= i^k (2z+1)^{-k/2} E_0\left(\frac{z}{2z+1}, s\right) \\
&= i^k E_{\frac{1}{2}}(z,s).
\end{aligned}$$

This completes the proof. $\square$

We now proceed to calculate the Fourier coefficients and Mellin tansforms of the Eisenstein Series. For the convenience of the reader, we include the proofs of some the facts. The techniques used for both the proofs given here and those propositions not proved herein can be found in more detail in [14].

Proposition 2.9. *For $Re(s) > 1 - k/4$, the following representations hold:*

$$E_\infty(z,s) = y^s + e^{\frac{\pi i k}{4}} y^s \sum_{(d,2c)=1, c>0} \frac{G(-d;4c)^k}{|4cz+d|^{2s}(4cz+d)^{k/2}}, \tag{2.13}$$

$$E_0(z,s) = (y/4)^s \sum_{(u,2v)=1, u>0} \frac{\left(\frac{-v}{u}\right)\epsilon_u^k}{|v+uz|^{2s}(v+uz)^{k/2}}, \tag{2.14}$$

$$E_{\frac{1}{2}}(z,s) = (y/4)^s e^{-\frac{\pi i k}{4}} \sum_{(d,2c)=1, d>0} \frac{G(d-2c;8d)^k}{|dz+c|^{2s}(dz+c)^{k/2}}. \tag{2.15}$$

Proof. Let W_0 denote the set of all ordered pairs of integers $\{c,d\}$ such that $(c,d) = 1$, $4|c$ and $d > 0$. Then there is a bijection between W_0 and the set of cosets of Γ_∞ over $\Gamma_0(4)$. First, since $-1 \in \Gamma_\infty$ (-1 fixes ∞), for every coset of Γ_∞ there is a representative $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $d > 0$. From this set of representatives, we consider the map taking $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty/\Gamma_0(4)$ with $d > 0$, to $\{c, d\}$. The first question is whether or not this map is well defined. Given γ such a representative, $4|c$, by the definition of $\Gamma_0(4)$. Furthermore, if $p|d$ and $p|c$, then $p|ad - bc = 1$, so $p = 1$. Thus (c, d)=1, and $\{c, d\} \in W_0$. Furthermore, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ are representatives of the same coset, then

$$\gamma\gamma'^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d' & -b' \\ -c' & a' \end{pmatrix} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$$

for some integer m. Equating entries, it is not hard to see that $c = c'$ and $d = d'$. So the map is well defined.

Suppose $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ are mapped to the same ordered pair $\{c, d\}$. Then $c = c'$ and $d = d'$. Furthermore

$$\gamma\gamma'^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b' \\ -c & a' \end{pmatrix} = \begin{pmatrix} 1 & a'b - ab' \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty,$$

since $ad - bc = 1$ and $a'd - b'c = 1$. So γ and γ' represent the same coset of Γ_∞, and hence the map is injective.

To see that the map is surjective, we notice that for any ordered pair $\{c, d\}$ in W_0, we can choose integers a and b such the $ad - bc = 1$ since c and d are relatively prime. Then the coset representative $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ maps to $\{c, d\}$.

So we have a one-to-one correspondence.

Thus, we can write E_∞ as follows:

$$E_\infty(z,s) = y^s \sum_{(c,d)=1,4|c,d>0} \frac{\left(\frac{c}{d}\right)^{-k} \epsilon_d^k}{|cz+d|^{2s}(cz+d)^{k/2}} \tag{2.16}$$

From (2.4), since $4|c$ and $(c,d) = 1$ implies that d must be odd, this becomes

$$E_\infty(z,s) = y^s \sum_{(c,d)=1,4|c,d>0} \frac{\left(\frac{c}{d}\right)^{-k} G(1;d)^k}{|cz+d|^{2s}(cz+d)^{k/2}}.$$

Using Lemma 2.4, and the fact $\left(\frac{c}{d}\right)^{-1} = \left(\frac{c}{d}\right)$, we have

$$E_\infty(z,s) = y^s \sum_{(c,d)=1,4|c,d>0} \frac{G(c;d)^k}{|cz+d|^{2s}(cz+d)^{k/2}},$$

and now we notice that when $c = 0$, the only corresponding d in the summand is 1. That term is then 1. Hence

$$E_\infty(z,s) = y^s + y^s \sum_{(c,d)=1,4|c,d>0,c\neq 0} \frac{G(c;d)^k}{|cz+d|^{2s}(cz+d)^{k/2}}.$$

Next, we make a simple change of index, taking c to $c/4$.

$$E_\infty(z,s) = y^s + y^s \sum_{(d,2c)=1,d>0,c\neq 0} \frac{G(4c;d)^k}{|4cz+d|^{2s}(4cz+d)^{k/2}}$$

By Lemma 2.4, and the fact that $\left(\frac{4}{d}\right) = \left(\frac{2}{d}\right)^2 = 1$, we have $G(4c;d) = G(c;d)$ since d and 4 are relatively prime. By Lemma 2.5, we get

$$E_\infty(z,s) = y^s + y^s \sum_{(d,2c)=1,d>0,c\neq 0} \frac{e^{\frac{\pi ik}{4}\operatorname{sgn}\left(\frac{c}{d}\right)} G(-d;4c)^k}{|4cz+d|^{2s}(4cz+d)^{k/2}}.$$

When $c > 0$ we get what we want in the summand, so we break the sum into two parts: when $c > 0$ and when $c < 0$. The first half, we leave as is. For the second, we instead sum over $c > 0$ and $d < 0$, and make the appropriate change of indices, getting

$$\sum_{(d,2c)=1,c>0,d<0} \frac{e^{\frac{-\pi ik}{4}} G(d;-4c)^k}{|-4cz-d|^{2s}(-4cz-d)^{k/2}}$$

.

As one might expect $g(m;-n) = g(-m;n)$, from a direct appeal to the definition, and so we can bring the negative sign up from the bottom in G as well. Furthermore, the -1 does not affect the modulus in the denominator. Multiplying both the numerator and the denominator by $(-i)^k = e^{-\frac{\pi i k}{2}}$, the negative sign inside the second factor in the denominator cancels. So the second sum becomes

$$\sum_{(d,2c)=1,c>0,d<0} \frac{e^{\pi i k(-\frac{1}{4}+\frac{1}{2})} G(-d;4c)^k}{|4cz+d|^{2s}(4cz+d)^{k/2}}$$

and the exponential is clearly equal to $e^{\frac{\pi i k}{4}}$.

Putting the two sums back together, we get (2.13).

To prove (2.14), we start again with (2.16) and use the definition of E_0. Notice that $\text{Im}(-1/4z) = \frac{y}{4|z|^2}$,

$$E_0(z,s) = \frac{y^s}{4^s|z|^{2s}z^{k/2}} \sum_{(c,d)=1,4|c,d>0} \frac{\left(\frac{c}{d}\right)^{-k} \epsilon_d^k}{|-c/4z+d|^{2s}(-c/4z+d)^{k/2}}.$$

Bringing the denominators together nicely, and making a change of index to drop the requirement that $4|c$, we get

$$E_0(z,s) = (y/4)^s \sum_{(d,2c)=1,d>0} \frac{\left(\frac{4c}{d}\right)^{-k} \epsilon_d^k}{|dz-c|^{2s}(dz-c)^{k/2}}.$$

Now we simply make the change of index $c = -v$, $d = u$, again noticing that $\left(\frac{4c}{d}\right) = \left(\frac{c}{d}\right)$, and, since k was chosen to be odd, $\left(\frac{c}{d}\right)^{-k} = \left(\frac{c}{d}\right)$, (2.14) holds.

For (2.15), we use the definition of $E_{\frac{1}{2}}$ and (2.14). Notice $\text{Im}(\frac{z}{2z+1}) = \frac{y}{|2z+1|^2}$. Direct computation gives

$$E_{\frac{1}{2}}(z,s) = (y/4)^s \sum_{(u,2v)=1,u>0} \frac{\left(\frac{-v}{u}\right) \epsilon_u^k}{|(u+2v)z+v|^{2s}((u+2v)z+v)^{k/2}}.$$

Since k is odd, it preserves the Jacobi symbol. We also have that $\left(\frac{2}{u}\right)^2 = 1$. So

$$\left(\frac{-v}{u}\right) = \left(\frac{-1}{u}\right)\left(\frac{2}{u}\right)\left(\frac{2v}{u}\right) = \left(\frac{-1}{u}\right)\left(\frac{2}{u}\right)\left(\frac{u+2v}{u}\right) = \left(\frac{-2(u+2v)}{v}\right).$$

Then, by (2.4), Lemma 2.4, and Lemma 2.5, we have

$$E_{\frac{1}{2}}(z,s) = (y/4)^s \sum_{(u,2v)=1,u>0} \frac{e^{\frac{\pi ik}{4}\operatorname{sgn}\left(\frac{-2(u+2v)}{u}\right)} G(u;8(u+2v))^k}{|(u+2v)z+v|^{2s}((u+2v)z+v)^{k/2}}.$$

Notice $(u,2v)=1$ if and only if $(u+2v,2v)=1$. We will again divide the sum into two parts. In the first we take only those terms in which $2v > -u$, so that these terms are exactly those terms in the sum where the sgn function yields a -1. We make the substitution $c = v$ and $d = u + 2v$. This gives us the sum

$$\sum_{(d,2c)=1,d>0,d>2c} \frac{e^{-\frac{\pi ik}{4}} G(d-2c;8d)^k}{|dz+c|^{2s}(dz+c)^{k/2}}.$$

In the second sum, we take those terms with $2v < -u$, so we have precisely those terms where the sgn function yields a 1. Here, we make the substitution $c = -v$ and $d = -(u+2v)$. This gives the sum

$$\sum_{(d,2c)=1,0<d<2c} \frac{e^{\frac{\pi ik}{4}} G(d-2c;8d)}{|-dz-c|^{2s}(-dz-c)^{k/2}}.$$

The -1 does not affect the modulus. Multiplying both top and bottom by $i^k = e^{\frac{\pi ik}{2}}$, we get

$$\sum_{(d,2c)=1,0<d<2c} \frac{e^{-\frac{\pi ik}{4}} G(d-2c;8d)}{|dz+c|^{2s}(dz+c)^{k/2}}.$$

Putting the two sums back together, we get (2.15). $\square$

2.4 Fourier Expansions for $E_\infty(z,s)$, $E_0(z,s)$ and $E_{\frac{1}{2}}(z,s)$

We are now ready to compute the Fourier expansions for the Eisenstein series.

Proposition 2.10. *We have*

$$E_0(z,s,k) = \sum_{m=-\infty}^{\infty} a_m(s,y,k)e^{2\pi imx}$$

where

$$a_m(s,y,k) = (y/4)^s \prod_{p\neq 2}\left[\sum_{l=0}^{\infty} \frac{(\epsilon_{p^l})^k g(-m,p^l)}{p^{l(2s+k/2)}}\right] K_m(s,y,k).$$

In the above

$$g(m,n) = \sum_{\substack{a(mod\ n)\\(a,n)=1}} \left(\frac{a}{n}\right) e^{\frac{2\pi iam}{n}}$$

$$K_m(s,y,k) = \int_{-\infty}^{\infty} \frac{e^{-2\pi imx}}{(x^2+y^2)^s(x+iy)^{k/2}}dx.$$

Proof. By definition, the Fourier coefficients for $E_0(z,s,k)$ are given by

$$a_m(s,y,k) = \int_0^1 E_0(z,s,k)e^{-2\pi imx}dx.$$

Using (2.14), we can interchange the order of the summation and the integration since the series converges uniformly. So we have

$$a_m(s,y,k) = (y/4)^s \sum_{\substack{(u,2v)=1\\u>0}} \left(\frac{-v}{u}\right) \epsilon_u^k \int_0^1 \frac{e^{-2\pi imx}}{|ux+v+iuy|^{2s}(ux+v+iuy)^{k/2}}dx.$$

We can factor out $u^{2s+k/2}$ and make the change of variables $x \to x+v/u$. This gives us

$$(y/4)^s \sum_{\substack{(u,2v)=1\\u>0}} u^{-(2s+k/2)}\epsilon_u^k \left(\frac{-v}{u}\right) e^{2\pi im\frac{v}{u}} \int_{\frac{v}{u}}^{1+\frac{v}{u}} \frac{e^{-2\pi imx}}{(x^2+y^2)^s(x+iy)^{k/2}}dx.$$

The sum is taken over those pairs of integers u and v where u is odd and positive and v is relatively prime to u. For a fixed u, as v ranges over those integers relatively prime to u, the product $\left(\frac{-v}{u}\right) e^{2\pi im\frac{v}{u}}$ depends only on the congruence class of v modulo u. Furthermore, as v varies over those integers relatively prime to u and congruent to a certain number a modulo u, the intervals of the form $[\frac{v}{u}, 1+\frac{v}{u})$ form a disjoint covering of the entire real line (we are essentially looking at all intervals of the form $[\frac{a+nu}{u}, 1+\frac{a+nu}{u})$ for $n \in \mathbb{Z}$ and for a fixed a with $(a, m) = 1$).

We now have

$$(y/4)^s \sum_{\text{odd } u>0} u^{-(2s+k/2)} \epsilon_u^k \left[\sum_{\substack{a \,(\mathrm{mod}\ u) \\ (a,u)=1}} \left(\frac{-a}{u}\right) e^{\frac{2\pi i a m}{u}} \right]$$
$$\times \int_{-\infty}^{\infty} \frac{e^{-2\pi i m x}}{(x^2+y^2)^s (x+iy)^{k/2}} dx.$$

It is easy to see that as a runs through a reduced residue system modulo u, so does $-a$. So we have

$$a_m(s, y, k) = (y/4)^s \sum_{\text{odd } u>0} \frac{\epsilon_u^k g(-m, u)}{u^{2s+k/2}} K_m(s, z, k). \tag{2.17}$$

We look at the function

$$\alpha(u) = \epsilon_u^k g(-m, u)$$

and notice that this function is multiplicative in the sense that $\alpha(u_1 u_2) = \alpha(u_1)\alpha(u_2)$ whenever u_1 and u_2 are two relatively prime odd positive integers. First, by quadratic reciprocity of the Jacobi symbol, we have

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{(p-1)(q-1)}{4}} = \begin{cases} -1, & \text{if } p \equiv q \equiv 3 \pmod 4 \\ 1, & \text{otherwise} \end{cases}$$

for positive odd integers p and q with $(p, q) = 1$. Also, it is easy to see that

$$\epsilon_p \epsilon_q = \begin{cases} 1, & \text{if } p \equiv q \equiv 1 \pmod 4 \\ -1, & \text{if } p \equiv q \equiv 3 \pmod 4 \\ i, & \text{otherwise} \end{cases}$$

for positive odd integers p and q.

This gives us the relation

$$\epsilon_{pq} = \left(\frac{p}{q}\right)\left(\frac{q}{p}\right)\epsilon_p\epsilon_q.$$

Now, if $(p,q)=1$, as a runs through a reduced residue system modulo p so does aq, and bp runs through a reduced residue system modulo q as b does. Now notice

$$\begin{aligned}\left(\frac{p}{q}\right)\left(\frac{q}{p}\right)\left(\frac{a}{p}\right)\left(\frac{b}{q}\right) &= \left(\frac{ap}{p}\right)\left(\frac{bp}{q}\right)\\ &= \left(\frac{bp+aq}{p}\right)\left(\frac{aq+bp}{q}\right)\\ &= \left(\frac{aq+bp}{pq}\right).\end{aligned}$$

So,

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right)g(m,p)g(m,q) = \sum_{\substack{a \,(\mathrm{mod}\ p)\\ (a,p)=1}}\sum_{\substack{b \,(\mathrm{mod}\ q)\\ (b,q)=1}}\left(\frac{ap+bp}{pq}\right)e^{2\pi i m\left(\frac{aq+bp}{pq}\right)}.$$

Now we simply notice that as a and b range over reduced residue systems modulo p and q, respectively, $aq+bp$ ranges over a reduced residue system modulo pq. And this double sum is then equal to $g(m,pq)$.

Using this together with the above relation for ϵ_{pq}, we get the desired multiplicative property for $\alpha(u)$. This allows us to change the sum in (2.17) into a product over all odd primes of the associated sum over all powers of that prime. The proposition follows. $\square$

Corollary 2.11. *We have*

$$a_0(s,y,k) = \frac{y^s}{4^s}\frac{\zeta(4s+k-2)}{\zeta(4s+k-1)}\frac{(1-2^{-4s-k+2})}{(1-2^{-4s-k+1})}K_0(s,y,k)$$

and for m not divisible by an odd square

$$a_m(s,y,k) = \frac{y^s}{4^s}\frac{L\left(2s+\frac{k-1}{2},\chi_m\right)}{\zeta(4s+k-1)}\frac{(1-\chi_m(2)2^{-2s-\frac{k-1}{2}})}{(1-2^{-4s-k+1})}K_m(s,y,k)$$

where χ_m is the real Dirichlet character associated to the quadratic field $\mathbb{Q}(\sqrt{\mu_k m})$, where $\mu_k = (-1)^{(k-1)/2}$.

Proof. The first result is almost immediate. For $l > 0$ odd, $g(0, p^l) = 0$. For l even, p^l is an odd square, and as such is congruent to 1 modulo 4. Thus $\epsilon_{p^l} = 1$. Furthermore, $\left(\frac{a}{p^l}\right) = \left(\frac{a}{p^{l/2}}\right)^2 = 1$, so $g(0, p^l) = \phi(p^l) = p^l - p^{l-1}$ (here ϕ is Euler's function). Thus the series inside the product becomes

$$1 + \left(1 - \frac{1}{p}\right) \sum_{l=1}^{\infty} p^{l(-4s-k+2)} = 1 + \left(1 - \frac{1}{p}\right) \frac{p^{-4s-k+2}}{1 - p^{-4s-k+2}} = \frac{1 - p^{-4s-k+1}}{1 - p^{-4s-k+2}}.$$

At this point we recall that the Euler products for $\zeta(s)$ and $\frac{1}{\zeta(s)}$ are, for Re$(s)> 1$,

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$

$$\frac{1}{\zeta(s)} = \prod_p (1 - p^{-s}).$$

The result follows.

For the second identity, we will proceed in a similar fashion. We will consider the sum

$$\sum_{l=0}^{\infty} \frac{(\epsilon_{p^l})^k g(-m, p^l)}{p^{l(2s+k/2)}}. \tag{2.18}$$

If p does not divide m, then, for the terms in which l is a positive even number, $\epsilon_{p^l} = 1$ and $\left(\frac{a}{p^l}\right) = \left(\frac{a}{p^{l/2}}\right)^2 = 1$ and $g(-m, p^l)$ reduces to Ramanujan's sum

$$c_{p^l}(n) = \sum_{\substack{a \bmod p^l \\ (a,p^l)=1}} e^{2\pi i a n / p^l}.$$

Elementary results about these sums give us $c_k(n) = \sum_{d|(n,k)} d\mu(k/d)$ (see p.

161 of Apostol [2]), so for $k = p^l$, we have

$$c_{p^l}(n) = \begin{cases} p^l - p^{l-1}, & \text{if } (n, p^l) = p^l \\ -p^{l-1}, & \text{if } (n, p^l) = p^{l-1} \\ 0, & \text{otherwise.} \end{cases} \tag{2.19}$$

So, for $l > 0$ even, $g(-m, p^l) = c_{p^l}(-m) = 0$ since p does not divide m.

The $l = 0$ term in the series is 1, and we now know that the terms corresponding to all other even values for l are 0.

If l is a positive odd integer, p^l is congruent to p modulo 4, so $\epsilon_{p^l} = \epsilon_p$. Furthermore, g is a Gauss sum, and as such is separable for n relatively prime to p^l, so $g(-m, p^l) = \left(\frac{-m}{p^l}\right) g(1, p^l)$.

Consider $g(1, p^l)^2$. Notice

$$g(1, p^l)^2 = \sum_{\substack{a \bmod p^l \\ (a,p^l)=1}} \sum_{\substack{b \bmod p^l \\ (b,p^l)=1}} \left(\frac{a}{p^l}\right) \left(\frac{b}{p^l}\right) e^{\frac{2\pi i(a+b)}{p^l}}.$$

For a fixed a, as b runs through a reduced residue system modulo p^l, so does ab. Thus there a unique (mod p^l) number t, also relatively prime to p^l such that $ta \equiv b \bmod p^l$. This gives $\left(\frac{a}{p^l}\right)\left(\frac{b}{p^l}\right) = \left(\frac{a}{p^l}\right)\left(\frac{ta}{p^l}\right) = \left(\frac{a^2}{p^l}\right)\left(\frac{t}{p^l}\right) = \left(\frac{t}{p^l}\right)$. So we have

$$g(1, p^l)^2 = \sum_{\substack{t \bmod p^l \\ (t,p^l)=1}} \sum_{\substack{a \bmod p^l \\ (a,p^l)=1}} \left(\frac{t}{p^l}\right) e^{\frac{2\pi i a(1+t)}{p^l}} = \sum_{\substack{t \bmod p^l \\ (t,p^l)=1}} \left(\frac{t}{p^l}\right) \sum_{\substack{a \bmod p^l \\ (a,p^l)=1}} e^{\frac{2\pi i a(1+t)}{p^l}}.$$

The last sum is again Ramanujan's sum $c_{p^l}(1 + t)$. As t runs through a reduced residue system modulo p^l, if $l > 1$, there will be exactly p values of t for which $p^{l-1}|(t + 1)$. These values can be represented modulo p^l by the integers $np^{l-1} - 1$ for $1 \le n \le p$, and exactly one of these will correspond to a t for which $p^l|(t+1)$, namely when $n = p$. Thus, by (2.19), we have

$$\begin{aligned} g(1,p^l)^2 &= \sum_{\substack{t \bmod p^l \\ (t,p^l)=1}} \left(\frac{t}{p^l}\right) c_{p^l}(t+1) \\ &= -p^{l-1}\sum_{n=1}^{p-1}\left(\frac{np^{l-1}-1}{p^l}\right) + \left(\frac{p^l-1}{p^l}\right)(p^l - p^{l-1}) \\ &= -p^{l-1}\sum_{n=1}^{p}\left(\frac{np^{l-1}-1}{p^l}\right) + p^l\left(\frac{p^l-1}{p^l}\right). \end{aligned}$$

Since l is odd, $\left(\frac{np^{l-1}-1}{p^l}\right) = \left(\frac{np^{l-1}-1}{p}\right) = \left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$. Now we have

$$g(1,p^l)^2 = -p^{l-1}(p(-1)^{\frac{p-1}{2}}) + p^l(-1)^{\frac{p-1}{2}} = 0.$$

If $l = 1$, by (2.19) $c_p(t+1)$ is -1 if p does not divide $(1+t)$, and is $p-1$ if p does divide $(1+t)$. We have

$$g(1,p)^2 = -\sum_{t=1}^{p-2}\left(\frac{t}{p}\right) + (p-1)\left(\frac{p-1}{p}\right) = -\sum_{t=1}^{p-1}\left(\frac{t}{p}\right) + p\left(\frac{-1}{p}\right).$$

Notice that the sum on the right hand side is 0. Thus $g(1,p)^2 = p\left(\frac{-1}{p}\right)$. So that

$$g(1,p) = \begin{cases} \sqrt{p}, & \text{if } p \equiv 1 \bmod 4; \\ i\sqrt{p}, & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

We now have that, in the case where p does not divide m, all of the terms of (2.18) are 0 except for $l = 0$ (which is 1) and when $l = 1$. Thus (2.18) reduces to

$$\begin{cases} 1 + \frac{\left(\frac{-m}{p}\right)\sqrt{p}}{p^{2s+k/2}}, & \text{if } p \equiv 1 \bmod 4; \\ 1 + \frac{i^{k+1}\left(\frac{-m}{p}\right)\sqrt{p}}{p^{2s+k/2}}, & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

If k is congruent to 1 modulo 4, $\chi_m(n) = \left(\frac{m}{n}\right)$, and, recalling that $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$, both cases reduce to

$$1 + \frac{\chi_m(p)\sqrt{p}}{p^{2s+k/2}}.$$

If k is congruent to 3 modulo 4, then $\chi_m(n) = \left(\frac{-m}{n}\right)$, and again, both cases reduce to $1 + \frac{\chi_m(p)\sqrt{p}}{p^{2s+k/2}}$.

Simple algebra can reduce this term even further so that (2.18) becomes

$$\frac{1 - p^{-2s-k+1}}{1 - \chi_m(p)p^{-2s-\left(\frac{k-1}{2}\right)}}.$$

Now consider the case where p divides m. Here $\chi_m(p) = 0$.

Again, the $l = 0$ term is 1. For l a positive even integer, $g(-m, p^l)$ again reduces to $c_{p^l}(-m)$, but this now evaluates to 0 unless $l = 2$, in which case, it is $-p$. Hence, the $l = 2$ term in (2.18) is $-p^{-4s-k+1}$, and all other even terms reduce to 0.

If $l = 1$, $e^{\frac{-2\pi iam}{p}} = 1$ for each a, since $p|m$. Therefore,

$$g(-m, p) = \sum_{\substack{a \bmod p \\ (a,p)=1}} \left(\frac{a}{p}\right) = 0.$$

So the $l = 1$ term is 0.

If l is a positive odd integer greater than 1, we do a computation very similar to that involved in computing $g(1, p^l)^2$ (essentially leaving in a multiple of $-m$ in the exponential), we see that

$$g(-m, p^l)^2 = \sum_{\substack{t \bmod p^l \\ (t,p^l)=1}} \left(\frac{t}{p^l}\right) \sum_{\substack{a \bmod p^l \\ (a,p^l)=1}} e^{\frac{-2\pi ia(1+t)m}{p^l}},$$

and the second sum again reduces to $c_{p^l}(-(1+t)m)$. Computing this we have, for $l > 1$,

$$c_{p^l}(-(1+t)m) = \begin{cases} p^l - p^{l-1}, & \text{if } p^{l-1}|(t+1); \\ -p^{l-1}, & \text{if } (t+1, p^l) = p^{l-2}; \\ 0, & \text{otherwise.} \end{cases}$$

Again, for $l \geq 3$, as t runs through a reduced residue system modulo p^l, there will be exactly p^2 values, corresponding to the integers $np^{l-2} - 1$, for $1 \leq n \leq p^2$, which are divisible by p^{l-2}, and each of these corresponding terms in the sums will be multiplied by a factor of $-p^{l-1}$. Furthermore, there will be exactly p values of t, corresponding to the integers $rp^{l-1} - 1$, for $1 \leq r \leq p$, which are divisible by p^{l-1} as well. These will be multiplied by an additional factor of p^l. Hence

$$\begin{aligned} g(-m, p^l)^2 &= \sum_{\substack{t \bmod p^l \\ (t,p^l)=1}} \left(\frac{t}{p^l}\right) c_{p^l}(-(1+t)m) \\ &= -p^{l-1} \sum_{n=1}^{p^2} \left(\frac{np^{l-2}-1}{p^l}\right) + p^l \sum_{r=1}^{p} \left(\frac{rp^{l-1}-1}{p^l}\right). \end{aligned}$$

Again, all of the Legendre symbols reduce to $\left(\frac{-1}{p}\right)$ and we get

$$g(-m, p^l)^2 = -p^{l-1}(p^2)\left(\frac{-1}{p}\right) + p^l(p)\left(\frac{-1}{p}\right) = 0.$$

Thus, all of the odd terms in (2.18) reduce to 0.

So in the case where p divides m, (2.18) also becomes

$$1 - p^{-4s-k+1} = \frac{1 - p^{-4s-k+1}}{1 - \chi_m(p)p^{-2s-\left(\frac{k-1}{2}\right)}}$$

Now we recall that the Euler product for $L(s, \chi)$ is

$$L(s, \chi) = \prod_p \frac{1}{1 - \chi(p)p^{-s}},$$

which is valid for $\mathrm{Re}(s) > 1$. The proposition now follows using this combined with the Euler product of $\frac{1}{\zeta(s)}$. $\square$

Proposition 2.12. *We have*

$$E_\infty(z, s) = \sum_{m=-\infty}^{\infty} b_m(s, y, k) e^{2\pi i m x}$$

where

$$b_0(s,y,k) = y^s + e^{\frac{\pi ik}{4}} 4^s c_0(s,k) a_0(s,y,k)$$

$$c_0(s,k) = \psi_8(k) 2^{-4s-k+\frac{1}{2}} (1 - 2^{-4s-k+2})^{-1}$$

and for $m \neq 0$

$$b_m(s,y,k) = (1+i^k) 4^s c_m(s,k) a_m(s,y,k)$$

where for $m \not\equiv 0\,(mod\ 4)$

$$c_m(s,k) = \begin{cases} -2^{-4s-k}, & (-1)^{(k-1)/2} m \not\equiv 1\,(mod\ 4); \\ 2^{-4s-k} + \psi_8(m) 2^{-6s-(3k/2)+(3/2)}, & (-1)^{(k-1)/2} \equiv 1\,(mod\ 4) \end{cases}$$

and for $m = 4^t m_0$, $m_0 \not\equiv 0\,(mod\ 4)$

$$c_m(s,k) = 2^{-4s-k} \frac{(1 - 2^{-t(4s+k-2)})}{(1 - 2^{-(4s+k-2)})} + 2^{-t(4s+k-2)} c_{m_0}(s,k).$$

Here, ψ_0 *denotes the real primitive Dirichlet character (mod 8).*

Proposition 2.13. *We have*

$$E_{\frac{1}{2}}(4z,s,k) = \sum_{m=-\infty}^{\infty} d_m(s,y,k) e^{2\pi im(x+\frac{1}{8})}$$

where

$$d_m(s,y,k) = \begin{cases} (1-i^k) a_m(s,y,k) 2^{-2s-k+\frac{3}{2}}, & m(-1)^{(k-1)/2} \equiv 1\,(mod\ 8); \\ -(1-i^k) a_m(s,y,k) 2^{-2s-k+\frac{3}{2}}, & m(-1)^{(k-1)/2} \equiv 5\,(mod\ 8); \\ 0, & otherwise. \end{cases}$$

2.5 Mellin Transforms

Now we will look at the transformation laws of these series at the cusps of $\Gamma_0(4)$, so that we can get functional equations for their Mellin transforms.

Proposition 2.14. *Let* $\alpha_{u,r} = \begin{pmatrix} 1 & u/r \\ 0 & 0 \end{pmatrix}$, $\tau_r = \begin{pmatrix} 0 & -1 \\ 4r^2 & 0 \end{pmatrix}$ *where* $r \in \mathbb{Z}^+$ *and* $(u,r) = 1$. *Then for* $r \equiv 1 \,(mod\ 2)$ *and* a *chosen so that* $-4ua \equiv 1 \,(mod\ r)$ *we have*

$$E_0(\alpha_{u,r}\tau_r z, s) = (4rz)^{k/2}(i\epsilon_r)^{-k}\left(\frac{a}{r}\right)E_\infty(\alpha_{a,r}z, s)$$

and

$$E_\infty(\alpha_{u,r}\tau_r z, s) = (rz)^{k/2}\epsilon_r^{-k}\left(\frac{a}{r}\right)E_0(\alpha_{a,r}z, s).$$

Proof. Notice $\alpha_{u,r}\tau_r z = \gamma\left(\frac{-1}{4\alpha_{a,r}z}\right)$ where $\gamma = \begin{pmatrix} \frac{4ua+1}{r} & u \\ 4a & r \end{pmatrix}$ which is an element of $\Gamma_0(4)$, since a was chosen so that $r|(4ua+1)$. (Again, this can be found by solving the equation $\alpha_{u,r}\tau_r = \gamma\begin{pmatrix} 0 & -1/2 \\ 2 & 0 \end{pmatrix}\alpha_{a,r}$ and cancelling common factors in the numerator and denominator of the associated linear fractional transformation.)

Using the properties in Lemma 2.8, we have

$$\begin{aligned} E_0(\alpha_{u,r}\tau_r z, s) &= E_0\left(\gamma\left(\frac{-1}{4\alpha_{a,r}z}\right), s\right) \\ &= j_\gamma\left(\frac{-1}{4\alpha_{a,r}z}\right)^k E_0\left(\frac{-1}{4\alpha_{a,r}z}, s\right) \\ &= j_\gamma\left(\frac{-1}{4\alpha_{a,r}z}\right)^k (4\alpha_{a,r}z)^{k/2} i^{-k} E_\infty(\alpha_{a,r}z, s) \end{aligned}$$

Since r is positive, we get

$$\begin{aligned} &= \left(\frac{4a}{r}\right)^k \epsilon_r^{-k}\left(4a\left(\frac{-1}{4\alpha_{a,r}z}\right) + r\right)^{k/2} (4\alpha_{a,r}z)^{k/2} i^{-k} E_\infty(\alpha_{a,r}z, s) \\ &= \left(\frac{a}{r}\right)(i\epsilon_r)^{-k}(4rz)^{k/2} E_\infty(\alpha_{a,r}z, s) \end{aligned}$$

which is the first transformation law (recall k is odd and that $\alpha_{a,r}z = z + \frac{a}{r}$).

For the second relation, we proceed in the same fashion, using the automorphic property of E_∞ in Lemma 2.8 and the definition of E_0. $\square$

Proposition 2.15. *Let τ_r and $\alpha_{u,r}$ be as in Proposition 2.14. For a chosen so that $au \equiv -1\,(mod\ 4r)$*

$$E_0(\alpha_{u,4r}\tau_{2r}z, s) = \left(\frac{4rz}{i}\right)^{k/2} (G(a;4r;))^{-k}\, E_0(\alpha_{a,4r}z, s).$$

Proof. Again, we notice $\alpha_{u,4r}\tau_{2r}z = \gamma\alpha_{a,4r}z$ where $\gamma = \begin{pmatrix} u & -\frac{ua+1}{4r} \\ 4r & -a \end{pmatrix}$ which is an element of $\Gamma_0(4)$. This gives us

$$E_0(\alpha_{u,4r}\tau_{2r}z, s) = E_0(\gamma\alpha_{a,4r}z, s) = j_\gamma(\alpha_{a,4r}z)^k E_0(\alpha_{a,4r}z, s).$$

Now, if $a < 0$, then $-a > 0$, and this becomes

$$E_0(\alpha_{u,4r}\tau_{2r}z, s) = \left(\frac{4r}{-a}\right)^k \epsilon_{-a}^{-k}(4r\alpha_{a,4r}z - a)^{k/2} E_0(\alpha_{a,4r}z, s).$$

By (2.4), $\epsilon_{-a} = G(1;-a)$. Also, $\left(\frac{4r}{-a}\right)^k = \left(\frac{r}{-a}\right)^k = \left(\frac{r}{-a}\right)^{-k}$, so by Lemma 2.4, we have

$$E_0(\alpha_{u,4r}\tau_{2r}z, s) = G(r;-a)^{-k}(4r\alpha_{a,4r}z - a)^{k/2} E_0(\alpha_{a,4r}z, s)$$

Using Lemma 2.5, where $\frac{r}{-a}$ is positive, and noting that $4r\alpha_{a,4r}z - a = 4rz$, we have

$$E_0(\alpha_{u,4r}\tau_{2r}z, s) = e^{-\frac{\pi ik}{4}} G(a;4r)^{-k}(4rz)^{k/2} E_0(\alpha_{a,4r}z, s).$$

Since $e^{-\frac{\pi ik}{4}} = i^{-k/2}$, we are done.

If $a < 0$, then we compute $j_{-\gamma}(\alpha_{a,4r}z)$ following exactly the same steps, but now $\frac{-r}{a}$ is negative. We get

$$E_0(\alpha_{u,4r}\tau_{2r}z, s) = e^{\frac{\pi ik}{4}} G(-a;-4r)^{-k}(-4rz)^{k/2} E_0(\alpha_{a,4r}z, s).$$

Recall that $G(-a;-4r) = G(a;4r)$. We point out that $e^{\frac{\pi ik}{4}}(-1)^{k/2} = (-i)^{k/2} = i^{-k/2}$. The relation then follows. ∎

Proposition 2.16. *Let τ_r, $\alpha_{u,r}$ be as in Proposition 2.14. Then for $r \equiv 1\,(mod\ 2)$ and a chosen so that $au \equiv -1\,(mod\ 2r)$ we have*

$$E_0(\alpha_{u,2r}\tau_r z, s) = \left(\frac{2rz}{i}\right)^{k/2} (G(-a;8r))^k\, E_{\frac{1}{2}}(\alpha_{a,2r}z, s)$$

and

$$E_{\frac{1}{2}}(\alpha_{a,2r}\tau_r z, s) = \left(\frac{2rz}{i}\right)^{k/2} (G(-a;8r))^{-k}\, E_0(\alpha_{u,2r}z, s).$$

Proof. We notice that

$$\alpha_{u,2r}\tau_r z = \gamma\frac{\alpha_{a,2r}z}{2\alpha_{a,2r}z+1},$$

where

$$\gamma = \begin{pmatrix} u+\frac{ua+1}{r} & -\frac{ua+1}{2r} \\ 2r+2a & -a \end{pmatrix} \in \Gamma_0(4)$$

since $2r|(ua+1)$ and both a and r are odd (so that $4|(2(r+a))$). Using the automorphic properties of E_0 in Lemma 2.8 and the definition of $E_{\frac{1}{2}}$, we have

$$\begin{aligned} E_0(\alpha_{u,2r}\tau_r z, s) &= E_0\left(\gamma\frac{\alpha_{a,2r}z}{2\alpha_{a,2r}z+1}, s\right) \\ &= j_\gamma\left(\frac{\alpha_{a,2r}z}{2\alpha_{a,2r}z+1}\right)^k E_0\left(\frac{\alpha_{a,2r}z}{2\alpha_{a,2r}z+1}, s\right) \\ &= j_\gamma\left(\frac{\alpha_{a,2r}z}{2\alpha_{a,2r}z+1}\right)^k (2\alpha_{a,2r}z+1)^{k/2} E_{\frac{1}{2}}(\alpha_{a,2r}z, s). \end{aligned}$$

Now, if $a<0$, so that $-a>0$, we have

$$j_\gamma\left(\frac{\alpha_{a,2r}z}{2\alpha_{a,2r}z+1}\right)^k = \left(\frac{2r+2a}{-a}\right)^k \epsilon_{-a}^{-k}\left((2r+2a)\frac{\alpha_{a,2r}z}{2\alpha_{a,2r}z+1}-a\right)^{k/2}.$$

A simple computation gives that

$$\left((2r+2a)\frac{\alpha_{a,2r}z}{2\alpha_{a,2r}z+1}-a\right)^{k/2}(2\alpha_{a,2r}z+1)^{k/2} = (2rz)^{k/2}.$$

Notice

$$\epsilon_{-a}^{-k} = \left(\frac{-1}{-a}\right)^k \epsilon_{-a}^k = \left(\frac{-1}{-a}\right)^k G(1;-a)^k$$

and $\left(\frac{2r+2a}{-a}\right)^k = \left(\frac{2r}{-a}\right)^k$, so that

$$\begin{aligned}
\left(\frac{2r+2a}{-a}\right)^k \epsilon_{-a}^{-k} &= \left(\frac{-2r}{-a}\right)^k G(1;-a)^k \\
&= G(-2r;-a)^k \\
&= G(2r;a)^k \\
&= e^{-\frac{\pi i k}{4}} G(-a;8r)^k.
\end{aligned}$$

Recalling that $e^{-\frac{\pi i k}{4}} = i^{-k/2}$, the first identity follows.

If $a > 0$, so that $-a < 0$, we have

$$\begin{aligned}
j_\gamma\left(\frac{\alpha_{a,2r}z}{2\alpha_{a,2r}z+1}\right) &= j_{-\gamma}\left(\frac{\alpha_{a,2r}z}{2\alpha_{a,2r}z+1}\right) \\
&= \left(\frac{-2r-2a}{a}\right)\epsilon_a^{-1}\left((-2r-2a)\frac{\alpha_{a,2r}z}{2\alpha_{a,2r}z+1}+a\right)^{1/2}.
\end{aligned}$$

As one might guess, the calculations are very similar, except now we have

$$\left((-2r-2a)\frac{\alpha_{a,2r}z}{2\alpha_{a,2r}z+1}+a\right)^{k/2}(2\alpha_{a,2r}z+1)^{k/2} = (-2rz)^{k/2}$$

and

$$\left(\frac{-2r-2a}{a}\right)^k \epsilon_a^{-k} = i^{k/2}G(-a;8r)^k$$

and the identity still holds since $-i = i^{-1}$.

For the second identity, notice $\tau_r\tau_r z = z$. Using this, the first identity reduces quite readily to the second. $\square$

We now introduce the Mellin Transforms of the Eisenstein series on $\Gamma_0(4)$. Let

$$\alpha_{a,r} = \begin{pmatrix} 1 & a/r \\ 0 & 1 \end{pmatrix}$$

as before. Then we define

$$\Phi_\infty(w, a/r; s, k) = \int_0^\infty (E_\infty(\alpha_{a,r}iy, s, k) - b_0(s, y, k))y^{w-1}dy$$

$$\Phi_0(w,a/r;s,k)=\int_0^\infty (E_0(\alpha_{a,r}iy,s,k)-a_0(s,y,k))y^{w-1}dy$$

$$\Phi_{\frac{1}{2}}(w,a/r;s,k)=\int_0^\infty E_{\frac{1}{2}}(\alpha_{a,r}iy,s,k)y^{w-1}dy$$

Since we are subtracting the constant term from the Eisenstein series it is clear that the above integrals converge absolutely for $\mathrm{Re}(w)$ sufficiently large.

The following propositions give the functional equations and locate the poles of the Mellin transforms of our Eisenstein series. Again, the reader is referred to [5] and [14] for the details of the proofs.

Proposition 2.17. *Let $r\equiv 1\,(mod\ 2)$ and let a, u be chosen so that $-4au\equiv 1\,(mod\ r)$. Then*

$$\Phi_0\left(w,s;\frac{u}{r},k\right)=\Lambda_\infty\left(w,\frac{a}{r},k\right)\Phi_\infty\left(\frac{k}{2}-w,s;\frac{a}{r},k\right)$$

where

$$\Lambda_\infty\left(w,\frac{a}{r},k\right)=(2r)^{(k/2)-2w}(-2i)^{k/2}\epsilon_r^{-k}\left(\frac{a}{r}\right).$$

Moreover $\Phi_0\left(w,s;\frac{u}{r},k\right)$ is holomorphic except for simple poles at $w=s+\frac{k}{2}$, $1-s,\ s+\frac{k}{2}-1$, with corresponding residues equal to

$$\Lambda_\infty\left(s+\frac{k}{2},\frac{a}{r},k\right),$$

$$c_1(s,k)\Lambda_\infty\left(1-s,\frac{a}{r},k\right),$$

$$-c_1(s,k)\psi_8(k)(i)^{-k/2}2^{2s+k-\frac{1}{2}}(1-2^{-4s-k+2})$$

where

$$c_1(s,k)=\frac{\psi_8(k)i^{k/2}\zeta(4s+k-2)K_0(s,1,k)}{2^{4s+k-\frac{1}{2}}\zeta(4s+k-1)(1-2^{-4s-k+1})}.$$

Proposition 2.18. *Let a, u be chosen so that $au \equiv -1 \,(mod\, 4r)$. Then*

$$\Phi_0\left(w, s; \frac{u}{4r}, k\right) = \Lambda_0\left(w, \frac{a}{4r}, k\right)\Phi_0\left(\frac{k}{2} - w, s; \frac{a}{4r}, k\right)$$

where

$$\Lambda_0\left(w, \frac{a}{4r}, k\right) = 2^{k-4w} r^{(k/2)-2w} G(a; 4r)^{-k}.$$

Moreover, $\Phi_0\left(w, \frac{u}{4r}; s, k\right)$ is holomorphic except for simple poles at $w = 1 - s,\ s + \frac{k}{2} - 1$. The residue at $w = 1 - s$ is $\Lambda_0\left(1 - s, \frac{a}{4r}, k\right) c_2(s,k)\zeta(4s + k - 1)^{-1}$ where $c_2(s,k)$ is given by

$$c_2(s,k) = \psi_8(k) i^{-k/2} 2^{2s+k-\frac{1}{2}} \zeta(4s + k - 1)(1 - 2^{-4s-k+2}) c_1(s,k).$$

Proposition 2.19. *Let r be odd and $au \equiv -1 \,(mod\, 2r)$. Then*

$$\Phi_0\left(w, s; \frac{u}{2r}, k\right) = \Lambda_{\frac{1}{2}}\left(w, \frac{a}{2r}, k\right)\Phi_{\frac{1}{2}}\left(\frac{k}{2} - w, s; \frac{a}{2r}, k\right)$$

where

$$\Lambda_{\frac{1}{2}}\left(w, \frac{a}{2r}, k\right) = 2^{(k/2)-2w} r^{(k/2)-2w} G(-a; 8r)^k.$$

Moreover, $\Phi_0\left(w, s; \frac{u}{2r}, k\right)$ is holomorphic except for a simple pole at

$$w = s + \frac{k}{2} - 1.$$

3 Asymptotics for $L(\rho, \chi_m)$

Corollary 2.10 suggests how these Eisenstein Series are related to the Dirichlet L-series for χ_m.

3.1 Whittaker Functions

We turn our attention now to the Whittaker functions $K_m(s, y, k)$ which occur in the Fourier expansions of our Eisenstein series of $\frac{1}{2}$-integral weight. They are essentially hypergeometric functions.

The following argument follows the proof of Lemma 1 in [14]. It is convenient at this time to define v^α for all complex numbers α and $v \neq 0$. We take

$$v^\alpha = e^{\alpha \log(v)}$$

where

$$-\pi < \mathrm{Im}(\log(v)) \leq \pi.$$

This gives us the convention $v^{\alpha+\beta} = v^\alpha v^\beta$, $v^{m\alpha} = (v^\alpha)^m$ for all $m \in \mathbb{Z}$, and $(uv)^\alpha = u^\alpha v^\alpha$ if $\arg(u)$, $\arg(v)$, and $\arg(u+v)$ are all contained in the interval $(-\pi, \pi]$.

We have

$$K_m(s,y,k) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i m x}}{(x^2+y^2)^s (x+iy)^{k/2}} dx.$$

Letting $z = x+iy$, this can be written another way:

$$K_m(s,y,k) = \int_{-\infty}^{\infty} z^{-(s+\frac{k}{2})} \bar{z}^{-s} e^{-2\pi i m x} dx.$$

Put $v = i\bar{z} = y + ix$. Then we can write

$$\begin{aligned}
K_m(s,y,k) &= -ie^{-\frac{\pi i k}{4}} e^{2\pi m y} \int_{y-i\infty}^{y+i\infty} v^{-s}(2y-v)^{-s-\frac{k}{2}} e^{-2\pi m v} dv \\
&= -ie^{-\frac{\pi i k}{4}} e^{2\pi m y} \Gamma\left(s+\frac{k}{2}\right)^{-1} \\
&\qquad \times \int_{y-i\infty}^{y+i\infty} v^{-s}(2y-v)^{-s-\frac{k}{2}} e^{-2\pi m v} \left\{ \int_0^\infty e^{-t} t^{s+\frac{k}{2}-1} dt \right\} dv \\
&= -ie^{-\frac{\pi i k}{4}} e^{2\pi m y} \Gamma\left(s+\frac{k}{2}\right)^{-1} \\
&\qquad \times \int_{y-i\infty}^{y+i\infty} v^{-s} \left\{ \int_0^\infty e^{-t(2y-v)} t^{s+\frac{k}{2}-1} dt \right\} dv \\
&= -ie^{-\frac{\pi i k}{4}} e^{2\pi m y} \Gamma\left(s+\frac{k}{2}\right)^{-1} \\
&\qquad \times \int_0^\infty t^{s+\frac{k}{2}-1} e^{-2yt} \left\{ \int_{y-i\infty}^{y+i\infty} v^{-s} e^{(t-2\pi m)v} dv \right\} dt.
\end{aligned}$$

We make use of the following identity:

$$\int_{y-i\infty}^{y+i\infty} v^{-\beta} e^{\lambda v} dv = \begin{cases} 2\pi i \Gamma(\beta)^{-1} \lambda^{\beta-1}, & \lambda > 0; \\ 0, & \lambda \leq 0. \end{cases}$$

This holds whenever $\mathrm{Re}(\beta) > 0$ and $y > 0$, for any real number λ.

So we have, taking $t = 2\pi p$,

$$\begin{aligned} K_m(s, y, k) = -ie^{-\frac{\pi ik}{4}} e^{2\pi my} \Gamma\left(s + \frac{k}{2}\right)^{-1} (2\pi)^{s+\frac{k}{2}} \\ \times \int_0^{\infty} p^{s+\frac{k}{2}-1} e^{-4\pi yp} \left\{ \int_{y-i\infty}^{y+i\infty} v^{-s} e^{2\pi(p-m)v} dv \right\} dp \\ = e^{-\frac{\pi ik}{4}} e^{2\pi my} \Gamma\left(s + \frac{k}{2}\right)^{-1} \Gamma(s)^{-1} (2\pi)^{2s+\frac{k}{2}} \\ \times \int_{\max(0,m)}^{\infty} p^{s+\frac{k}{2}-1} (p-m)^{s-1} e^{-4\pi yp} dp \end{aligned}$$

We examine the integral $\int_{\max(0,m)}^{\infty} p^{s+\frac{k}{2}-1} (p-m)^{s-1} e^{-4\pi yp} dp$.

If $m > 0$, we let $(p - m) = mq$. The integral becomes

$$m^{2s+\frac{k}{2}-1} e^{-4\pi my} \int_0^{\infty} (q+1)^{s+\frac{k}{2}-1} (q)^{s-1} e^{-4\pi myq} dq.$$

If $m < 0$, we let $p - m = -mq$. The integral then becomes

$$|m|^{2s+\frac{k}{2}-1} \int_1^{\infty} (q-1)^{s+\frac{k}{2}-1} q^{s-1} e^{-4\pi|m|(q-1)} dq.$$

We make another change of variables, letting $u = q - 1$, and get

$$|m|^{2s+\frac{k}{2}-1} \int_0^{\infty} u^{s+\frac{k}{2}-1} (u+1)^{s-1} e^{-4\pi|m|yu} du.$$

If $m = 0$, the integral reduces to

$$\int_0^{\infty} p^{2s+\frac{k}{2}-2} e^{-4\pi yp} dp = (4\pi y)^{1-2s-\frac{k}{2}} \Gamma\left(2s + \frac{k}{2} - 1\right).$$

In summary, we have

$$
\begin{aligned}
K_m(s,y,k) =& e^{\frac{-\pi i k}{4}}\Gamma\left(s+\frac{k}{2}\right)^{-1}\Gamma(s)^{-1}(2\pi)^{2s+\frac{k}{2}}e^{2\pi m y} \\
&* \begin{cases} m^{2s+\frac{k}{2}-1}e^{-4\pi m y}\int_0^\infty (q+1)^{s+\frac{k}{2}-1}(q)^{s-1}e^{-4\pi m y q}dq & m>0 \\ |m|^{2s+\frac{k}{2}-1}\int_0^\infty u^{s+\frac{k}{2}-1}(u+1)^{s-1}e^{-4\pi|m|yu}du & m<0 \\ (4\pi y)^{1-2s-\frac{k}{2}}\Gamma(2s+\frac{k}{2}-1) & m=0. \end{cases}
\end{aligned}
\tag{3.1}
$$

Proposition 3.1. *For $Re(w) > Re(s) + \frac{k}{2} - 1$, $m \neq 0$*

$$\int_0^\infty K_m(s,y,k)y^{w+s-1}dy = \frac{e^{-\frac{\pi i k}{4}}F_\pm(w,s,k)}{2^{2w-k/2}\pi^{w-s-k/2}|m|^{w+1-s-k/2}}$$

where

$$
\begin{aligned}
F_\pm(w,s,k) =& \Gamma\left(w-s+1-\frac{k}{2}\right)\Gamma(w+s) \\
&* \begin{cases} \frac{F\left(w+s,w-s+1-\frac{k}{2},w+1-\frac{k}{2};\frac{1}{2}\right)}{\Gamma\left(s+\frac{k}{2}\right)\Gamma\left(w+1-\frac{k}{2}\right)} & m>0(+) \\ \frac{F\left(w+s,w-s+1-\frac{k}{2},w+1;\frac{1}{2}\right)}{\Gamma(s)\Gamma(w+1)} & m<0(-) \end{cases}
\end{aligned}
$$

and

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^\infty \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!}z^n$$

is the Gaussian hypergeometric function.

Proof. Using (3.1) when we take the Mellin Transform for $m > 0$, and making the change of variable $4\pi m y \to y$, we are left with

$$
\begin{aligned}
\int_0^\infty K_m(s,y,k)y^{w+s-1}dy =& \frac{e^{-\frac{\pi i k}{4}}}{\Gamma\left(s+\frac{k}{2}\right)\Gamma(s)2^{2w-k/2}\pi^{w-s-k/2}m^{w+1-s-k/2}} \\
&\times \int_0^\infty e^{-\frac{y}{2}}y^{w+s-1}\left\{\int_0^\infty (q+1)^{s+\frac{k}{2}-1}q^{s-1}e^{-yq}dq\right\}dy.
\end{aligned}
\tag{3.2}
$$

The integral above is the same as

$$\int_0^\infty e^{\frac{y}{2}}y^{w+s-1}\left\{\int_0^\infty (q+1)^{s+\frac{k}{2}-1}q^{s-1}e^{-y(q+1)}dq\right\}dy.$$

Using the power series expansion of $e^{\frac{y}{2}}$ and switching the order of summation and integration, this becomes

$$\sum_{n=0}^{\infty} \frac{1}{2^n n!} \int_0^{\infty} (q+1)^{s+\frac{k}{2}-1} q^{s-1} \left\{ \int_0^{\infty} e^{-y(q+1)} y^{w+s+n-1} dy \right\} dq$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^n n!} \Gamma(w+s+n) \int_0^{\infty} (q+1)^{-w-n+\frac{k}{2}-1} q^{s-1} dq.$$

Let $u = \frac{1-t}{t}$. We have

$$= \sum_{n=0}^{\infty} \frac{1}{2^n n!} \Gamma(w+s+n) \int_0^1 t^{w+n-s-\frac{k}{2}} (1-t)^{s-1} dt.$$

Using the well known identity

$$\int_0^1 (t)^{\alpha-1}(1-t)^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

the sum becomes

$$\sum_{n=0}^{\infty} \frac{\Gamma(w+s+n)\Gamma\left(w+n-s+1-\frac{k}{2}\right)\Gamma(s)}{\Gamma\left(w+n+1-\frac{k}{2}\right) 2^n n!}.$$

Substituting this back into (3.2) gives the desired result for $m > 0$.

If $m < 0$, we again take the Mellin Transform of (3.1) and making the transformation $4\pi|m|y \to y$, keeping in mind that now $|m| = -m$, we have

$$\int_0^{\infty} K_m(s,y,k) y^{w+s-1} dy = \frac{e^{-\frac{\pi i k}{4}}}{\Gamma\left(s+\frac{k}{2}\right)\Gamma(s) 2^{2w-k/2} \pi^{w-s-k/2} |m|^{w+1-s-k/2}} \times \int_0^{\infty} e^{-\frac{y}{2}} y^{w+s-1} \left\{ \int_0^{\infty} u^{s+\frac{k}{2}-1} (u+1)^{s-1} e^{-yu} du \right\} dy. \tag{3.3}$$

As before, this integral is the same as

$$\int_0^{\infty} e^{\frac{y}{2}} y^{w+s-1} \left\{ \int_0^{\infty} u^{s+\frac{k}{2}-1} (u+1)^{s-1} e^{-y(u+1)} du \right\} dy$$

and again using the power series expansion of $e^{\frac{y}{2}}$ and changing the order of summation and integration, we get

$$\sum_{n=0}^{\infty} \frac{1}{2^n n!} \Gamma(w+s+n) \int_0^{\infty} u^{s+\frac{k}{2}-1}(u+1)^{-w-n-1} du.$$

Substituting $\frac{1-t}{t}$ for u, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1}{2^n n!} \Gamma(w+s+n) \int_0^{1} (1-t)^{s+\frac{k}{2}-1} t^{w+n-s-\frac{k}{2}} dt \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(w+s+n)\Gamma\left(s+\frac{k}{2}\right)\Gamma\left(w+n+1-s-\frac{k}{2}\right)}{\Gamma(w+n+1)2^n n!}.
\end{aligned}$$

Substituting this back into (3.3), we get the result for $m<0$. □

From now on, let

$$\begin{aligned}
\rho &= 2s + \frac{k-1}{2}, \\
w &= s + \frac{k}{2} + \delta.
\end{aligned}$$

Define

$$\begin{aligned}
G_k(\rho,\delta) &= F_+\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}+\delta, \frac{\rho}{2}+\frac{1}{4}-\frac{k}{4}\right) \\
G_{-k}(\rho,\delta) &= F_-\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}+\delta, \frac{\rho}{2}+\frac{1}{4}-\frac{k}{4}\right).
\end{aligned}$$

There is no ambiguity in this definition, because one can easily check and see that

$$G_k(\rho,\delta) = G_{-(-k)}(\rho,\delta).$$

Proposition 3.2. *Let $G_k(\rho,\delta)$ be defined as above. Then for ρ fixed, $|\delta|$ sufficiently large, we have*

$$G_k(\rho,\delta) = \frac{2^{\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}+\delta}\Gamma\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}+\delta\right)}{\Gamma\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}\right)}\left(1+O\left(\frac{1}{|\delta|}\right)\right)$$

where the O-symbol depends at most on ρ and k.

Proof. We make use of the following transformation (see [1] page 559):

$$F(a,b,c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b,a+b-c+1;1-z)$$
$$+\frac{(1-z)^{c-a-b}\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a,c-b,c-a-b+1;1-z),$$

which is valid for $|\arg(1-z)| < \pi$.

Let $\alpha = \frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}+\delta$ and $\beta = \frac{\rho}{2}+\frac{1}{4}-\frac{k}{4}$. Our definition of $G_k(\rho,\delta)$ gives

$$\begin{aligned}G_k(\rho,\delta) &= F_+(\alpha,\beta,k)\\ &= \frac{\Gamma\left(\alpha-\beta+1-\frac{k}{2}\right)\Gamma(\alpha+\beta)}{\Gamma\left(\beta+\frac{k}{2}\right)\Gamma\left(\alpha+1-\frac{k}{2}\right)}F\left(\alpha+\beta,\alpha-\beta+1-\frac{k}{2},\alpha+1-\frac{k}{2};\frac{1}{2}\right),\end{aligned}$$

and

$$\begin{aligned}G_{-k}(\rho,\delta) &= F_-(\alpha,\beta,k)\\ &= \frac{\Gamma\left(\alpha-\beta+1-\frac{k}{2}\right)\Gamma(\alpha+\beta)}{\Gamma(\beta)\Gamma(\alpha+1)}F\left(\alpha+\beta,\alpha-\beta+1-\frac{k}{2},\alpha+1;\frac{1}{2}\right).\end{aligned}$$

Using the transformation above, we get

$$\begin{aligned}G_k(\rho,\delta) =&\Upsilon\left\{\frac{\Gamma\left(\alpha+1-\frac{k}{2}\right)\Gamma(-\alpha)}{\Gamma\left(1-\beta-\frac{k}{2}\right)\Gamma(\beta)}F\left(\alpha+\beta,\alpha-\beta+1-\frac{k}{2},\alpha+1;\frac{1}{2}\right)\right\}\\ &+\Upsilon\left\{\frac{2^{\alpha}\Gamma\left(\alpha+1-\frac{k}{2}\right)\Gamma(\alpha)}{\Gamma(\alpha+\beta)\Gamma\left(\alpha-\beta+1-\frac{k}{2}\right)}F\left(1-\beta-\frac{k}{2},\beta,1-\alpha;\frac{1}{2}\right)\right\}\end{aligned}$$

where

$$\Upsilon = \frac{\Gamma\left(\alpha-\beta+1-\frac{k}{2}\right)\Gamma(\alpha+\beta)}{\Gamma\left(\beta+\frac{k}{2}\right)\Gamma\left(\alpha+1-\frac{k}{2}\right)}.$$

This simplifies to

$$\begin{aligned}G_k(\rho,\delta) =&\frac{\Gamma(\alpha+1)\Gamma(-\alpha)}{\Gamma\left(\beta+\frac{k}{2}\right)\Gamma\left(1-\beta-\frac{k}{2}\right)}G_{-k}(\rho,\delta)\\ &+\frac{2^{\alpha}\Gamma(\alpha)}{\Gamma\left(\beta+\frac{k}{2}\right)}F\left(1-\beta-\frac{k}{2},\beta,1-\alpha;\frac{1}{2}\right).\end{aligned}$$

Plugging back in for α and β, we get

$$G_k(\rho,\delta) = \frac{\Gamma\left(\frac{5}{4}+\delta+\frac{\rho}{2}+\frac{k}{4}\right)\Gamma\left(-\frac{1}{4}-\delta-\frac{\rho}{2}-\frac{k}{4}\right)}{\Gamma\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}\right)\Gamma\left(\frac{3}{4}-\frac{\rho}{2}-\frac{k}{4}\right)}G_{-k}(\rho,\delta)+ \tag{3.4}$$

$$\frac{2^{\frac{1}{4}+\delta+\frac{\rho}{2}+\frac{k}{4}}\Gamma\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}+\delta\right)}{\Gamma\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}\right)}F\left(\frac{3}{4}-\frac{\rho}{2}-\frac{k}{4},\frac{1}{4}+\frac{\rho}{2}-\frac{k}{4},\frac{3}{4}-\delta-\frac{\rho}{2}-\frac{k}{4};\frac{1}{2}\right). \tag{3.5}$$

We look at the expression $F\left(\frac{3}{4}-\frac{\rho}{2}-\frac{k}{4},\frac{1}{4}+\frac{\rho}{2}-\frac{k}{4},\frac{3}{4}-\delta-\frac{\rho}{2}-\frac{k}{4};\frac{1}{2}\right)$.

For any complex number s and any positive integer n, let $(s:n)$ be the product $s(s+1)(s+2)...(s+n)$. Thus, the functional equation $\Gamma(s+1)=s\Gamma(s)$ gives us $\Gamma(s+n)=(s:n)\Gamma(s)$. Then $F(a,b,c;z)$ can also be expressed as

$$F(a,b,c;z)=1+\sum_{n=0}^{\infty}\frac{(a:n)(b:n)}{(c:n)(n+1)!}z^{n+1}.$$

So we have

$$\begin{aligned}&F\left(\frac{3}{4}-\frac{\rho}{2}-\frac{k}{4},\frac{1}{4}+\frac{\rho}{2}-\frac{k}{4},\frac{3}{4}-\delta-\frac{\rho}{2}-\frac{k}{4};\frac{1}{2}\right)\\&=1+\sum_{n=0}^{\infty}\frac{\left(\frac{3}{4}-\frac{\rho}{2}-\frac{k}{4}:n\right)\left(\frac{1}{4}+\frac{\rho}{2}-\frac{k}{4}:n\right)}{\left(\frac{3}{4}-\delta-\frac{\rho}{2}-\frac{k}{4}:n\right)2^{n+1}(n+1)!}.\end{aligned}$$

Consider the expression

$$\frac{(a:n)(b:n)}{(c:n)(n+1)!}.$$

This is the same as

$$\frac{ab}{c}\prod_{u=1}^{n}\frac{(a+u)(b+u)}{(c+u)(u+1)}=\frac{ab}{c}\prod_{u=1}^{n}\frac{ab+u(a+b)+u^2}{c+u(c+1)+u^2}.$$

It is clear that if $|c|$ is chosen to be sufficiently large, each factor in the product is less than 1 in absolute value, independent of u. Therefore

$$\frac{(a:n)(c:n)}{(c:n)(n+1)!}=O\left(\frac{1}{|c|}\right)$$

where the $O-$constant is dependant at most on a and b. So for $|\delta|$ sufficiently large, we have

$$F\left(\frac{3}{4}-\frac{\rho}{2}-\frac{k}{4},\frac{1}{4}+\frac{\rho}{2}-\frac{k}{4},\frac{3}{4}-\delta-\frac{\rho}{2}-\frac{k}{4};\frac{1}{2}\right)$$

$$= 1 + \sum_{n=0}^{\infty} O\left(\frac{1}{|\delta|}\right)\left(\frac{1}{2}\right)^n = 1 + O\left(\frac{1}{|\delta|}\right)$$

where the $O-$constant depends at most on ρ and k.

Then (3.5) becomes

$$\begin{aligned} G_k(\rho,\delta) =& \frac{\Gamma\left(\frac{5}{4}+\delta+\frac{\rho}{2}+\frac{k}{4}\right)\Gamma\left(-\frac{1}{4}-\delta-\frac{\rho}{2}-\frac{k}{4}\right)}{\Gamma\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}\right)\Gamma\left(\frac{3}{4}-\frac{\rho}{2}-\frac{k}{4}\right)} G_{-k}(\rho,\delta) \\ &+ \frac{2^{\frac{1}{4}+\delta+\frac{\rho}{2}+\frac{k}{4}}\Gamma\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}+\delta\right)}{\Gamma\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}\right)}\left(1+O\left(\frac{1}{|\delta|}\right)\right). \end{aligned} \tag{3.6}$$

Recall that $G_k(\rho,\delta) = G_{-(-k)}(\rho,\delta)$.

Using (3.6) with $-k$ in place of k substituted back into (3.6) for $G_{-k}(\rho,\delta)$ we get an expression involving $G_k(\rho,\delta)$. If we solve this expression for $G_k(\rho,\delta)$, we get

$$G_k(\rho,\delta) = \frac{2^{\frac{1}{4}+\delta+\frac{\rho}{2}+\frac{k}{4}}\Gamma\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}+\delta\right)}{\Gamma\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}\right)}\left[\Omega_k(\rho,\delta)\right]\left(1+O\left(\frac{1}{|\delta|}\right)\right)$$

where

$$\Omega_k(\rho,\delta) = \frac{\frac{2^{-k/2}\left(\frac{1}{4}+\delta+\frac{\rho}{2}+\frac{k}{4}\right)\Gamma\left(-\frac{1}{4}-\delta-\frac{\rho}{2}-\frac{k}{4}\right)\Gamma\left(\frac{\rho}{2}+\frac{5}{4}-\frac{k}{4}+\delta\right)}{\left(\frac{\rho}{2}+\frac{1}{4}-\frac{k}{4}+\delta\right)\Gamma\left(\frac{3}{4}-\frac{\rho}{2}-\frac{k}{4}\right)\Gamma\left(\frac{\rho}{2}+\frac{1}{4}-\frac{k}{4}\right)}+1}{1-\frac{\Gamma\left(\frac{5}{4}+\delta+\frac{\rho}{2}+\frac{k}{4}\right)\Gamma\left(-\frac{1}{4}-\delta-\frac{\rho}{2}-\frac{k}{4}\right)\Gamma\left(\frac{5}{4}+\delta+\frac{\rho}{2}-\frac{k}{4}\right)\Gamma\left(-\frac{1}{4}-\delta-\frac{\rho}{2}+\frac{k}{4}\right)}{\Gamma\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}\right)\Gamma\left(\frac{3}{4}-\frac{\rho}{2}-\frac{k}{4}\right)\Gamma\left(\frac{\rho}{2}+\frac{1}{4}-\frac{k}{4}\right)\Gamma\left(\frac{3}{4}-\frac{\rho}{2}+\frac{k}{4}\right)}}.$$

We are interested in what happens as $|\delta| \to \infty$. We examine the product of gamma functions involving δ in the first term of the numerator. The reader is reminded of Stirling's formula (see p.47 of [4]):

$$\Gamma(z) \sim e^{-z}e^{(z-1/2)\log z}(2\pi)^{1/2} \text{ as } |z| \to \infty.$$

Using this, we get

$$\Gamma\left(-\frac{1}{4}-\delta-\frac{\rho}{2}-\frac{k}{4}\right)\Gamma\left(\frac{\rho}{2}+\frac{5}{4}-\frac{k}{4}+\delta\right) \sim$$

$$e^{\frac{k}{2}-1}(2\pi)e^{\left(\frac{\rho}{2}+\frac{3}{4}\right)\log\left(\frac{\frac{\rho}{2}+\frac{5}{4}-\frac{k}{4}+\delta}{-\frac{1}{4}-\delta-\frac{\rho}{2}-\frac{k}{4}}\right)}e^{\delta\log\left(\frac{\frac{\rho}{2}+\frac{5}{4}-\frac{k}{4}+\delta}{-\frac{1}{4}-\delta-\frac{\rho}{2}-\frac{k}{4}}\right)}$$

$$\times e^{-\frac{k}{4}\log\left(\left(\frac{\rho}{2}+\frac{5}{4}-\frac{k}{4}+\delta\right)\left(-\frac{1}{4}-\delta-\frac{\rho}{2}-\frac{k}{4}\right)\right)}.$$

The second exponential is clearly approaching 1 as $|\delta|$ approaches ∞. Using basic calculus techniques, one can see that the second exponential is approaching $e^{1-\frac{k}{2}}$. Finally, the last exponential is approaching 0.

Hence, the numerator of $\Omega_k(\rho, \delta)$ approaches 1 as $|\delta| \to \infty$.

Examining the denominator, we pair the middle two Gamma functions in the numerator of the second term together and we pair the outer two Gamma functions together, and the same calculations show that each of these products is approaching 0 as $|\delta| \to \infty$. Hence the denominator is approaching 1.

Therefore, for $|\delta|$ sufficiently large, $\Omega_k(\rho, \delta) \sim 1$ and the result follows. $\square$

Proposition 3.3. *Let*

$$D_k(\rho, \delta) = G_k(\rho, \delta)G_{-k-4}(\rho, \delta) - G_{k+4}(\rho, \delta)G_{-k}(\rho, \delta).$$

Then for fixed ρ, k and $|\delta| \to \infty$, we have

$$D_k(\rho, \delta) \sim 2^{2\delta+\frac{3}{2}+\rho}\frac{\Gamma\left(\delta+\frac{\rho}{2}+\frac{5}{4}+\frac{k}{4}\right)\Gamma\left(\delta+\frac{\rho}{2}+\frac{1}{4}-\frac{k}{4}\right)}{\Gamma\left(\frac{\rho}{2}+\frac{5}{4}+\frac{k}{4}\right)\Gamma\left(\frac{\rho}{2}+\frac{1}{4}-\frac{k}{4}\right)}.$$

Furthermore, if ρ, δ are fixed, there exist infinitely many integers $k \equiv 1 \pmod 4$ and also infinitely many $k \equiv 3 \pmod 4$ such that $D_k(\rho, \delta) \neq 0$.

Proof. To get the first result we make use of Proposition 3.2. Looking at the quotient, we have

$$\frac{D_k(\rho, \delta)}{2^{2\delta+\frac{3}{2}+\rho}\frac{\Gamma\left(\delta+\frac{\rho}{2}+\frac{5}{4}+\frac{k}{4}\right)\Gamma\left(\delta+\frac{\rho}{2}+\frac{1}{4}-\frac{k}{4}\right)}{\Gamma\left(\frac{\rho}{2}+\frac{5}{4}+\frac{k}{4}\right)\Gamma\left(\frac{\rho}{2}+\frac{1}{4}-\frac{k}{4}\right)}}$$
$$= \frac{1}{4}\frac{\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}\right)\left(\frac{\rho}{2}-\frac{3}{4}-\frac{k}{4}\right)}{\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}+\delta\right)\left(\frac{\rho}{2}-\frac{3}{4}-\frac{k}{4}+\delta\right)}\left(1+O\left(\frac{1}{|\delta|}\right)\right) - \left(1+O\left(\frac{1}{|\delta|}\right)\right).$$

This clearly $\to -1$ as $|\delta| \to \infty$.

Using the definition of $G_{\pm k}(\rho,\delta)$ together with the definition of $F_\pm$ (see Proposition 3.1), we have

$$G_{\pm k}(\rho,\delta)=\Gamma(1+\delta)\Gamma\left(\rho+\frac{1}{2}+\delta\right)\frac{F\left(\rho+\frac{1}{2}+\delta,1+\delta,\frac{\rho}{2}+\frac{5}{4}\mp\frac{k}{4}+\delta;\frac{1}{2}\right)}{\Gamma\left(\frac{\rho}{2}+\frac{1}{4}\pm\frac{k}{4}\right)\Gamma\left(\frac{\rho}{2}+\frac{5}{4}\mp\frac{k}{4}+\delta\right)}. \tag{3.7}$$

Using a method similar to that found in the proof of Proposition 3.2, it easily seen that, for ρ, δ fixed and $k\to\pm\infty$, $\frac{\rho}{2}+\frac{5}{4}\mp\frac{k}{4}+\delta$ not an integer,

$$\begin{aligned}&F\left(\rho+\frac{1}{2}+\delta,1+\delta,\frac{\rho}{2}+\frac{5}{4}\mp\frac{k}{4}+\delta;\frac{1}{2}\right)\\&=1+\frac{1}{2}\frac{\left(\rho+\frac{1}{2}+\delta\right)(1+\delta)}{\left(\frac{\rho}{2}+\frac{5}{4}\mp\frac{k}{4}+\delta\right)}+O\left(\frac{1}{k^2}\right).\end{aligned}$$

Substituting this into (3.7) and substituting into the definition of $D_k(\rho,\delta)$, we get

$$\begin{aligned}&D_k(\rho,\delta)=\\&\frac{\left(1+\frac{1}{2}\frac{\left(\rho+\frac{1}{2}\right)(1+\delta)}{\left(\frac{\rho}{2}+\frac{5}{4}-\frac{k}{4}+\delta\right)}+O\left(\frac{1}{k^2}\right)\right)\left(1+\frac{1}{2}\frac{\left(\rho+\frac{1}{2}\right)(1+\delta)}{\left(\frac{\rho}{2}+\frac{5}{4}+\frac{k}{4}+1+\delta\right)}+O\left(\frac{1}{k^2}\right)\right)}{\Gamma\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}\right)\Gamma\left(\frac{\rho}{2}+\frac{5}{4}-\frac{k}{4}+\delta\right)\Gamma\left(\frac{\rho}{2}+\frac{1}{4}-\frac{k}{4}-1\right)\Gamma\left(\frac{\rho}{2}+\frac{5}{4}+\frac{k}{4}+1+\delta\right)}\\&-\frac{\left(1+\frac{1}{2}\frac{\left(\rho+\frac{1}{2}\right)(1+\delta)}{\left(\frac{\rho}{2}+\frac{5}{4}-\frac{k}{4}-1+\delta\right)}+O\left(\frac{1}{k^2}\right)\right)\left(1+\frac{1}{2}\frac{\left(\rho+\frac{1}{2}\right)(1+\delta)}{\left(\frac{\rho}{2}+\frac{5}{4}+\frac{k}{4}+\delta\right)}+O\left(\frac{1}{k^2}\right)\right)}{\Gamma\left(\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}+1\right)\Gamma\left(\frac{\rho}{2}+\frac{5}{4}-\frac{k}{4}-1+\delta\right)\Gamma\left(\frac{\rho}{2}+\frac{1}{4}-\frac{k}{4}\right)\Gamma\left(\frac{\rho}{2}+\frac{5}{4}+\frac{k}{4}+\delta\right)}.\end{aligned} \tag{3.8}$$

Again, we use Stirling's Formula (see Proposition 3.2) on the denominators. A consequence of Stirling's formula is the asymptotic formula (p. 47 of [4])

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)}\sim z^{\alpha-\beta}.$$

We also have the well known functional equation

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}.$$

Combining these two, we have the formula

$$\Gamma(\alpha+z)\Gamma(\beta-z)=\frac{\pi}{\sin(\pi(\beta-z))}\frac{\Gamma(\alpha+z)}{\Gamma(1-\beta+z)}\sim\frac{\pi}{\sin(\pi(\beta-z))}z^{\alpha+\beta-1}.$$

Using this with $z = \frac{k}{4}$, the denominator of the first term in (3.8) becomes

$$\frac{\pi}{\sin(\pi(\frac{\rho}{2}+\frac{5}{4}-\frac{k}{4}+\delta))}\left(\frac{k}{4}\right)^{\rho+\frac{3}{2}+\delta-1}\frac{\pi}{\sin(\pi(\frac{\rho}{2}+\frac{1}{4}-\frac{k}{4}-1))}\left(\frac{k}{4}\right)^{\rho+\frac{3}{2}+\delta-1}$$

and the denominator of the second term becomes

$$\frac{\pi}{\sin(\pi(\frac{\rho}{2}+\frac{5}{4}-\frac{k}{4}+\delta-1))}\left(\frac{k}{4}\right)^{\rho+\frac{3}{2}+\delta-1}\frac{\pi}{\sin(\pi(\frac{\rho}{2}+\frac{1}{4}-\frac{k}{4}))}\left(\frac{k}{4}\right)^{\rho+\frac{3}{2}+\delta-1}.$$

Notice, these are equal (the -1 in the sine functions can be factored out as a sign change).

So, for k sufficiently large, the denominators in (3.8) are essentially the same, so now we look at the difference of the numerators. Multiplying things out, getting a common denominator, and simplifying we get this difference to be

$$\frac{-\left(\rho+\frac{1}{2}+\delta\right)(1+\delta)\left(\left(\frac{\rho}{2}+\frac{5}{4}+\delta\right)^2+\left(\frac{k}{4}\right)^2+\left(\frac{k}{4}\right)\right)}{\left(\frac{\rho}{2}+\frac{5}{4}-\frac{k}{4}+\delta\right)\left(\frac{\rho}{2}+\frac{5}{4}+\frac{k}{4}+1+\delta\right)\left(\frac{\rho}{2}+\frac{5}{4}-\frac{k}{4}-1+\delta\right)\left(\frac{\rho}{2}+\frac{5}{4}+\frac{k}{4}+\delta\right)}$$
$$+O\left(\frac{1}{k^2}\right).$$

We notice at this point that the O term does not cancel with the first term for infinitely many k, proving the proposition. □

Proposition 3.4. *We have*

$$K_0(s,y,k) = y^{1-2s-k/2}2^{k/2}\sqrt{\pi}e^{\frac{-\pi ik}{4}}\frac{\Gamma\left(s+\frac{k}{2}-\frac{1}{2}\right)\Gamma\left(2s+\frac{k}{2}-1\right)}{\Gamma(2s+k-1)\Gamma(s)}.$$

Proof. By (3.1), when $m = 0$ we have

$$K_0(s,y,k) = y^{1-2s-k/2}2^{2-2s-k/2}\pi e^{\frac{-\pi ik}{4}}\frac{\Gamma\left(2s+\frac{k}{2}-1\right)}{\Gamma\left(s+\frac{k}{2}\right)\Gamma(s)}. \tag{3.9}$$

Using the well-known transformation formula

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$$

with $z = s + \frac{k}{2} - \frac{1}{2}$, we get

$$\frac{1}{\Gamma\left(s+\frac{k}{2}\right)} = \frac{1}{2^{2-2s-k}\sqrt{\pi}} \frac{\Gamma\left(s+\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(2s+k-1\right)}.$$

Substituting back into (3.9), we get the result. □

3.2 Growth Estimates

Recall that the Fourier coefficients $a(s, y, k)$ of $E_0(z, s, k)$ can be evaluated as a product over the odd primes. Again letting $\rho = 2s + \frac{k-1}{2}$, we define

$$A_m(\rho, k) = \prod_{p\neq 2} \left(\sum_{l=0}^{\infty} \frac{(\epsilon_p)^k g(-m, p^l)}{p^{l(\rho+\frac{1}{2})}} \right).$$

By Proposition 2.10 we have

$$a_m(s, y, k) = (y/4)^s A_m(\rho, k) K_m(s, y, k) \tag{3.10}$$

and by Corollary 2.11, if m is squarefree, we have

$$A_m(s, y, k) = \frac{L(\rho, \chi_m)(1 - \chi_m(2)2^{-\rho})}{\zeta(2\rho)(1 - 2^{-2\rho})} \tag{3.11}$$

where χ_m is the real primitive Dirichlet character associated to $Q(\sqrt{\mu_k m})$ where $\mu_k = (-1)^{(k-1)/2}$.

For $\mathrm{Re}(\rho) \geq \frac{1}{2}$ and $\mathrm{Re}(\delta) > 0$, we define the zeta function

$$Z_{\pm}(\rho, \delta, r, k) = \sum_{\substack{\pm m>0 \\ m\equiv 0 \,(\mathrm{mod}\ r)}} A_m(\rho, k) |m|^{-1-\delta}. \tag{3.12}$$

The series on the right converges absolutely since the Rankin-Selberg zeta function (see [9])

$$\sum_{m\neq 0} |A_m(\rho, k)|^2 |m|^{-1-\delta}$$

has its first pole at $\delta = 0$.

The function $A_m(\rho, k)$ satisfies

$$A_m(\rho, k) = A_m(\rho, k \pm 4) = A_{-m}(\rho, k + 2).$$

Therefore $Z_\pm(\rho, \delta, r, k)$ satisfies

$$Z_\pm(\rho, \delta, r, k) = Z_\pm(\rho, \delta, r, k \pm 4) = Z_\mp(\rho, \delta, r, k + 2).$$

For simplicity, we shall henceforth assume that $k \equiv 1 \pmod{4}$.

Our main purpose in this section will be to show that for fixed values ρ, k, the zeta function in (3.12) has a meromorhic continuation to the entire complex δ-plane and satisfies certain growth estimates uniformly in δ and r.

Proposition 3.5. *Let m be squarefree. Then for $Re(\rho) \geq 0$, $Re(w)$ sufficiently large, we have*

$$\sum_{n=1}^{\infty} A_{mn^2}(\rho, k) n^{-w} = A_m(\rho, k) \frac{\zeta(w)\zeta(w + 2\rho - 1)(1 - 2^{-w-2\rho+1})}{L(\rho + w, \chi_m)(1 - \chi_m(2)2^{-\rho-w})}$$

where χ_m is the real Dirichlet character associated to $Q(\sqrt{\mu_k m})$.

Proof. We follow the proof of a more general result by Shimura in [14] pages 87-89. We need a couple of preliminary results. First we have the identity

$$g(1, r) = \epsilon_r \sqrt{r} \tag{3.13}$$

whenever r is a positive squarefree odd integer.

We will prove this by induction on the the number of prime factors of r. The case where r is prime was done in the proof of Corollary 2.11 where we showed

$$g(1, p) = \begin{cases} \sqrt{p}, & \text{if } \equiv 1 \pmod{4}; \\ i\sqrt{p}, & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

If the assertion holds for odd squarefree numbers with $j-1$ prime factors, suppose r has j prime factors, and let p be one of them. Then r/p has $j-1$ prime factors.

Recall that in the proof of Proposition 2.10, we showed that

$$g(m, ab) = \left(\frac{a}{b}\right)\left(\frac{b}{a}\right) g(m, a) g(m, b) \tag{3.14}$$

whenever a and b are relatively prime odd positive integers. Also, we showed that

$$\epsilon_{ab} = \left(\frac{a}{b}\right)\left(\frac{b}{a}\right) \epsilon_a \epsilon_b. \tag{3.15}$$

Therefore

$$\begin{aligned} g(1, r) &= \left(\frac{r/p}{p}\right)\left(\frac{p}{r/p}\right) g(1, r/p) g(1, p) \\ &= \left(\frac{r/p}{p}\right)\left(\frac{p}{r/p}\right) \epsilon_{r/p}\epsilon_p \sqrt{r/p}\sqrt{p} \\ &= \epsilon_r \sqrt{r}. \end{aligned}$$

Next, we claim that for r an odd positive squarefree integer, $g(m, r)$ is separable for all m, i.e.

$$g(m, r) = \left(\frac{m}{r}\right) g(1, r). \tag{3.16}$$

Again, we prove this by induction on the number of prime factors of r. Here, the case where r is prime is a well known fact from analytic number theory ($g(r, r) = \left(\frac{r}{r}\right) = 0$ in this case). If the assertion holds for $j-1$ prime factors, suppose r has j prime factors, and let p be one of them. Then r/p has $j-1$ prime factors and by (3.14) and (3.13) we have

$$\begin{aligned} g(m, r) &= \left(\frac{r/p}{p}\right)\left(\frac{p}{r/p}\right) g(m, r/p) g(m, p) \\ &= \left(\frac{r/p}{p}\right)\left(\frac{p}{r/p}\right)\left(\frac{m}{r/p}\right)\left(\frac{m}{p}\right) g(1, r/p) g(1, p) \\ &= \left(\frac{m}{r}\right) g(1, r). \end{aligned}$$

(3.13) and (3.16) together give us the identity

$$g(m,r) = \left(\frac{m}{r}\right)\epsilon_r\sqrt{r} \tag{3.17}$$

for all integers m and all odd positive squarefree integers r.

Let r be an odd squarefree positive integer, and let c be any integer, and m any integer. Then we have

$$\sum_{n=1}^{rc}\left(\frac{n}{r}\right)e^{\frac{2\pi inm}{rc}} = \begin{cases} 0, & \text{if } c \nmid m; \\ c\left(\frac{m/c}{r}\right)g(1,r), & \text{if } c|m. \end{cases} \tag{3.18}$$

To show this, we start with

$$\sum_{n=1}^{rc}\left(\frac{n}{r}\right)e^{\frac{2\pi inm}{rc}} = \sum_{s=1}^{c}e^{\frac{2\pi ism}{c}}\sum_{t=1}^{r}\left(\frac{t}{r}\right)e^{\frac{2\pi itm}{rc}},$$

where we set $n = sr + t$. The first sum is c if $c|m$ and 0 otherwise. The second sum is $g(m/c, r)$ which by (3.16) is separable, giving (3.18).

We use (3.18) to get the following identity. For r a squarefree odd integer, u any odd positive integer, and any integer m, we have

$$g(m, ru^2) = \epsilon_r\sqrt{r}\sum_{c|(u^2,m)} c\mu(u^2/c)\left(\frac{u^2/c}{r}\right)\left(\frac{m/c}{r}\right), \tag{3.19}$$

where μ is the Möbius function.

Recall that

$$\sum_{d|n}\mu(d) = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore,

$$\left(\frac{n}{ru^2}\right) = \begin{cases} \left(\frac{n}{r}\right), & \text{if } (n,u^2) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

This gives

$$g(m,ru^2) = \sum_{n=1}^{ru^2} \left(\frac{n}{ru^2}\right) e^{\frac{2\pi imn}{ru^2}}$$
$$= \sum_{n=1}^{ru^2} \left(\sum_{d|(n,u^2)} \mu(d)\right) \left(\frac{n}{r}\right) e^{\frac{2\pi imn}{ru^2}}$$
$$= \sum_{d|u^2} \mu(d) \left(\frac{d}{r}\right) \sum_{t=1}^{r\frac{u^2}{r}} \left(\frac{t}{r}\right) e^{\frac{2\pi imt}{r\frac{u^2}{d}}}.$$

In the last step, we have put $n = td$. Putting $c = u^2/d$ in the second sum in the last step above, as d cycles through the divisors of u^2, so does c. Using (3.18), we get

$$g(m,ru^2) = \left(\sum_{s=1}^{r} \left(\frac{s}{r}\right) e^{\frac{2\pi is}{r}}\right) \left(\sum_{c|(u^2,m)} c\mu\left(\frac{u^2}{c}\right)\left(\frac{u^2/c}{r}\right)\left(\frac{m/c}{r}\right)\right)$$
$$= g(1,r) \left(\sum_{c|(u^2,m)} c\mu\left(\frac{u^2}{c}\right)\left(\frac{u^2/c}{r}\right)\left(\frac{m/c}{r}\right)\right)$$
$$= \epsilon_r\sqrt{r} \left(\sum_{c|(u^2,m)} c\mu\left(\frac{u^2}{c}\right)\left(\frac{u^2/c}{r}\right)\left(\frac{m/c}{r}\right)\right).$$

This proves (3.19). In (3.19), put $u^2 = ac$. Since $\mu(a) = 0$ unless s is squarefree, we can assume a is in fact a divisor of u, and put $u = ab$ (b a positive integer) so that $u^2/c = a$ and $c = ab^2$. We have

$$g(m,ru^2) = \epsilon_r\sqrt{r} \sum_{\substack{a,b \\ ab=u, ab^2|m}} ab^2\mu(a)\left(\frac{m/b^2}{r}\right). \tag{3.20}$$

Here we have noticed $\left(\frac{a}{r}\right)\left(\frac{m/ab^2}{r}\right) = \left(\frac{m/b^2}{r}\right)$.

From the proof of Proposition 2.10, we recall that $A_m(\rho,k)$ is also equal to

$$\sum_{u>0 \text{ odd}} \frac{\epsilon_u^k g(-m,u)}{u^{\rho+\frac{1}{2}}}$$

which can also be written

$$\sum_{\substack{r>0 \text{ squarefree, odd} \\ u>0 \text{ odd}}} \frac{\epsilon_{ru^2}^k g(-m, ru^2)}{(r^{\rho+\frac{1}{2}} u^{2\rho+1})}.$$

Now, $u^2 \equiv 1 \pmod 4$, so that $ru^2 \equiv r \pmod 4$, giving us $\epsilon_{ru^2} = \epsilon_r$. Using this and (3.20) we have

$$A_m(\rho, k) = \sum_{\substack{r>0 \text{ squarefree, odd} \\ a,b \text{ odd, with } ab^2|m}} \frac{\epsilon_r^{k+1} ab^2 \mu(a) \left(\frac{-m/b^2}{r}\right)}{r^\rho a^{2\rho+1} b^{2\rho+1}}. \tag{3.21}$$

Recall that $\epsilon_r = \sqrt{\left(\frac{-1}{r}\right)}$, so that

$$\epsilon_r^{k+1} = \left(\frac{-1}{r}\right)^{\frac{k+1}{2}}.$$

Put $m = tn^2$ with t squarefree. Then $ab^2|m$ implies that $b|n$, so we put $n = bh$. If $\mu(a) \neq 0$ (i.e. a is squarefree), then $a|th$.

Thus

$$\begin{aligned}
\epsilon_r^{k+1}\left(\frac{-m/b^2}{r}\right) &= \epsilon_r^{k+1}\left(\frac{-th^2}{r}\right) \\
&= \left(\frac{-1}{r}\right)^{\frac{k+3}{2}} \left(\frac{th^2}{r}\right) \\
&= \begin{cases} 0, & \text{if } (th, r) \neq 1; \\ \left(\frac{-1}{r}\right)^{\frac{k+3}{2}} \left(\frac{t}{r}\right), & \text{if } (th, r) = 1. \end{cases}
\end{aligned}$$

Notice, the quantity in the second case is precisely $\chi_t(r)$ (see proof of Corollary 2.11), so that we have

$$\epsilon_r^{k+1}\left(\frac{-m/b^2}{r}\right) = \begin{cases} 0, & \text{if } (th, r) \neq 1; \\ \chi_t(r), & \text{if } (th, r) = 1. \end{cases}$$

So (3.21) becomes

$$A_{tn^2}(\rho,k) = \sum_{\substack{b|n \\ b \text{ odd}}} \sum_{\substack{a|th \\ a \text{ odd}}} \sum_{\substack{(r,th)=1 \\ r \text{ squarefree}}} \frac{\chi_t(r)ab^2\mu(a)}{r^\rho a^{2\rho+1}b^{2\rho+1}}$$

$$= \left(\sum_{\substack{b|n \\ b \text{ odd}}} \frac{1}{b^{2\rho-1}}\right)\left(\sum_{\substack{a|th \\ a \text{ odd}}} \frac{\mu(a)}{a^{2\rho}}\right)\left(\sum_{\substack{(r,th)=1 \\ r \text{ squarefree}}} \frac{\chi_t(r)}{r^\rho}\right).$$

The second sum is $\prod_{p|th,p\neq 2}\left(1-p^{-(2\rho)}\right)$ (see page 37 [2]). The third sum is

$$\prod_{\substack{p\nmid th \\ p\neq 2}}(1+\chi_t(p)p^{-\rho}) = \prod_{\substack{p\nmid th \\ p\neq 2}} \frac{(1-p^{-2\rho})}{(1-\chi_t(r)p^{-\rho})}.$$

So we have

$$A_{tn^2}(\rho,k) = \left(\sum_{\substack{b|n \\ b \text{ odd}}} \frac{1}{b^{2\rho-1}}\right)\prod_{\substack{p|th \\ p\neq 2}}(1-p^{-(2\rho)})\prod_{\substack{p\nmid th \\ p\neq 2}}\frac{(1-p^{-2\rho})}{(1-\chi_t(p)p^{-\rho})}$$

$$= \left(\sum_{\substack{b|n \\ b \text{ odd}}} \frac{1}{b^{2\rho-1}}\right)\prod_{p\neq 2}(1-p^{-(2\rho)})\prod_{\substack{p\nmid th \\ p\neq 2}}\frac{1}{(1-\chi_t(p)p^{-\rho})}$$

$$= \left(\sum_{\substack{b|n \\ b \text{ odd}}} \frac{1}{b^{2\rho-1}}\right)\frac{L(\rho,\chi_t)(1-\chi_t(2)2^{-\rho})}{\zeta(2\rho)(1-2^{-2\rho})}\prod_{\substack{p|th \\ p\neq 2}}(1-\chi_t(p)p^{-\rho})$$

$$= \left(\sum_{\substack{b|n \\ b \text{ odd}}} \frac{1}{b^{2\rho-1}}\right)A_t(\rho,k)\prod_{\substack{p|th \\ p\neq 2}}(1-\chi_t(p)p^{-\rho}).$$

(Notice, at this point, if we take $n=1$ we have an alternate proof for Corollary 2.11.)

Now, $\chi_t(p)=0$ if $p|t$, so the last product can actually be taken over those primes dividing only h. The resulting product is the same as

$$\sum_{\substack{c|h \\ c \text{ odd}}} \frac{\mu(c)\chi_t(c)}{c^\rho}.$$

Recalling that $n = bh$, we put the two sums back together and we have

$$A_{tn^2}(\rho, k) = A_t(\rho, k) \sum_{\substack{ab|n \\ a,b \text{ odd}}} \frac{\mu(a)\chi_t(a)}{a^\rho b^{2\rho-1}}. \tag{3.22}$$

This gives

$$\sum_{n=1}^{\infty} \frac{A_{tn^2}(\rho, k)n^{-w}}{A_t(\rho, k)} = \sum_{n=1}^{\infty} \sum_{\substack{ab|n \\ a,b \text{ odd}}} \frac{\mu(a)\chi_t(a)}{a^\rho b^{2\rho-1}} n^{-w}.$$

Putting $n = abc$, we get

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{A_{tn^2}(\rho, k)n^{-w}}{A_t(\rho, k)} &= \sum_{n=1}^{\infty} \sum_{\substack{ab|n \\ a,b \text{ odd}}} \frac{\mu(a)\chi_t(a)}{a^{\rho+w} b^{2\rho-1+w} c^w} \\ &= \left(\sum_{a \text{ odd}} \frac{\mu(a)\chi_t(a)}{a^{\rho+w}}\right) \left(\sum_{b \text{ odd}} \frac{1}{b^{2\rho-1+w}}\right) \left(\sum_{c=1}^{\infty} \frac{1}{c^w}\right).\end{aligned}$$

The last series is $\zeta(w)$. The middle series is $\zeta(2\rho - 1 + w)(1 - 2^{-(2\rho-1+w)})$. And finally, the first series is the reciprocal of $L(\rho + w, \chi_t)(1 - \chi_t(2)2^{-(\rho+w)})$ (see page 229 of [2]). □

Proposition 3.6. *Let $m = m_0 n^2$ where m_0 is squarefree. Then for $Re(\rho) \geq \frac{1}{2}$*

$$|A_{m_0 n^2}(\rho, k)| \leq |A_{m_0}(\rho, k)| \sigma(n) 2^{w(n)}$$

where $w(n)$ is the number of distinct prime divisors of n and $\sigma(n)$ is the number of divisors of n.

Proof. The right hand Dirichlet series in Proposition 3.5 is

$$A_{m_0}(\rho, k) \left(\sum_{a \text{ odd}} \frac{\mu(a)\chi_{m_0}(a)}{a^{\rho+w}}\right) \left(\sum_{b \text{ odd}} \frac{1}{b^{2\rho-1+w}}\right) \left(\sum_{c=1}^{\infty} \frac{1}{c^w}\right).$$

Choosing a particular n, the coefficient of n^{-w} is

$$A_{m_0}(\rho, k) \sum_{\substack{d_1 d_2 d_3 = n \\ (2, d_2 d_3) = 1}} d_2^{1-2\rho} \chi_{m_0}(d_3) \mu(d_3) d_3^{-\rho}.$$

So, equating the coefficients of the Dirichlet series of Proposition 3.5 we have

$$A_{m_0 n^2}(\rho, k) = A_{m_0}(\rho, k) \sum_{\substack{d_1 d_2 d_3 = n \\ (2, d_2 d_3) = 1}} d_2^{1-2\rho} \chi_{m_0}(d_3) \mu(d_3) d_3^{-\rho}.$$

Now we examine the terms inside this last sum. Notice that $\mu(d_3) = 0$ unless d_3 is squarefree. If d_3 is squarefree, we have

$$|d_2^{1-2\rho} \chi_{m_0}(d_3) \mu(d_3) d_3^{-\rho}| \leq |d_2^{1-2\rho} d_3^{-\rho}| \leq d_3^{-\frac{1}{2}} \leq 1,$$

since $\mathrm{Re}(\rho) \geq \frac{1}{2}$.

We only need to count the possible choices of d_1, d_2, and d_3. Certainly, there are exactly $\sigma(n)$ choices for d_1. This leaves us with counting the choices for d_3 with d_3 squarefree (choosing d_1 and d_3 determines d_2). If $w(n)$ is the number of the distinct prime divisors of n, there are at most

$$\binom{w(n)}{0} + \binom{w(n)}{1} + \cdots\cdots\cdots + \binom{w(n)}{w(n)} = 2^{w(n)}$$

possible choices for d_3, where $\binom{a}{b}$ denotes the standard binomial coefficient, and the proposition follows. $\square$

Proposition 3.7. *The function $\zeta(2\rho) Z_{\pm}(\rho, \delta, r, k)$ defined in (3.12) has for fixed ρ with $Re(\rho) \geq \frac{1}{2}$ a meromorphic continuation to the whole complex δ-plane. Moreover, it has simple poles at $\delta = 0$, $\frac{1}{2} - \rho$, -1.*

Proof. For $\mathrm{Re}(\delta) \geq 0$, the right hand side of (3.12) converges absolutely and defines a holomorphic function. We want to extend this to the lower half plane, so we now consider the case when $\mathrm{Re}(\delta) \leq 0$. Recall the Mellin transform

$$\Phi_0\left(w, s; \frac{a}{r}, k\right) = \int_0^\infty [E_0(\alpha_{a,r} iy; s, k) - a_0(s, y, k)] y^{w-1} dy$$

where $\alpha_{a,r} = \begin{pmatrix} 1 & a/r \\ 0 & 1 \end{pmatrix}$.

In order to simplify notation, we set

$$\Phi_c^*\left(\rho, \delta; \frac{a}{r}, k\right) = \Phi_c\left(\frac{\rho}{2} + \frac{1}{4} + \frac{k}{4} + \delta, \frac{\rho}{2} - \frac{k}{4} + \frac{1}{4}; \frac{a}{r}, k\right)$$

for $c = 0,\ \infty,\ \frac{1}{2}$.

Substituting the Fourier expansion for E_0 and recognizing that $\alpha_{a,r} iy = \frac{a}{r} + iy$, we have

$$\Phi_0^*\left(\rho, \delta; \frac{a}{r}, k\right) =$$
$$\int_0^\infty \left[\sum_{m\neq 0} \left(\frac{y}{4}\right)^{\frac{\rho}{2}-\frac{k}{4}+\frac{1}{4}} A_m(\rho,k) K_m\left(\frac{\rho}{2} - \frac{k}{4} + \frac{1}{4}, y, k\right) e^{\frac{2\pi i m a}{r}}\right] y^{\frac{\rho}{2}+\frac{1}{4}+\frac{k}{4}+\delta} dy.$$

Reversing the order of integration and summation and simplifying, this becomes

$$\Phi_0^*\left(\rho, \delta; \frac{a}{r}, k\right) =$$
$$2^{\frac{k}{2}-\frac{1}{2}-\rho} \sum_{m\neq 0} A_m(\rho,k) e^{\frac{2\pi i m a}{r}} \int_0^\infty K_m\left(\frac{\rho}{2} - \frac{k}{4} + \frac{1}{4}, y, k\right) y^{\rho+\frac{1}{2}+\delta-1} dy.$$

Applying Proposition 3.1 to the integral, we have

$$\begin{aligned}
&\Phi_0^*\left(\rho, \delta; \frac{a}{r}, k\right) \\
&= 2^{\frac{k}{2}-\frac{1}{2}-\rho} \sum_{m\neq 0} A_m(\rho,k) e^{\frac{2\pi i m a}{r}} \frac{e^{-\frac{\pi i k}{4}} F_\pm\left(\frac{\rho}{2} + \frac{1}{4} + \frac{k}{4} + \delta, \frac{\rho}{2} - \frac{k}{4} + \frac{1}{4}, k\right)}{2^{\rho+\frac{1}{2}+3\delta} \pi^\delta |m|^{1+\delta}} \\
&= c_k(\rho,\delta) G_k(\rho,\delta) \sum_{m>0} A_m(\rho,k) e^{\frac{2\pi i m a}{r}} |m|^{-1-\delta} \\
&\quad + c_k(\rho,\delta) G_{-k}(\rho,\delta) \sum_{m<0} A_m(\rho,k) e^{\frac{2\pi i m a}{r}} |m|^{-1-\delta}
\end{aligned}$$

where

$$c_k(\rho,\delta) = i^{-\frac{k}{2}} \pi^{-\delta} 2^{\frac{k}{2}-2\rho-2\delta-1}$$

and $F_\pm$ and $G_{\pm k}(\rho,\delta)$ are defined as in Section 3.1.

We define

$$\phi_0(\rho,\delta;r,k) = r^{-1} \sum_{a \,(\mathrm{mod}\ r)} \Phi_0^*\left(\rho,\delta;\frac{a}{r},k\right).$$

Plugging in for Φ_0^* and summing, we use the fact that

$$\sum_{a \bmod r} e^{\frac{2\pi i m a}{r}} = \begin{cases} 0, & \text{if } r \nmid m; \\ r, & \text{if } r|m \end{cases}$$

to get

$$\phi_0(\rho,\delta,r,k) = c_k(\rho,\delta)\left[G_k(\rho,\delta)Z_+(\rho,\delta,r,k) + G_{-k}(\rho,\delta)Z_-(\rho,\delta,r,k)\right]. \tag{3.23}$$

Now, substitute $k+4$ in for k in (3.23) (recall that $Z_\pm(\rho,\delta,r,k\pm 4) = Z_\pm(\rho,\delta,r,k)$) and we get

$$\begin{gathered} \phi_0(\rho,\delta,r,k+4) = \\ c_{k+4}(\rho,\delta)\left[G_{k+4}(\rho,\delta)Z_+(\rho,\delta,r,k) + G_{-k-4}(\rho,\delta)Z_-(\rho,\delta,r,k)\right]. \end{gathered} \tag{3.24}$$

Solving these two equations simultaneously for $Z_+(\rho,\delta,r,k)$ and $Z_-(\rho,\delta,r,k)$, we get the identity

$$\begin{aligned} Z_\pm(\rho,\delta,r,k) = &\pm D_k(\rho,\delta)^{-1}\phi_0(\rho,\delta;r,k)G_{\mp(k+4)}(\rho,\delta)c_k(\rho,\delta)^{-1} \\ &\mp D_k(\rho,\delta)^{-1}\phi_0(\rho,\delta;r,k+4)G_{\mp k}(\rho,\delta)c_{k+4}(\rho,\delta)^{-1}, \end{aligned} \tag{3.25}$$

where

$$D_k(\rho,\delta) = G_k(\rho,\delta)G_{-k-4}(\rho,\delta) - G_{k+4}(\rho,\delta)G_{-k}(\rho,\delta)$$

is the discriminant discussed in Proposition 3.3.

The representation in (3.25) is the meromorphic continuation we are after. The functions $c_k(\rho,\delta)$ and $G_k(\rho,\delta)$ are certainly holomorphic so the only possible singularities are at the zeros of $D_k(\rho,\delta)$ and the poles of $\phi_0(\rho,\delta;r,k)$. By

Proposition 2.17, $\phi_0(\rho, \delta; r, k)$, which is a linear combination of $\Phi_0^*\left(\rho, \delta; \frac{a}{r}, k\right)$ with various values of a, has simple poles at $\delta = 0$, $\delta = \frac{1}{2} - \rho$, and $\delta = -1$. Also $Z_{\pm}(\rho, \delta, r, k) = Z_{\pm}(\rho, \delta, r, k+4)$ so if δ is such that $\delta \neq 0$, $\frac{1}{2} - \rho$, -1 and $Z_{\pm}(\rho, \delta, r, k)$ has a singularity at δ, then it must needs be that $D_k(\rho, \delta) = 0$ and $D_{k_1}(\rho, \delta) = 0$ for all $k_1 \equiv k \pmod 4$. But this contradicts Proposition 3.3. $\square$

Proposition 3.8. *Let $\epsilon > 0$, $Re(\rho) \geq \frac{1}{2}$ be fixed. Let $r > 0$ be squarefree. Let $\delta = \sigma + it$ satisfy $|\delta| > \epsilon$, $|\delta - \frac{1}{2} + \rho| > \epsilon$ and $-Re(\rho) - \frac{1}{2} - \epsilon \leq \sigma \leq \epsilon$. Then there exists a positive function $h(r)$ (independent of t, σ) such that*

$$|Z_{\pm}(\rho, \delta, r^2, k)| \ll r^{-2\sigma - 2 + 2\epsilon} |t|^{-\sigma + \epsilon} h(r)$$

where the $\ll$ constant depends at most on ρ, σ, k, ϵ. Moreover, the function $h(r)$ satisfies the condition

$$\sum_{r=1}^{\infty} h(r) r^{-1-\epsilon} \ll 1$$

where the $\ll$-symbol depends at most on ϵ.

Proof. By definition

$$Z_{\pm}(\rho, \epsilon_i t, r^2, k) = \sum_{\substack{\pm m > 0 \\ m \equiv 0 \pmod{r^2}}} A_m(\rho, k) |m|^{-1-\epsilon-it}.$$

Taking absolute values we have

$$\begin{aligned} |Z_{\pm}(\rho, \epsilon + it, r^2, k)| &\leq \sum_{\substack{\pm m > 0 \\ m \equiv 0 \pmod{r^2}}} |A_m(\rho, k)| |m|^{-1-\epsilon} \\ &= r^{-2-2\epsilon} \sum_{\pm m > 0} |A_{mr^2}(\rho, k)| |m|^{-1-\epsilon} \\ &\leq r^{-2-2\epsilon} \sum_{m \neq 0} |A_{mr^2}(\rho, k)| |m|^{-1-\epsilon}. \end{aligned}$$

Set

$$h_1(r) = \sum_{m \neq 0} |A_{mr^2}(\rho, k)||m|^{-1-\epsilon}.$$

Then, writing $m = m_0 n^2$ with m_0 square-free, and using Proposition 3.6, we have

$$\begin{aligned}|A_{mr^2}(\rho, k) &\leq |A_{m_0}(\rho, k)|\sigma(nr)2^{w(nr)} \\ &\ll |A_{m_0}(\rho, k)(nr)^{\epsilon/2}.\end{aligned}$$

Here we have used the fact that $\sigma(n) \ll n^{\epsilon/4}$ and $2^{w(n)} \ll n^{\epsilon/4}$.

Now

$$\begin{aligned}\sum_{r=1}^{\infty} \frac{h_1(r)}{r^{1+\epsilon}} &= \sum_{r=1}^{\infty} r^{-1-\epsilon} \sum_{m \neq 0} |A_{mr^2}(\rho, k)|m|^{-1-\epsilon} \\ &\ll \sum_{r=1}^{\infty} r^{-1-\epsilon/2} \sum_{n \neq 0} n^{-2-\frac{3}{2}\epsilon} \sum_{m_0 \text{ squarefree}} |A_{m_0}(\rho, k)||m_0|^{-1-\epsilon} \\ &\ll \sum_{m_0 \text{ squarefree}} |A_{m_0}(\rho, k)||m_0|^{-1-\epsilon} \\ &\ll 1,\end{aligned} \tag{3.26}$$

making use of the well known fact that for m_0 squarefree, $L(\rho, \chi_{m_0}) \ll {m_0}^{\epsilon'}$ for any positive number ϵ'.

This proves the proposition when $\sigma = \epsilon$.

Now we consider the other edge of the strip, setting

$$\delta = -\rho - \frac{1}{2} - \epsilon - it$$

with ρ and ϵ satisfying the conditions in the statement of the proposition.

We will consider first the case when r is odd.

We consider

$$\phi_0(\rho, \delta, r^2, k) = r^{-2} \sum_{a \pmod{r^2}} \Phi_o^*\left(\rho, \delta; \frac{a}{r^2}, k\right) \tag{3.27}$$

as defined in the proof of Proposition 3.7. Now, reducing the fraction $\frac{a}{r^2}$ to lowest terms, we can rewrite this as

$$\phi_0(\rho, \delta, r^2, k) = r^{-2} \sum_{d|r^2} \sum_{\substack{u \,(\mathrm{mod}\ d) \\ (u,d)=1}} \Phi_0^* \left(\rho, \delta; \frac{u}{d}, k\right).$$

Using Proposition 2.17 (this is where we need r odd), with $w = \frac{\rho}{2} + \frac{1}{4} + \frac{k}{4} + \delta$ and $s = \frac{\rho}{2} - \frac{k}{4} + \frac{1}{4}$, we get

$$\begin{aligned} \Phi_0^* \left(\rho, \delta; \frac{u}{d}, k\right) &= \Phi_0 \left(w, s; \frac{u}{d}, k\right) \\ &= \Lambda_\infty \left(w, \frac{a}{d}, k\right) \Phi_\infty \left(\frac{k}{2} - w, s; \frac{a}{d}, k\right) \qquad (3.28) \\ &= \Lambda_\infty \left(w, \frac{a}{d}, k\right) \Phi_\infty^* \left(\rho, \epsilon + it; \frac{a}{d}, k\right) \end{aligned}$$

where

$$-4au \equiv 1 \,(\mathrm{mod}\ d)$$

and

$$\Lambda_\infty \left(w, \frac{a}{d}, k\right) = (2d)^{k/2-2w}(-2i)^{k/2}\epsilon_d^{-k} \left(\frac{a}{d}\right).$$

Put $\delta' = \epsilon + it$, $w' = \frac{\rho}{2} + \frac{1}{4} + \frac{k}{4} + \delta'$, and $s = \frac{\rho}{2} - \frac{k}{4} + \frac{1}{4}$. Using Proposition 2.12, we have

$$\begin{aligned} \Phi_\infty^* \left(\rho, \delta'; \frac{a}{d}, k\right) &= \Phi_\infty \left(\frac{\rho}{2} + \frac{1}{4} + \frac{k}{4} + \delta', \frac{\rho}{2} - \frac{k}{4} + \frac{1}{4}; \frac{a}{d}, k\right) \\ &= \int_0^\infty \left[E_\infty(\alpha_{a,d} iy, s, k) - b_0(s, y, k)\right] y^{w'-1} dy \\ &= \int_0^\infty \left[\sum_{m \neq 0} b_m(s, y, k) e^{2\pi i m \frac{a}{d}}\right] y^{w'-1} dy \\ &= \int_0^\infty \sum_{m \neq 0} (1 + i^k) c_m(s, k) A_m(\rho, k) K_m(s, y, k) e^{2\pi i m \frac{a}{d}} y^{w'+s-1} dy \\ &= (1 + i^k) \sum_{m \neq 0} c_m(s, k) A_m(\rho, k) e^{2\pi i m \frac{a}{d}} \int_0^\infty K_m(s, y, k) y^{w'+s-1} dy. \end{aligned}$$

By Proposition 3.1, this becomes

$$\begin{aligned}
&\Phi^*_\infty\left(\rho,\delta';\frac{a}{d},k\right)\\
=&(1+i^k)\sum_{m>0} c_m(s,k)A_m(\rho,k)e^{2\pi i m\frac{a}{d}}\frac{i^{-k/2}G_k(\rho,\delta')}{2^{\rho+\frac{1}{2}+2\delta'}\pi^{\delta'}|m|^{1+\delta'}}\\
&+(1+i^k)\sum_{m<0} c_m(s,k)A_m(\rho,k)e^{2\pi i m\frac{a}{d}}\frac{i^{-k/2}G_{-k}(\rho,\delta')}{2^{\rho+\frac{1}{2}+2\delta'}\pi^{\delta'}|m|^{1+\delta'}}\\
=&(1+i^k)2^{\rho+\frac{1}{2}-\frac{k}{2}}c_k(\rho,\delta')\sum_{m>0}A_m(\rho,k)c_m(s,k)e^{2\pi i m\frac{a}{d}}G_k(\rho,\delta')|m|^{-1-\delta'}\\
&+(1+i^k)2^{\rho+\frac{1}{2}-\frac{k}{2}}c_k(\rho,\delta')\sum_{m<0}A_m(\rho,k)c_m(s,k)e^{2\pi i m\frac{a}{d}}G_{-k}(\rho,\delta')|m|^{-1-\delta'},
\end{aligned} \tag{3.29}$$

where

$$c_k(\rho,\delta') = i^{-k/2}\pi^{-\delta'}2^{k/2-2\rho-2\delta'-1}.$$

Referring to the definition of $c_m(s,k)$ given in Proposition 2.12 (recall that $s=\frac{\rho}{2}-\frac{k}{4}+\frac{1}{4}$), it is easy to see that $c_m(s,k)\ll 1$ if $4\nmid m$ and if $4|m$, $c_m(s,k)\ll \log_4|m|$. In particular,

$$c_m(s,k)\ll \log|m|$$

and the $\ll$-symbol depends only on k, and $\mathrm{Re}(s)$ (which depends only on ρ, ϵ and k).

We substitute (3.29) into (3.28) and notice that summing over u modulo d with $(u,d)=1$ is the same as summing over a modulo d with $(a,d)=1$ and $-4au\equiv 1 \pmod d$ (since inverses are unique modulo d).

Also we have

$$|c_k(\rho,\delta')|\ll 1$$

$$|(1+i^k)|\ll 1$$

$$|2^{\rho-k/2+1/2}|\ll 1$$

$$|c_m(s,k)| \ll \log|m| \ll |m|^{\epsilon/2}$$

and

$$\left|\Lambda\left(w, \frac{a}{d}, k\right)\right| \ll d^{\rho+\frac{1}{2}+2\epsilon}.$$

(Recall $w = \frac{\rho}{2} + \frac{1}{4} + \frac{k}{4} + \delta$ and $\delta = -\rho - \frac{1}{2} - \epsilon - it$.) In all the above cases, the $\ll$ constants depend only on ρ, ϵ, and k.

In the last asymptotic, factoring out the term $\left(\frac{a}{d}\right)$ from Λ_∞ does not affect the inequality. We will borrow this term from Λ_∞ and group it with the $e^{2\pi i m \frac{a}{d}}$. Doing this and summing over a modulo d with $(a,d)=1$ yields the Gauss sum $g(m,d)$.

In summary we have

$$\begin{aligned}
&\sum_{\substack{u \,(\mathrm{mod}\ d)\\ (u,d)=1}} \Phi_0^*\left(\rho, \delta; \frac{u}{d}, k\right)\\
&\quad \ll G_k(\rho, \epsilon+it)\sum_{m>0} |A_m(\rho,k)||g(m,d)||m|^{-1-\epsilon/2} d^{\rho+\frac{1}{2}+2\epsilon}\\
&\qquad + G_{-k}(\rho, \epsilon+it)\sum_{m<0} |A_m(\rho,k)||g(m,d)||m|^{-1-\epsilon/2} d^{\rho+\frac{1}{2}+2\epsilon}.
\end{aligned} \tag{3.30}$$

Here the $\ll$ constant is independent of d and t.

The Gauss sum

$$g(m,d) = \sum_{a \,(\mathrm{mod}\ d)} \left(\frac{a}{d}\right) e^{2\pi i a m/d}$$

satisfies the bound

$$|g(m,d)| \le \begin{cases} 0, & r_2 \nmid m;\\ r_1^{\frac{1}{2}} r_2 \left(\frac{m}{r_2}, r_2\right), & r_2 | m \end{cases} \tag{3.31}$$

where $d = r_1 r_2^2$, r_1 is squarefree, and $(r_1, r_2) = 1$.

To see this, we use (3.14) (since d is odd) and (3.17) together to immediately get

$$|g(m,d)| = r_1^{\frac{1}{2}} |g(m, r_2^2)|.$$

Now

$$g(m, r_2^2) = \sum_{a=1}^{r_2^2} \left(\frac{a}{r_2^2}\right) e^{\frac{2\pi i a m}{r_2^2}}.$$

Writing $a = br_2 + c$ this becomes

$$g(m, r_2^2) = \sum_{b=1}^{r_2} \sum_{c=1}^{r_2} \left(\frac{br + c}{r_2^2}\right) e^{\frac{2\pi i (br+c)m}{r_2^2}} = \sum_{b=1}^{r_2} e^{\frac{2\pi i b m}{r_2}} \sum_{c=1}^{r_2} \left(\frac{c}{r_2^2}\right) e^{\frac{2\pi i c m}{r_2^2}}.$$

The first sum is 0 if $r_2 \nmid m$ and is r_2 otherwise. For the second sum, let $u = \left(\frac{m}{r_2}, r_2\right)$, $v = r_2/s$, and $z = \frac{m}{r_2}/s$. Writing $c = pv + q$ and doing the usual algebraic manipulations, the second sum becomes

$$\sum_{q=1}^{v} \left(\frac{q}{v^2}\right) e^{\frac{2\pi i q z}{v}} \sum_{p=1}^{u} \left(\frac{pv + q}{u^2}\right).$$

Taking absolute values, this is

$$\leq u \left| \sum_{q=1}^{v} \left(\frac{q}{v^2}\right) e^{\frac{2\pi i z}{v}} \right|.$$

We notice that since

$$\left(\frac{q}{v^2}\right) = \begin{cases} 1, & (q, v) = 1; \\ 0, & (q, v) \neq 1, \end{cases}$$

the sum in absolute values is simply a sum of the primitive v^{th} roots of unity which is $\mu(v)$, giving us the bound for $g(m, d)$.

By Proposition 3.2, for $|t|$ sufficiently large, we have

$$G_{\pm k}(\rho, \epsilon + it) \ll \Gamma\left(\frac{\rho}{2} + \frac{1}{4} \pm \frac{k}{4} + \epsilon + it\right). \tag{3.32}$$

Now we use (3.27), (3.30), and (3.32) and we get the following:

$$\begin{aligned} \phi_0(\rho, \delta, r^2, k) =& r^{-2} \sum_{a \,(\text{mod } r^2)} \Phi_o^*\left(\rho, \delta; \frac{a}{r^2}, k\right) \\ \ll& r^{-2} \Gamma\left(\frac{\rho}{2} + \frac{1}{4} \pm \frac{k}{4} + \epsilon + it\right) \\ & \times \sum_{d | r^2} \left[\sum_{m \neq 0} |A_m(\rho, k)||g(m, d)||m|^{-1-\epsilon/2}\right] d^{\rho + \frac{1}{2} + 2\epsilon}. \end{aligned}$$

Recall that r is squarefree. For any $d|r^2$, we can write $d = r_1 r_2^2$ with $r_1|r$ (so r_1 is squarefree), $r_2|r$, and $(r_1, r_2) = 1$. Doing this, we also put $r = r_1 r_2 r_3$. By (3.31), we get

$$\phi_0(\rho, \delta, r^2, k) \ll r^{2\rho-1+4\epsilon}\Gamma\left(\frac{\rho}{2} + \frac{1}{4} + \frac{|k|}{4} + \epsilon + it\right) h_2(r) \tag{3.33}$$

where

$$h_2(r) = \sum_{r_1 r_2 r_3 = r} r_1^{-\rho-2\epsilon} r_2^{-\epsilon/2} r_3^{-2\rho-1-4\epsilon} \left[\sum_{n\neq 0} |A_{nr_2}(\rho, k)|(n, r_2)|n|^{-1-\epsilon/2}\right]. \tag{3.34}$$

Now, Stirling's Formula implies the asymptotic (see page 47 of [4])

$$|t|^{\sigma-\frac{1}{2}} e^{-(\pi/2)|t|} \ll \Gamma(\sigma + it) \ll |t|^{\sigma-\frac{1}{2}} e^{-(\pi/2)|t|}, \ (|t| \to \infty).$$

Using the left hand side of this, from Proposition 3.3, we easily see that

$$D_k(\rho, \delta) \gg |t|^{-\rho-\frac{1}{2}-2\epsilon} e^{-\pi|t|}$$

(recall that $\delta = -\rho - \frac{1}{2} - \epsilon - it$).

For $k > 0$, using the right hand side together with (3.32) and (3.33), we get

$$\phi_0(\rho, \delta, r^2, k) G_{-k-4}(\rho, \delta) \ll r^{2\rho-1+4\epsilon} |t|^{-2} e^{-\pi|t|} h_2(r)$$

and

$$\phi_0(\rho, \delta, r^2, k+4) G_{-k}(\rho, \delta) \ll r^{2\rho-1+4\epsilon} e^{-\pi|t|} h_2(r).$$

Putting these together, we get

$$\frac{\phi_0(\rho, \delta, r^2, k) G_{-k-4}(\rho, \delta)}{D_k(\rho, \delta)} \ll r^{2\rho-1+4\epsilon} |t|^{\rho-\frac{3}{2}+2\epsilon} h_2(r)$$

and

$$\frac{\phi_0(\rho, \delta, r^2, k+4) G_{-k}(\rho, \delta)}{D_k(\rho, \delta)} \ll r^{2\rho-1+4\epsilon} |t|^{\rho+\frac{1}{2}+2\epsilon} h_2(r).$$

We put these into (3.25) and get

$$Z_+\left(\rho, -\rho - \frac{1}{2} - \epsilon - it, r^2, k\right) \ll r^{2\rho-1+4\epsilon}|t|^{\rho+\frac{1}{2}+2\epsilon}h_2(r) \tag{3.35}$$

since the term $c_k(\rho,\delta)^{-1} \ll 1$.

Because $Z_\pm$ depends only on k modulo 4, we can remove the restriction $k > 0$, and the bound still holds.

For Z_-, we use a similar argument, except this time around we assume $k < -3$. We get the same bound for $D_k(\rho,\delta)$ but we have now

$$\phi_0(\rho, \delta, r^2, k)G_{k+4}(\rho, \delta) \ll r^{2\rho-1+4\epsilon}e^{-\pi|t|}h_2(r)$$

and

$$\phi_0(\rho, \delta, r^2, k+4)G_k(\rho, \delta) \ll r^{2\rho-1+4\epsilon}|t|^{-2}e^{-\pi|t|}h_2(r)$$

$(|k+4| = -k-4)$.

This gives

$$\frac{\phi_0(\rho, \delta, r^2, k)G_{k+4}(\rho, \delta)}{D_k(\rho, \delta)} \ll r^{2\rho-1+4\epsilon}|t|^{\rho+\frac{1}{2}+2\epsilon}h_2(r)$$

and

$$\frac{\phi_0(\rho, \delta, r^2, k+4)G_k(\rho, \delta)}{D_k(\rho, \delta)} \ll r^{2\rho-1+4\epsilon}|t|^{\rho-\frac{3}{2}+2\epsilon}h_2(r).$$

Putting these into (3.25), we get the bound

$$Z_-\left(\rho, -\rho - \frac{1}{2} - \epsilon - it, r^2, k\right) \ll r^{2\rho-1+4\epsilon}|t|^{\rho+\frac{1}{2}+2\epsilon}h_2(r) \tag{3.36}$$

which again must also hold for all k since Z_- depends only on the congruence class of k modulo 4.

So we have the following bounds along the edges of the strip

$$Z_\pm(\rho, \epsilon + it, r^2, k) \ll r^{-2-2\epsilon}h(r),$$

$$Z_{\pm}\left(\rho, -\rho-\frac{1}{2}-\epsilon-it, r^2, k\right) \ll r^{2\rho-1+4\epsilon}|t|^{\rho+\frac{1}{2}+2\epsilon}h(r)$$

where

$$h(r) \ll h_1(r)+h_2(r).$$

We apply the Phragmen-Lindelöf Principle to obtain a bound inside the strip. The reader is referred to page 15 of Hardy [6] for a proof of the Theorem. We apply the result to $Z_{\pm}(\rho, \sigma+it, r^2, k)$ and get

$$Z_{\pm}(\rho, \sigma+it, r^2, k) \ll r^{-2\sigma-2+2\epsilon}|t|^{-\sigma+\epsilon}h(r)$$

proving the first part of the proposition.

For the second part of the theorem, (3.26) tells us it is enough to show

$$\sum_{r=1}^{\infty} h_2(r)r^{-1-\epsilon} \ll 1. \tag{3.37}$$

To see this we substitute (3.34) into the left hand side (3.37) and get

$$\begin{aligned}
\sum_{r=1}^{\infty} h_2(r)r^{-1-\epsilon} &= \sum_{r=1}^{\infty}\sum_{r_1r_2r_3=r} r_1^{-\rho-1-3\epsilon}r_2^{-1-\frac{3}{2}\epsilon}r_3^{-2\rho-2-5\epsilon}\sum_{n\neq 0}\frac{|A_{nr_2}(\rho,k)|(n,r_2)}{|n|^{1+\epsilon/2}}\\
&\leq \sum_{r_1=1}^{\infty} r_1^{-\rho-1-3\epsilon}\sum_{r_3=1}^{\infty} r_3^{-2\rho-2-5\epsilon}\sum_{n\neq 0}\sum_{r_2=1}^{\infty} r_2^{-\epsilon}\frac{|A_{nr_2}(\rho,k)|(n,r_2)}{|nr_2|^{1+\epsilon/2}}\\
&\ll \sum_{n\neq 0}\sum_{r_2=1}^{\infty}\frac{|A_{nr_2}(\rho,k)|(n,r_2)}{|nr_2|^{1+\epsilon/2}}\\
&= \sum_{m\neq 0}\sum_{d_1d_2=m}\frac{|A_m(\rho,k)|(d_1,d_2)}{|m|^{1+\epsilon/2}}.
\end{aligned} \tag{3.38}$$

Write $m = m_0n^2$ where m_0 is squarefree and $n > 0$. Then $(d_1, d_2)|n$, and hence $(d_1, d_2) \leq n$. This implies that

$$(d_1, d_2)|m|^{-1-\epsilon/2} \leq |m_0|^{-1-\epsilon/2}n^{-1-\epsilon}. \tag{3.39}$$

So we have

$$\sum_{r=1}^{\infty} h_2(r) r^{-1-\epsilon} \ll \sum_{m_0 \text{ squarefree}} \sum_{n=1}^{\infty} \sum_{d_1 d_2 = m_0 n^2} \frac{|A_{m_0 n^2}(\rho, k)|}{|m_0|^{1+\epsilon/2} n^{1+\epsilon}}$$
$$\ll \sum_{m_0 \text{ squarefree}} \sum_{n=1}^{\infty} \sum_{d_1 d_2 = m_0 n^2} \frac{|A_{m_0 n^2}(\rho, k)|}{|m_0|^{1+\epsilon/4} n^{1+\epsilon/2}}$$

since $\sum_{d_1 d_2 = m} 1 \ll m^{\epsilon/4}$. Using Proposition 3.6 and the fact that $\sigma(n) \ll n^{\epsilon/8}$ and $2^{w(n)} \ll n^{\epsilon/8}$ we have

$$\sum_{r=1}^{\infty} h_2(r) r^{-1-\epsilon} \ll \sum_{m_0 \text{ squarefree}} \frac{|A_{m_0}(\rho, k)|}{|m_0|^{1+\epsilon/4}} \sum_{n=1}^{\infty} \frac{\sigma(n) 2^{w(n)}}{n^{1+\epsilon/2}}$$
$$\ll \sum_{m_0 \text{ squarefree}} \frac{|A_{m_0}(\rho, k)|}{|m_0|^{1+\epsilon/4}}$$
$$\ll 1.$$

This proves the proposition when r is odd. To generalize to the case where r is even (so that $4|r^2$), instead of using Proposition 2.17 to get (3.28) we must consider the three possible cases: where d is odd, where $4|d$, or where d is even but $4 \nmid d$. We must then instead use Propositions 2.17, 2.18, or 2.19 respectively. The arguments where d is odd are of course the same, and the arguments in the other two cases are very similar to the odd case, using Propositions 2.10 and 2.13 in place of Proposition 2.12.

We will omit the details. □

3.3 Asymptotics for Dirichlet Series

It was shown in Section 3.2 that for $\mathrm{Re}(\rho) \geq \frac{1}{2}$, the function $\zeta(2\rho) Z_{\pm}(\rho, \delta, r^2, k)$ has a meromorphic continuation to the whole complex δ-plane whose only singularities, for $\rho \neq \frac{1}{2}$, are simple poles at $\delta = 0$, $\frac{1}{2} - \rho$, -1. For $\rho = \frac{1}{2}$, it has a double pole at $\delta = 0$ and a simple pole at $\delta = -1$.

We now give the residue at 0.

Proposition 3.9. *For $r > 0$ and squarefree, $Re(\rho) \geq \frac{1}{2}$, $\rho \neq \frac{1}{2}$, let $R_{\pm}(\rho, a, r^2, k)$ denote the residue of $\zeta(2\rho)Z_{\pm}(\rho, \delta, r^2, k)$ at $\delta = a$. We have*

$$R_{\pm}(\rho, 0, r^2, k) = \begin{cases} \zeta(2\rho)r^{-2}\sum_{d|r} d^{-2\rho-1}\phi(d^2) & (r,2) = 1 \\ \zeta(2\rho)r^{-2}\sum_{d|r_0} d^{-2\rho-1}\phi(d^2) & r = 2r_0. \end{cases}$$

Proof. The meromorphic continuation of $Z_{\pm}(\rho, \delta, r^2, k)$ is given in (3.25). The only poles occur in connection with the function $\phi_0(\rho, \delta, r^2, k)$ which is defined in (3.27) as a linear combination of the functions $\Phi_0^*\left(\rho, \delta, \frac{a}{r^2}, k\right)$, whose poles and residues are given in Propositions 2.17, 2.18, and 2.19.

So, we write

$$\phi_0(\rho, \delta, r^2, k) = r^{-2}\sum_{d|r^2}\sum_{\substack{u \,(\mathrm{mod}\ d)\\(u,d)=1}} \Phi_0^*\left(\rho, \delta; \frac{u}{d}, k\right)$$

and add up the residues. Recall that the pole at $\delta = 0$ corresponds to the pole $w = s + \frac{k}{2}$ in Propositions 2.17, 2.18, and 2.19.

Let us first consider the case when r is odd.

If d is a divisor of r^2, and if u coprime to d, by Proposition 2.17, the residue of $\Phi_0^*\left(\rho, \delta; \frac{u}{d}, k\right)$ is

$$\Lambda_\infty\left(\frac{\rho}{2} + \frac{k}{4} + \frac{1}{4}, \frac{u}{d}, k\right) = (2d)^{-\rho-\frac{1}{2}}(-2i)^{k/2}\epsilon_d^{-k}\left(\frac{u}{d}\right).$$

Recall that we have assumed that $k \equiv 1 \,(\mathrm{mod}\ 4)$, so $\epsilon_d^{-k} = \epsilon_d$.

Summing over u, the only part that is dependent on u is the quadratic residue symbol. Since d is odd, this is a character modulo d unless d is a perfect square, so the summing over all u gives 0 if d is not a perfect square. If d is a perfect square, then summing over u gives

$$(2d)^{-\rho-\frac{1}{2}}(-2i)^{k/2}\epsilon_d\phi(d).$$

If d is a square divisor of r^2, then we can write $d = d'^2$ where d' is a divisor of r. Thus, summing over all d we get

$$r^{-2}2^{k/2-\rho-\frac{1}{2}}i^{-k/2}\sum_{d|r} d^{-2\rho-1}\phi(d^2)$$

(recall that $\epsilon_{d^2} = 1$) as the total residue of $\phi_0(\rho, \delta, r^2, k)$ at $\delta = 0$.

Recall that

$$c_k(\rho, 0) = i^{-k/2}2^{k/2-2\rho-1}.$$

Substituting back into 3.27 we have

$$R_\pm(\rho, 0, r^2, k) = \zeta(2\rho)r^{-2}\sum_{d|r} d^{-2\rho-1}\phi(d^2)\frac{2^{\rho+\frac{1}{2}}\left(G_{\mp(k+4)}(\rho, 0) - G_{\mp k}(\rho, 0)\right)}{\pm D_k(\rho, 0)}.$$

We use (3.5) with $\delta = 0$ and easily get the identity

$$\begin{aligned} G_{\pm k}(\rho, 0) + G_{\mp k}(\rho, 0) &= 2^{\frac{1}{4}+\frac{\rho}{2}+\frac{k}{4}}F\left(\frac{3}{4} - \frac{\rho}{2} - \frac{k}{4}, \frac{1}{4} + \frac{\rho}{2} - \frac{k}{4}, \frac{3}{4} - \frac{\rho}{2} - \frac{k}{4}; \frac{1}{2}\right) \\ &= 2^{\frac{1}{4}+\frac{\rho}{2}+\frac{k}{4}}\frac{1}{\Gamma\left(\frac{1}{4} + \frac{\rho}{2} - \frac{k}{4}\right)}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{1}{4} + \frac{\rho}{2} - \frac{k}{4} + n\right)}{n!2^n}. \end{aligned}$$

We use the following identity:

$$\begin{aligned} \frac{1}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)}{n!2^n} &= \sum_{n=0}^{\infty}\frac{\alpha(\alpha+1)\cdots(\alpha+(n-1))}{n!2^n} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n(-\alpha)(-\alpha-1)\cdots(-\alpha-(n-1))}{n!2^n} \\ &= \sum_{n=0}^{\infty}\binom{-\alpha}{n}\left(-\frac{1}{2}\right)^n \\ &= \left(1 - \frac{1}{2}\right)^{-\alpha} \\ &= 2^\alpha. \end{aligned}$$

Here, the last series is just a Binomial Series. This identity gives us

$$G_{\pm k}(\rho, 0) + G_{\mp k}(\rho, 0) = 2^{\rho+\frac{1}{2}}$$

which is independent of k.

Using this identity with k and $k+4$, we get

$$2^{\rho+\frac{1}{2}}\left(G_{\mp(k+4)}(\rho,0)-G_{\mp k}(\rho,0)\right)$$
$$=G_{\pm k}(\rho,0)G_{\mp(k+4)}(\rho,0)-G_{\pm(k+4)}(\rho,0)G_{\mp k}(\rho,0)$$
$$=\pm D_k(\rho,0)$$

proving the proposition in the case where r is odd.

For the case where r is even, we simply notice that the functions

$$\Phi_0^*\left(\rho,\delta;\frac{u}{d},k\right)$$

have residue 0 at $\delta=0$ if d is even by Propositions 2.18 and 2.19. Thus we need only compute the residues for divisors d of r^2 which are odd (so that d is a divisor of r_0^2 where $r=2r_0$). The subsequent computations are the same as before. $\square$

For simplicity, we now restrict our work to the case where $\mathrm{Re}(\rho)\geq 1$ and $k=1$.

Proposition 3.10. *Let $\rho=\beta+it$ with $\beta\geq 1$. Let ϵ be fixed with $0<\epsilon<\frac{1}{2}$. Then for $x\geq r^2$ with r squarefree*

$$\zeta(2\rho)\sum_{\substack{0<\pm m<x\\ m\equiv 0\,(mod\ r^2)}}A_m(\rho,1)=R_\pm(\rho,0,r^2,1)x+O((x/r^2)^{\frac{2}{3}+\epsilon}x^\epsilon h(r))$$

where $R_\pm(\rho,0,r^2,1)$ is given in Proposition 3.9 and $h(r)$ is given in Proposition 3.8.

Proof. Fix T with $1\leq T<x/r^2$. Let

$$I=\frac{1}{2\pi i}\int_{\Re}\zeta(2\rho)Z_\pm(\rho,\delta,r^2,k)\frac{x^{1+\delta}}{1+\delta}d\delta$$

where $\Re$ is the rectangle with corners $\epsilon \pm iT$ and $\epsilon - \frac{1}{2} \pm iT$, traversed in a counter clockwise direction. Then by Cauchy's theorem and Proposition 3.9, we have

$$I = R_{\pm}(\rho, 0, r^2, 1)x.$$

On the other hand, we can compute the contribution of each side in the following manner.

First, the left hand edge is

$$-\int_{\epsilon-\frac{1}{2}-iT}^{\epsilon-\frac{1}{2}+iT} \zeta(2\rho) Z_{\pm}(\rho, \delta, r^2, k) \frac{x^{1+\delta}}{1+\delta} d\delta.$$

Using Proposition 3.8, this is

$$\begin{aligned}
&\ll \zeta(2\rho) h(r) r^{-1} x^{\frac{1}{2}+\epsilon} \int_{-T}^{T} \frac{|t|^{\frac{1}{2}}}{|\frac{1}{2}+\epsilon+it|} dt \\
&\leq \zeta(2\rho) h(r) r^{-1} x^{\frac{1}{2}+\epsilon} 4\sqrt{T}.
\end{aligned}$$

Next, the bottom edge is given by

$$\int_{\epsilon-\frac{1}{2}-iT}^{\epsilon-iT} \zeta(2\rho) Z_{\pm}(\rho, \delta, r^2, k) \frac{x^{1+\delta}}{1+\delta} d\delta.$$

Again, using Proposition 3.8, this is

$$\begin{aligned}
&\ll \zeta(2\rho) h(r) r^{-2+2\epsilon} T^{\epsilon} x \int_{\epsilon-\frac{1}{2}}^{\epsilon} \frac{\left(\frac{x}{Tr^2}\right)^{\sigma}}{|1+\sigma-iT|} d\sigma \\
&\leq h(r) r^{-2+2\epsilon} x \frac{1}{T} \int_{\epsilon-\frac{1}{2}}^{\epsilon} \left(\frac{x}{Tr^2}\right)^{\sigma} d\sigma \\
&= h(r) r^{-2+2\epsilon} x \left[\frac{\left(\frac{x}{r^2}\right)^{\epsilon}}{T \log\left(\frac{x}{Tr^2}\right)} - \frac{\left(\frac{x}{r^2}\right)^{\epsilon-\frac{1}{2}}}{\sqrt{T} \log\left(\frac{x}{Tr^2}\right)} \right].
\end{aligned}$$

We get a similar expression for the top edge.

Along the right hand edge, $\mathrm{Re}(\delta) > 0$ so we can use the series representation of $Z_{\pm}(\rho, \delta, r^2, 1)$. The integral then is

$$\int_{\epsilon-iT}^{\epsilon_i T} \zeta(2\rho) Z_{\pm}(\rho, \delta, r^2, k) \frac{x^{1+\delta}}{1+\delta} d\delta$$

$$= \zeta(2\rho) \sum_{\substack{\pm m>0 \\ m\equiv 0 \,(\mathrm{mod}\ r^2)}} A_m(\rho,1) \int_{1+\epsilon-iT}^{1+\epsilon+iT} \frac{\left(\frac{x}{|m|}\right)^s}{s} ds.$$

It is not hard to show that (see pages 105-6 of Davenport [3])

$$\left| \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{y^s}{s} ds \right| < \begin{cases} \frac{y^c}{T|\log y|}, & \text{if } y<1; \\ \frac{1}{2}+\frac{c}{T}, & \text{if } y=1; \\ 1+\frac{y^c}{T\log y}, & \text{if } y>1. \end{cases}$$

So the above integral is

$$\ll \zeta(2\rho) \sum_{\substack{0<\pm m<x \\ m\equiv 0 \,(\mathrm{mod}\ r^2)}} A_m(\rho,1) + \zeta(2\rho) \sum_{\substack{\pm m>0 \\ m\equiv 0 \,(\mathrm{mod}\ r^2)}} A_m(\rho,1) \frac{\left(\frac{x}{m}\right)^{1+\epsilon}}{T\left|\log\left(\frac{x}{m}\right)\right|}.$$

Consider the sum on the right. Writing $m = m_0 n^2 r^2$ with m_0 squarefree, by Proposition 3.6, we see that

$$\begin{aligned}
&\left| \sum_{\substack{\pm m>0 \\ m\equiv 0 \,(\mathrm{mod}\ r^2)}} A_m(\rho,1) \frac{\left(\frac{x}{m}\right)^{1+\epsilon}}{T\left|\log\left(\frac{x}{m}\right)\right|} \right| \\
&\le \frac{x^{1+\epsilon}}{T} \sum_{m_0 \text{ squarefree}} \frac{|A_{m_0}(\rho,1)|}{m_0^{1+\epsilon}} \sum_{n=1}^{\infty} \frac{\sigma(nr)2^{w(nr)}(nr)^{-2-2\epsilon}}{|\log x - \log|m_0 n^2 r^2||} \\
&\ll \frac{x^{1+\epsilon}}{T} \sum_{m_0 \text{ squarefree}} \frac{|A_{m_0}(\rho,1)|}{m_0^{1+\epsilon}} \sum_{n=1}^{\infty} (nr)^{-2-\frac{3}{2}\epsilon} \\
&\ll \frac{x^{1+\epsilon}}{T} r^{-2-\frac{3}{2}\epsilon} \sum_{m_0 \text{ squarefree}} \frac{|A_{m_0}(\rho,1)|}{m_0^{1+\epsilon}} \\
&\ll \frac{x^{1+\epsilon}}{Tr^{2+\frac{3}{2}\epsilon}}.
\end{aligned}$$

Again, we have used the fact that $\sigma(n) \ll n^{\epsilon/4}$ and $2^{w(n)} \ll n^{\epsilon/4}$.

We now have that the contribution of the right hand side is

$$\sum_{\substack{0<\pm m<x \\ m\equiv 0 \,(\mathrm{mod}\ r^2)}} A_m(\rho,1) + O\left(\frac{x^{1+\epsilon}}{Tr^{2+\frac{3}{2}\epsilon}}\right).$$

Now take

$$T = \left(\frac{x}{r^2}\right)^{1/3}.$$

One easily checks that the contribution on the right hand side is now

$$\sum_{\substack{0<\pm m<x \\ m\equiv 0\,(\mathrm{mod}\ r^2)}} A_m(\rho,1) + O\left(\left(\frac{x}{r^2}\right)^{\frac{2}{3}+\epsilon}(r^2)^{\epsilon/4}\right).$$

The contribution on the left becomes

$$O\left(\left(\frac{x}{r^2}\right)^{\frac{2}{3}+\epsilon}(r^2)^{\epsilon}h(r)\right)$$

and the contributions from the top and the bottom become

$$O\left(\left(\frac{x}{r^2}\right)^{\frac{2}{3}+\epsilon}(r^2)^{\epsilon}h(r)\right).$$

Since $x \geq r^2$, putting these together gives the proposition. $\square$

We now prove the asymptotic relations. Again, they are not better than other known results (see [5]). It is the techniques used up to this point and hereafter that are of interest.

Theorem 3.11. *Let $\epsilon > 0$ be fixed. Let χ_m for m squarefree be defined as*

$$\chi_m(n) = \begin{cases} \left(\frac{m}{n}\right), & m \equiv 1\,(mod\ 4); \\ \left(\frac{4m}{n}\right), & m \equiv 2,3\,(mod\ 4). \end{cases}$$

For $Re(\rho) \geq 1$, we have

$$\sum_{\substack{1<\pm m<x \\ m \text{ squarefree}}} L(\rho,\chi_m) = c(\rho)x + O(x^{\frac{2}{3}+\epsilon}),$$

where

$$c(\rho) = \frac{3}{4}(1-2^{-2\rho})\zeta(2\rho)\prod_{p\neq 2}(1-p^{-2}-p^{-2\rho-1}+p^{-2\rho-2})$$

and the O-constant depends at most on ρ and ϵ.

Theorem 3.12. *Let $Re(\rho) \geq 1$. The Dirichlet series*

$$Z_{\pm}(\rho, w) = \sum_{\substack{\pm m>1 \\ m \text{ squarefree}}} L(\rho, \chi_m)|m|^{-w}$$

converges absolutely for $Re(w) > 1$. It has a meromorphic continuation to the half-plane $Re(w) > \frac{1}{2}$ with a simple pole at $w = 1$. The residue at $w = 1$ is $c(\rho)$. Finally, for $\frac{1}{2} <Re(w) \leq 1$ and $\epsilon > 0$, we have the growth estimate

$$Z_{\pm}(\rho, w) \ll |Im(w)|^{1-Re(w)+\epsilon}, \quad |Im(w)| \to \infty$$

where the $\ll$ constant depends at most on ρ, ϵ.

Proof of Theorems 3.11 and 3.12. Notice

$$\begin{aligned}\zeta(2\rho) \sum_{\substack{0<\pm m<x \\ m \text{ squarefree}}} A_m(\rho, 1) &= \zeta(2\rho) \sum_{0<\pm m<x} A_m(\rho, 1) \sum_{r^2|m} \mu(r) \\ &= \zeta(2\rho) \sum_{r<\sqrt{x}} \mu(r) \sum_{\substack{0<\pm m<x \\ m\equiv 0 \,(\mathrm{mod}\ r^2)}} A_m(\rho, 1).\end{aligned}$$

By Proposition 3.9, we have

$$\begin{aligned}&\sum_{\substack{0<\pm m<x \\ m \text{ squarefree}}} A_m(\rho, 1) \\ &\qquad = \sum_{r<\sqrt{x}} \mu(r) R_{\pm}(\rho, 0, r^2, 1)x + O\left(\sum_{r<\sqrt{x}} \left[\left(\frac{x}{r^2}\right)^{\frac{2}{3}+\epsilon} x^{\epsilon} h(r) \right] \right).\end{aligned} \tag{3.40}$$

By Proposition 3.8,

$$\begin{aligned}\sum_{r<\sqrt{x}} \left(\frac{x}{r^2}\right)^{\frac{2}{3}+\epsilon} x^{\epsilon} h(r) &= x^{\frac{2}{3}+2\epsilon} \sum_{r<\sqrt{x}} \frac{h(r)}{r^{\frac{4}{3}+2\epsilon}} \\ &\leq x^{\frac{2}{3}+2\epsilon} \sum_{r<\sqrt{x}} \frac{h(r)}{r^{1+2\epsilon}} \\ &\ll x^{\frac{2}{3}+2\epsilon}.\end{aligned} \tag{3.41}$$

Let

$$\alpha_{\pm}(\rho,0)=\sum_{r=1}^{\infty}\mu(r)R_{\pm}(\rho,0,r^2,1). \tag{3.42}$$

Then by Proposition 3.9,

$$\begin{aligned}\alpha_{\pm}(\rho,0)&=\zeta(2\rho)\left[\sum_{r\text{ odd}}\frac{\mu(r)}{r^{-2}}\sum_{d|r}d^{-2\rho-1}\phi(d^2)-\sum_{r\text{ odd}}\frac{\mu(r)}{(2r)^{-2}}\sum_{d|r}d^{-2\rho-1}\phi(d^2)\right]\\&=\frac{3}{4}\zeta(2\rho)\sum_{r\text{ odd}}\frac{\mu(r)}{r^2}\sum_{d|r}d^{-2\rho-1}\phi(d^2).\end{aligned}$$

Let

$$f(r)=\mu(r)\sum_{d|r}d^{-2\rho-1}\phi(d^2).$$

Then one easily checks that f defines a multiplicative arithmetic function and that

$$|f(n)|\le\sigma_{1-2\beta}(n)$$

where β =Re(ρ) and $\sigma_s(n)=\sum_{d|n}d^s$. So the Dirichlet series $\sum_{r\text{ odd}}\frac{f(r)}{r^2}$ converges absolutely by comparison with the series $\sum_{n=1}^{\infty}\frac{\sigma_{1-2\beta}(n)}{n^2}$ which converges absolutely (see page 229 of [2]). We can therefore use the Euler product representation of the Dirichlet series and get that

$$\sum_{r\text{ odd}}\frac{f(r)}{r^2}=\prod_{p\neq 2}\left(1-\frac{1+p^{-2\rho+1}-p^{-2\rho}}{p^2}\right).$$

Thus

$$\alpha_{\pm}(\rho,0)=\frac{3}{4}\zeta(2\rho)P(3-2\rho), \tag{3.43}$$

where

$$P(w)=\prod_{p\neq 2}(1-p^{-2}-p^{-w}+p^{-w+1}).$$

Furthermore, since $f(r)\le\sigma_{1-2\beta}(r)\ll r^{\epsilon}$ we have the following estimate

$$\sum_{r\ge\sqrt{x}}\mu(r)R_{\pm}(\rho,0,r^2,1)\ll\sum_{r\ge\sqrt{x}}r^{-2+\epsilon}\ll x^{-\frac{1}{2}+\epsilon}; \tag{3.44}$$

see page 55 of [2].

Putting (3.40), (3.41) and (3.44) together, we have

$$\sum_{\substack{0<\pm m<x \\ m \text{ squarefree}}} \zeta(2\rho) A_m(\rho, 1) = \alpha_{\pm}(\rho, 0)x + O(x^{\frac{2}{3}+\epsilon}). \tag{3.45}$$

If we simply multiply by $1 - 2^{-\rho}$, this gives Theorem 3.11 with $L_2(\rho, \chi)$ instead of $L(\rho, \chi)$. In the special case $\rho = 1$, it is necessary to remove the term with $m = 1$

To get Theorem 3.11 proper, we can modify the above arguments in the following manner.

First

$$\sum_{\substack{1<\pm m<x \\ m \text{ squarefree}}} L_2(\rho, \chi_m) = \sum_{\substack{1<\pm m<x \\ m \text{ squarefree}}} (1 - \chi_m(2)2^{-\rho}) L(\rho, \chi_m)$$

$$= \sum_{\substack{1<\pm m<x \\ m \text{ squarefree}}} L(\rho, \chi_m) - 2^{-\rho} \left[\sum_{\substack{1<\pm m<x \\ \pm m\equiv 1 \,(\mathrm{mod}\ 8) \\ m \text{ squarefree}}} L(\rho, \chi_m) - \sum_{\substack{1<\pm m<x \\ \pm m\equiv 5 \,(\mathrm{mod}\ 8) \\ m \text{ squarefree}}} L(\rho, \chi_m) \right]$$

since

$$\chi_m(2) = \begin{cases} +1, & \text{if } m \equiv 1 \,(\mathrm{mod}\ 8); \\ -1, & \text{if } m \equiv 5 \,(\mathrm{mod}\ 8); \\ 0, & \text{otherwise.} \end{cases}$$

Let S denote the bracketed difference of two sums above (i.e. the coefficient of $2^{-\rho}$). To complete the proof, it is enough to show that, for $\mathrm{Re}(\rho) \geq 1$,

$$S = O(x^{\frac{2}{3}+\epsilon}). \tag{3.46}$$

To do this, we consider the new function

$$E^*_{\frac{1}{2}}(z, s) = (2z+1)^{-k/2} E_\infty\left(\frac{z}{2z+1}, s\right) = 2^{-k/2} E_{\frac{1}{2}}\left(z + \frac{1}{2}, s\right).$$

If $r \equiv 1 \pmod 2$ and $-4au \equiv 1 \pmod r$, then $(2u+r)(2a-r) \equiv -1 \pmod{2r}$ and by Proposition 2.16, we have the transformation formula

$$E^*_{\frac{1}{2}}(\alpha_{u,r}\tau_r z, s) = i^{-k/2}(rz)^{k/2}\left(G(r+2u;8r)\right)^k E_0\left(\alpha_{a,r}z - \frac{1}{2}\right). \tag{3.47}$$

Taking Mellin Transforms and using Proposition 2.13 to get the fourier expansion of $E^*_{\frac{1}{2}}$, we can then follow the same procedures as before applied to $E^*_{\frac{1}{2}}$ to obtain (3.46). Conveniently, there is no term corresponding to x in (3.46) due to the fact that $E^*_{\frac{1}{2}}$ transforms to E_0 and not E_∞, so that the Mellin transform has no pole at $w = s + \frac{k}{2}$.

Now we turn our attention to Theorem 3.12. That the series $Z_\pm(\rho, w)$ converges absolutely for $\mathrm{Re}(w) > 1$ actually follows from the well-known fact that $L(\rho, \chi_m) \ll m^\epsilon$ for m squarefree, but we can readily see it by applying Abel's identity (page 77 of [2]) to the quantity

$$\sum_{0<\pm m<x} \zeta(2\rho)\mu(|m|)^2 A_m(\rho, 1)|m|^{-w}$$

and using (3.45).

We notice that

$$Z_\pm(\rho, w) = \zeta(2\rho)\sum_{r=1}^{\infty} \mu(r) Z_\pm(\rho, w-1, r^2, 1) \tag{3.48}$$

assuming the series on the right hand side converges. To see that this converges absolutely for $\mathrm{Re}(w) > 1$ we simply utilize the first part of the proof of Proposition 3.8 with $\epsilon = \mathrm{Re}(w) - 1$, so that

$$\begin{aligned}\sum_{r=1}^{\infty} |Z_\pm(\rho, w-1, r^2, 1)| &\ll \sum_{r=1}^{\infty} r^{-2-2\epsilon} h_1(r) \\ &\ll 1.\end{aligned}$$

We also notice that this series converges absolutely for $\frac{1}{2} < \mathrm{Re}(w) \leq 1$, since for any $\epsilon > 0$, we have, by Proposition 3.8,

$$\begin{aligned}\sum_{r=1}^{\infty} |Z_{\pm}(\rho, w-1, r^2, 1)| &\ll \sum_{r=1}^{\infty} r^{-2\mathrm{Re}(w)+2\epsilon} |\mathrm{Im}(w)|^{1-\mathrm{Re}(w)+\epsilon} h(r) \\ &\ll |\mathrm{Im}(w)|^{1-\mathrm{Re}(w)+\epsilon}.\end{aligned}$$

Thus, (3.48) gives us our meromorphic continuation, and for $\mathrm{Re}(\rho) \geq 1$, the only pole is at $w = 1$. By (3.42), the residue is given by $\alpha_{\pm}(\rho, 0)$. This, together with the comments concerning (3.47), completes the proof of the theorem. □

4 Selberg's Eigenvalue Conjecture

Now we return to the discussion we began in the introduction. Our goal is to develop a formula relating the eigenvalues λ with the Eisenstein Series $E(z, s)$ for $\Gamma_0(N)$.

4.1 Preliminary Results Concerning the Cusp Form $\psi(z)$

Let ψ be a cusp form associated with a positive discrete eigenvalue λ for $\Gamma_0(N)$. Since λ is a positive number, we can assume that ψ is a real-valued function. We have the Fourier expansion [7]

$$\psi(z) = \sqrt{y} \sum_{m \neq 0} \rho(m) K_{i\kappa}(2\pi |m| y) e^{2m\pi i x} \tag{4.1}$$

at the cusp ∞, where $\kappa = \sqrt{\lambda - 1/4}$ and $K_{\nu}(y)$ is given by

$$K_{\nu}(y) = \frac{1}{2} \int_0^{\infty} \exp\left(-\frac{y}{2}\left(t + \frac{1}{t}\right)\right) \frac{dt}{t^{\nu+1}}. \tag{4.2}$$

We can also write

$$K_\nu(y) = \frac{\pi}{2}\frac{I_{-\nu}(y) - I_\nu(y)}{\sin \pi\nu} \tag{4.3}$$

with

$$I_\nu(y) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+1+\nu)} \left(\frac{y}{2}\right)^{\nu+2k}.$$

Lemma 4.1. *Let $\psi(z)$ be a cusp form associated with a positive discrete eigenvalue λ for $\Gamma_0(N)$. Then we have*

$$\int_0^1 \int_0^\infty \Delta|\psi(z)|^2 \, y^{s-2} dx dy = s(s-1) \int_0^1 \int_0^\infty |\psi(z)|^2 \, y^{s-2} dx dy$$

for $Re(s) > 1$.

Proof. Since

$$|\psi(z)|^2 = y \sum_{m,n\neq 0} \rho(m)\bar{\rho}(n) K_{i\kappa}(2\pi|m|y)\bar{K}_{i\kappa}(2\pi|n|y) e^{2\pi i(m-n)x}$$

by (4.1), we have

$$\Delta|\psi(z)|^2 = \sum_{m,n\neq 0} \rho(m)\bar{\rho}(n) y^2 \frac{\partial^2}{\partial y^2} \left[y K_{i\kappa}(2\pi|m|y)\bar{K}_{i\kappa}(2\pi|n|y) \right] e^{2\pi i(m-n)x}$$

$$- 4\pi^2 y^3 \sum_{m,n\neq 0} (m-n)^2 \rho(m)\bar{\rho}(n) K_{i\kappa}(2\pi|m|y)\bar{K}_{i\kappa}(2\pi|n|y) e^{2\pi i(m-n)x}.$$

Since, for every fixed positive y, the series

$$\sum_{m,n\neq 0} \rho(m)\bar{\rho}(n) y^2 \frac{\partial^2}{\partial y^2} \left[y K_{i\kappa}(2\pi|m|y)\bar{K}_{i\kappa}(2\pi|n|y) \right] e^{2\pi i(m-n)x}$$

$$- 4\pi^2 y^3 \sum_{m,n\neq 0} (m-n)^2 \rho(m)\bar{\rho}(n) K_{i\kappa}(2\pi|m|y)\bar{K}_{i\kappa}(2\pi|n|y) e^{2\pi i(m-n)x}$$

converges uniformly for x in the interval $[0,1]$, we can interchange integration and summation in the following integral and obtain that

$$\int_0^1 \{ \sum_{m,n\neq 0} \rho(m)\bar{\rho}(n) y^2 \frac{\partial^2}{\partial y^2} \left[y K_{i\kappa}(2\pi|m|y)\bar{K}_{i\kappa}(2\pi|n|y) \right] e^{2\pi i(m-n)x}$$

$$- 4\pi^2 y^3 \sum_{m,n\neq 0} (m-n)^2 \rho(m)\bar{\rho}(n) K_{i\kappa}(2\pi|m|y)\bar{K}_{i\kappa}(2\pi|n|y) e^{2\pi i(m-n)x} \} dx$$

$$= \sum_{m\neq 0} |\rho(m)|^2 y^2 \frac{\partial^2}{\partial y^2} \left[y |K_{i\kappa}(2\pi|m|y)|^2 \right],$$

recalling that

$$\int_0^1 e^{2\pi i(m-n)x}dx = \begin{cases} 1, & \text{if } m = n; \\ 0, & \text{otherwise.} \end{cases}$$

By (8.17) on page 110 of Iwaniec [7], we can interchange integration and summation in the following integral and get

$$\int_0^\infty \int_0^1 \Delta|\psi(z)|^2 \, y^{s-2}dxdy = \sum_{m\neq 0} |\rho(m)|^2 \int_0^\infty y^s \frac{\partial^2}{\partial y^2} \left[y|K_{i\kappa}(2\pi|m|y)|^2\right] dy \tag{4.4}$$

for $\mathrm{Re}(s) > 1$. Since $\lambda \geq 975/4096$ according to Kim and Sarnak [8], $i\kappa$ is either an imaginary number or a positive number less than $11/100$.

Using representation (4.3), it is clear that

$$y^s|K_{i\kappa}(2\pi|m|y)|^2$$

converges to 0 as y approaches 0 for $\mathrm{Re}(s) \geq \frac{1}{2}$. Using fundamental results about absolutely convergent power series, we have

$$\begin{aligned} \frac{\partial}{\partial y}\left[y|K_{i\kappa}(2\pi|m|y)|^2\right] =& |K_{i\kappa}(2\pi|m|y)|^2 \\ &+ 2\pi|m|y \sum_{k=0} \frac{1}{k!\Gamma(k+1+i\kappa)} \left(\frac{2\pi|m|y}{2}\right)^{i\kappa+2k} \\ &\times \sum_{k=0} \frac{\nu'+2k}{k!\bar{\Gamma}(k+1+i\kappa)} \left(\frac{2\pi|m|y}{2}\right)^{\nu'+2k-1} \\ &+ 2\pi|m|y \sum_{k=0} \frac{i\kappa+2k}{k!\Gamma(k+1+\nu)} \left(\frac{2\pi|m|y}{2}\right)^{i\kappa+2k-1} \\ &\times \sum_{k=0} \frac{1}{k!\bar{\Gamma}(k+1+i\kappa)} \left(\frac{2\pi|m|y}{2}\right)^{\nu'+2k} \end{aligned}$$

where

$$\nu' = \begin{cases} -i\kappa & \text{if } i\kappa \text{ is imaginary} \\ i\kappa & \text{otherwise.} \end{cases}$$

Thus, the expression

$$y^s \frac{\partial}{\partial y} \left[y |K_{i\kappa}(2\pi|m|y)|^2 \right]$$

also converges to 0 as y approaches 0 for $\mathrm{Re}(s) \geq \frac{1}{2}$.

Using representation (4.2), we see that

$$\begin{aligned}
|K_{i\kappa}(2\pi|m|y)| &\leq \left| \frac{1}{2} \int_0^1 e^{-\left(\frac{2\pi|m|y}{2}\right)\left(t+\frac{1}{t}\right)} \frac{dt}{t^{i\kappa+1}} \right| + \left| \frac{1}{2} \int_1^\infty e^{-\left(\frac{2\pi|m|y}{2}\right)\left(t+\frac{1}{t}\right)} \frac{dt}{t^{-i\kappa+1}} \right| \\
&= \left| \frac{1}{2} \int_1^\infty e^{-\left(\frac{2\pi|m|y}{2}\right)\left(t+\frac{1}{t}\right)} \frac{dt}{t^{-i\kappa+1}} \right| + \left| \frac{1}{2} \int_1^\infty e^{-\left(\frac{2\pi|m|y}{2}\right)\left(t+\frac{1}{t}\right)} \frac{dt}{t^{i\kappa+1}} \right| \\
&\leq \int_1^\infty e^{-2\pi|m|yt} dt \\
&= \frac{1}{2\pi|m|y} e^{-2\pi|m|y}
\end{aligned}$$

so that

$$y^s |K_{i\kappa}(2\pi|m|y)|^2$$

approaches 0 as y approaches ∞ for all s.

Now, we wish to take the derivative of (4.2) with respect to y. Recall that

$$\frac{e^{-\left(\frac{y+h}{2}\right)\left(t+\frac{1}{t}\right)} - e^{-\left(\frac{y}{2}\right)\left(t+\frac{1}{t}\right)}}{h} \to -\frac{1}{2}\left(t+\frac{1}{t}\right) e^{-\left(\frac{y}{2}\right)\left(t+\frac{1}{t}\right)} \text{ as } h \to 0^+$$

monotonically, and the same is true as $h \to 0^-$. Thus, using Lebesgue Monotone Convergence, it is enough to differentiate inside the integrand with respect to y, and we have

$$\frac{\partial}{\partial y} K_{i\kappa}(2\pi|m|y) = \int_0^\infty -\frac{1}{2}\left(t+\frac{1}{t}\right) e^{\left(-\frac{y}{2}\left(1+\frac{1}{t}\right)\right)} \frac{dt}{t^{i\kappa+1}}$$

and a calculation similar to that above shows that

$$\left| \frac{\partial}{\partial y} K_{i\kappa}(2\pi|m|y) \right| \leq \frac{1}{\pi|m|y} e^{-y}.$$

Therefore, we see that

$$y^s \frac{\partial}{\partial y} \left[y |K_{i\kappa}(2\pi|m|y)|^2 \right]$$

converges to 0 as y approaches ∞ for all s.

Hence, we find by partial integration that

$$\int_0^\infty y^s \frac{\partial^2}{\partial y^2}\left[y|K_{i\kappa}(2\pi|m|y)|^2\right] dy = s(s-1)\int_0^\infty |K_{i\kappa}(2\pi|m|y)|^2 y^{s-1} dy \quad (4.5)$$

for $\mathrm{Re}(s) \geq 1/2$. Putting (4.5) into (4.4), we obtain that

$$\int_0^1 \int_0^\infty \Delta|\psi(z)|^2\, y^{s-2} dx dy = s(s-1)\int_0^1 \int_0^\infty |\psi(z)|^2\, y^{s-2} dx dy$$

for $\mathrm{Re}(s) > 1$.

This completes the proof of the lemma. $\square$

Lemma 4.2. *Let $\psi(z)$ be a cusp form associated with a positive discrete eigenvalue λ for $\Gamma_0(N)$. Then the identity*

$$\left|y\frac{\partial\psi}{\partial y} + iy\frac{\partial\psi}{\partial x}\right|^2 = \frac{1}{2}\Delta|\psi(z)|^2 + \lambda|\psi(z)|^2$$

holds.

Proof. Since ψ is a real-valued function, $|\psi(z)|^2 = \psi(z)^2$ and we have

$$\begin{aligned} \frac{1}{2}\Delta|\psi(z)|^2 &= y^2\left(\psi_x^2 + \psi_y^2 + \psi[\psi_{xx} + \psi_{yy}]\right) \\ &= y^2\left(\psi_x^2 + \psi_y^2\right) + \psi\Delta\psi \\ &= |y\psi_y + iy\psi_x|^2 - \lambda\psi^2. \end{aligned}$$

That is,

$$\left|y\frac{\partial\psi}{\partial y} + iy\frac{\partial\psi}{\partial x}\right|^2 = \frac{1}{2}\Delta|\psi(z)|^2 + \lambda|\psi(z)|^2. \quad (4.6)$$

$\square$

Lemma 4.3. *Let $\psi(z)$ be a cusp form associated with a positive discrete eigenvalue λ for $\Gamma_0(N)$. Then we have*

$$\int_0^1 \int_0^\infty \left|y\frac{\partial\psi}{\partial y} + iy\frac{\partial\psi}{\partial x}\right|^2 y^{s-2} dx dy = \left[\lambda + \frac{s(s-1)}{2}\right]\int_0^1 \int_0^\infty |\psi(z)|^2 y^{s-2} dx dy$$

for $Re(s) > 1$.

Proof. By Lemma 4.2 we have

$$\begin{aligned}\int_0^1\int_0^\infty &\left|y\frac{\partial\psi}{\partial y}+iy\frac{\partial\psi}{\partial x}\right|^2 y^{s-2}dxdy\\ =\int_0^1\int_0^\infty &\frac{1}{2}\Delta|\psi(z)|^2y^{s-2}dxdy+\lambda\int_0^1\int_0^\infty|\psi(z)|^2y^{s-2}dxdy.\end{aligned}\tag{4.7}$$

By Lemma 4.1 we have

$$\frac{1}{2}\int_0^1\int_0^\infty \Delta|\psi(z)|^2\,y^{s-2}dxdy=\frac{s(s-1)}{2}\int_0^1\int_0^\infty|\psi(z)|^2\,y^{s-2}dxdy.\tag{4.8}$$

Then the stated identity follows from (4.7) and (4.8). □

Lemma 4.4. *Let $\psi(z)$ be a cusp form associated with a positive discrete eigenvalue λ for $\Gamma_0(N)$. Then we have*

$$\left(y\left[\frac{\partial}{\partial y}+i\frac{\partial}{\partial x}\right]\psi\right)(\gamma z)=\frac{(cz+d)^2}{|cz+d|^2}\left(y\left[\frac{\partial}{\partial y}+i\frac{\partial}{\partial x}\right]\psi\right)(z)$$

for all elements $\gamma\in\Gamma_0(N)$.

Proof. Let $w(z)=\gamma z$. Then w is an analytic function. Write

$$w(z)=u(z)+iv(z)$$

where u and v are real valued functions. Then, by the Cauchy-Riemann equations, we have

$$\begin{aligned}w'(z)&=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}\\&=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}.\end{aligned}$$

But we also know that

$$w'(z)=\frac{1}{(cz+d)^2}.$$

Since ψ is a cusp form, we have

$$\psi(z)=\psi(\gamma z)\tag{4.9}$$

for all $\gamma \in \Gamma_0(N)$. Differentiating both sides of (4.9) and using the chain rule, we have

$$\frac{\partial \psi}{\partial y}(z) = \frac{\partial \psi}{\partial y}(\gamma z)\frac{\partial v}{\partial y} + \frac{\partial \psi}{\partial x}(\gamma z)\frac{\partial u}{\partial y}$$

and

$$\frac{\partial \psi}{\partial x}(z) = \frac{\partial \psi}{\partial y}(\gamma z)\frac{\partial v}{\partial x} + \frac{\partial \psi}{\partial x}(\gamma z)\frac{\partial u}{\partial x}.$$

Putting these together we have

$$\begin{aligned}\left[\frac{\partial \psi}{\partial y} + i\frac{\partial \psi}{\partial x}\right](z) &= \frac{\partial \psi}{\partial y}(\gamma z)\left(\frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}\right) + i\frac{\partial \psi}{\partial x}(\gamma z)\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)\\ &= \frac{\partial \psi}{\partial y}(\gamma z)(w'(z)) + i\frac{\partial \psi}{\partial x}(\gamma z)(w'(z))\\ &= \frac{1}{(cz+d)^2}\left[\frac{\partial \psi}{\partial y} + i\frac{\partial \psi}{\partial x}\right](\gamma z).\end{aligned}$$

Recalling that $\mathrm{Im}(\gamma z) = \frac{y}{|cz+d|^2}$, this completes the proof of the lemma.

□

4.2 Eisenstein Series for $\Gamma_0(N)$

Let

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z} \right\}$$

be the stabilizer of ∞. The Eisenstein series for the cusp ∞ of $\Gamma_0(N)$ is defined to be

$$E(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \frac{y^s}{|cz+d|^{2s}}$$

which converges absolutely for $\mathrm{Re}(s) > 1$. Here, we of course write $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let D denote the fundamental domain of $\Gamma_0(N)$. Let P denote the strip $P = \{z = x + iy \,|\, 0 < x < 1,\, y > 0\}$. By choosing an appropriate set of coset representatives $\gamma \in \Gamma_\infty \backslash \Gamma_0(N)$, the sets γD cover P exactly once.

Lemma 4.5. *Let $\psi(z)$ be a cusp form associated with a positive discrete eigenvalue λ for $\Gamma_0(N)$. Then we have*

$$\int_D \left| y\frac{\partial\psi}{\partial y} + iy\frac{\partial\psi}{\partial x}\right|^2 E(z,s)dz = \left(\lambda + \frac{s(s-1)}{2}\right)\int_D |\psi(z)|^2 E(z,s)dz$$

for $Re(s) > 1$.

Proof. By the above arguments we have

$$\begin{aligned}
&\int_D \left| y\frac{\partial\psi}{\partial y} + iy\frac{\partial\psi}{\partial x}(z)\right|^2 E(z,s)dz \\
&\qquad = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \int_D \left| y\frac{\partial\psi}{\partial y} + iy\frac{\partial\psi}{\partial x}(z)\right|^2 \frac{y^s}{|cz+d|^{2s}}dz \\
&\qquad = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \int_{\gamma D} \left| y\frac{\partial\psi}{\partial y} + iy\frac{\partial\psi}{\partial x}(z)\right|^2 y^2 dz \\
&\qquad = \int_P \left| y\frac{\partial\psi}{\partial y} + iy\frac{\partial\psi}{\partial x}(z)\right|^2 y^s dz \\
&\qquad = \int_0^1 \int_0^\infty \left| y\frac{\partial\psi}{\partial y} + iy\frac{\partial\psi}{\partial x}(z)\right|^2 y^{s-2} dydx
\end{aligned}$$

where we have made the change of variables $z \to \gamma^{-1}z$ so that $\frac{y^s}{|cz+d|^{2s}}dz \to y^s dz$, and by Lemma 4.4, $\left| y\frac{\partial\psi}{\partial y} + iy\frac{\partial\psi}{\partial x}(z)\right|^2$ is invariant under this transformation.

Using Lemma 4.3 and reversing the change of variables, this becomes

$$\begin{aligned}
&\int_D \left| y\frac{\partial\psi}{\partial y} + iy\frac{\partial\psi}{\partial x}(z)\right|^2 E(z,s)dz \\
&\qquad = \left(\lambda + \frac{s(s-1)}{2}\right)\int_0^1\int_0^\infty |\psi(z)|^2 y^{s-2}dydx \\
&\qquad = \left(\lambda + \frac{s(s-1)}{2}\right)\int_P |\psi(z)|^2 y^s dz \\
&\qquad = \left(\lambda + \frac{s(s-1)}{2}\right)\int_D |\psi(z)|^2 E(z,s)dz.
\end{aligned}$$

This proves the lemma. □

Let W_N be the set of all couples $\{c, d\}$ of integers such that

$$(c, d) = 1, \ c \equiv 0 \pmod N,$$
$$c > 0 \text{ if } c \neq 0,$$
$$d = 1 \text{ if } c = 0.$$

By Lemma 3.2 of Shimura [13], the Eisenstein series for the cusp ∞ is given by

$$E(z, s) = \sum_{\{c,d\} \in W_N} \frac{y^s}{|cz + d|^{2s}}$$

for $\mathrm{Re}(s) > 1$, where $z = x + iy$.

Let

$$K(t, z) = \sum_{c=-\infty}^{\infty} \sum_{d=-\infty, \{c,d\} \neq \{0,0\}}^{\infty} \exp\left(-\frac{\pi}{yN}|cNz + d|^2 t\right).$$

We have the integral expression

$$\begin{aligned}
&2\pi^{-s} N^s \Gamma(s) \zeta(2s) E(z, s) \\
&\quad = 2\pi^{-s} N^s \sum_{\{c,d\} \in W_N} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{1}{n^{2s}} \frac{y^s}{|cz + d|^{2s}} e^{-u} u^{s-1} du \\
&\quad = 2\pi^{-s} N^s \sum_{\{c,d\} \neq \{0,0\}, c \geq 0} \int_0^{\infty} \frac{y^s}{|cNz + d|^{2s}} e^{-u} u^{s-1} du \\
&\quad = 2 \sum_{\{c,d\} \neq \{0,0\}, c \geq 0} \int_0^{\infty} e^{-\frac{\pi}{yN}|cN + d|^{2s} t} t^{s-1} dt \\
&\quad = \sum_{\{c,d\} \neq \{0,0\}} \int_0^{\infty} e^{-\frac{\pi}{yN}|cN + d|^{2s} t} t^{s-1} dt \\
&\quad = \int_0^{\infty} K(t, z) t^{s-1} dt
\end{aligned}$$

for $\mathrm{Re}(s) > 1$ and for $y > 0$. By Lemma 2 of Rankin [9], the identity

$$1 + K(t, z) = \frac{1}{t}\left(1 + K\left(\frac{1}{t}, z\right)\right) \tag{4.10}$$

holds for positive t and y. It follows from this identity that

$$\begin{aligned}
&2\pi^{-s}N^s\Gamma(s)\zeta(2s)E(z,s)\\
&\qquad = \int_0^\infty K(t,z)t^{s-1}dt\\
&\qquad = \int_0^1 K(t,z)t^{s-1}dt + \int_1^\infty K(t,z)t^{s-1}dt\\
&\qquad = \int_1^\infty K\left(\frac{1}{t},z\right)t^{-s-1}dt + \int_1^\infty K(t,z)t^{s-1}dt\\
&\qquad = \int_1^\infty K(t,z)t^{-s}dt + \int_1^\infty [t^{-s}+t^{-s-1}]dt + \int_1^\infty K(t,z)t^{s-1}dt\\
&\qquad = \int_1^\infty K(t,z)\left(t^{s-1}+t^{-s}\right)\,dt + \frac{1}{s(s-1)}
\end{aligned}$$

for $\mathrm{Re}(s) > 1$.

Let

$$\Xi(s,z) = 4s(1-s)\pi^{-s}N^s\Gamma(s)\zeta(2s)E(z,s).$$

Then

$$\Xi(s,z) = 2s(1-s)\int_1^\infty K(t,z)\left(t^{s-1}+t^{-s}\right)\,dt - 2 \tag{4.11}$$

for $\mathrm{Re}(s) > 1$. This integral expression of $\Xi(s,z)$ implies that $\Xi(s,z)$ is an entire function of s for each fixed complex z in the upper half-plane.

Theorem 4.6. *Let $\psi(z)$ be a cusp form associated with a positive discrete eigenvalue λ for $\Gamma_0(N)$. Then we have*

$$\begin{aligned}
&\int_D \left| y\left(\frac{\partial\psi}{\partial y} + i\frac{\partial\psi}{\partial x}\right)\right|^2 \Xi\left(\frac{1+i}{2},z\right)\,dz\\
&\qquad\qquad = \left(\lambda - \frac{1}{4}\right)\int_D |\psi(z)|^2\Xi\left(\frac{1+i}{2},z\right)\,dz.
\end{aligned}$$

Proof. By Lemma 4.5 and the definition of $\Xi(s,z)$ we have

$$\int_D \left| y\frac{\partial\psi}{\partial y} + iy\frac{\partial\psi}{\partial x}\right|^2 \Xi(s,z)dz = \left(\lambda + \frac{s(s-1)}{2}\right)\int_D |\psi(z)|^2\Xi(s,z)dz \tag{4.12}$$

for $\mathrm{Re}(s) > 1$.

We will now use an argument similar to the argument of page 363 of Rankin [9]. By Lemma 3 of Rankin [9], we have

$$K(t,z) \leq A(1+t^{-1}+y^{-\frac{1}{2}}t^{-\frac{1}{2}}+y^{\frac{1}{2}}t^{-\frac{1}{2}})(e^{-\frac{\pi Nyt}{2}}+e^{-\frac{\pi t}{2Ny}}) \tag{4.13}$$

for some constant A which depends only on N and not z or t. (This inequality follows easily using simple methods of approximation.) Also, using integration by parts, it is easy to see

$$\int_1^\infty t^w e^{-\alpha t}dt \ll \frac{1}{\alpha}e^{-\alpha} \tag{4.14}$$

where the $\ll$ constant depends at most on w.

(4.13) and (4.14) together show that

$$\int_1^\infty K(t,z)(t^{s-1}+t^{-s})dt \ll y^{-\frac{3}{2}}+y^{-1}+y^{-\frac{1}{2}}+y^{\frac{1}{2}}+y+y^{\frac{3}{2}}, \tag{4.15}$$

where the $\ll$ constant depends at most on N and s.

Recall that ψ is a cusp form, so by definition, ψ decays rapidly to 0 as z approaches the rational cusps of D. In particular, the integral

$$\int_D y^\beta |\psi(z)|^2 dz$$

converges absolutely for all β (see also Lemma 1 of [9]). Furthermore, differentiating (4.1) using (4.2) and (4.3), we see that $y\frac{\partial\psi}{\partial y}+iy\frac{\partial\psi}{\partial x}$ also decays rapidly as z approaches the rational cusps of D, so that we also have that the integral

$$\int_D y^\beta \left| y\frac{\partial\psi}{\partial y}+iy\frac{\partial\psi}{\partial x}\right|^2 dz$$

converges absolutely for all β. This, together with (4.11) and (4.15), shows that both sides of (4.12) converge absolutely for all s.

Thus both sides of (4.12) represent entire functions of s, and hence the identity (4.12) holds for all complex s. In particular, the stated identity follows when we choose $s = \frac{1+i}{2}$. $\square$

Theorem 4.6 gives the desired formula. It is important to note that, using $s = \frac{1+i}{2}$ in (4.11), $\Xi\left(\frac{1+i}{2}, z\right)$ is in fact a real-valued function ($s-1$ and $-s$ are complex conjugates). Thus Theorem 4.6 reduces Selberg's Eigenvalue Conjecture to the statement that $\Xi\left(\frac{1+i}{2}, z\right) \neq 0$ for all $z \in D$. However, by definition, we have

$$\Xi\left(\frac{1+i}{2}, z\right) = 2\pi^{-\frac{1+i}{2}} N^{\frac{1+i}{2}} \Gamma\left(\frac{1+i}{2}\right) \zeta(1+i) E\left(z, \frac{1+i}{2}\right).$$

Notice that the coefficient of $E\left(z, \frac{1+i}{2}\right)$ is a nonzero constant independent of z. Thus $\Xi\left(\frac{1+i}{2}, z\right) = 0$ if and only if $E\left(z, \frac{1+i}{2}\right) = 0$, reducing Selberg's Eigenvalue Conjecture to the non-vanishing of $E\left(z, \frac{1+i}{2}\right)$ in the fundamental domain D.

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