2006-07-06

On the Combinatorics of Certain Garside Semigroups

Christopher R. Cornwell
Brigham Young University - Provo

ON THE COMBINATORICS OF
CERTAIN GARSIDE SEMIGROUPS

by

Christopher Cornwell

A thesis submitted to the faculty of

Brigham Young University

in partial fulfillment of the requirements for the degree of

Master of Science

Department of Mathematics

Brigham Young University

August 2006

BRIGHAM YOUNG UNIVERSITY

GRADUATE COMMITTEE APPROVAL

of a thesis submitted by

Christopher Cornwell

This thesis has been read by each member of the following graduate committee and by majority vote has been found to be satisfactory.

Date	Stephen Humphries, Chair
Date	Darrin Doud
Date	Rodney Forcade

BRIGHAM YOUNG UNIVERSITY

As chair of the candidate's graduate committee, I have read the thesis of Christopher Cornwell in its final form and have found that (1) its format, citations, and bibliographical style are consistent and acceptable and fulfill university and department style requirements; (2) its illustrative materials including figures, tables, and charts are in place; and (3) the final manuscript is satisfactory to the graduate committee and is ready for submission to the university library.

____
Date

____
Stephen Humphries
Chair, Graduate Committee

Accepted for the Department

____
Date

____
Tyler Jarvis
Department Chair

Accepted for the College

____
Date

____
Thomas Sederberg, Associate Dean
College of Physical and Mathematical Sciences

ABSTRACT

ON THE COMBINATORICS OF CERTAIN GARSIDE SEMIGROUPS

Christopher Cornwell

Department of Mathematics

Master of Science

In his dissertation, F.A. Garside provided a solution to the word and conjugacy problems in the braid group on n-strands, using a particular element that he called the fundamental word. Others have since defined fundamental words in the generalized setting of Artin groups, and even more recently in Garside groups. We consider the problem of finding the number of representations of a power of the fundamental word in these settings. In the process, we find a Pascal-like identity that is satisfied in a certain class of Garside groups.

ACKNOWLEDGMENTS

I greatly appreciate the help of my advisor, Stephen Humphries. His guidance and criticism were invaluable. I also want to thank the members of my committee and other faculty in the BYU Mathematics department for their time and feedback. They saved me many hours by pointing me to appropriate sources. Thank you to all who have taken part in proofreading and editing this work. And finally, I thank everyone whose support is not manifested in the final work; your help may be invisible to the reader, but not to me.

Table of Contents

List of Figures

1 Introduction

The braid group on n strands was expressly introduced by Emil Artin in 1925, though it has been recognized that the interpretation of the braid group as the fundamental group of a configuration space was implicit in the work of Hurwitz as early as 1891. The braid group has many interesting manifestations in topological settings, such as in knot theory and the study of moduli spaces. This chapter gives a brief introduction to Artin's geometric interpretation of the braid group. We begin by defining braids and the product on which they form a group; then we will prove some basic results on that group.

The second chapter discusses an important element of the braid group, known as the fundamental word, as defined by F.A. Garside in his doctoral work in 1969. By use of this fundamental word, Garside was able to solve the word and conjugacy problems for the braid group.

In Chapters 3 and 4 we investigate one generalization of the braid group, known as Artin groups. Specifically we consider Artin groups associated to the dihedral groups and prove several results related to representations of the fundamental word in this setting.

Finally, we consider a generalization of Artin groups known as Garside groups. These groups have recently been defined in a way to preserve many of the nice properties of Artin groups. Chapter 5 shows how the results of the previous chapter can be interpreted in order to generalize to a certain class of Garside groups.

1.1 Defining a Braid and the Braid Group B_n

We begin with a definition of the braid group that follows the geometric interpretation of Artin. This much of the formulation is similar to that of Dehornoy in [4, p. 4]. In what follows, S_n is the symmetric group on n elements.

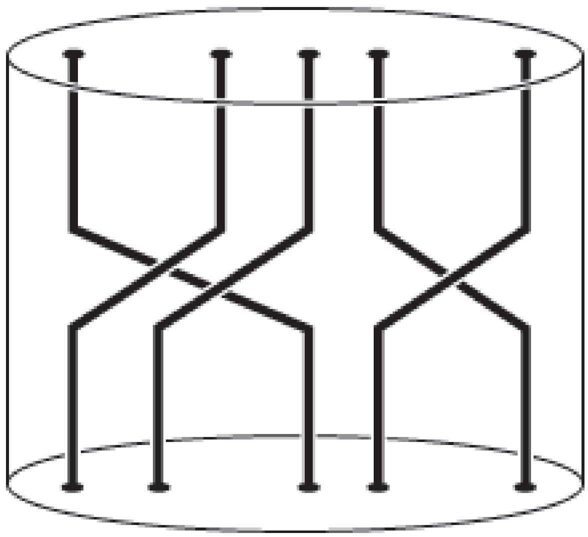


Figure 1: A 5-strand braid

Definition 1.1. Let $\tau \in S_n$. An *n-strand braid*(or n-braid) is the union of n disjoint polygonal arcs $d_1, d_2, \ldots, d_n \in \mathbb{R}^2 \times [0, 1]$, where each d_i satisfies the following:

(1) d_i has endpoints $(i, 0, 0)$ and $(\tau(i), 0, 1)$

(2) d_i is monotonically increasing in the z-coordinate.

If β is an n-braid and τ its associated permutation, we define $perm(\beta) = \tau$.

Figure 1 displays an example of a 5-braid. If we call this braid β, then $perm(\beta) = (123)(45)$.

The z-coordinate of this 5-braid is vertically oriented, as it will be in all subsequent figures. Also, in the figure a break in a strand denotes that the y-coordinate of the strand is greater than that of the line that it crosses. Formally, if $1 \leq i \leq n$, one may parameterize d_i by coordinate functions $d_i(t) = (x_i(t), y_i(t), t)$, where $d_i(0)$ and $d_i(1)$ are as specified in Definition 1.1. If d_i and d_j are a pair of distinct arcs of the same braid such that $x_i(t_0) = x_j(t_0)$ for $0 < t_0 < 1$, call the pair $\{d_i(t_0), d_j(t_0)\}$ a *crossing*. Due to an equivalence relation on braids that will be introduced shortly, we may assume that the derivatives $\frac{dx_i}{dz}$ and $\frac{dx_j}{dz}$ exist at t_0 and are not equal. Furthermore, suppose that, at t_0, $\frac{dx_i}{dx} > 0$ and $\frac{dx_i}{dz} - \frac{dx_j}{dz} > 0$. By Definition 1.1, $y_i(t_0) \neq y_j(t_0)$. If $y_i(t_0) < y_j(t_0)$ we say the crossing is a *right crossing*. If $y_i(t_0) > y(t_0)$ it is a *left crossing*.

There is a natural equivalence relation that presents itself given this geometric concept of a braid. If we imagine the arcs to be strings that are connected on one end to the plane $z = 0$ and to $z = 1$ on the other end, then it seems reasonable to say that if one braid can be made into another by pulling at these strings (but leaving the strings attached at the ends), then the braids are the same. We will show that this relation is, in fact, an equivalence relation. First we need the following definition.

Definition 1.2. Given a line segment AB on a polygonal arc of some braid, β, and a point C not on AB, let δ take the segment AB to $AC \cup CB$. If no other strand of β intersects the convex interior bounded by AB, AC, CB, as shown in Figure 2, we say δ is a *fundamental move*.

If the arrow in Figure 2 is reversed and we take $AC \cup CB$ to AB, we will denote this move by δ^{-1} and it is also a fundamental move.

Now if β and β' are two n-braids, we say β is *isotopic* to β' if there are a finite number of fundamental moves, $\delta_1, \delta_2, \ldots, \delta_k$ such that

$$\beta = \beta_0 \xrightarrow{\delta_1} \beta_1 \xrightarrow{\delta_2} \dots \xrightarrow{\delta_k} \beta_k = \beta'.$$

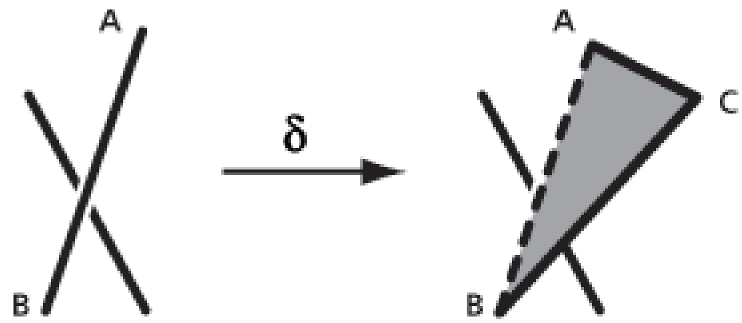


Figure 2: A fundamental move

Proposition 1.3. *Isotopy (denoted by $\sim$) on n-braids is an equivalence relation.*

Proof.

It is clear that any braid is isotopic to itself. Also, if β and β' are two n-braids such that $\beta \sim \beta'$, let $\delta_1, \delta_2, \dots, \delta_k$ be a set of fundamental moves that takes β to β'. Then, since for each i, δ_i^{-1} is another fundamental move, $\delta_k^{-1}, \dots, \delta_1^{-1}$ gives $\beta' \sim \beta$.

Finally, suppose α, β, and γ are n-braids with $\alpha \sim \beta$ and $\beta \sim \gamma$. Then,

$$\alpha = \beta_0 \xrightarrow{\delta_1} \beta_1 \xrightarrow{\delta_2} \dots \xrightarrow{\delta_k} \beta_k = \beta = \beta_k \xrightarrow{\delta_{k+1}} \beta_{k+1} \xrightarrow{\delta_{k+2}} \dots \xrightarrow{\delta_{k+m}} \beta_{k+m} = \gamma$$

for some $k, m \in \mathbb{Z}$.

This gives that $\alpha \sim \gamma$. $\square$

Remark 1. It is important to note that given an n-braid β, the definition of a fundamental move requires that the endpoints of any arc d_i of β remain fixed. Therefore, $perm(\beta)$ is invariant under the isotopy.

We wish to define a product on the set of n-braids. However, this will only be possible by means of an important lemma.

Lemma 1.4. *For any piecewise affine bijection t of $[0,1]$ onto itself, let $\widetilde{t} : \mathbb{R}^2 \times [0,1] \longrightarrow \mathbb{R}^2 \times [0,1]$ be the function defined by $\widetilde{t}(x,y,z) = (x,y,t(z))$. If $\widetilde{t}$ takes any polygonal arc in $\mathbb{R}^2 \times [0,1]$ to another polygonal arc in $\mathbb{R}^2 \times [t(0),t(1)]$, then any n-braid is isotopic to its image under $\widetilde{t}$.*

Proof.

Let β be an n-braid. It is clear that the statement is true if for any two distinct points $(x_1,y_1,z_1),(x_2,y_2,z_2) \in \beta, \quad z_1 < z_2$ implies $t(z_1) < t(z_2)$. Also, this condition is true for *any* two points in β if and only if it is true for any two points lying on the same arc of β (since $\widetilde{t}$ acts only on z). Note also that this condition ensures that the image of any arc $d_i \in \beta$ will be monotonically increasing.

Let $[a,b],[b,c] \in [0,1]$ be two of the subintervals used in the definition of t. Since $t|_{[a,b]}(resp.\ t|_{[b,c]})$ is strictly affine, for any $z,z' \in [a,b](or\ [b,c])$

$$z < z' \Rightarrow t(z) < t(z').$$

Let $z_1 \in [a,b)$ and $z_2 \in (b,c]$. Since the image of an arc under $\widetilde{t}$ is connected, we also have

$$t|_{[a,b]}(b) = t|_{[b,c]}(b).$$

Hence, $t|_{[a,b]}(z_1) < t|_{[a,b]}(b) = t|_{[b,c]}(b) < t|_{[b,c]}(z_2)$, which gives the result.

□

Now we are able introduce a well-defined product. In the following definition, let $t(x,y,z) = (x,y,\frac{z}{2})$.

Definition 1.5. Let β, β' be n-braids. The product of β and β' is the braid that results by applying $t(x, y, z) + (0, 0, \frac{1}{2})$ to every point in β, $t(x, y, z)$ to every point of β' and "pasting" β on top of β' as in Figure 3. We write this product as $\beta * \beta'$, or $\beta\beta'$.

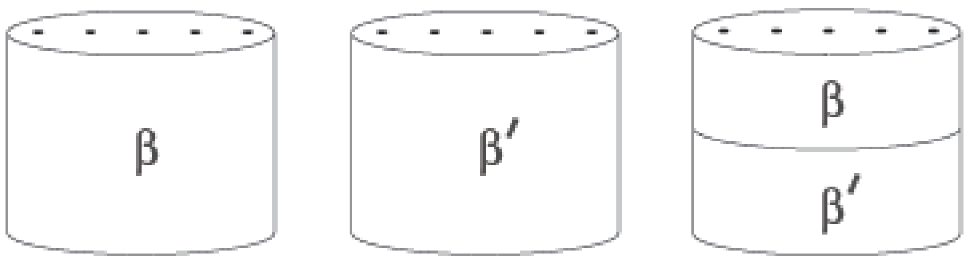


Figure 3: The product of two braids

We would like that the product $\beta * \beta'$ also be an n-braid. If $d_1, \ldots, d_n$ are the n polygonal arcs of β, and $e_1, \ldots, e_n$ the arcs that comprise β' then for $1 \leq i \leq n$, $d_{perm(\beta')(i)}$ is attached by the product to e_i at the point $(perm(\beta')(i), 0, \frac{1}{2})$. So $\beta * \beta'$ has n polygonal arcs (any permutation is a bijection) and

$$perm(\beta * \beta') = perm(\beta) \circ perm(\beta') \tag{1.1}$$

is its associated permutation. It is clear that the arcs $d_{perm(\beta')(i)} \cup e_i$ satisfy conditions (1) and (2) of Definition 1.1. Hence, $\beta * \beta'$ is an n-braid. Also, since t is affine and $\sim$ has the property of transitivity, Lemma 1.4 shows the product to be well-defined.

Theorem 1.6. *If $\mathcal{B}_n$ is the set of all n-braids, let $B_n = \mathcal{B}_n/\!\sim$. Then B_n is a group under the product defined on n-braids.*

Proof. The remarks above show that B_n is closed under $*$. As an identity, let ι denote the equivalence class in B_n of the braid in $\mathcal{B}_n$ with d_i being the vertical segment from $(i,0,0)$ to $(i,0,1)$, for each $1 \leq i \leq n$. It is easy to see that $\beta\iota = \iota\beta = \beta$ for all $\beta \in B_n$.

Also, if β is a representative of an equivalence class in B_n, define β^{-1} such that

$$\beta^{-1} = \{(x,y,z) | (x,y,1-z) \text{ is a point in } \beta\}.$$

It is clear then that $\beta\beta^{-1} \sim \beta^{-1}\beta \sim \iota$, making β^{-1} the inverse of β in B_n.

To see that the product is associative, define $t : [0,1] \longrightarrow [0,1]$ such that

$$t(z) = \begin{cases} \frac{z}{2} & \text{if } 0 \leq z \leq \frac{1}{2} \\ z - \frac{1}{4} & \text{if } \frac{1}{2} \leq z \leq \frac{3}{4} \\ 2z - 1 & \text{if } \frac{3}{4} \leq z \leq 1 \end{cases}$$

Now if $f : \mathbb{R}^2 \times [0,1] \longrightarrow \mathbb{R}^2 \times [0,1]$ is defined by $f(x,y,z) = (x,y,t(z))$, then applying f to the points of $(\beta_1\beta_2)\beta_3$ gives $\beta_1(\beta_2\beta_3)$ for $\beta_i \in \mathcal{B}_n$. This shows that

$$(\beta_1\beta_2)\beta_3 \sim \beta_1(\beta_2\beta_3)$$

which implies that they are equal in B_n.

Thus, we have that B_n is a group under the operation $*$.

□

1.2 Properties of B_n

For the present an n-braid, β, will refer to its equivalence class in B_n. We wish to better understand the structure of B_n; in doing so, it will be convenient to give a name to a special type of braid.

Definition 1.7. Define σ_i, $1 \leq i \leq n$, to be the braid in B_n with exactly one right

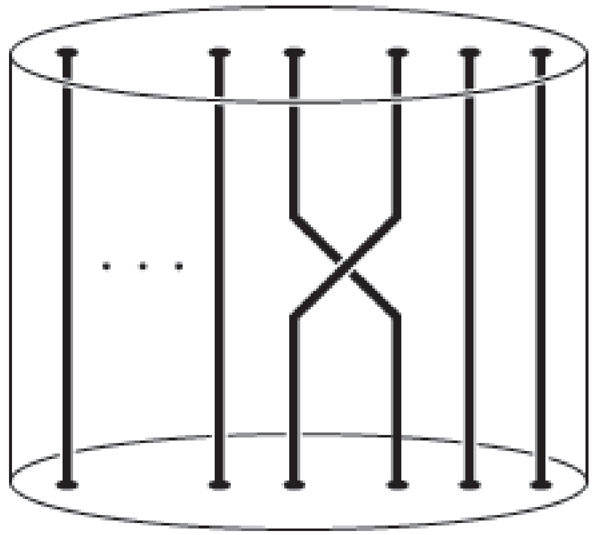


Figure 4: σ_i

crossing at the arcs d_i and d_{i+1}, as seen in Figure 4. Of course, σ_i^{-1} is the braid with one left crossing at d_i, d_{i+1}.

Now consider the set of pure braids $P_n = \{\beta \in B_n | perm(\beta) = id\}$, where id is the identity permutation. Equation (1.1) above shows that

$$perm : B_n \longrightarrow S_n$$

is a homomorphism. Since any transposition of the form $(i, i+1)$ in S_n is the image of σ_i, we see that $perm$ is surjective. Also, by its definition, P_n is the kernel of $perm$. This gives the result that P_n is a normal subgroup of B_n and the factor group B_n/P_n is isomorphic to S_n.

The preceding paragraph uses the well-known fact from group theory that for $n > 1$, the group S_n may be generated by the transpositions $(1,2), (2,3), \ldots, (n-1, n)$. Finding a set of generators for the group B_n will surely be of great benefit.

Theorem 1.8. *The set* $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ *generates the group* B_n.

Proof.

Let $\beta \in B_n$ be a braid with k crossings. Since k is finite, if two crossings are equal in the z-coordinate, a fundamental move may be defined that takes one of the crossings to a z-value on which no other crossing is defined. Since this may be done at most $k-1$ times, β is equivalent in B_n to the resulting braid. Hence, β can be partitioned by $k-1$ planes parallel to $z=0$ such that exactly one crossing of β lies in the section between any two planes ($z=0$ and $z=1$ inclusive). From Lemma 1.4, it is clear that any one of these k sections is isotopic to $\sigma_i^{\pm 1}$ for some $1 \leq i \leq n-1$. Thus, $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ generates B_n.

□

It is clear that for any $1 \leq i \leq n$, the generator σ_i has infinite order. Also, according to Theorem 1.8, B_2 is generated by σ_1. This gives that B_2 is cyclic of infinite order and hence must be isomorphic to $\mathbb{Z}$, the isomorphism being given by

$$\phi : B_2 \longrightarrow \mathbb{Z} \text{ such that } \phi(\sigma_1^a) = a \text{ for } a \in \mathbb{Z}.$$

But B_2 is clearly a subgroup of B_n for $n \geq 2$. Hence B_n is an infinite group if $n \geq 2$.

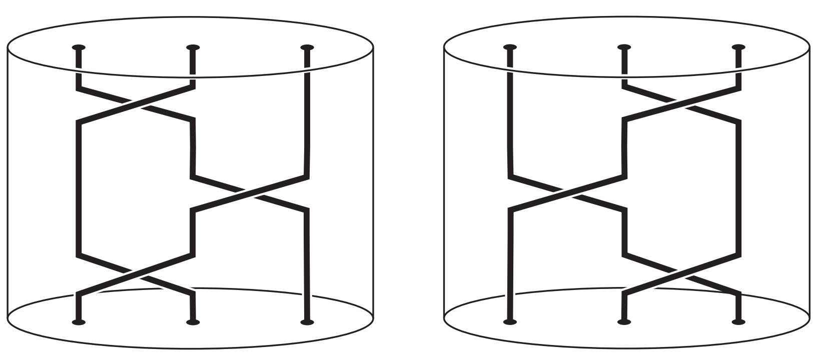


Figure 5: $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

It is clear that for any $1 \leq i < j \leq n-1$ with $|i-j| \geq 2$,

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

since $j > i+1$, the crossings occur at distinct arcs.

Now, consider the case that $|i-j| = 1$, that is $j = i+1$. We know that $\sigma_i \sigma_{i+1} \neq \sigma_{i+1} \sigma_i$ for

$perm(\sigma_i \sigma_{i+1}) = (i, i+2, i+1)$ but $perm(\sigma_{i+1} \sigma_i) = (i, i+1, i+2)$. However, we do find the property that

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}. \tag{1.2}$$

Justification for equation 1.2 is given visually by Figure 5.

These observations lead to a general presentation of the group B_n that we state as a theorem but do not prove. A full proof is given in [4, pp. 7-12].

Theorem 1.9. *The group B_n admits the presentation*

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \ |\sigma_i \sigma_j = \sigma_j \sigma_i \ \textit{for} \ |i-j| \geq 2 \ \textit{and} \ \ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle$$

Given a presentation of a group, any element of the group can be represented as a string of positive or negative powers of the generators. For example, $\sigma_1^2 \sigma_3^{-1} \sigma_2$ is a word that represents an element of B_4. If the string contains only positive powers of the generators, it is called a *positive word.* If a word contains no substrings of the form $\sigma\sigma^{-1}$, it is called a *reduced word.*

2 The Fundamental Word

In the study of B_n, a particular element is found to play an essential role in establishing important properties of the group. In [5], Garside named this element the *fundamental word* Δ and described several of the properties it induces. Some of these properties apply specifically to the semigroup with the same presentation as B_n, which we will write as B_n^+. We list just a few of his results, beginning with a definition.

Definition 2.1. For $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ generators as described in Theorem 1.9; for $i < n$ we define

$$\Pi_i = \sigma_1 \sigma_2 \ldots \sigma_i.$$

Then, the *fundamental word of* B_n is

$$\Delta = \Pi_{n-1} \Pi_{n-2} \ldots \Pi_1.$$

Garside used the definition $\Delta_r = \Pi_r \Pi_{r-1} \ldots \Pi_1$ and defined the fundamental word of B_n to be $\Delta = \Delta_{n-1}$; however, due to the interest of this investigation in the more general setting of Garside groups, we will denote the fundamental word of B_n by $\Delta(B_n)$. This will be shortened to Δ if there is no ambiguity.

Now it is necessary to describe a map that extends to an automorphism on B_n. To this end, let $\mathcal{R} : B_n \to B_n$ be the set map that sends σ_i to σ_{n-i} and extends to any word in the generators of B_n in the usual way. If $|i - j| \geq 2$, then $\mathcal{R}(\sigma_i \sigma_j) = \sigma_{n-i} \sigma_{n-j}$. But $|(n-i) - (n-j)| = |i - j| \geq 2$, so $\mathcal{R}(\sigma_i \sigma_j) = \mathcal{R}(\sigma_j \sigma_i)$. Also,

$$\begin{aligned} \mathcal{R}(\sigma_i \sigma_{i+1} \sigma_i) &= \sigma_{n-i} \sigma_{n-i-1} \sigma_{n-i} \\ &= \sigma_{n-i-1} \sigma_{n-i} \sigma_{n-i-1} \\ &= \mathcal{R}(\sigma_{i+1} \sigma_i \sigma_{i+1}). \end{aligned}$$

These two facts together are sufficient to show that $\mathcal{R}$ is a well-defined endomorphism of B_n. Since $\mathcal{R}$ is also a permutation of the generators of B_n, it must be an automorphism of B_n. This allows us to state an important result of Garside.

Theorem 2.2. *If Δ is the fundamental word in B_n, then*
(i) $P\Delta^{2m} = \Delta^{2m}P,\ P\Delta^{2m+1} = \Delta^{2m+1}\mathcal{R}P$ *for all* $m \geq 0$ *and* $P \in B_n^+$,
(ii) $Q\Delta^{2m} = \Delta^{2m}Q,\ Q\Delta^{2m+1} = \Delta^{2m+1}\mathcal{R}Q$ *for all* $m \in \mathbb{Z}$ *and* $Q \in B_n$.

The property encapsulated in the preceding theorem that allows a power of Δ to be "pushed" to either side of a word is very useful. In fact, the theorem gives the immediate result that if Δ is the fundamental word in B_n, the set $S = \left\{\Delta^{2m} | m \in \mathbb{Z}\right\}$ is contained in the center of B_n. Even more, S is actually the whole of the center of B_n, see [5, p. 246].

To see this in another way, we refer to another result from [5], omitting proof.

Lemma 2.3. *In B_n, for $1 < s \leq t \leq n-1$, we have $\sigma_s\Pi_t = \Pi_t\sigma_{s-1}$.*

With this lemma, it is possible to prove the following result:

Lemma 2.4. *For $n \geq 2$, $\Pi_{n-1}^n = \Pi_{n-1}\Pi_{n-2}\ldots\Pi_1\Pi_{n-1}\Pi_{n-2}\ldots\Pi_1 = \Delta(B_n)^2$.*

Proof. We note that the result is true if $n = 2$. Consider the product Π_{n-k}^2 for $n > 2$ and $1 \leq k < n$. Repeated application of Lemma 2.3 shows,

$$\begin{aligned}\Pi_{n-k}\Pi_{n-k} = \Pi_{n-k-1}\sigma_{n-k}\Pi_{n-k} = \Pi_{n-k-1}\Pi_{n-k}\sigma_{n-k-1} &= \Pi_{n-k-2}\Pi_{n-k}\sigma_{n-k-2}\sigma_{n-k-1} \\ &= \sigma_1\Pi_{n-k}\Pi_{n-k-1}.\end{aligned}$$

In particular, $\Pi_{n-1}^2 = \Pi_1\Pi_{n-1}\Pi_{n-2}$. One may continue by induction on the power of Π_{n-1}. That is, suppose that for $k < n$, the equality $\Pi_{n-1}^k = \Pi_{k-1}\ldots\Pi_1\Pi_{n-1}\ldots\Pi_{n-k}$

holds. Then

$$\begin{aligned}
\Pi_{n-1}^{k+1} &= \Pi_{n-1}(\Pi_{n-1}^{k}) \\
&= \Pi_{n-1}(\Pi_{k-1}\ldots\Pi_1\Pi_{n-1}\ldots\Pi_{n-k}) \\
&= \sigma_2\ldots\sigma_k\Pi_{n-1}(\Pi_{k-2}\ldots\Pi_1\Pi_{n-1}\ldots\Pi_{n-(k-1)})\Pi_{n-k} \\
&= \sigma_2\ldots\sigma_k\Pi_{n-1}^{k}\Pi_{n-k} \\
&= \sigma_2\ldots\sigma_k\Pi_{k-1}\ldots\Pi_1\Pi_{n-1}\ldots\Pi_{n-k}\Pi_{n-k} \\
&= \sigma_2\ldots\sigma_k\Pi_{k-1}\ldots\Pi_1\Pi_{n-1}\ldots\sigma_1\Pi_{n-k}\Pi_{n-k-1} \\
&= (\sigma_2\ldots\sigma_k\Pi_{k-1}\ldots\Pi_1\sigma_k)\Pi_{n-1}\ldots\Pi_{n-k-1} \\
&= \Pi_k\Pi_{k-1}\ldots\Pi_1\Pi_{n-1}\ldots\Pi_{n-k-1}.
\end{aligned}$$

The last equality is given by a repeated use of the relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ and the fact that σ_i and σ_j commute if $|i-j| \geq 2$. □

If $k = n-1$ then we see $\Pi_{n-1}^{n} = \Pi_{n-1}\Pi_{n-2}\cdots\Pi_1\Pi_{n-1}\Pi_{n-2}\cdots\Pi_1 = \Delta^2$. Using this result, we see from [7] that Π_{n-1}^{n} generates the center of B_n, and so Δ^2 does also.

2.1 Representations of $\Delta(B_n)$ in $\mathcal{B}_n$

We define two concepts that are necessary for our combinatorial discussion of the group B_n. Recall that $\mathcal{B}_n$ refers to the set of braids *without* isotopy equivalence.

Definition 2.5. Let W be a braid with only right crossings in $\mathcal{B}_n$. We may represent any braid that is isotopic to W by a series $a_1a_2\cdots a_r$ where σ_{a_i} is isotopic to $\sigma_j \in B_n$ for some $1 \leq j \leq n-1$. If $\omega \sim W$ such that $\omega = \sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_s}$ in B_n and s is minimal, then we say ω is a *reduced representation* of W.

Definition 2.6. Let $\omega = \sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_s}$ be a reduced representation of W. Then the *length* of W is s.

Remark 2. We note that given W in $\mathcal{B}_n$, any two reduced representations of W are equal in the group B_n, since they are isotopic to W.

Let $\tau \in S_n$. Then τ may be represented in the form $a_1a_2a_3\ldots a_n$ where $\tau(i) = a_i$. Written this way, $123\ldots n$ is the identity permutation and for any permutation τ in S_n, τ may be constructed from $123\ldots n$ through a series of *adjacent transpositions* of the form

$$a_1a_2\ldots a_{i-1}a_ia_{i+1}a_{i+2}\ldots a_n \longrightarrow a_1a_2\ldots a_{i-1}a_{i+1}a_ia_{i+2}\ldots a_n.$$

Any adjacent transposition of this form will be written α_i.

Definition 2.7. A *reduced decomposition* of $\tau \in S_n$ is some sequence $\alpha_{k_1}, \alpha_{k_2}, \ldots, \alpha_{k_s}$ with the $k_i \in \{1, 2, \ldots, n-1\}$ such that

$$123\ldots n \xrightarrow{\alpha_{k_1}} \ldots \xrightarrow{\alpha_{k_s}} \tau$$

and s is minimal.

For any given $\tau \in S_n$, there may be more than one reduced decomposition of τ. For example, two reduced decompositions exist for the permutation 321. Explicitly we have,

$$123 \xrightarrow{\alpha_1} 213 \xrightarrow{\alpha_2} 231 \xrightarrow{\alpha_1} 321 \quad \text{and} \tag{2.1}$$

$$123 \xrightarrow{\alpha_2} 132 \xrightarrow{\alpha_1} 312 \xrightarrow{\alpha_2} 321. \tag{2.2}$$

In fact, for any permutation of longest length (i.e. a permutation $\tau \in S_n$ such that $\tau(i) \neq i$ for each $1 \leq i \leq n$) Richard P. Stanley showed the number of reduced decompositions of the permutation to be equal to the number of standard Young tableaux of shape $\lambda = (n-1, n-2, \ldots, 1)$ [8]. Using the Hook formula, this number is well-known to be

$$\frac{\binom{n}{2}!}{1^{n-1}3^{n-2}5^{n-3}\cdots(2n-3)^1}.$$

This leads us to a remarkable theorem.

Theorem 2.8. *Let $\mathcal{F}_n$ be the free group generated by the braid elements $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$. Denote by C_n the set of elements in $\mathcal{F}_n$ which, in B_n, are equal to the fundamental word Δ. Then we have*

$$|C_n| = \frac{\binom{n}{2}!}{1^{n-1}3^{n-2}5^{n-3}\cdots(2n-3)^1}.$$

Proof.

By the remarks that precede Theorem 2.8, all that remains to prove the statement is to exhibit a bijection between the set R_n of reduced decompositions of the permutation $n-1, n-2, \ldots, 2, 1$ and the set C_n.

Note that the sequence representation $\alpha_{k_1}, \ldots, \alpha_{k_s}$ given in Definition 2.7 of a reduced decomposition is unique; hence, if $\alpha_{k_1}, \ldots, \alpha_{k_s}$ is an element of the set R_n, define $\psi : R_n \longrightarrow C_n$ by the rule

$$\psi(\alpha_{k_1}, \ldots, \alpha_{k_s}) = \sigma_{k_1} \cdots \sigma_{k_s}.$$

It is clear that ψ is injective by its construction. Showing that the map is onto will also make it clear that Imψ is in fact contained in C_n. To see this, define χ_i to be the sequence $\alpha_1, \alpha_2, \ldots, \alpha_i$. Then note that $\chi_{n-1}, \chi_{n-2}, \ldots, \chi_1$ is a sequence in R_n and

$$\psi(\chi_{n-1}, \chi_{n-2}, \ldots, \chi_1) = \Pi_{n-1}\Pi_{n-2}\cdots\Pi_1$$

which is the fundamental word in B_n.

Now if ω is a word in C_n, then $\Pi_{n-1}\Pi_{n-2}\cdots\Pi_1$ may be transformed into ω through a finite sequence of moves of the form

(1) $\sigma_i\sigma_j \longrightarrow \sigma_j\sigma_i$ with $|i-j| \geq 2$, or

(2) $\sigma_i\sigma_{i+1}\sigma_i \longrightarrow \sigma_{i+1}\sigma_i\sigma_{i+1}$.

Also, if $\alpha_{k_1}, \alpha_{k_2}, \ldots, \alpha_{k_s}$ is a sequence in R_n and $k_m = i, k_{m+1} = j$ and $|i-j| \geq 2$ then the order of α_{k_m} and $\alpha_{k_{m+1}}$ may be changed to no effect. In other words, $\alpha_{k_1}, \ldots, \alpha_{k_{m-1}}, \alpha_{k_{m+1}}, \alpha_{k_m}, \alpha_{k_{m+2}}, \ldots, \alpha_{k_s} \in R_n$.

In addition, equations 2.3 and 2.4 below show that if

$$\alpha_{k_1}, \ldots, \alpha_{k_i}, \alpha_{k_i+1}, \alpha_{k_i}, \alpha_{k_{i+3}}, \ldots, \alpha_{k_s}$$

is in R_n then so is $\alpha_{k_1}, \ldots, \alpha_{k_i+1}, \alpha_{k_i}, \alpha_{k_i+1}, \alpha_{k_{i+3}}, \ldots, \alpha_{k_s}$.

$$\cdots a_i a_{i+1} a_{i+2} \cdots \xrightarrow{\alpha_i} \cdots a_{i+1} a_i a_{i+2} \cdots \xrightarrow{\alpha_{i+1}} \cdots a_{i+1} a_{i+2} a_i \cdots \xrightarrow{\alpha_i} \cdots a_{i+2} a_{i+1} a_i \cdots \quad (2.3)$$

$$\cdots a_i a_{i+1} a_{i+2} \cdots \xrightarrow{\alpha_{i+1}} \cdots a_i a_{i+2} a_{i+1} \cdots \xrightarrow{\alpha_i} \cdots a_{i+2} a_i a_{i+1} \cdots \xrightarrow{\alpha_{i+1}} \cdots a_{i+2} a_{i+1} a_i \cdots \quad (2.4)$$

Hence, any element in C_n is the image of some element in R_n, and ψ is surjective. It is easy to see that a similar argument may be reproduced to show that Im$\psi \subset C_n$, implying that Im$\psi = C_n$. So $\psi : R_n \longrightarrow C_n$ is a bijective function. □

This determines the number of representations of $\Delta(B_n)$ in the generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$. However, counting representations of arbitrary powers of $\Delta(B_n)$ appears to be a more subtle problem. In the next section, we turn instead to a set of Artin groups associated to dihedral groups, of which B_3 is a particular example. We then provide a solution to the counting problem in the fourth section.

3 Artin Groups associated to Dihedral Groups

In this section we define the Artin group, denoted A_k, associated to a dihedral group of order $2k$. We then define another class of groups, G_k, that have the same growth series as A_k which is calculated using a sequence of Fibonacci-like numbers. What is more, the groups A_k and G_k have isomorphic Cayley graphs, and this fact is used to simplify the problem of finding the number of representations of $\Delta^n(A_k)$ in two generators.

3.1 Artin Groups

Recall that $B_3/P_3 \cong S_3$. Moreover, the symmetric group S_3 is isomorphic to the group of isometries of a triangle. In this setting, $S_3 = D_6$ is a particular example of a dihedral group, D_{2k}, where $2k$ is the order of the group that acts by isometries on a regular k-gon. Any dihedral group, D_{2k}, can be generated by two reflections, a and b, such that ab represents a rotation of the k-gon by $2\pi/k$. This says that ab has order k. In fact, D_{2k} can be represented by the presentation

$$D_{2k} = \left\langle a, b \; \middle| a^2 = b^2 = (ab)^k = e \right\rangle .$$

Defining $p_k(x, y)$ to be the word of length k with the form $xyxyxyxy\ldots$, since a, b are their own inverses, the presentation says that $p_k(a, b) = p_k(b, a)$ in D_{2k}.

Definition 3.1. For $k \geq 1$, define the *Artin group associated to* D_{2k} to be the group with the presentation

$$A_k = \langle a, b \; | p_k(a, b) = p_k(b, a) \rangle .$$

As the generators are no longer involutions, but of infinite order, A_k is an infinite group. There are analogous ways to define the Artin group associated to any finite reflection group; for now we restrict our attention to A_k, but mention that B_n is the Artin group associated to S_n. Also, let $\mathcal{R} : A_k \to A_k$ be the map sending $a \to b$ and $b \to a$. This map is analogous to the previous $\mathcal{R}$, and is an automorphism.

Note that if k is even,

$$bp_k(a,b) = p_k(b,a)b = p_k(a,b)b \quad \text{and} \quad ap_k(a,b) = ap_k(b,a) = p_k(a,b)a.$$

If k is odd,

$$bp_k(a,b) = p_k(b,a)a = p_k(a,b)\mathcal{R}b \quad \text{and} \quad ap_k(a,b) = ap_k(b,a) = p_k(a,b)\mathcal{R}a.$$

The result is similar for $b^{-1}p_k(a,b)$ and $a^{-1}p_k(a,b)$. This implies that $p_k(a,b)$ expresses the same "pushing" property in A_k as Δ in Theorem 2.2. Hence, we call $p_k(a,b)$ the fundamental word of A_k, denoted $\Delta(A_k)$.

3.2 Growth Series

In this section we see that the growth series of A_2^+ is equivalent to that of G_2^+ by a natural equivalence of the Cayley graph of each group. This serves to motivate the use of the Cayley graph of G_k^+, in place of that of A_k^+, in order to find the number of geodesics from the identity to $\Delta^n(A_k)$.

If necessary, the reader may wish to become familiar with the construction of the Cayley graph of a presentation of a group. In [1], Cannon develops the notion of a labeled graph that has both the connectedness and permutation properties. He then shows how to recognize that such a graph is the Cayley graph of a group with a given presentation.

Roughly, a Cayley graph is a directed, labeled graph, Γ, that is both connected and homogeneous with respect to the labels at every vertex. Vertices of Γ correspond to elements of the group. The edges of Γ are labeled by the generators, and movement along an edge labeled a represents multiplication by a on the right. Also, cycles in the graph correspond to relations in the group.

A metric can be defined on the vertices of Γ by letting $d(v_1, v_2)$ be the infimum of the number of edges in any path from v_1 to v_2. A *geodesic* in Γ is a path from v_1 to v_2 with exactly $d(v_1, v_2)$ edges. This precludes a geodesic in Γ from containing any cycles.

Also, note that our presentation of A_k is such that for any positive word w_1, the equality $w_1 = w_2$, where w_2 is a reduced word, implies w_2 is a positive word. This establishes, in the case of Artin groups, a one to one correspondence between representations of $\Delta^n(A_k)$ as a positive word in the generators, and geodesics from the identity vertex of Γ to $\Delta^n(A_k)$.

Let G be a finitely generated group with presentation

$$G = \langle x_1, x_2, \ldots, x_n \mid R(x_1, x_2, \ldots, x_n) \rangle .$$

For any $g \in G$, if $g = x_{i_1} x_{i_2} \ldots x_{i_k}$, and k is minimal (this is a shortest representation of g), let $\mathrm{len}(g) = k$.

Definition 3.2. For any finitely generated group G as described above, the *growth series* of G is

$$\gamma(t) = \sum_{g \in G} t^{\mathrm{len}(g)}.$$

Note that this is a formal series and not a function. We also use the same definition for the growth series of a finitely generated semigroup.

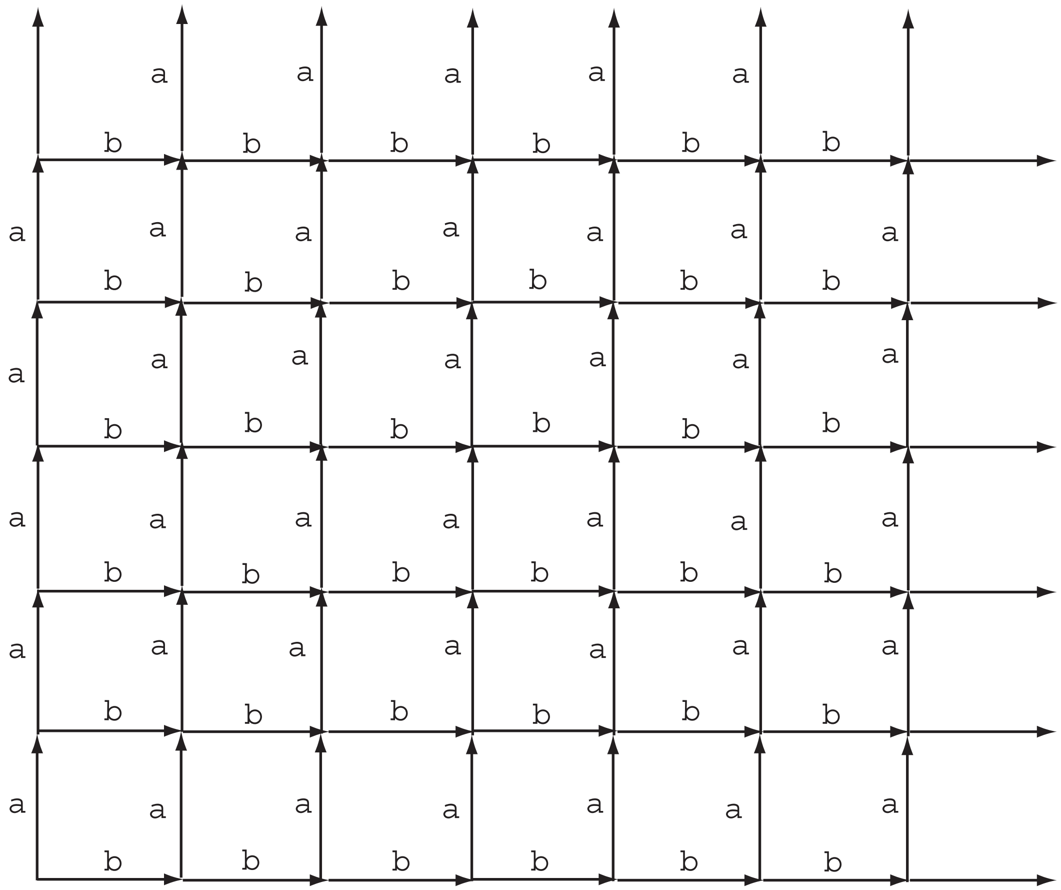


Figure 6: The Cayley graph of A_2^+

In this consideration, the semigroup A_2^+ is of interest, for it has a simple growth series,

$$\gamma(t) = \sum_{n=0}^{\infty}(n+1)t^n.$$

There is, admittedly, a very straightforward way to see that this is, in fact, the correct growth series of A_2^+; however, we appeal to the corresponding Cayley graph in order to develop the subsequent discussion. Recall that $A_2^+ = \langle a, b \mid ab = ba \rangle$. The Cayley graph of this semigroup can be described by the first quadrant of the integral lattice ($\left(\mathbb{Z}^+ \cup \{0\}\right)^2$), labelling the vertical edges a and horizontal edges b (see Figure 6).

One can see that the coefficient of t^n in $\gamma(t)$ is the number of points in the intersection of the lattice with $y = n - x$. It is clear that this number is $n + 1$. Certainly this is true for $n = 0$. But for each point (x_0, y_0) on the lattice that is intersected by $y = n - x$, the point $(x_0, y_0 + 1)$ will be on the line $y = n + 1 - x$ and this describes all the points $y = n + 1 - x$ intersects except at $y = 0$.

There is another Cayley graph that may be constructed in the following way. Take the same part of the lattice. We will define rules for labelling the edges:

(1) For any vertical edge connecting $(x, 0)$ and $(x, 1)$, label the edge a if x is even and b if x is odd.

(2) For any horizontal edge from $(0, y)$ to $(1, y)$, label the edge a if y is odd and b if y is even.

Once (1) and (2) are complete, we finish the labelling inductively. For an unlabelled horizontal edge h between (n, y) and $(n+1, y)$, if the edge between $(n-1, y)$ and (n, y) is labelled a, label edge h with b, otherwise label h with a. Also label vertical edges based on the vertical edge below them in the analagous way.

We note that this is the Cayley graph of a semigroup, which we call G_2^+, that has presentation

$$G_2^+ = \left\langle a, b \;|a^2 = b^2 \right\rangle .$$

An examination of the Cayley graph of G_2^+ shows it to have the same growth series as A_2^+.

We would like to see if there is some similar relation between the semigroups A_n^+ and G_n^+, where

$$G_n^+ = \langle a, b \;|a^n = b^n \rangle .$$

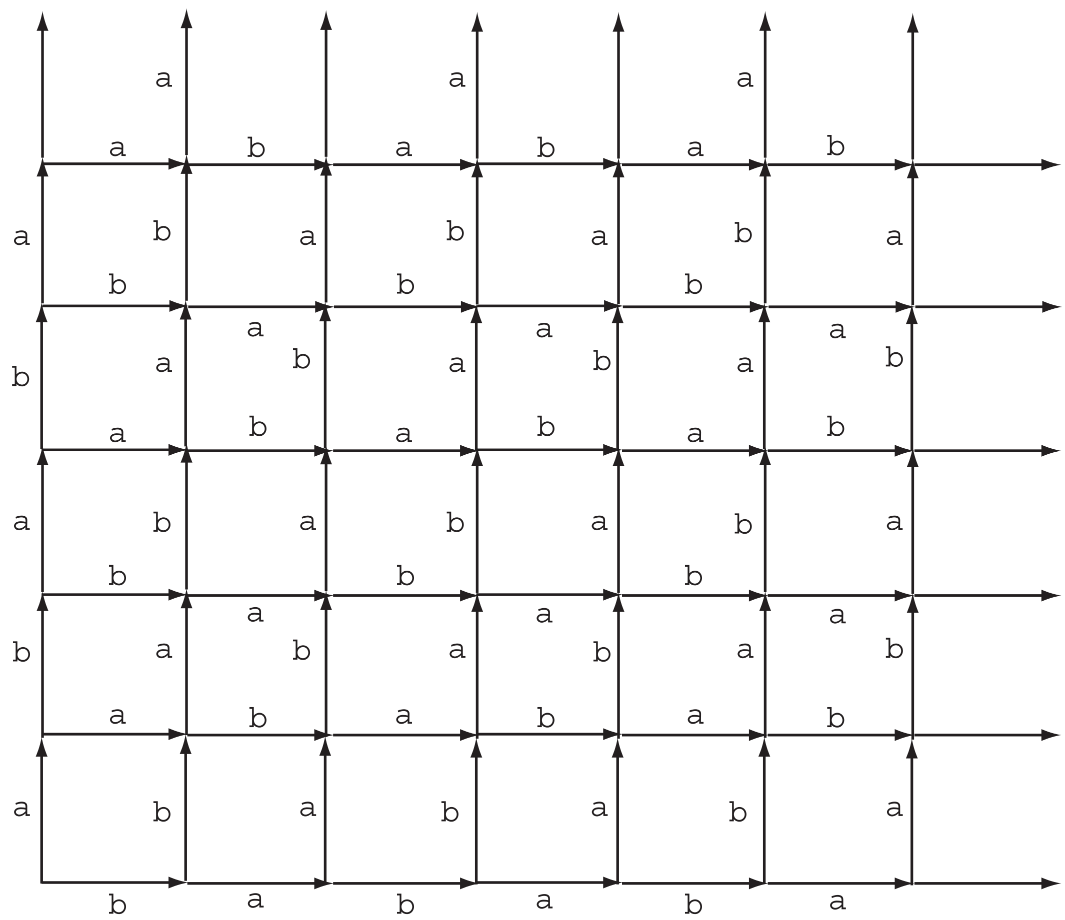


Figure 7: The Cayley graph of G_2^+

In general, the Cayley graphs are difficult to construct, and so we will resort to other methods for now and postpone their construction until the following section. Particularly, we desire to construct a bijective map between the two semigroups that preserves length. In order to do this, however, it is necessary that we know more about the elements in G_n^+. In fact, we hope that G_n^+ admits a fundamental word exhibiting the same "pushing" principle seen in braid groups and Artin groups, that was expressed in Theorem 2.2.

Stated another way, this principle states that, given any $g \in G_n^+$, g has a (unique) representation of the form

$$g = g'\Delta^k,$$

for some word $\Delta \in G_n^+$ such that g' is Δ-free (i.e., no word equivalent to g' in G_n^+ has Δ as a subword). This we now prove, letting $\Delta = a^n$.

Theorem 3.3. *Given any $g \in G_n^+$, g has a unique representation of the form*

$$g = g'a^{nk}$$

where $a^n \nmid g'$, i.e. g' is a^n-free.

Proof. If g is a^n-free, its word representation is unique. Otherwise, we must show that

$$(a^n)a = a(a^n) \quad \text{and} \quad (a^n)b = b(a^n) \tag{3.1}$$

in order to show that g can be written as $g'a^{nk}$, since the operation in G_n^+ is associative.

The first half of (3.1) is quite clear. For the second half, $(a^n)b = (b^n)b = b(b^n) = b(a^n)$. To see that this representation is unique, we first note that G_n^+ has no relations that will shorten the length of a word. Thus if $g = h$ in G_n^+, their lengths are equal. Also, let $g = g'a^{nk_1}$ and $h = h'a^{nk_2}$. Since G_n^+ inherits the cancellation property from the group G_n and any a^n-free word cannot be equal to a word that a^n divides, it must be that $k_1 = k_2$. Also, this implies $g' = h'$, and these words have only one representation. Thus the representation of g as $g'a^{nk}$ is unique. □

We now construct a map $\theta : G_n^+ \longrightarrow A_n^+$ in the following manner. Recall the map $\mathcal{R}$ introduced prior to Theorem 2.2. Let $x_1 = a$ and $x_2 = b$ in G_n^+. Then if

$g = x_{i_1}x_{i_2}\ldots x_{i_k}$, $i_j \in \{1,2\}$ for all j, is an element of G_n^+, let

$$\theta(g) = \begin{cases} x_{i_1}\mathcal{R}(x_{i_2})x_{i_3}\ldots\mathcal{R}(x_{i_{2m}})x_{i_{2m+1}}\ldots x_{i_k} \text{ if } k \text{ is odd,} \\ x_{i_1}\mathcal{R}(x_{i_2})x_{i_3}\ldots\mathcal{R}(x_{i_{2m}})x_{i_{2m+1}}\ldots\mathcal{R}(x_{i_k}) \text{ if } k \text{ is even.} \end{cases}$$

Note that θ is a map into A_n^+, so the products indicated are products in A_n^+. Moreover, θ is certainly a bijection on the free semigroups associated to G_n^+ and A_n^+, for if $x_{i_1}x_{i_2}\ldots x_{i_k}$ (suppose k is odd) is any element of A_n^+ in the generators a and b, then

$$x_{i_1}\mathcal{R}(x_{i_2})\ldots\mathcal{R}(x_{i_{2m}})x_{i_{2m+1}}\ldots x_{i_k}$$

is its preimage under θ. Injectivity is also clear. Thus, if we are able to show that θ is well-defined from G_n^+ to A_n^+, since it preserves the length of words, this will show G_n^+ to have the same growth series as A_n^+.

To show that θ is well-defined, we need two lemmas. In the following, we let Δ_n denote the fundamental word in A_n^+. First we show

Lemma 3.4. *If g and h are words in G_n^+ such that len(g) is odd, then*

$$\theta(gh) = \theta(g)\mathcal{R}(\theta(h)).$$

If len(g) is even

$$\theta(gh) = \theta(g)\theta(h).$$

This lemma is clear when the definition of θ is considered, so we omit the proof. Also recall that if $\Delta_n = p_n(a,b)$ has n odd, then $\Delta_n w = \mathcal{R}(w)\Delta_n$ for any $w \in A_n^+$. Moreover, we give the following lemma.

Lemma 3.5. *$\theta(wa^{nk}) = \theta(w)\Delta_n^k$ for any $w \in G_n^+$.*

Proof. If $\text{len}(w)$ is even, this is clear for in that case $\theta(wa^{nk}) = \theta(w)\theta(a^{nk})$ and $\theta(a^{nk}) = \Delta_n^k$ in A_n^+ for any n and power k. Now, if $\text{len}(w)$ is odd, then

$$\theta(wa^{nk}) = \theta(w)\mathcal{R}(\theta(a^{nk})) = \theta(w)\mathcal{R}(\Delta_n^k) = \theta(w)\Delta_n^k. \tag{3.2}$$

The last equality, that $\mathcal{R}(\Delta_n^k) = \Delta_n^k$, is a straightforward check.

□

We now prove that θ is well-defined. Let $g \in G_n^+$ be such that only one copy of a^n appears in any representation of g; we write $g = wa^n w'$. Then we consider two cases: $\text{len}(wa^n)$ is odd or even. Note that in both cases

$$g = ww'a^n \text{ and } \theta(ww'a^n) = \theta(ww')\Delta_n.$$

1. If $\text{len}(wa^n)$ is even, $\theta(g) = \theta(wa^n)\theta(w') = \theta(w)\Delta_n\theta(w')$. Now $\text{len}(wa^n)$ being even implies $\text{len}(w)$ and $\text{len}(a^n)$ have the same parity. If both are even, then using Lemma 3.4 and Lemma 3.5,

   $$\theta(g) = \theta(w)\Delta_n\theta(w') = \theta(w)\theta(w')\Delta_n = \theta(ww')\Delta_n,$$

   since n in this case is even. If both are odd, then

   $$\theta(g) = \theta(w)\Delta_n\theta(w') = \theta(w)\mathcal{R}(\theta(w'))\Delta_n = \theta(ww')\Delta_n$$

   by both Lemmas as well.

2. If $\text{len}(wa^n)$ is odd, $\theta(g) = \theta(wa^n)\mathcal{R}(\theta(w')) = \theta(w)\Delta_n\mathcal{R}(\theta(w'))$. But also, in this case, $\text{len}(w)$ and $\text{len}(a^n)$ have opposite parity. Suppose $\text{len}(w)$ is odd.

Then

$$\theta(g) = \theta(w)\Delta_n\mathcal{R}(\theta(w')) = \theta(w)\mathcal{R}(\theta(w'))\Delta_n = \theta(ww')\Delta_n$$

since $\operatorname{len}(\Delta_n)$ is even. And if $\operatorname{len}(a^n)$ is odd, then

$$\theta(g) = \theta(w)\Delta_n\mathcal{R}(\theta(w')) = \theta(w)\theta(w')\Delta_n = \theta(ww')\Delta_n$$

since $\operatorname{len}(w)$ is even.

This shows that θ is well-defined on any word in which one copy of a^n appears. We don't want to go through so many cases again. So assume through induction that for $k \geq 1$ and $k < l$, θ is well-defined on any word containing k copies of a^n. That is, if g has the representation $g'(a^{nk})$ with g' being a^n-free, then $\theta(g)$ and $\theta(g'(a^{nk})) = \theta(g')\Delta_n^k$ are equal for $k < l$.

Suppose g has l copies of a^n. Then we can write g as $g'a^ng''$ where g' has $l-1$ copies of a^n. Note that in the proof that θ is well-defined on a word with one copy of a^n we never used the fact that w was a^n free. Thus we know that

$$\theta(g) = \theta(g'a^ng'') = \theta(g'g''a^n) = \theta(g'g'')\Delta_n.$$

But by induction, $\theta(g'g'') = \theta(w)\Delta_n^{l-1}$ where w is a^n-free. Thus, θ is well-defined on all of G_n^+.

Remark 3. The previous discussion shows G_n^+ and A_n^+ to have the same growth series. It does not, however, show θ to be an isomorphism. For example, if $aaaabba \in G_4^+$ then $\theta(aaaabba) = ababbaa = \Delta_4 baa$. But unfortunately,

$$\theta(aaa)\theta(abba) = abaaabb$$

which is a Δ_4-free word.

3.3 Calculating the Growth Series of A_k^+

It is possible, in fact to calculate explicitly the coefficients of t^{3n} in the growth series of A_k^+. The result is given in terms of a Fibonacci-like sequence. Define μ_n to be the number of positive words in the generators $\{a, b\}$ of length $3n$ in $A_3^+ = B_3^+$. Also, denote the n^{th} Fibonacci number by F_n and use the convention that

$$F_0 = 1,\ F_1 = 1,\ F_2 = 2,\ \ldots$$

Theorem 3.6. *For $n \geq 1$, we have*

$$\mu_n = 2\sum_{m=0}^{n} F_{3n-3m} - 1.$$

Proof. Let $\Delta_k = \Delta(A_k)$, and define a Δ_k-free word in A_k^+ to be a word that does not contain any subword equivalent to Δ_k. The proof of the theorem is accomplished mainly by the means of the lemma:

Lemma 3.7. *The number of words of length $n \geq 1$ that are Δ_3-free is $2F_n$.*

Proof. The proof will be by induction on $n \geq 1$. There are $2 = 2F_1$ words of length one, namely a and b, and $4 = 2F_2$ words of length two. Listed, these words are:

$$aa, \quad ab, \quad ba, \quad bb.$$

Assume that for $m < n$, the number of Δ_3-free words of length m is $2F_m$ and let ω be a Δ_3-free word of length n. Then $\omega = \omega' a \quad or \quad \omega' b$, where ω' is a Δ_3-free word of length $n - 1$. Suppose $\omega = \omega' a$.

Half the Δ_3-free words of any length end in a since $\mathcal{R}$ is an automorphism. Thus, by induction, F_{n-1} words of length $n-1$ end in a. If ω' is such a word, the last two letters of ω' are aa or ba.

Also, since to any word of length $n-2$ that ends in b we can attach another b, there are exactly F_{n-2} Δ_3-free words of length $n-1$ ending in bb. It is clear that ω' does not end in ab. Hence there are $F_{n-2} + F_{n-1} = F_n$ possible ω''s, implying that there are $2F_n$ Δ_3-free words of length n.

□

Any element in A_3^+ of length $3m$ is divisible by Δ_3 at most m times. Since we have seen that any copy of Δ_3 in a word ω can be "pushed" to the left, Lemma 3.7 shows that there are $2F_{3m-3k}$ words in A_3 of length $3m$ that are k-times divisible by Δ_3, excepting the case where $k = m$. Here of course there is only one equivalent word in the group, namely Δ_3^m. This proves Theorem 3.6. □

This proof may be extended to the more general Artin group A_k. To generalize F_n, define the k-nacci sequence as the sequence of integers $F_{k,n}$ with $F_{k,0} = 1$ and $F_{k,n} = 0$ for $n < 0$ and the relation $F_{k,n} = F_{k,n-1} + F_{k,n-2} + \ldots + F_{k,n-k}$.

Lemma 3.8. *The number of words of length $n \geq 1$ that are Δ_k-free in A_k, $k > 1$ is $2F_{k-1,n}$.*

Proof. To see that the statement holds for $k = 2$ note that $F_{1,n} = 1$ for all positive n, and A_2 has only two Δ_2-free words of any given length. Theorem 3.6 shows that the result also holds for $k = 3$. The proof of the general statement will be similar to that of Theorem 3.6.

Begin by noting that $F_{k-1,n} = 2^{n-1}$ for $1 \leq n < k$. Also, in A_k no word of length less than k will contain Δ_k as a subword. This gives 2^n words of length $1 \leq n < k$, and

$$2^n = 2(2^{n-1}) = 2F_{k-1,n}.$$

Assume that for $m < n$ the number of Δ_k-free words of length m is $2F_{k-1,m}$ and let ω be a Δ_k-free word of length $n \geq k$. To find the number of Δ_k words of length n, we take a Δ_k-free word of length $n-1$ and find the number of ways to multiply on the right without producing a Δ_k. To do this one only needs consider the k rightmost letters. Suppose $\omega = \omega' a$ with ω' a Δ_k-free word of length $n-1$. To any ω' ending in a we may multiply on the right by a. This gives $F_{k-1,n-1}$ possibilities. To find the number of possible ω''s ending in b, let $\omega' = \omega'' b$. Then there are $F_{k-1,n-2}$ ω'''s ending in b from which we are free to choose. Iterating the process k times, there are

$$F_{k-1,n-k+1} + \ldots + F_{k-1,n-2} + F_{k-1,n-1} = F_{k-1,n}$$

Δ_k-free words of length n ending in a. These are half the permissible words, giving the statement. □

With the same reasoning as above we have the result,

Theorem 3.9. *The number of elements in A_k of length kn is*

$$\sum_{m=0}^{n} 2F_{k-1,kn-km} - 1.$$

4 Counting Fundamental Words in A_k^+

The goal of this section is to find the number of geodesics in the positive part of the Cayley graph of A_k that begin at the identity vertex $(\hat{0})$ and end at $(\Delta(A_k))^n$. Any such geodesic can be represented as an element of the free group on two generators, say a and b. To achieve this, we first construct the Cayley graph of A_k and then show that geodesics of the interval $[\hat{0}, (\Delta(A_k))^n]$ in this graph have a bijective correspondence to certain admissible northeast walks on the integer lattice. We define later what is meant here by admissible.

4.1 The Cayley graph of A_k

In fact, we construct a two-dimensional complex that has the Cayley graph of A_k as its 1-skeleton. Begin by considering the uniformly branching k-valent tree, T. Index the vertices of T in the following manner. Choose some vertex of T and call this vertex v. There are k edges with v as an endpoint. Choose one of them and label its other endpoint v_0. Now order the remaining $k-1$ edges at v in a counter-clockwise direction, labeling the other vertex on the i^{th} edge v_i. We complete the labeling of the vertices inductively. If $v_{i_1,i_2,\ldots,i_s}$ is a labeled vertex of T, $s \geq 1$ and it has $k-1$ unlabeled vertices connected to it by a single edge, order the edges counter-clockwise and call the unlabeled vertex on the i^{th} edge $v_{i_1,i_2,\ldots,i_s,i_{s+1}}$. This is shown in Figure 8 for the case of $k = 3$ near v.

Now, consider the space $T \times \mathbb{R}$, with $\mathbb{R}$ oriented vertically and viewed as a simplicial complex with $\mathbb{Z}$ as its vertex set and the intervals $[i, i+1]$ $(i \in \mathbb{Z})$ as edges. We want to sew in vertices and edges at certain parts of this space. For every $u = v_{i_1,i_2,\ldots,i_s}$, a vertex of T, put a vertex in $T \times \mathbb{R}$ at (u, n) for every $n \in \mathbb{Z}$. Also, put in an edge between (u, n) and $(u, n+1)$. Additionally, given an edge $e = [v_{i_1,i_2,\ldots,i_s}, v_{i_1,i_2,\ldots,i_s,i_{s+1}}]$ in T, place an horizontal edge at $\left(e, \left((\sum_{j=1}^{s+1} i_j) \mod k\right) + kn\right)$ for all $n \in \mathbb{Z}$.

Call the 1-skeleton of the resulting complex $\overline{\Gamma}_k$. It has an infinite strip of $1 \times k$ rectangles for each edge of T. Note that every vertex in $\overline{\Gamma}_k$ is connected to

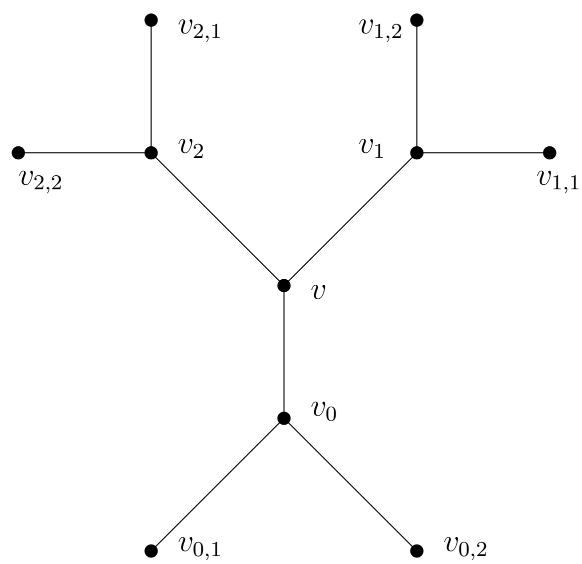


Figure 8: An example of T near v when $k = 3$.

exactly one horizontal edge. Also, we wish to orient the horizontal edges such that movement from $(v_{i_1,\ldots,i_s}, n)$ to $(v_{i_1,\ldots,i_{s+1}}, n)$ is considered movement in the positive direction. A portion of $\overline{\Gamma}_3$ near $(v, 0)$ is shown in Figure 9.

Finally, label a vertical edge $v_{i_1,i_2,\ldots,i_s} \times [n, n+1]$ by b if $n+s$ is even and by a if $n+s$ is odd. Call two points p, q of $\overline{\Gamma}_k$ equivalent, or $p \sim q$, if p and q are contained in the same horizontal edge. In fact, every 0-cell of $\overline{\Gamma}_k$ is equivalent to exactly one other 0-cell (e.g. $(v, 0) \sim (v_0, 0)$). Define $\Gamma_k = \overline{\Gamma}_k / \sim$, so that Γ_k is the complex obtained by retracting each horizontal edge of $\overline{\Gamma}_k$ to a point. All other parts of the graph $\overline{\Gamma}_k$ are left unchanged by this identification, so we consider the labeling just defined to be the same in Γ_k. Also, define the image of the 0-cell $(v_0, 0)$ under the identification to be $\hat{0}$, the identity vertex, in Γ_k–which turns out to be a Cayley graph.

Theorem 4.1. *The 1-skeleton of Γ_k, or $(\Gamma_k)_{(1)}$, is the Cayley graph of A_k. Thus, the corresponding 2-complex is the universal cover of the standard presentation 2-*

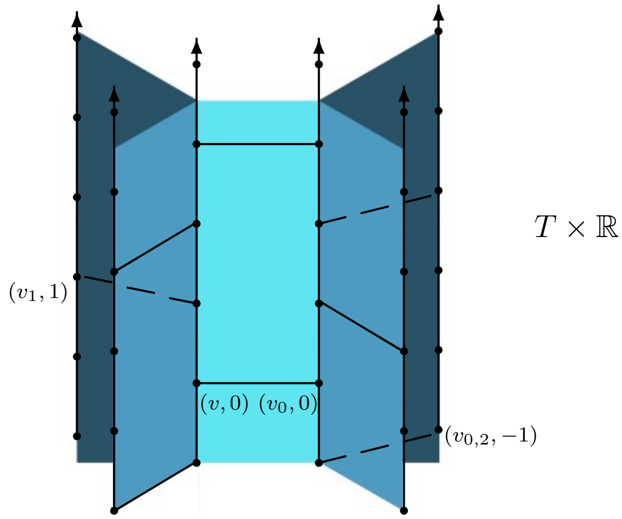


Figure 9: A section of Γ_3.

complex for A_k.

Proof. This result can be checked using Cannon's recognition theorem in [1, p. 277]. □

Let $v_{i_1,i_2,\ldots,i_s}$ be a vertex of T and change the $\{a,b\}$-labeling of $\overline{\Gamma}_k$ by labeling any vertical 1-cell on $v_{i_1,\ldots,i_s} \times \mathbb{R}$ with b if s is even, and with a if s is odd. Then, passing to Γ_k, the 1-skeleton is now equal to the Cayley graph of G_k, which is defined by the presentation

$$G_k = \left\langle a, b \mid a^k = b^k \right\rangle.$$

Thus the Cayley graphs of these two groups are isomorphic as unlabeled graphs. This implies that any result concerning geodesics in one of the graphs implies the same result is true on the other graph. In fact, this phenomenon (that certain results on geodesics depend only on the number of generators and the length of the relators) will manifest itself in more generality in the result of Theorem 5.13. Throughout the section that follows, Γ_k refers to the Cayley graph of G_k, as this labeling Γ_k provides somewhat simpler notation. Note that it was shown earlier that a^k has the same property in G_k (the "pushing principle") as $\Delta(A_k)$ does in A_k.

4.2 A correspondence to lattice walks

The Cayley graph Γ_k is now defined. Let Γ_k^+ be the subgraph determined by vertices (v, n), such that if $v = v_{i_1,\ldots,i_s}$, then $n \geq \sum i_j$ and all edges of Γ_k that are between such vertices. This corresponds to the Cayley graph of the semigroup $G_k^+ = \langle a, b \,| a^k = b^k \rangle^+$ of positive words in G_k. This section will prove that the set of geodesics in Γ_k^+ from $\hat{0}$ to a^{kn} is in bijective correspondence to a set of admissible integer lattice walks, $Q_k(n)$ that is defined below.

Definition 4.2. Let L_{k-1} be the line in $\mathbb{R}^2$ of slope $\dfrac{1}{k-1}$ that passes through $(0, 0)$ and call the integer lattice, $Q = \mathbb{Z} \times \mathbb{Z}$. Define $Q_k(n)$ to be the set of northeast paths in Q which start at (0,0), end at $((k-1)n, n) \in L_{k-1}$, and only cross L_{k-1} at vertices of Q. A path in $Q_k(n)$ will be called *admissible*.

Admissible paths will be denoted by a sequence of elements of the set $\{u, r\}$ (which we will write as a word in u, r). Note that such a path has length kn. For example, Figure 10 illustrates the path $ururrrrru \in Q_3(3)$ by a sequence of arrows on the edges of Q starting at (0,0).

Definition 4.3. Define $\Gamma_k(n)$ to be the set of positive words in $\{a, b\}$ (i.e. elements of the free semigroup $F(a, b)^+$, which we will mostly think of as paths in Γ_k^+ starting at the identity vertex) that are equal in G_k^+ to a^{kn}. Thus, for example, $\Gamma_k(1) = \{a^k, b^k\}$ for all $k \geq 2$.

Theorem 4.4. *For all $k \geq 2$ and $n \geq 0$, we have $|Q_k(n)| = |\Gamma_k(n)|$.*

To prove this theorem, we define a function,

$$\beta_k = \beta_{k,n} : Q_k(n) \to \Gamma_k(n),$$

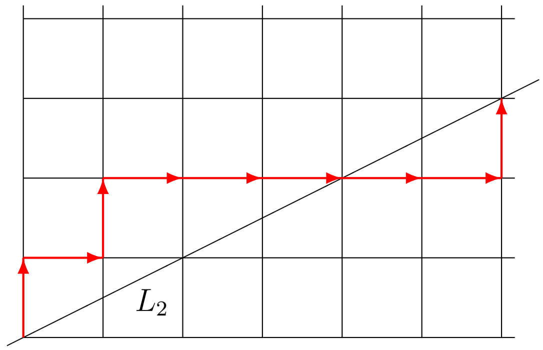


Figure 10: The path $urur rrrru \in Q_3(3)$.

which we will show is a bijection.

We will first define $\overline{\beta}_{k,n} : Q_k(n) \to \overline{\Gamma}_k^+$. Then, having done so, $\beta_{k,n}$ will be the composition of $\overline{\beta}_{k,n}$ and the identification map $\overline{\Gamma}_k^+ \to \Gamma_k^+$. We then show that Im $\beta_{k,n}$ is in fact contained in $\Gamma_k(n)$ and that $\beta_{k,n}$ is a bijection.

Let Q_k^+ denote the part of Q which is on or above L_{k-1} and let Q_k^- be the part on or below L_{k-1}. For any $\pi = \pi_1\pi_2 \ldots \pi_{kn} \in Q_k(n)$, $\pi_i \in \{u, r\}$, we define $\overline{\beta}_{k,n}(\pi)$ inductively. To begin, define $\overline{\beta}_{k,n}$ on a sequence $\pi = \pi_1\pi_2 \ldots \pi_s$ that defines a path in Q_k^+ starting at $(0, 0)$. In such a case, $\pi_1 = u$. Define $\overline{\beta}_{k,n}(u)$ to be the path that begins at $(v, 0)$, moves horizontally to $(v_0, 0)$, and up one vertical edge to $(v_0, 1)$. This makes $\overline{\beta}_{k,n}(u) = a$. Now assume (inductively) that the path $\overline{\beta}_{k,n}(\pi_1\pi_2 \ldots \pi_{s-1})$ is defined and let $z \in \Gamma_k^+$ be the end point of this path. If $\pi_s = u$, move along the (unique) horizontal edge at z, in a direction away from $v \times \mathbb{R}$. Then follow the vertical edge up to the next vertex. If $\pi_s = r$, then simply follow the vertical edge to the next vertex up. Also, if ever the vertex *arrived at* has a horizontal edge that leads closer to $v \times \mathbb{R}$, cross this edge. As an example, note that $\overline{\beta}_{3,3}(uuurrrrrr) = abaaabbaa$, while $\overline{\beta}_{3,3}(urur rurrr) = aabbbbbba$. The path $ururrurrr \in Q_3(3)$ and its image in $\overline{\Gamma}_3^+$ are shown in Figure 11.

This completes the definition of $\overline{\beta}_{k,n}$ on any path completely in Q_k^+ that begins at $(0,0)$. Note that if π is such a path, then $\overline{\beta}_{k,n}(\pi)$ will end at (v, kn) for some positive integer n. If instead $\pi'\pi$ is a path in $Q_k(n)$ such that π' ends at $(x, y) \in L_{k-1}$ and π begins on (x, y) and π is contained in Q_k^+, then define $\overline{\beta}_{k,n}(\pi'\pi) = \overline{\beta}_{k,n}(\pi')\overline{\beta}_{k,n}(\pi)$, where this multiplication of paths in $\overline{\Gamma}_k^+$ is simply concatenation.

But wait, you say, we haven't yet defined $\overline{\beta}_{k,n}(\pi')$ if π' is not completely in Q_k^+. This is a valid point, and will now be remedied. Let $\pi = \eta_1\eta_2 \ldots \eta_s$, where every η_i is either completely in Q_k^+ or in Q_k^-. We have defined $\overline{\beta}_{k,n}(\eta_i)$ if $\eta_i \in Q_k^+$. However, if $\eta_i \in Q_k^-$, let w_1, w_2 denote the endpoints of η_i on L_{k-1} and let $w_3 = (w_1 + w_2)/2$ be their midpoint. Then let $\rho(w_3)$ be the rotation of $\mathbb{R}^2$ by π radians, centered at w_3. Then $\rho(w_3)(\eta_i)$ is a path in Q_k^+ from w_2 to w_1; let θ be the reverse of this path, so that θ goes from w_1 to w_2. Then θ begins with r. Define $\overline{\beta}_{k,n}(\theta)$ in the same way as before. This defines $\overline{\beta}_{k,n}(\pi)$ for any $\pi \in Q_k(n)$. Finally, let $\beta_{k,n}$ be the composition

$$\beta_{k,n} : Q_k(n) \xrightarrow{\overline{\beta}_{k,n}} \overline{\Gamma}_k^+ \longrightarrow \Gamma_k^+.$$

We now proceed to prove Theorem 4.4 by showing that $\beta_{k,n}$ is a bijection onto $\Gamma_k(n) \subset \Gamma_k^+$.

Proof of Theorem 4.4. Note that if $p \in Q_k(n)$ and $\text{len}(p) = s$, then since the projection $\overline{\Gamma}_k \to \Gamma_k$ is injective on vertically oriented edges of $\overline{\Gamma}_k$, we have that $\text{len}(\beta_{k,n}(p)) = s$ (as a length in Γ_k). Thus, $\beta_{k,n}(p) = \beta_{k,n}(p')$ implies that $\text{len}(p) = \text{len}(p')$.

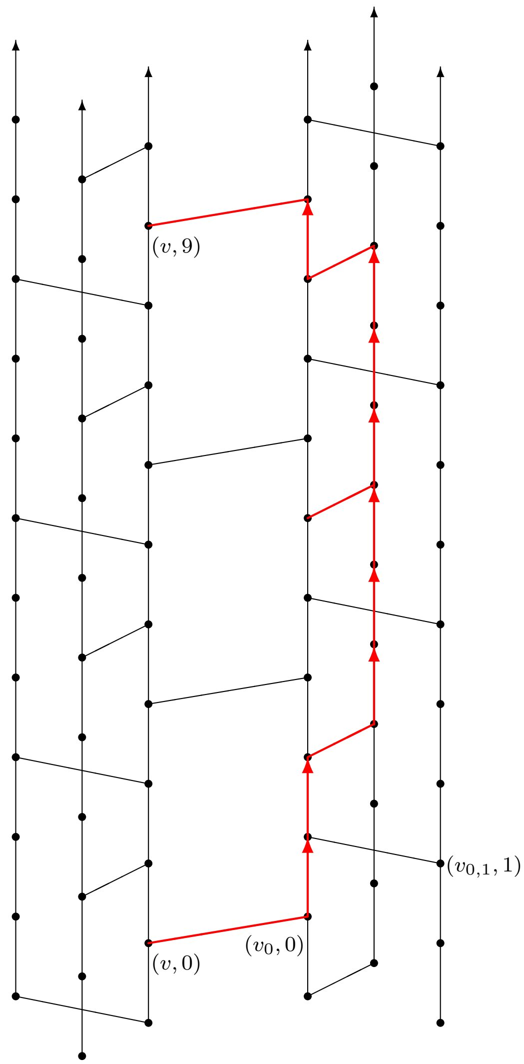


Figure 11: $\overline{\beta}_{3,3}(urur rur rr)$ in $\overline{\Gamma}_3^+$.

Suppose that $p = p_1 p_2 \ldots p_s$ and $p' = p'_1 p'_2 \ldots p'_s$ are in $Q_k(n)$, and are paths such that $\beta_{k,n}(p) = \beta_{k,n}(p')$. Then, as words in $\{a, b\}$, $\beta_{k,n}(p)$ and $\beta_{k,n}(p')$ are the same, i.e. if $\beta_{k,n}(p) = c_1 c_2 \ldots c_s$ and $\beta_{k,n}(p') = d_1 d_2 \ldots d_s$, then $c_i = d_i$ for all i. Since $c_1 = d_1$, by the definition of $\beta_{k,n}$, we have that $p_1 = p'_1$. This says that $\overline{\beta}_{k,n}(p_1)$ and $\overline{\beta}_{k,n}(p'_1)$ both end at the same point. Thus, since $c_2 = d_2$, the definition of the map makes clear that $p_2 = p'_2$ as well. After s steps like this, we see that $p = p'$, so $\beta_{k,n}$ is injective.

The proof that $\beta_{k,n}$ is surjective requires three lemmas.

Lemma 4.5. *Let w be a word in $\{a, b\}$ which is equal in G_k^+ to a^{kn}, $n \geq 1$. Then w contains a subword equal to a^k or a subword equal to b^k; i.e. as words in a, b we have: $w = w_1 a^k w_2$ or $w = w_1 b^k w_2$ for words w_1, w_2 in a, b.*

Proof. Since $w = a^{kn}$ and $n \geq 1$ one can get from a^{kn} to w in a finite number of steps by replacing subwords equal to a^k by b^k or vice versa. The last replacement to get to w is a subword of w, and must have been either a^k or b^k, so (i) follows. □

Let k be fixed and abbreviate $\beta_{k,n}$ as β_n. We use induction on $n \geq 1$ to show that $\beta_n : Q_k(n) \to \Gamma_k^+$ is surjective. If $n = 1$, then $w = a^k$ or $w = b^k$. Since

$$\beta_k(ur^{k-1}) = a^k, \quad \text{and} \quad \beta_k(r^{k-1}u) = b^k,$$

β_1 is surjective.

Assume that β_{n-1} is surjective onto $\Gamma_k(n-1)$ and let $w \in \Gamma_k(n)$. By Lemma 4.5, either $w = w_1 a^k w_2$ or $w = w_1 b^k w_2$, where w_1, w_2 are words in a, b. The cases are similar, so assume that $w = w_1 a^k w_2$. As mentioned previously (in the proof of Theorem 3.3),

$$w = w_1 a^k w_2 = w_1 w_2 a^k.$$

Since $w = a^{kn}$, the cancellative property of G_k^+ says that $w_1 w_2 = a^{k(n-1)} \in \Gamma_k(n-1)$. By induction, there is a $p \in Q_k(n-1)$ such that $\beta_{n-1}(p) = w_1 w_2$. This implies that $\beta_n(pur^{k-1}) = w_1 w_2 a^k$. Thus, to show that β_n is surjective, we only need show that if $w_1 w_2 a^k$ is in the image of β_n, then so is $w_1 a^k w_2$. This is shown by the following two lemmas.

Lemma 4.6. *(i) If $z \in \Gamma_k^+$ where z is the endpoint of the path $\beta(\pi)$, $\pi \in Q_k(n)$, then any path equal to $\beta(\pi)$ followed by a^k has the form $\beta(\pi r^m u r^{k-1-m})$ for some $0 \leq m < k$.*

(ii) Let w_1, w_2 be words in $\{a, b\}$. Then $w_1 a^k w_2 \in Im\ \beta_n \iff w_1 b^k w_2 \in Im\ \beta_n$.

Proof. We prove the result for paths in the space $\overline{\Gamma}_k$ and conclude that it holds in Γ_k^+. If z is not on a vertical line labeled with a's, then $m = 0$ or $m = k-1$, and it is clear that the statement holds. If this is not the case, then there is a minimal m with $0 < m < k-1$, such that the vertex m edges above z is connected to a horizontal edge that leads towards $v \times \mathbb{R}$. Thus $\overline{\beta}(\pi r^m u r^{k-1-m})$ will consist of the first k vertical edges above z, which is $\overline{\beta}(\pi) a^k$. This proves (i).

We prove (ii), supposing that, if z is the endpoint of w_1, then z is in Im $(Q_k^+ \cap Q_k(n))$. The other case follows by symmetry. So, let π, σ_1, σ_2 be words in u, r such that $\beta_n(\sigma_1 \pi \sigma_2) = w_1 a^k w_2$, $\beta_n(\sigma_1) = w_1$ and $\beta_n(\sigma_1 \pi) = w_1 a^k$. Then π must be of the form $r^m u r^{k-1-m}$. If $m > 0$ then $\beta_n(\sigma_1 u r^{k-1} \sigma_2) = w_1 b^k w_2$. If $m = 0$, then there is some $v = ur^l$ such that, as words, $\sigma_1 = \sigma_1' v$. Then we have that $\beta_n(\sigma_1 r^{k-1-l} v \sigma_2) = w_1 b^k w_2$. The proof of the other direction in the implication is, in fact, exactly the same with the a's and b's interchanged. □

Lemma 4.7. *Suppose that there are words u_1, u_2 (in a, b) and $p \in Q_k(n)$ with*

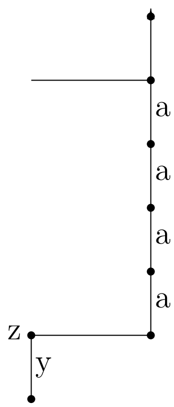
$\beta_n(p) = u_1 a^k u_2$. *Also, let* $u_1 = u_1' y$, $y \in \{a, b\}$. *Then there is* $p' \in Q_k(n)$ *with*

$$\beta_n(p') = u_1' a^k y u_2.$$

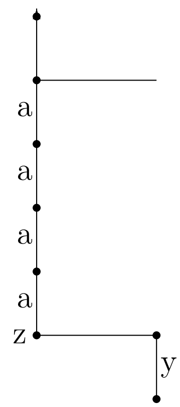
Proof. Let $p = p_1 p_2 p_3$, where p_i is a word in u and r, such that $\beta_n(p_1) = u_1$ and $\beta_n(p_1 p_2) = u_1 a^k$. Furthermore, let z be the vertex that is the endpoint of $\beta_n(p_1)$. The proof is split up into two cases: when $y = b$ and when $y = a$.

Case 1: If $y = b$, then the k edges labeled by a that will follow $\beta_n(p_1)$ are on a different vertical line then the edge labeled by y. Either z is a vertex on the same vertical line as the edge labeled by y or it is on a vertical line closer to $v \times \mathbb{R}$ than the edge labeled by y (this happens if $u_1 = u_1'' b^k$) These two subcases (for $k = 4$) are shown in Figure 12 and labeled as Case 1a and 1b, respectively. In the first case, $p_2 = ur^{k-1}$. In the second $p_2 = r^{k-1}u$. (Note that we used a conclusion in the proof of Lemma 4.6(i), in the case $m = 0, k-1$). Let $p_1 = p_1' x$, where $\beta_n(p_1') = u_1'$. Then $\beta_n(p_1' ur^{k-1} x p_3)$ is $u_1' b^k y u_2$ if $x = u$ and $u_1' a^k y u_2$ if $x = r$. In either case, by Lemma 4.6(ii), the statement of this lemma is satisfied.

Case 2: If $y = a$, then $u_1' y a^k u_2 = u_1' a^k y u_2$ as a word, however as paths in $\overline{\Gamma}_k^+$ (and therefore in Γ_k^+), it is not necessarily true that $\beta_n(p_1' x p_2 p_3) = \beta_n(p_1' p_2 x p_3)$. However, we know that the edge labeled by y is on the same vertical line in the complex as the k edges labeled by a that follow y in $\beta_n(p)$. By Lemma 4.6(i), there is an m with $0 \leq m \leq k-1$ such that $p_2 = r^m u r^{k-m-1}$. If $x = u$ then we have $\beta_n(p_1' ur^{k-1} x p_3) = u_1' a^k y u_2$. However, if $x = r$, we can say that $\beta_n(p_1' r^{m+1} u r^{k-m-2} x p_3) = u_1' a^k y u_2$. This proves the lemma.



Case 1a: In a section of Γ_4



Case 1b: In a section of Γ_4

Figure 12: Case 1

□

By the remarks made previous to Lemma 4.6, this concludes the proof of Theorem 4.4.

□

4.3 Enumerating Fundamental Paths of $\Gamma_k(n)$

Having shown that $|\Gamma_k(n)| = |Q_k(n)|$, one is able to enumerate explicitly the number of representations of $\Delta(A_k)^n$ as a positive word in the generators a, b, by calculating the number of paths in $Q_k(n)$. The purpose of this section is to show this calculation, using standard results from generating function theory, as shown in [2], [9], and [11].

Let $k \geq 2$ be fixed and denote by A_n the admissible paths in $Q_k(n)$ that are always weakly above (on or above) the line L_{k-1}. Also, define $a_n = |A_n|$. We may think of paths in A_n as sequences of numbers of the form $(x_1, x_2, \ldots, x_{kn})$, where each x_i is either $k-1$ or -1, that satisfy the two conditions

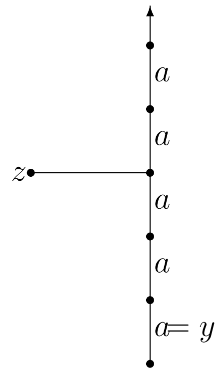


Figure 13: Case 2 ($m = 2$): A section of Γ_4.

- $\sum_{i=1}^{kn} x_i = 0$,
- $\sum_{i=1}^{m} x_i \geq 0$, for any $1 \leq m \leq kn$,

where the correspondence is given by letting $x_i = k-1$ if the i^{th} step is upward and -1 otherwise.

Define $P(t)$ to be the generating function $P(t) = \sum_{n=0}^{\infty} a_n t^n$, where $a_0 = 1$. To find an explicit form for a_n, one needs the following lemma:

Lemma 4.8. *For non-negative integers $n_1, n_2, \ldots, n_k$,*

$$a_n = \sum_{n_1+\cdots+n_k=n-1} a_{n_1} a_{n_2} \cdots a_{n_k}.$$

Proof. Let B_n be the set $\{(x_1, \ldots, x_{kn}) \in A_n \mid x_1 + \ldots + x_m > 0 \text{ if } m < kn\}$and let $\mathbf{x} \in A_{n-1}$ be a sequence equal to $(x_1, \ldots, x_{k(n-1)})$. We adopt the convention that y_0 is an empty sequence and that if y_i and y_j are two sequences with $y_i = (\cdot, \cdot, \ldots, \cdot)$ and $y_j = (*, *, \ldots, *)$ then $(y_i, y_j) = (\cdot, \cdot, \ldots, \cdot, *, \ldots, *)$. There is a minimal $r_1 > 0$

such that $x_1 + \cdots + x_{kr_1} = 0$. This implies that $x_{kr_1+1} = k-1$ and there is again a minimal $r_2 > 0$ such that $x_{kr_1+1} + \cdots + x_{kr_1+kr_2} = 0$. This process continues until $r_1 + \cdots + r_m = n-1$ for some m.

Since $r_i > 0$, for each i, it must be that $m \leq k$. Thus $\mathbf{x}$ can be written in the form $\mathbf{x} = (y_{n_1}, \ldots, y_{n_k})$ where $(n_1, \ldots, n_k)$ is some permutation of $(r_1, \ldots, r_m, \overbrace{0, \ldots, 0}^{k-m})$, and

$$y_{r_i} = (x_{k(r_1+\cdots+r_{i-1})+1}, \ldots, x_{k(r_1+\cdots+r_i)}).$$

Note that $y_{n_i} \in B_{n_i}$ for each i. This representation is not unique; however, if $\overline{A}_{n-1}$ is the set of all such representations of sequences in A_{n-1}, then $\left|\overline{A}_{n-1}\right| = \sum_{n_1+\cdots+n_k=n-1} a_{n_1} \cdots a_{n_k}$. So the claim is that there is a bijection between $\overline{A}_{n-1}$ and A_n.

Define $f : \overline{A}_{n-1} \to A_n$ by

$$f(y_{n_1}, \ldots, y_{n_k}) = (k-1, y_{n_1}, -1, y_{n_2}, -1, \ldots, -1, y_{n_k}).$$

Since $y_{n_i} \in B_{n_i} \subseteq A_{n_i}$, one can see that $\operatorname{Im} f \subseteq A_n$. Moreover, the definition of f shows that it is not a triviality to distinguish between sequences that only differ by where the empty sequences, y_0, occur in the sequence. For example, if $k = 3$ and $n = 4$ then $\mathbf{x} = (2, -1, 2, -1, -1, -1, 2, -1, -1) \in A_{n-1}$ and represents the path *ururrrurr* in Figure 14.

In this case $r_1 = 2$ and $r_2 = 1$. So let (y_2, y_0, y_1) and (y_0, y_2, y_1) be two representations of $\mathbf{x}$ in $\overline{A}_{n-1}$. Then $f(y_2, y_0, y_1)$ and $f(y_0, y_2, y_1)$ are shown in Figure 15.

Now, given some $\mathbf{x} = (x_1, \ldots, x_{kn}) \in A_n$, it must be that $x_1 = k-1$. Crop $\mathbf{x}$ to $(x_2, \ldots, x_{kn})$. If $x_2 = -1$, crop it as well. Continue to crop off the first component of the remaining sequence until $x_i = k-1$ or k croppings have been made. If

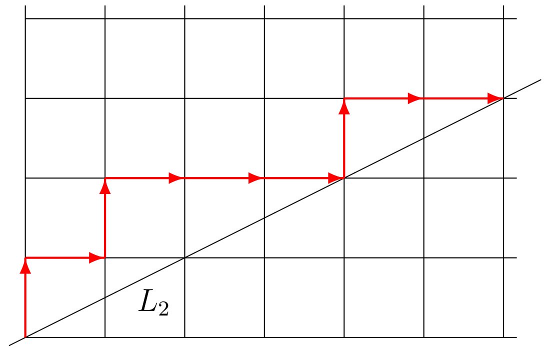


Figure 14: The path $urur rrurr \in Q_3(3)$.

$x_i = k-1$, find the smallest integer m_i such that $x_i + \cdots + x_{m_i} = 0$. Then if $x_{m_i+1} = -1$, crop the components until a $k-1$ is arrived at. When this occurs, say at x_j, find minimal m_j such that $x_j + \cdots + x_{m_j} = 0$. Continue this process until k croppings have been made in total, which is guaranteed to occur since $k-1$ was cropped off first and $\mathbf{x} \in A_n$. This gives a sequence in $\overline{A}_{n-1}$, that has $r_i = m_i - i$ and is an inverse for f. $\square$

With the result of Lemma 4.8, one can find an explicit form for a_n through the use of generating functions. Letting P^k denote the k^{th} power of the generating function $P(t)$, the results of [11, Chapter 2] imply that the n^{th} coefficient of P^k is $\sum_{n_1+\cdots+n_k=n} a_{n_1}\cdots a_{n_k}$; in [11], Wilf uses the symbol $\xleftrightarrow{ops}$ to denote this correspondence between a generating function and its n^{th} coefficient. Thus the previous statement is written,

$$P^k \xleftrightarrow{ops} \left\{ \sum_{n_1+\cdots+n_k=n} a_{n_1}\cdots a_{n_k} \right\}.$$

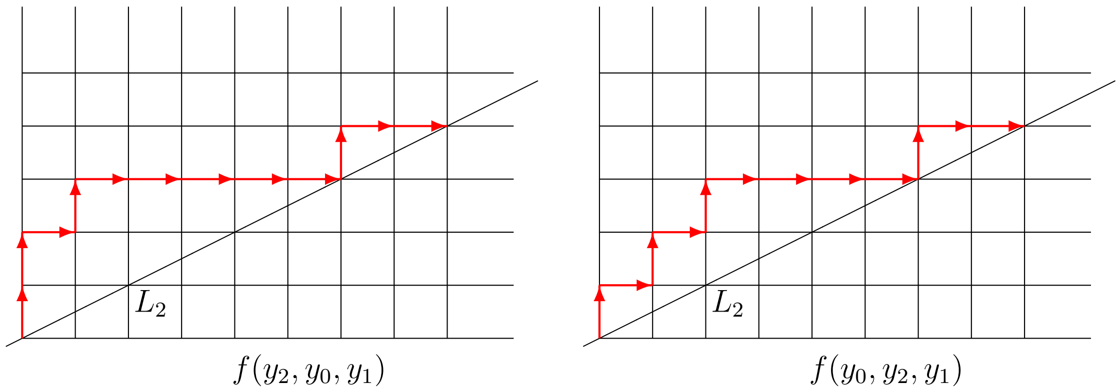


Figure 15: Paths represented by $f(y_2, y_0, y_1)$ and $f(y_0, y_2, y_1)$.

Other standard results of [11, Chapter 2] as well as Lemma 4.8 give $P^k \overset{ops}{\longleftrightarrow} \{a_{n+1}\} \overset{ops}{\longleftrightarrow} \dfrac{P-1}{t}$. Thus,

$$\frac{P(t)-1}{t} = P^k(t) \quad \text{i.e.,} \quad P(t) = 1 + tP^k(t). \tag{4.1}$$

By the Lagrange Inversion formula as discussed in [9] and [10, Theorem 14.3], the equation (4.1) implies that

$$P(t) = \sum_{n=0}^{\infty} \frac{1}{(k-1)n+1} \binom{kn}{n} t^n. \tag{4.2}$$

We now turn to a discussion of the number of representations of $\Delta(A_k)^n$, which we call $\delta_{k,n} = |Q_k(n)|$. Again, we let $k \geq 2$ be fixed, so we will refer to $\delta_{k,n}$ as simply δ_n. Define the generating function $C(t) = \sum_{n=0}^{\infty} \delta_n t^n$. For any path p in $Q_k(n+1)$ there exists an associated minimal integer $r \geq 1$ such that p is the concatenation of some $q \in Q_k(n-r+1)$ followed by a path of length kr that stays either above L_{k-1}

until $((k-1)n, n)$ or stays below it. Since the number of paths that always stay above L_{k-1} is clearly the same as the number that stay below it, one may use the same reasoning as in Lemma 4.8 to see that the number of paths with associated integer r is

$$\delta_{n-r+1}\left(\sum_{n_1+\cdots+n_{k-1}=r-1} 2a_{n_1}\cdots a_{n_{k-1}}\right).$$

This implies that

$$\begin{aligned}\delta_{n+1} &= \sum_{r=1}^{n+1}\delta_{n-r+1}\left(\sum_{n_1+\cdots+n_{k-1}=r-1} 2a_{n_1}\cdots a_{n_{k-1}}\right)\\ &= 2\sum_{r=1}^{n+1}\delta_{n-r+1}c_{k-1,r-1}\\ &= 2\sum_{r=0}^{n}\delta_{n-r}c_{k-1,r}\end{aligned}$$

where $\{c_{k-1,r}\} \overset{ops}{\longleftrightarrow} P^{k-1}$. This says that

$$\frac{C-1}{t} \overset{ops}{\longleftrightarrow} \{\delta_{n+1}\} = \left\{2\sum_{r=0}^{n}\delta_{n-r}c_{k-1,r}\right\} \overset{ops}{\longleftrightarrow} 2P^{k-1}C,$$

which gives the equation

$$C(t) = 1 + 2tP^{k-1}(t)C(t). \tag{4.3}$$

From the results of equations (4.1) and (4.3),

$$C(t) = \frac{P(t)}{2-P(t)} = \frac{P(t)}{1-(P(t)-1)},$$

and so $C(t) = P(t)\sum_{m=0}^{\infty}(P(t)-1)^m$. Recalling (4.2), we can use the Lagrange Inversion formula in the more general case, as in [2, (6.22), p.205] to find the coefficients δ_n of $C(t)$. Begin by setting $t = \frac{P-1}{P^k}$ and $z = P-1$, $f(z) = z^m$, and $g(z) = (z+1)^k$. Then we have the formula

$$f(z) = f(0) + \sum_{i=1}^{\infty}\left[\frac{d^{i-1}}{dz^{i-1}}\left(g^i(z)\frac{d}{dz}f(z)\right)\right]_{z=0}\frac{t^i}{i!} \tag{4.4}$$

from (6.22) of [2].

Note that

$$\frac{1}{i!}\left[\frac{d^{i-1}}{dz^{i-1}}((z+1)^{ki}mz^{m-1})\right]_{z=0} = \frac{1}{i!}\binom{i-1}{i-m}\frac{m!(ki)!}{(ki-(i-m))!}(1+z)^{ki-(i-m)}\bigg|_{z=0}$$
$$= \frac{m}{i}\binom{ki}{i-m}.$$

From this and (4.4) we see that $(P(t)-1)^m = \sum_{i=1}^{\infty} \frac{m}{i+m}\binom{k(i+m)}{i}t^{i+m}$. And so we have

$$\begin{aligned} C(t) &= P(t)\sum_{m=0}^{\infty}(P(t)-1)^m \\ &= \left(\sum_{n=0}^{\infty}\frac{1}{(k-1)n+1}\binom{kn}{n}t^n\right)\sum_{m=0}^{\infty}\left(\sum_{i=0}^{\infty}\frac{m}{i+m}\binom{k(i+m)}{i}t^{i+m}\right) \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\frac{1}{(k-1)n+1}\frac{m}{i+m}\binom{kn}{n}\binom{k(i+m)}{i}t^{i+m+n}. \end{aligned}$$

5 Pascal's Identity in Certain Garside Groups

5.1 Pascal's Triangle and Paths in $\Gamma_k(n)$

The correspondence of geodesics in Γ_k^+ to admissible paths in the integer lattice suggests that the combinatorics of representations of $\Delta(A_k)$ has a relationship with Pascal's triangle. Specifically, if $k = 2$ (this is $A_2 = \mathbb{Z}^2$), L_{k-1} is the line $y = x$ and $\delta_{k,n}$ is given by the central binomial coefficients, which satisfy the interesting identity $2\sum_{i=1}^{n}\binom{2n-i-1}{n-i} = \binom{2n}{n}$. If one defines $d_{n,i}$ to be the number of northeast paths on the lattice that begin with exactly i upward steps and end at $(2n, n)$, then the above identity comes directly from noticing that $2\sum_{i=1}^{n} d_{n,i} = \binom{2n}{n}$, since $d_{n,i} = \binom{2n-i-1}{n-i}$. Consider the lower triangular matrix, M_2, that has $2d_{n,i}$ as

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & \dots \\ 4 & 2 & 0 & 0 & 0 & \dots \\ 12 & 6 & 2 & 0 & 0 & \dots \\ 40 & 20 & 8 & 2 & 0 & \dots \\ 140 & 70 & 30 & 10 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Figure 16: The matrix M_2

its $(n,i)^{th}$ entry. The first few rows of this matrix are given in Figure 16. Though the entries in the matrix are not the standard Pascal's triangle numbers, if one examines M_2 as a right adjusted matrix (see Figure 17), then it becomes clear that if x is an entry to the left of y and below $z \neq 0$, then $x = y + z$.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 12 & 6 & 2 \\ 0 & 0 & 40 & 20 & 8 & 2 \\ 0 & 140 & 70 & 30 & 10 & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Figure 17: The right-adjusted matrix of M_2

We would expect something like this from the way the entries $2d_{n,i}$ were defined; indeed, since $d_{n,i} = \binom{2n-i-1}{n-i}$, the assertion that $x = y + z$ is the equality

$$\binom{2n-i-1}{n-i} = \binom{2n-i-2}{n-i-1} + \binom{2n-i-2}{n-i}.$$

Writing $m = 2n - i - 1$ and $r = n - i$, this is the familiar Pascal's identity

$\binom{m}{r} = \binom{m-1}{r-1} + \binom{m-1}{r}$. Moreover, the sum of the entries in any row is $\binom{2n}{n} = \delta_{2,n}$.

Geodesics in $\Gamma_k(n)$ can be shown to exhibit a similar property. We will state this property and prove it. We will then note that from this observation, one may derive an alternate proof of the bijective correspondence between $\Gamma_k(n)$ and $Q_k(n)$.

Let $w \in \Gamma_k(n)$. As any word in $\Gamma_k(n)$ has a^k or b^k as a subword, define $D_{k,n,i}$ to be the set of words in $\Gamma_k(n)$ with the first occurrence of a^k or b^k being in the i^{th} position. For example, $bbbbbb \in D_{3,2,1}$, but $bbaaab \in D_{3,2,3}$. Also, define $d_{k,n,i} = |D_{k,n,i}|$. If we also let $\delta_{k,n} = |\Gamma_k(n)|$, then we see that

$$\delta_{k,n} = \sum_{i=1}^{\infty} d_{k,n,i}.$$

Of course, $d_{k,n,i} = 0$ if $i > (k-1)n$, but we can do even better than this as seen in part (d) of the following theorem.

Theorem 5.1. *If $d_{k,n,i}$ is defined as above, then the following rules hold:*

(a) For all k, n, i such that $i \geq k-1$,

$$d_{k,n,i} = \sum_{j=i-k+1}^{kn} d_{k,n-1,j} = d_{k,n,i+1} + d_{k,n-1,i-k+1}.$$

(b) $d_{k,n,1} = 2\delta_{k,n-1}$.

(c) $d_{k,n,i} = \delta_{k,n-1}$ for $2 \leq i \leq k$.

(d) $d_{k,n,(k-1)n-(k-2)} = 2$ and $d_{k,n,i} = 0$ for $i > (k-1)n - (k-2)$.

Proof. (a) Suppose $w \in D_{k,n,i}$ and $i \geq k-1$. Then assume w has the form $w_1 a^k w_2$, where $\mathrm{len}(w_1) = i-1$ and w_1 has no factors of a^k or b^k (the case where

$w = w_1 b^k w_2$ is similar). Also, $w_1 a^k w_2 = w_1 w_2 a^k$ in G_k^+, so by cancellation, $w_1 w_2 \in \Gamma_k(n-1)$. Then $w_1 w_2 \in D_{k,n-1,j}$ for some j. Since $w \in D_{k,n,i}$, it must be that w_1 ends with a b, i.e. $w_1 = w_1' b$. So w_1 has b^m as a suffix with $m \geq 1$; but this sequence cannot be b^k, so $m \leq k-1$. If $m = k-1$, and w_2 begins with b, then $j = i-k+1$. If w_2 begins with a or $m < k-1$, then $j > i-k+1$. Thus

$$w_1 w_2 \in \bigcup_{j \geq i-k+1} D_{k,n-1,j}. \tag{5.1}$$

It is clear by an analogous argument, that if $w = w_1 b^k w_2$, then statement (5.1) still holds. Note that in this case, w_1 must end in a.

Conversely, if $v \in \bigcup_{j \geq i-k+1} D_{k,n-1,j}$, write v as $v_1 v_2$ such that $\text{len}(v_1) = i-1$. If v_1 ends in b then $v_1 a^k v_2' \in D_{k,n,i}$ and if v_1 ends in a then $v_1 b^k v_2' \in D_{k,n,i}$.

In fact the map $D_{k,n,i} \to \bigcup_{j \geq i-k+1} D_{k,n,j}$ is a set bijection, for if w_1 ends in b, then

$$w = w_1 a^k w_2 \mapsto w_1 w_2 \mapsto w_1 a^k w_2$$

and otherwise,

$$w = w_1 b^k w_2 \mapsto w_1 w_2 \mapsto w_1 b^k w_2.$$

This bijection shows that $d_{k,n,i} = \sum_{j=i-k+1}^{kn} d_{k,n-1,j}$. Replacing i by $i+1$, we see $d_{k,n,i+1} = \sum_{j=i-k+2}^{kn} d_{k,n-1,j}$. Thus $d_{k,n,i} = d_{k,n,i+1} + d_{k,n-1,i-k+1}$. This proves (a).

(b) If $w \in D_{k,n,1}$, then $w = a^k w'$ or $w = b^k w'$. By cancellation and the fact that $w \in \Gamma_k(n)$, we know that $w' \in \Gamma_k(n-1)$. However, for any $w' \in \Gamma_k(n-1)$, it is clear that $a^k w'$ and $b^k w'$ are each in $D_{k,n,1}$. This shows a 2-1 correspondence between $D_{k,n,1}$ and $\Gamma_k(n-1)$.

(c) If $w \in D_{k,n,i}$ for $2 \leq i \leq k$, then the situation is similar to that in (a), however $1 \leq \text{len}(w_1) \leq k-1$. So there is no restriction on what w_1 could be (only on its length). Thus $D_{k,n,i}$ is in 1-1 correspondence with $\bigcup_{j \geq 1} D_{k,n-1,j} = \Gamma_k(n-1)$, and so $d_{k,n,i} = \delta_{k,n-1}$.

(d) This part may be proved by induction. If $n = 1$, then the statement reads $d_{k,1,1} = 2$ and $d_{k,1,i} = 0$ for $i \geq 1$, which statements are clearly true. Now, suppose that, given $n > 1$, (d) is true for $n-1$. Note that $(k-1)(n-1) - (k-2) = [(k-1)n - (k-2)] - k + 1$, so if $i = (k-1)n - (k-2)$ then by (a) and by the induction hypothesis,

$$d_{k,n,i} = d_{k,n-1,i-k+1} + d_{k,n,i+1} = 2 + \sum_{j=i-k+2}^{kn} d_{k,n-1,j} = 2 + 0.$$

Also, the above statement makes it clear that for any $i > (k-1)n - (k-2)$, one has that $d_{k,n,i} = \sum_{j=i-k+1}^{kn} d_{k,n-1,j} = 0$, by the induction hypothesis.

□

5.2 Garside Groups

In an effort to better understand braid groups and their generalizations a theory of what are known as Garside groups has recently been developed. Garside groups are defined in such a way that many of the desirable qualities of braid groups have been retained. The following definition of a Garside monoid was given in [3]; a Garside group is defined to be any group of fractions of a Garside monoid. In the following, given a monoid M, an *atom* is any $a \in M$ such that $a \neq 1$ and $a = bc$ implies $b = 1$ or $c = 1$.

The length of an element $m \in M$, denoted $\text{len}(m)$, is the supremum of all lengths of representations of m as a word in the atoms of M. The monoid M is *atomic* if M

is generated by its atoms and for any $m \in M$, len(m) is finite. In such an M, a is a *left divisor* of c (and c is a *right multiple* of a) if there is some $b \in M$ with $ab = c$. Right divisors and left multiples are defined analogously. Now for the definition:

Definition 5.2. If M is an atomic monoid, it is a *Garside monoid* if

(1) M is left and right cancellative,

(2) any two elements of M admit both a least common multiple and a greatest common divisor on the left and the right,

(3) there is $\Delta \in M$ such that the set of left divisors of Δ equals the set of right divisors, these sets are finite, and they are a set of generators of M.

The element Δ is called the *Garside element*.

As Ruth Charney notes in [3], the fact that the sets of left divisors of Δ and right divisors of Δ are the same leads to the following fact.

Proposition 5.3. *If M is a Garside monoid, with Garside element Δ, let $D = \{x_1, \ldots, x_n\}$ be the set of divisors of Δ. Then there is some permutation σ of elements of D such that for every $i \leq n$,*

$$\Delta x_i = \sigma(x_i)\Delta.$$

In [6], Jon McCammond defines Garside groups through particular labeled partially ordered sets he calls combinatorial Garside structures. If P is such a structure, there is a natural monoid, M, associated to P, as well as a group G, which is the group of fractions of M. McCammond proves that the Hasse diagram of P embeds into the Cayley graph of M (and that of G). He also shows M to be a Garside monoid (and so G is a Garside group). Given the definition of a combinatorial Garside structure, one can also see that any Garside monoid will have an associated combinatorial Garside structure. The following section defines a combinatorial Garside structure, as described in [6]. Thereafter, a certain class of Garside groups is given that exhibits similar combinatorics as those seen in Theorem 5.1.

5.3 Combinatorial Garside Structures

Before a proper definition of a combinatorial Garside structure can be given, it is necessary to define several properties of posets that will be important to the discussion.

Definition 5.4. A poset, P is said to be *bounded* if there exist two elements $\hat{0}, \hat{1} \in P$ such that $x \geq \hat{0}$ and $x \leq \hat{1}$ for all $x \in P$.

Given a bounded poset P, and any $x \in P$, the set $C_0 = \{\hat{0}, x\}$ is a chain in P from $\hat{0}$ to x. Ordering chains in P by inclusion, we can find a maximal chain from $\hat{0}$ to x as follows:

Define A to be the set of all chains in P from $\hat{0}$ to x that contain C_0. It is clear that $A \neq \emptyset$ since $C_0 \in A$. Also, if $\mathscr{C}$ is a chain of elements in A (i.e. $\mathscr{C}$ is a chain of chains), then $\bigcup_{C \in \mathscr{C}} C$ is in A. It is clear that $C_0 \subseteq \bigcup_{C \in \mathscr{C}} C$. It is a chain in P from $\hat{0}$ to x since, if $x_\alpha, x_\beta \in \bigcup_{C \in \mathscr{C}} C$, there is some C_α such that $x_\alpha \in C_\alpha$ and $x_\beta \in C_\alpha$. So, without loss of generality, $\hat{0} \leq x_\alpha \leq x_\beta \leq x$. Thus $\mathscr{C}$ is bounded in A. By Zorn's Lemma, A has a maximal element.

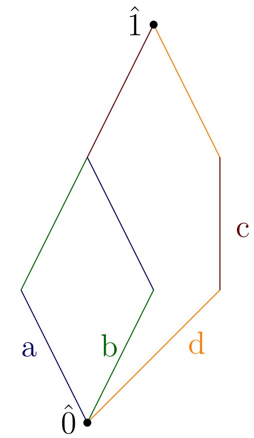


Figure 18: A labeled poset P.

Definition 5.5. If for any $x \in P$, every maximal chain from $\hat{0}$ to x is finite, then P is said to be of *finite height*. If, in addition, every maximal chain to x has the same length, then P is a *graded* poset (and the length of a maximal chain is called the *height* of x).

In this section, P is always a bounded, graded poset of finite height. An *interval* $[x, y]$ in P is the set of all $z \in P$, where $x \leq z \leq y$. The set of all intervals in P is denoted $I(P)$. If $x \leq y$ and the height of x and y differ by one, then $[x, y]$ is called a *covering relation*.

Given a set S, define S^* to be the set of words in the alphabet S. A map from the covering relations of P to S is called a *labeling* of P. This labeling can be extended to a map $I(P) \to \mathscr{P}(S^*)$, where $\mathscr{P}(S^*)$ is the power set of S^*. That is, each interval of P is assigned a subset of S^* (or a language in S^*) where the language assigned is the set of words that correspond to each chain from x to y in the interval $[x, y]$. This language will be represented by $\lambda([x, y])$. For example, given a labeled poset P as in Figure 18, the language assigned to $[\hat{0}, \hat{1}]$ is $\{abc, bac, dcd\}$. Such labeled posets can be associated to a monoid: if S is the set of labels for the covering relations of P, define

$$M(P) = \langle S \mid w_1 = w_2 \text{ if there are } x, y \in P \text{ with } w_1, w_2 \in \lambda([x, y]) \rangle \,,$$

where multiplication in $M(P)$ is given by concatenation in S^*. Also define $G(P)$ to be the group of fractions of $M(P)$. That P contains the minimal element $\hat{0}$ guarantees that $M(P)$ has an identity. Since P is graded and of finite height, $M(P)$ is atomic. Requiring the following properties of P guarantees that $M(P)$ defines a Garside monoid with Δ being any word in $\lambda([\hat{0}, \hat{1}])$.

Definition 5.6. A poset is a *lattice* if any two elements have a supremum and an infimum.

This requirement guarantees that $M(P)$ admits least common multiples and greatest common divisors.

Definition 5.7. A labeled poset P is *group-like* if given any two triplets x, y, z and x', y', z' of elements of P, such that $x \le y \le z$ and $x' \le y' \le z'$, the following condition holds:

> Let $I_1 = [x, y]$, $I_2 = [x, z]$ and $I_3 = [y, z]$ and similarly define I'_i using x', y', z'. If ever there is $i \neq j$, such that $\lambda(I_i) = \lambda(I'_i)$ and $\lambda(I_j) = \lambda(I'_j)$, then the pair of remaining intervals also have the same language.

Thus, if P is a group-like labeled poset, and if $a \in S$ and $w_1, w_2 \in M(P)$, then $w_1 a = w_2 a$ implies $w_1 = w_2$. Likewise, $a w_1 = a w_2$ implies $w_1 = w_2$. Since $M(P)$ is atomic, this shows it is left and right cancellative. An example of a labeled poset that is not group-like is shown in Figure 19. In the monoid associated to the poset of Figure 19, one has that $ab = ca$ and $ab = cd$. So $ca = cd$ but $a \neq d$.

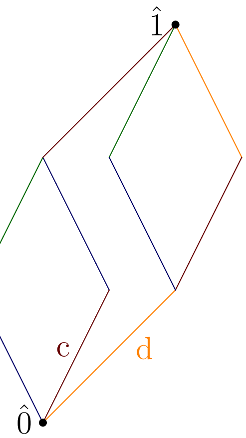


Figure 19: A labeled poset that is not group-like.

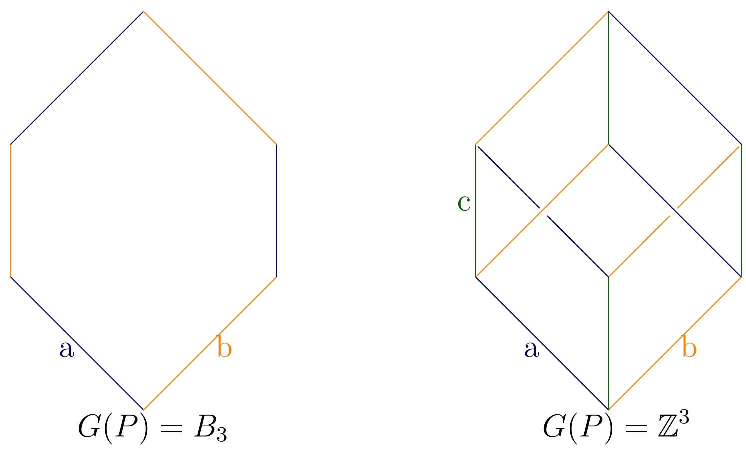


Figure 20: Combinatorial Garside structures associated to B_3 and $\mathbb{Z}^3$

Definition 5.8. Define the two sets $L(P) = \{\lambda([\hat{0}, x]) \mid x \in P\}$ and $R(P) = \{\lambda([x, \hat{1}]) \mid x \in P\}$. A labeled poset P is said to be *balanced* if $L(P) = R(P)$.

If Δ is defined to be any of the words in $\lambda([\hat{0}, \hat{1}])$, then balanced is exactly property (3) of Definition 5.2. Thus we make the following definition:

Definition 5.9. A bounded, graded, labeled poset of finite height is a *combinatorial Garside structure* if it is a group-like, balanced, lattice.

Examples of combinatorial Garside structures are shown in Figure 20. The groups associated to these structures are B_3 and $\mathbb{Z}^3$ with the presentations

$$B_3 = \langle a, b \mid aba = bab \rangle \quad \text{and}$$

$$\mathbb{Z}^3 = \langle a, b, c \mid ab = ba, ac = ca, bc = cb \rangle .$$

5.4 Longitudinal Groups

We now define a class of groups through their corresponding labeled posets. Each poset is given by a set of parameters $k, n_1, n_2, \ldots, n_r$, where k and n_i are positive integers. The corresponding poset is denoted by $P_{k;n_1,n_2,\ldots,n_r}$, and its associated monoid is $M_{k;n_1,n_2,\ldots,n_r}$. To define $P_{k;n_1,\ldots,n_r}$, we will first define $P_{k;n_i}$. Let $S_i = \{x_1, x_2, \ldots, x_{n_i}\}$ be a set of n_i elements. If $k > 1$, define $P_{k;n_i}$ to be the (bounded, graded, finite height) poset with S_i as its set of covering relations such that,

- $\lambda([\hat{0}, \hat{1}]) = \{w = x_j x_{j+1} \ldots x_{n_i} x_1 x_2 \ldots \mid 1 \leq j \leq n_i \text{ and } \mathrm{len}(w) = k\}$
- $\lambda([x, y])$ is a single word if either $x \neq \hat{0}$ or $y \neq \hat{1}$.

This determines $P_{k;n_i}$ and it can be seen to be a combinatorial Garside structure. We would also like to combine posets in the following way:

Definition 5.10. If P and P' are two bounded, graded posets of finite height, then let $P * P'$ be the poset that is made by identifying the minimal elements, $\hat{0}_P$ and $\hat{0}_{P'}$, and identifying the maximal elements, $\hat{1}_P$ and $\hat{1}_{P'}$.

The posets $P_{k;n_i}$ and $P_{k;n_j}$ are combinatorial Garside structures. If $S_i \cap S_j = \emptyset$, then $P_{k;n_i} * P_{k;n_j}$ is also a combinatorial Garside structure with $S_i \cup S_j$ as its set of covering relations. Let $\{S_1, S_2, \ldots, S_r\}$ be such that $S_i \cap S_j = \emptyset$ if $i \neq j$. Define

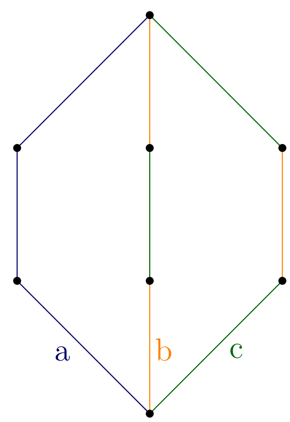


Figure 21: The Hasse Diagram of $P_{3;1,2}$.

$P_{k;n_1,\ldots,n_r} = P_{k;n_1} * \ldots * P_{k;n_r}$. For example, let $r = 2$ with $S_1 = \{a\}$ and $S_2 = \{b, c\}$. Then the diagram of $P_{3;1,2}$ is shown in the Figure 21:

The fundamental element $\Delta(M_{k;n_1,\ldots,n_r})$ is defined to be equivalent to any of the words $w \in \lambda([\hat{0}, \hat{1}])$. It is clear that such a Δ will have the properties enumerated in Definition 5.2(3).

Definition 5.11. The group of fractions of any monoid of the form $M_{k;n_1,\ldots,n_r}$ is called a *longitudinal group*.

The reasoning for the name *longitudinal group* is seen by the general shape of the combinatorial Garside structure of such a group, which resembles longitudinal lines on a pointed globe. Figure 21 illustrates this fact.

The next result follows from Proposition 5.3 and the fact that Δ satisfies Definition 5.2(3).

Proposition 5.12. *If x is any generator of $M_{k;n_1,\ldots,n_r}$, then $x \in S_i$ for some $i = 1, 2, \ldots, r$. Thus, if $S_i = \{x_1, x_2, \ldots, x_{j-1}, x = x_j, x_{j+1}, \ldots, x_{n_i}\}$, then there is some s such that*

$$x\Delta(M_{k;n_1,\ldots,n_r}) = \Delta(M_{k;n_1,\ldots,n_r})x_s$$

in $M_{k;n_1,n_2,\ldots,n_r}$.

Remark 4. In particular, note that $x = x_j \in S_i$, and that there exists some word $w \in \lambda([\hat{0}, \hat{1}])$ such that w begins with x_{j+1}. Let $x_s \in S_i$ be the last letter of w. Then $x\Delta = xw = (x_j x_{j+1} \ldots x_{s-1})x_s = \Delta x_s$ in $M_{k;n_1,\ldots,n_r}$. This defines a bijection $\phi : S \to S$ by $\phi(x_j) = x_s$. Note that $\phi|_{S_i}$ is a permutation of S_i for each i.

The function ϕ will be used in Theorem 5.13. Much of the notation of Theorem 5.1 will also be used, as the groups A_k are particular examples of longitudinal groups with $r = 1$ and $|S| = 2$.

Theorem 5.13. *Let* $G_{k;n_1,\ldots,n_r} = G$ *be a longitudinal group with generating set* S. *Define* $D_{k,n,i}$ *to be the set of representations of* $\Delta(G)^n$ *as a positive word in* S^* *such that the first occurrence of a word equivalent to* Δ, *in* G, *is at the position* i. *Also, let* $d_{k,n,i} = |D_{k,n,i}|$. *Then the following rules hold for all* $n \geq 1$ *and* $k \geq 2$*:*

(a) For k, n, i *such that* $i \geq k - 1$,

$$d_{k,n,i} = (|S| - 1) \sum_{j=i-k+1}^{kn} d_{k,n-1,j}.$$

(b) $d_{k,n,1} = |S| \, \delta_{k,n-1}$.

(c) $d_{k,n,i} = (|S| - 1) \, \delta_{k,n-1}$ *for* $2 \leq i \leq k$.

(d) $d_{k,n,(k-1)n-(k-2)} = |S| \, (|S| - 1)^{n-1}$ *and* $d_{k,n,i} = 0$ *for* $i > (k-1)n - (k-2)$.

Proof. (a) Suppose $w \in D_{k,n,i}$ and $i \geq k - 1$. We will write $\Delta(x_r)$ to denote the representation of $\Delta(G)$ that begins with x_r. Since G is longitudinal, there is only one such representation. Write w in the form $w_1 \Delta(x_r) w_2$, where $1 \leq r \leq |S|$, $\text{len}(w_1) = i - 1$ and Δ does not occur in any form in w before the i^{th} letter. In G there is the equality

$$w = w_1 \Delta(x_r) w_2 = w_1 \phi(w_2) \Delta(x_r),$$

which implies that $w_1 \phi(w_2) \in D_{k,n-1,j}$ for some j. In fact, it must be that $j \geq i-(k-1)$, for otherwise, $w = w_1 \Delta(x_r) w_2 \notin D_{k,n,i}$ since $\text{len}(\Delta) = k$. Thus there is a set map

$$D_{k,n,i} \xrightarrow{\rho} \bigcup_{j \geq i-(k-1)} D_{k,n-1,j}$$

But, for any $w \in D_{k,n-1,j}$ with $j \geq i-(k-1)$, it is possible to write w as $w_1 w_2$ where $\text{len}(w_1) = i-1$ and w_1 not having Δ as a subword. Suppose $w_1 = w_1' x_r$. Then choose any $x_s \in S$ such that $\Delta(x_s)$ does *not* have x_r as its last letter. There are exactly $|S|-1$ many such choices. This gives part (a).

(b) If $w \in D_{k,n,1}$, then it has the form $\Delta(x_r) w'$ for $1 \leq r \leq |S|$, where $w' = \Delta^{n-1}$ in G. Thus $d_{k,n,1} = |S| \, \delta_{k,n-1}$.

(c) If w is equivalent to Δ^{n-1} in G, then for any i such that $2 \leq i \leq k$, write $w = w_1 w_2$ with $\text{len}(w_1) = i-1$. If $w_1 = w_1' x_r$, then for any x_s such that x_r is not the last letter of $\Delta(x_s)$,

$$w_1 w_2 \Delta(x_s) = w_1 \Delta(x_s) \phi^{-1}(w_2) \in D_{k,n,i}.$$

Again, there are $|S|-1$ such choices of x_s. On the other hand, if $w \in D_{k,n,i}$, for $2 \leq i \leq k$, then $w = w_1 \Delta(x_s) w_2$ for some x_s such that $\Delta(x_s)$ doesn't end in the same letter as w_1, $\text{len}(w_1) = i-1$ and w_1 contains no subwords equivalent to Δ. Moreover, $w_1 \Delta(x_s) w_2 = w_1 \phi(w_2) \Delta(x_s)$ shows there is a $|S|-1$ to one relationship between $D_{k,n,i}$ and representations of Δ^{n-1}.

(d) If $n = 1$, then $(k-1)n - (k-2) = 1$. So the first part of the statement reads $d_{k,1,1} = |S|$, which is true by (b). Also, $d_{k,1,i}$ is clearly 0 for $i > 1$. Now, suppose that, given $n > 1$, (d) is true for $n-1$. Note that $(k-1)(n-1) - (k-2) = [(k-1)n - (k-2)] - k + 1$. So if $i = (k-1)n - (k-2) \geq k-1$ then by (a) and by the induction hypothesis,

$$d_{k,n,i} = (|S|-1) \sum_{j=i-k+1}^{kn} d_{k,n-1,j} = (|S|-1) \left[d_{k,n-1,i-k+1} + \sum_{j=i-k+2}^{kn} d_{k,n-1,j} \right]$$
$$= (|S|-1) \left[|S| \left(|S|-1\right)^{n-2} + 0 \right].$$

In fact, this makes both parts of (d) clear, since

$$d_{k,n,i+m} = \sum_{j=i-k+1+m}^{kn} d_{k,n-1,j} = 0$$

by (a) and by induction.

□

Thus, if G is any longitudinal group, the number of geodesics in the Cayley graph from the identity to $\Delta^n(G)$ does not depend on the way that S is partitioned, but only on $|S|$, n and k–the height of $\hat{1}$ in the related combinatorial Garside structure. Moreover, this result shows that longitudinal groups also exhibit a Pascal-like identity with regard to representations of $\Delta^n(G)$.

References

[1] J. W. Cannon. *Geometric group theory.* North-Holland, Amsterdam, 2002. 3.2, 4.1

[2] Charalambos A. Charalambides. *Enumerative combinatorics.* CRC Press Series on Discrete Mathematics and its Applications. Chapman & Hall/CRC, Boca Raton, FL, 2002. 4.3, 4.3, 4.3

[3] Ruth Charney and John Meier. The language of geodesics for garside groups. *Mathematische Zeitschrift*, 2004. 5.2, 5.2

[4] Patrick Dehornoy. *Braids and self-distributivity*, volume 192 of *Progress in Mathematics.* Birkhäuser Verlag, Basel, 2000. 1.1, 1.2

[5] F. A. Garside. The braid group and other groups. *Quart. J. Math. Oxford Ser. (2)*, 20:235–254, 1969. 2, 2

[6] Jon McCammond. Noncrossing partitions in surprising locations. 5.2

[7] Kunio Murasugi and Bohdan I. Kurpita. *A study of braids*, volume 484 of *Mathematics and its Applications.* Kluwer Academic Publishers, Dordrecht, 1999. 2

[8] Richard P. Stanley. On the number of reduced decompositions of elements of Coxeter groups. *European J. Combin.*, 5(4):359–372, 1984. 2.1

[9] Richard P. Stanley. *Enumerative combinatorics. Vol. 1*, volume 49 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original. 4.3, 4.3

[10] J. H. van Lint and R. M. Wilson. *A course in combinatorics.* Cambridge University Press, Cambridge, second edition, 2001. 4.3

[11] Herbert S. Wilf. *Generatingfunctionology.* Academic Press Inc., Boston, MA, second edition, 1994. 4.3, 4.3