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High Order Convergence with a Low Order Discretization of the 2D MFIE

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Abstract—Moment method solutions to the MFIE are often less accurate for a given grid than corresponding solutions to the EFIE. We propose that the cause of this observation is the identity operator in the MFIE and show how regularizing the identity increases the convergence rate of the discretized 2D MFIE by three orders.

Index Terms—Regularization, boundary integral equations, electromagnetic scattering, error analysis, moment methods, numerical analysis, high order methods

I. INTRODUCTION

It has been observed that moment method solutions of the MFIE are less accurate than their EFIE counterparts. At present, the cause of the larger error is unknown. A recent paper [1] investigates two potential error sources, quadrature error associated with a logarithmic singularity and the proper computation of moment matrix elements near sharp edges, concluding that neither of these two sources causes the observed inaccuracy, see also [11]. In this letter, we propose an explanation for this unexpectedly large error and show how moment method solutions to the MFIE for two dimensional problems may be significantly improved.

It is generally thought that the numerical scattering amplitude approximation converges more quickly than the current approximation. This belief is founded on the fact that undesirable, high frequency content in a numerical current solution does not radiate to the far field and on variational expressions of the form [2]–[6]

$$\Delta S = \langle L\Delta J, \Delta J^a \rangle \quad (1)$$

where L is either the EFIE or MFIE operator and ΔS and ΔJ represent the scattering amplitude and current errors, respectively. The ΔJ^a is similarly defined for an auxiliary adjoint equation. Based on this expression it might be argued that ΔS is smaller than ΔJ because there are two Δ 's in the scattering amplitude error. While the above expression is correct, Dudley [6] points out that it does not guarantee a “second order” scattering amplitude error, for a small ΔJ may still lead to a large $L\Delta J$. It should also be noted that the expression (1)

applies only to ideal moment method implementations, so error sources seen in practice, such as quadrature error in evaluating moment matrix elements and geometrical error in modeling the scatterer surface, invalidate Eq. (1).

Detailed studies of 2D problems have borne out Dudley's contention. Although Eq. (1) is formerly second order, we see in Table I that for moment method solutions to scattering from a PEC, infinite circular cylinder, ΔS rarely decays at twice the rate of the current error as the mesh is refined.

Of special note in Table I is the slow convergence of scattering amplitude solutions to the MFIE as compared to the EFIE for low order discretizations. The MFIE scattering amplitude solution is one order less accurate than the EFIE solution for both polarizations. One would expect better convergence from the MFIE, especially since the MFIE kernel is continuous, whereas the EFIE kernels are singular. It is the purpose of this letter to show that although the identity operator associated with the MFIE is trivial to discretize, in determining solution error it behaves as an integral operator with a highly singular kernel and leads to unexpectedly low convergence rates.

It is well known that solution accuracy may be improved by increasing the polynomial order of the testing and basis functions. As an alternative, we show how to use a regularization approach [9], [10] on the identity operator associated with the MFIE to obtain high order convergence with a low order discretization. This regularization greatly improves solution accuracy, while maintaining the simpler implementation and smaller matrix fill time of a low order discretization. In fact, we will show analytically for the circular cylinder and numerically for other scatterers that this regularization gives scattering amplitude solutions three orders more accurate than their unregularized analogues. For example, fifth order convergence may be obtained with a piecewise constant (pulse) basis and point-matching discretization.

Because both the 3D studies [1], [11] and this study involve low order bases of local support, it is hoped that observations in 2D will lend themselves directly to the

TABLE I

THEORETICAL CONVERGENCE RATES FOR MOMENT METHOD SOLUTIONS TO SCATTERING FROM A CIRCULAR CYLINDER [7], [8]. HERE, h IS THE DISCRETIZATION WIDTH OF THE BASIS FUNCTIONS. WHILE CURRENT ERROR IS GIVEN FOR THE L^2 NORM, IT SHOULD BE NOTED THAT THE CURRENT ERROR AT THE MESH NODES CONVERGES AS h^2 FOR ALL DISCRETIZATIONS SHOWN.

testing/basis ($\uparrow$ Dirac delta, $\square$ piecewise constant, $\wedge$ piecewise linear)	$\uparrow \square$	$\square \square$	$\uparrow \wedge$	$\square \wedge$	$\wedge \square$	$\wedge \wedge$
combined polynomial order $p + p'$	-1	0	0	1	1	2
EFIE TM current error (L^2)	h	h	h^2	h^2	h	h^2
EFIE TM scattering error	h^3	h^3	h^3	h^4	h^4	h^5
MFIE TM/TE current error (L^2)	h	h	h^2	h^2	h	h^2
MFIE TM/TE scattering error	h^2	h^2	h^2	h^4	h^4	h^4
EFIE TE current error (L^2)	h	h	h^2	h^2	h	h^2
EFIE TE scattering error	h^3	h^3	h^3	h^3	h^3	h^3

3D problem.

II. REGULARIZING THE IDENTITY

In this section, we motivate the need for a regularization of the identity operator in the MFIE and derive an explicit form for the regularizer. For an arbitrary, closed scatterer, we first assume that the integral operator is entirely dominated by the identity term:

$$\frac{1}{2}J = H_t^i \quad (2)$$

where J is the scalar electric current and H_t^i is the known tangential component of the incident magnetic field on the scatterer. The quantity of interest is the scattering amplitude $S = \langle E^s, J \rangle$, where E^s is a scattered plane wave. We will show that MoM solutions for S are surprisingly inaccurate despite the triviality of discretizing Eq. (2).

To describe the scatterer surface, we will use the arc length s measured from an arbitrary point on the scatterer. We also define the parameter $t = (2\pi/C)s$, where C is the total perimeter of the scatterer, so that all quantities evaluated on the scatterer surface are periodic in t with period 2π . We can then express all quantities on the surface as Fourier series in t .

Since we are interested in low-order discretizations, consider only the first three discretizations in Table I ($p + p' = -1, 0$). For these discretizations, the moment matrix is the identity ($\langle t_m, f_n \rangle = \delta_{mn}$) and we immediately have the current unknowns as

$$J_n = \langle t_n, H_t^i \rangle = \sum_q T_q B_q e^{iq t_n} \quad (3)$$

where T_q and B_q are the Fourier coefficients of the testing function and $2H_t^i$, respectively. The error $\Delta S = S - \hat{S}$ is found to be

$$\Delta S = \sum_q \sum_s (-1)^s A_q^* B_{q+sN} [\delta_{s0} - F_q T_{q+sN}] \quad (4)$$

where A_q and F_q are the Fourier coefficients of E^s and the basis function, respectively. To derive Eq. (4), a uniform discretization was assumed. The order of the error is determined by the $s = 0$ terms, which include the factor

$$1 - F_q T_q = 1 - \text{sinc}^{p+p'+2}(q/N) = O(h^2). \quad (5)$$

This error is surprisingly large, despite the ease of discretizing the identity equation (2). From Table I, we see that the identity equation leads to a less accurate scattering solution than even the TM/TE EFIE operators, which have highly singular kernels.

We now seek to improve the accuracy of the MFIE by regularizing the identity operator. This has the form of integration with a filter D , so that the regularization is described by

$$J \rightarrow \int_S D(\rho, \rho') J(\rho') d\rho \quad (6)$$

The moment matrix associated with this operator has elements

$$Z_{mn} = \frac{C}{2\pi} \int_0^{2\pi} t_m^*(t) \frac{C}{2\pi} \int_0^{2\pi} D(t, t') f_n(t') dt' dt \quad (7)$$

Assuming that D has a Fourier series of the form

$$D(\rho, \rho') = \frac{1}{C} \sum_q D_q e^{iq(t-t')} \quad (8)$$

and substituting in the Fourier series of all surface quantities yields

$$Z_{mn} = \frac{1}{N} \sum_q D_q T_q F_q e^{iq(t_m - t_n)}. \quad (9)$$

Following [7], if we apply the moment matrix to its eigenvector $[e^{iq t_1}, e^{iq t_2}, \dots, e^{iq t_N}]^T$, where q is any integer in the interval $(-N/2, N/2)$, we obtain

$$\sum_{n=1}^N Z_{mn} e^{iq t_n} = \hat{\Lambda}_q e^{iq t_m} \quad (10)$$

where

$$\hat{\Lambda}_q = \sum_s D_{q+sN} F_{q+sN} T_{q+sN} \quad (11)$$

and we have again assumed a uniform discretization. We immediately recognize Eq. (11) as the eigenvalues of the moment matrix. The right hand side of the MoM system is Eq. (3). Since it also involves the eigenvectors of the moment matrix, we can find the current unknowns by dividing by the eigenvalues to give

$$J_n = \sum_q \frac{T_q B_q}{\hat{\Lambda}_q} e^{i q t_n} \quad (12)$$

The approximate current and scattering amplitude solutions can then be computed as before, yielding the scattering amplitude error

$$\Delta S = \sum_q \sum_s (-1)^s A_q^* B_{q+sN} \left[\delta_{s0} - \frac{F_q T_{q+sN}}{\hat{\Lambda}_{q+sN}} \right] \quad (13)$$

The leading order behavior of ΔS is again determined by the $s = 0$ terms of Eq. (13), which include the factor

$$1 - \frac{F_q T_q}{\hat{\Lambda}_q} = (D_q - 1) \frac{F_q T_q}{\hat{\Lambda}_q} + \sum_{s \neq 0} D_{q+sN} \frac{F_{q+sN} T_{q+sN}}{\hat{\Lambda}_{q+sN}} \quad (14)$$

We can force Eq. (14) to zero by taking $D_q = 1$ for the first N modes and 0 otherwise. This gives a very small ΔS which now comprises only the very high order terms ($s \neq 0$) in Eq. (13). Evaluating Eq. (8) gives an explicit expression for the filter D :

$$D(\rho, \rho') = \frac{1}{C} \frac{\sin \frac{\pi}{h}(s-s')}{\sin \frac{\pi}{C}(s-s')} e^{i \frac{\pi}{C}(s-s')} \quad (15)$$

which is a periodic sinc function with height $1/h$ and main lobe width $2h$, centered at $\rho = \rho'$. The effect of D is to filter out the high frequency content in the basis functions, which increases their effective smoothness.

III. APPLICATION TO MFIE

We now use this regularization in the identity term of the MFIE:

$$\frac{1}{2} \int_S D(\rho, \rho') J(\rho') d\rho' + \int_S K(\rho, \rho') J(\rho') d\rho' = H_t^i \quad (16)$$

where K is the 2D MFIE kernel. To give a rigorous expression for the scattering amplitude error with the regularized MFIE (16), we assume the scatterer to be a circle and follow a derivation similar to that in [8]. The scattering amplitude error is found to be

$$\Delta S = \sum_q \frac{A_q^* B_q}{\Lambda_q^K} \sum_{s \neq 0} \Lambda_{q+sN}^K F_{q+sN} T_{q+sN} \quad (17)$$

where Λ_q^K are the eigenvalues of the operator $\int_S K(\rho, \rho')$. Substituting in their asymptotic expansions

[12], and assuming the testing/basis sets in Table I, we obtain

$$\Delta S = \frac{(ka)^2}{4} \sum_q \frac{A_q^* B_q}{\Lambda_q^K} \sum_{s \neq 0} \frac{\text{sinc}^{p+p'+2}(s + \frac{q}{N})}{|q + sN|^3} \quad (18)$$

This expression applies for all the discretizations in Table I, and is $O(h^5)$ for the $p+p' = -1, 0$ discretizations and $O(h^7)$ for the $p+p' = 1, 2$ discretizations (3 orders better than their unregularized counterparts), so even for the simplest discretization, pulse expansion and point matching, a scattering amplitude error of $O(h^5)$ is obtained. Thus, the fast solution convergence associated with a high order discretization is achieved while retaining the convenience of a low order discretization.

Figure (1) shows how different low-order discretizations relate for a circular cylinder scatterer with a TM polarized incident plane wave. Each implementation uses a point-pulse discretization ($p + p' = -1$) and a simple, 32 point Euler quadrature rule. The error shown is the relative backscattering error and the RMS error for bistatic scattering is very similar. The largest error, which is $O(h)$, was generated using a flat-facet mesh and the error decaying as h^2 uses a curved mesh and an unregularized MFIE. The error decaying as h^3 was generated identically to the h^2 implementation except that a single quadrature point was used, causing a special cancellation in the error [8]. The most accurate curve (fifth order) was generated with a curved mesh and the regularization described in this letter.

Fig. 1. Error curves for different low-order discretizations of the MFIE. The fifth order method was obtained by regularizing the identity term in the MFIE.

For non-circular scatterers, we generate numerical convergence plots for a set of nine scatterers also studied in [8]. Figure (2) shows numerical solution convergence for these nine scatterers using the same point-pulse discretization and quadrature rule of Fig. 1. Both

regularized and unregularized identity operators were used. As suggested by the theoretical results presented, the unregularized solutions converge as h^2 , whereas the regularized solutions converge as h^5 . Results are very similar for the TE polarization.

Fig. 2. TM scattering amplitude error for nine different scatterers studied also in [8]. The errors that are decaying as h^2 were generated from a pulse expansion, point-matching discretization of the unregularized MFIE. The errors that are decaying as h^5 were generated from an identical discretization, but with a regularized identity operator. TE results are similar.

As an implementation note, better error is achieved if both integrations in Eq. (16) are performed using the same quadrature rule. This allows the leading term of the quadrature error to cancel in the final solution, making the solution largely insensitive to the accuracy of the numerical integration.

IV. CONCLUSIONS

We prove that the identity operator associated with the MFIE, though easily discretized, causes inaccuracy in moment method solutions of the MFIE. We show how the identity operator may be regularized and support through analytical and numerical means that it can lead to convergence rates three orders faster than the MFIE as typically discretized. This allows high order convergence with the advantages of a low order discretization.

It is hoped that these findings will extend directly to three dimensions.

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