Decoherence of Open Systems of Coupled Oscillators and Qubits

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I. INTRODUCTION

The field of quantum dynamics, that is how quantum systems evolve in time, is central to current research in quantum systems with applications in quantum information, computing, and cryptography. Progress in these areas could revolutionize computer science, making it possible to solve problems that are currently unattainable. Recent experiments\[1\] have demonstrated that manipulation of small numbers of qubits is feasible. Yet these technologies are difficult to scale up to the macroscopic scale without disrupting desired quantum behaviors, such as superposition. That is, understanding how to maintain coherence in quantum technologies is necessary for them to be scalable and hence useful. As such, it is necessary to study quantum dynamics to gain an understanding of how to minimize or mitigate decoherence\[2\].

We focus on simple harmonic oscillator (SHO) and qubit systems that can be solved analytically. SHOs are applicable in the modeling of Electromagnetic radiation confined to a cavity, vibrations in an ion trap, covalent bonds, and other systems with restoring forces while qubits are the building blocks of quantum computers. Due to the utility of SHOs and qubits in modeling quantum systems we consider open systems of SHOs and/or qubits coupled to environments of SHOs and/or qubits. Thus, by solving for the dynamics of the system we see the effect of interacting with the environment, especially its effect on decoherence.

II. METHODOLOGY

A. Procedure

The consideration of open systems requires describing both the system and environment quantum mechanically, which can be done by considering their corresponding Hilbert spaces $\mathcal{H}_{\text{sys}}$ and $\mathcal{H}_{\text{env}}$, respectively. An arbitrary initial state of the system and environment can be expressed as the density operator $\hat{\rho}_0 = \sum_i p_i |\Psi_i\rangle \langle \Psi_i|$, where $\{|\Psi_i\rangle\langle \Psi_i|\}$ forms a basis and $p_i$ is the probability that a state corresponding to the ensemble given by $\hat{\rho}_0$
will be found in the state \(|\Psi_i\rangle\langle\Psi_i|\) corresponding to the pure state \(|\Psi_i\rangle \in \mathcal{H}_{\text{sys}} \otimes \mathcal{H}_{\text{env}}\). To find the state of the system and environment at subsequent times we use the time evolution operator \(\hat{U}(t)\) which evolves a state to a later time as \(\hat{\rho}(t) = \hat{U}\hat{\rho}_{0}\hat{U}^\dagger\). Assuming \(\hat{U}\) is invertible it satisfies the Schrödinger-like equation

\[
\frac{i\hbar}{\partial t} \hat{U} = \hat{H}. \tag{1}
\]

Thus, by finding \(\hat{U}(t)\) one can obtain the dynamics any state subject to the Hamiltonian \(\hat{H}(t)\). To solve for the time evolution operator we use the Wei-Norman method\(^\text{1}\) in which one extract the dynamics of the system from its underlying Lie algebra (see Appendix \(\text{A}\) for a development of this method). In order to quickly solve for \(\hat{U}(t)\) using the Wei-Norman method we use a Mathematica program developed by Beu\(^\text{2}\) that deals with the algebra of operators we are concerned with allowing for both analytic and numerical results to be computed.

From \(\hat{\rho}(t)\) one obtains the state of the system with environmental effects on it, given by the reduced density operator \(\hat{\rho}_{\text{sys}}\), by partial tracing over the degrees of freedom of the environment:

\[
\hat{\rho}_{\text{sys}}(t) = \text{Tr}_{\text{env}}[\hat{\rho}(t)]. \tag{2}
\]

Tracing over a subspace (here the environment) is defined by the linear operator \(\text{Tr}_{\text{env}}\):

\[
\text{Tr}_{\text{env}} (A \otimes B) = A \text{Tr} B, \tag{3}
\]

for all matrices \(A\) and \(B\) on \(\mathcal{H}_{\text{sys}}\) and \(\mathcal{H}_{\text{env}}\), respectively. For a SHO environment, which is all we consider, this partial trace can be computed for an arbitrary state \(\hat{\rho}(t)\) with respect to a basis \(|a_n\rangle\) of the system as

\[
\text{Tr}_{\text{env}}[\hat{\rho}(t)] = \sum_{n=0}^{\infty} \langle a_n | \hat{\rho}(t) | a_n \rangle, \tag{4}
\]

where for a continuous basis, such as the coherent states, this can be done via an integral. Upon finding \(\hat{\rho}_{\text{sys}}(t)\) we compute the linear entropy

\[
\zeta(t) = 1 - \text{Tr} [\hat{\rho}^2_{\text{sys}}(t)] \tag{5}
\]

which is a measure of decoherence. It is bounded below by zero and above by one, corresponding to pure and completely mixed states, respectively.

### B. Systems and Environments

In order to develop the time evolution operator analytically we consider the system to be either a single SHO (Sec. \(\text{III}\)) or qubit (Sec. \(\text{IV}\)) coupled to the minimal environment of a single SHO. In this paper when we refer to a coupled system and environment we will do so with the system on the left and the environment on the right joined by a hyphen, for example, in Sec. \(\text{III A}\) we use “Coherent-Thermal” to refer to a coherent state system in a thermal environment.

As we will be considering SHOs and qubits throughout this paper it is appropriate to mention some of operators related to them. For a SHO of angular frequency \(\omega\) the Hamiltonian is

\[
\hat{H}_{\text{SHO}} = \hbar \omega (\hat{N} + 1/2), \tag{6}
\]

where \(\hat{N} = \hat{a}^\dagger \hat{a}\) and acts on a number state as \(\hat{N} |n\rangle = n |n\rangle\). Here \(\hat{a}^\dagger\) is the raising operator which raises a number state as \(\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle \forall n \in \{0, 1, ...\}\) and \(\hat{a}\) is the lowering operator which lowers a number state as \(\hat{a} |n\rangle = \sqrt{n-1} |n-1\rangle \forall n \in \{1, 2, ...\}\) while \(\hat{a}|0\rangle = 0\). The Hamiltonian of a qubit of angular frequency \(\omega_\sigma\) is

\[
\hat{H}_Q = \frac{\hbar \omega_\sigma}{2} \hat{\sigma}_z, \tag{7}
\]

where \(\hat{\sigma}_z\) can be represented by the third Pauli matrix

\[
\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8}
\]

We also commonly encounter the spin ladder operators \(\hat{\sigma}^+\) and \(\hat{\sigma}^-\) which have matrix representations of \(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\) and \(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\), respectively. \(\hat{\sigma}^+\) and \(\hat{\sigma}^-\) act like \(\hat{a}^\dagger\) and \(\hat{a}\) in that they raise and lower energy eigenstates yet one critical difference is that the spin-1/2 ladder only has two levels so they are nilpotent with (\(\hat{\sigma}^+\)^2 = \(\hat{\sigma}^-\)^2 = 0. Note that we use the standard convention for qubits where the ground and excited states are given by \(|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\) and \(|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\), respectively.

### C. Oscillator-Oscillator Coupling

In our research we have focused on the amplitude coupling of SHOs which involves a spring like interaction term. For two classical SHOs this would correspond to a spring connecting the two oscillators giving rise the interaction potential \(\hat{U}_{\text{int}} = \frac{1}{2} m_2 \omega^2 (x_1 - x_2)^2\). This interaction can be used to motivate the quantum mechanical case where we let \(x_1\) and \(x_2\), the positions of the first and second oscillating masses, respectively, go to their corresponding operators. This motivates a Hamiltonian of the form

\[
\hat{H}_{\text{AC}}/\hbar = \omega_a \hat{N}_a + \omega_b \hat{N}_b + \gamma (\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger) + \mu (\hat{a} \hat{b} + \hat{a}^\dagger \hat{b}^\dagger), \tag{9}
\]

where \(\omega_a\) and \(\omega_b\) are the angular frequencies of the two oscillators and \(\gamma\) and \(\mu\) are coupling strengths controlling the interaction between the oscillators. Here we use “a” and “b” to refer to the SHOs of the system and environment, respectively. That is, \(\{\hat{a}^\dagger, \hat{a}, \hat{N}_a\}\) act only in the
The second approximation entails letting \( \gamma \) where \( \alpha \) is the natural frequency of the subspace of the system while \( \{ \hat{b}^\dagger, \hat{b}, \hat{N}_b \} \) act only in the subspace of the environment.

The Lie algebra corresponding to \( \hat{H}_{AC} \) has the basis \( \{ \hat{a}^\dagger \hat{b}, \hat{a} \hat{b}^\dagger, \hat{a}^\dagger, \hat{a}^2, \hat{b}^2, \hat{N}_a, \hat{N}_b, I \} \), where \( I \) is the identity, which is not practical to work with so we consider two approximations. The first entails letting \( \mu = 0 \) and is called the rotating wave approximation (RWA) corresponding to the Hamiltonian

\[
\hat{H}_{R}/\hbar = \omega_a \hat{N}_a + \omega_b \hat{N}_b + \gamma (\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger) \tag{10}
\]

with the algebra basis \( \mathfrak{A} = \{ \hat{a}^\dagger \hat{b}, \hat{a} \hat{b}^\dagger, \hat{N}_a, \hat{N}_b \} \). Using the Wei-Norman method we find that the time evolution operator is given by

\[
\hat{U}(t) = \prod_{j=1}^{4} e^{\alpha_{R_j}(t)\mathfrak{A}}, \tag{11}
\]

with

\[
\alpha_{R_1}(t) = \frac{\gamma}{\Delta - i \Gamma \cot (\Gamma t)} \tag{12a}
\]

\[
\alpha_{R_2}(t) = \frac{\gamma}{\Gamma} \left[ \frac{i}{2} \sin (2 \Gamma t) - \Delta \sin^2 (\Gamma t) \right] \tag{12b}
\]

\[
\alpha_{R_3}(t) = -i \omega_t + c_R(t) \tag{12c}
\]

\[
\alpha_{R_4}(t) = -i \omega_t - c_R(t), \tag{12d}
\]

where \( c_R(t) = \ln [\cos (\Gamma t) + \frac{i \Delta}{\gamma} \sin (\Gamma t)] \), the detuning is \( \Delta = \frac{\omega_a - \omega_b}{2} \), the natural frequency of the \( \alpha_R(t) \)’s is \( \Gamma = \sqrt{\gamma^2 + \Delta^2} \), and the average frequency is \( \overline{\omega} = \frac{\omega_a + \omega_b}{2} \). The second approximation entails letting \( \gamma = 0 \) and is called the anti-rotating wave approximation (ARWA) corresponding to the Hamiltonian

\[
\hat{H}_{A}/\hbar = \omega_a \hat{N}_a + \omega_b \hat{N}_b + \mu (\hat{a} \hat{b}^\dagger + \hat{a}^\dagger \hat{b}) \tag{13}
\]

with the algebra basis \( \mathfrak{A} = \{ \hat{a} \hat{b}^\dagger, \hat{a}^\dagger \hat{b}, \hat{N}_a, \hat{N}_b, I \} \). Using the Wei-Norman method we find that the time evolution operator is given by

\[
\hat{U}(t) = \prod_{j=1}^{5} e^{\alpha_{A_j}(t)\mathfrak{A}}, \tag{14}
\]

with

\[
\alpha_{A_1}(t) = \frac{-\mu}{\overline{\omega} - i \kappa \cot (\kappa t)} \tag{15a}
\]

\[
\alpha_{A_2}(t) = \frac{\mu}{\kappa \overline{\omega}} \left[ \frac{\overline{\omega}}{\kappa} \sin (\kappa t) - i \kappa \cos (\kappa t) \right] \sin (\kappa t) \tag{15b}
\]

\[
\alpha_{A_3}(t) = i \Delta t + c_A(t) \tag{15c}
\]

\[
\alpha_{A_4}(t) = -i \Delta t + c_A(t) \tag{15d}
\]

\[
\alpha_{A_5}(t) = i \omega_t + c_A(t), \tag{15e}
\]

where \( c_A(t) = -\ln [\cos (\kappa t) + \frac{i \Delta}{\overline{\omega}} \sin (\kappa t)] \), \( \kappa = \sqrt{\overline{\omega}^2 - \mu^2} \) is the natural frequency of the \( \alpha_A(t) \)’s, and the other variables have been defined above.

### D. Qubit-Oscillator Coupling

Jaynes-Cummings involves the coupling of a qubit and SHO under the RWA and corresponds to the Hamiltonian

\[
\hat{H}_{JC}/\hbar = \omega_a (\hat{N} + 1/2) + \omega_\sigma \hat{\sigma}_z + \gamma (\hat{a}^\dagger \hat{\sigma}^- + \hat{a} \hat{\sigma}^+). \tag{16}
\]

In its current form we cannot close the Lie algebra corresponding to the operators in this Hamiltonian due to the commutation relation

\[
[\hat{a} \hat{\sigma}^+, [\hat{a}^\dagger \hat{\sigma}^-, \hat{N}^n \hat{\sigma}_z]] = 2 \hat{N}^{n+1} \hat{\sigma}_z + \text{LOT}, \tag{17}
\]

for \( n \in \{0, 1, \ldots\} \), where LOT are lower order terms containing smaller powers of \( \hat{N} \) (which can be shown by induction). Thus, starting with the operators \( \{ \hat{N}, \hat{\sigma}_z, \hat{a}^\dagger \hat{\sigma}^-, \hat{a} \hat{\sigma}^+ \} \) taking the commutator of Eq. \( 17 \) for \( n = 0 \) we obtain the linearly independent operator \( \hat{N} \hat{\sigma}_z \) which must be added to the set, then must consider the commutator of Eq. \( 17 \) for \( n = 1 \) which generates \( \hat{N}^2 \hat{\sigma}_z \), and so forth leading to basis elements \( \hat{N}^n \hat{\sigma}_z \) with arbitrarily high powers of \( n \). Hence, the algebra corresponding to the operators selected does not close.

#### 1. Closing the algebra

To close the algebra we can express the Hamiltonian using different operators that close under commutation. The key to doing this is defining the operator \( \tilde{M} \)

\[
\tilde{M} = \hat{N} + \frac{1 + \hat{\sigma}_z}{2} \tag{18a}
\]

\[
\tilde{B} = \frac{\hat{a} \hat{\sigma}^+}{\sqrt{\tilde{M}}} \tag{18b}
\]

\[
\tilde{B}^\dagger = \frac{\hat{a}^\dagger \hat{\sigma}^-}{\sqrt{\tilde{M}}}, \tag{18c}
\]

such that

\[
\hat{H}_{JC}/\hbar = \omega_a \tilde{M} + \Delta \hat{\sigma}_z + \gamma \sqrt{\tilde{M}} (\tilde{B}^\dagger \tilde{B}), \tag{19}
\]

corresponding to the Lie algebra basis \( \{ \tilde{M}, \tilde{B}^\dagger, \tilde{B}, \hat{\sigma}_z \} \). The utility of \( \tilde{M} \) can be seen by considering how it acts in the subspace of number states with qubits allowing us to evaluate

\[
f(\tilde{M}) \ket{n} \ket{0} = f(n) \ket{n} \ket{0} \tag{20a}
\]

\[
f(\tilde{M}) \ket{n} \ket{1} = f(n + 1) \ket{n + 1} \ket{1}, \tag{20b}
\]

for \( f \) any arbitrary function. Thus, we can evaluate \( \tilde{M} \) on an arbitrary state of system and environment by decomposing it into the \( \{ \ket{0}, \ket{1} \} \) basis.
2. Anti-Jaynes-Cummings

Anti-Jaynes-Cummings involves the amplitude coupling of qubits and SHOs under the ARWA and corresponds to the Hamiltonian
\[
\hat{H}_{AJC}/\hbar = \omega_0 (N + 1/2) + \omega_a \hat{\sigma}_z/2 + \mu (\hat{a}^\dagger \hat{\sigma}^- + \hat{a} \hat{\sigma}^+). \tag{21}
\]
Similar to JC the algebra corresponding to this Hamiltonian does not close in its current form yet we can employ a similar tactic defining the operators
\[
\hat{M}_A = N + \frac{1 - \hat{\sigma}_z}{2}, \quad \hat{C} = \frac{\hat{a} \hat{\sigma}^-}{\sqrt{M_A}}, \quad \hat{C}^\dagger = \frac{\hat{a}^\dagger \hat{\sigma}^+}{\sqrt{M_A}}. \tag{22a}
\]
Using these operators we can express the Hamiltonian as
\[
\hat{H}_{AJC}/\hbar = \omega_0 \hat{M}_A + \Delta \hat{\sigma}_z/2 + \mu \sqrt{M_A} (\hat{C}^\dagger + \hat{C}), \tag{23}
\]
corresponding to the Lie algebra basis \( \{ \hat{M}_A, \hat{C}^\dagger, \hat{C}, \hat{\sigma}_z \} \).

III. COUPLED OSCILLATOR RESULTS

In this section we consider several initial states of the system and environment for amplitude coupled oscillators under the RWA and ARWA.

A. Coherent-Thermal

Here we consider the initial state
\[
\rho_0 = |\alpha\rangle_a \langle \alpha|_a \otimes \rho_{Th,b}, \tag{24}
\]
where \( \alpha \) and \( b \) subscripts signify states of the system's and environment's subspace, respectively. Note that \( |\alpha\rangle \) is a coherent state which is, in a sense, the most classical like quantum state\(^{[2]} \) and can be expressed in terms of the number states as
\[
|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{25}
\]
Also, \( \rho_{Th} \) is a thermal state with average photon number \( \bar{n} \) which is given by
\[
\rho_{Th} = \frac{1}{1 + \bar{n}} \sum_{n=0}^{\infty} \left( \frac{\bar{n}}{1 + \bar{n}} \right)^n |n\rangle \langle n| \tag{26}
\]
or alternatively can be written in the exponential form
\[
\rho_{Th} = \frac{e^{\bar{n} N}}{1 + \bar{n}}, \tag{27}
\]
where \( \eta = -\hbar \omega/k_B T \) and \( e^n = \frac{\pi}{\Gamma(1 + \bar{n})} \).

Under the RWA, using the procedure described above, we find that the linear entropy is
\[
\zeta_{R,Co-Th}(t) = 1 - \left( 1 + \frac{2 \bar{n}^2 \pi^2}{\Gamma^2} \sin^2 (\Gamma t) \right)^{-1}. \tag{28}
\]
This linear entropy agrees with the results of Vidiella\(^{[3]} \) as computed via solving the Heisenberg equations with no detuning \( (\Delta = 0) \). Note that this linear entropy starts out at zero, corresponding to the initial pure state of the system and is periodic with period \( \pi/\Gamma \). The linear entropy has a maximum of \( \frac{\pi^2}{2} \) which only depends on the strength of the thermal field and the relative strength of the coupling to the detuning. Increasing the natural frequency \( \Gamma \) by increasing \( |\alpha| \) (see Fig. 1) or \( |\Delta| \) leads to decoherence occurring more rapidly as period of the linear entropy decreases yet increasing \( |\Delta| \) also results in a smaller maximum linear entropy, corresponding to a less mixed state.

Under the ARWA we find that the linear entropy is
\[
\zeta_{A,Co-Th}(t) = \left( 1 + \frac{\kappa^2 \csc^2 (\kappa t)}{2 \gamma^2 (1 + \pi \bar{n})} \right)^{-1}. \tag{29}
\]
This linear entropy is zero at \( t = 0 \) and has two distinctly different behaviors depending on whether \( \kappa = \sqrt{\omega^2 - \mu^2} \) is real or imaginary corresponding to \( \omega > |\mu| \) and \( \omega < |\mu| \), respectively. For small couplings \( (\omega > |\mu|) \) \( \kappa \) is real giving rise to oscillatory solutions with period \( \pi/\kappa \) and maximum amplitude of \( \frac{\omega^2 - \mu^2}{2 \omega^2 (1 + \pi \bar{n})} \). For large couplings \( (\omega < |\mu|) \) \( \kappa \) is pure imaginary leading to a hyperbolic linear entropy that is monotonically increasing, approaching one as \( t \to \infty \). This transition of \( \zeta_{A,Co-Th}(t) \) from periodic to hyperbolic as \( \mu \) is increased, for fixed \( \omega \), is shown in Fig. 1. Note that the linear entropies for both RWA and ARWA are independent of the initial coherent state \( |\alpha\rangle \) of the system.

B. Cat-Thermal

Here we consider the initial state
\[
\rho_0 = |\psi\rangle_a \langle \psi|_a \otimes \rho_{Th,b}, \tag{30}
\]
where \( |\psi\rangle \) is a cat state given by superposing two coherent states as
\[
|\psi\rangle = \frac{|\alpha\rangle + e^{i\phi} |\alpha\rangle}{\sqrt{2(1 + e^{-2|\alpha|^2} \cos \phi)}}, \tag{31}
\]
where these coherent states approach orthogonality for large \( \alpha \) and \( \rho_{Th} \) is a thermal state. Under the RWA, the linear entropy of the system corresponding to the state of Eq. (27) has a natural frequency of \( \Gamma \) as for the coherent-thermal case and is given by
\[
\zeta_{R,Cat-Th}(t) = 1 - \frac{\Gamma^2 \left[ e^{4|\alpha|^2} + \cos (2\phi) + 2e^{2|\alpha|^2} \chi(t) \right]}{2 \left( e^{2|\alpha|^2} + \cos \phi \right)^2 \left[ \Gamma^2 + 2\gamma^2 \pi \sin^2 (\Gamma t) \right]} \tag{32}
\]
FIG. 1: Plots of the linear entropies under the RWA and ARWA for a coherent state in a thermal environment for varying coupling strengths $\mu = \gamma$. Here $\omega_a = \omega_b = 1$ and $n = 5$.

where

$$\chi(t) = 2 \cos \phi + \cosh \left[ \frac{2|\alpha|^2 (1 + 2\pi)}{\Gamma^2 \csc^2(\Gamma t) + 2 \gamma^2 \pi - 1} \right].$$

(33)

It is useful to note that like $\zeta_{R,Co-Th}(t)$ this linear entropy is a periodic function of time $t$ with period $T = \pi / \Gamma$ depending on the coupling $\gamma$, the frequency $\Gamma$, and the average photon number of the thermal field $\pi$. The results obtained from changing these parameters ($\gamma$, $\Gamma$, and $\pi$) are qualitatively analogous to the results for RWA with a coherent system in a thermal environment discussed above. Unlike the coherent-thermal case, here there is a dependence on the initial configuration of system in phase space through $|\alpha|$ and $\phi$ which are specified by the cat state of the system.

Increasing $|\alpha|$ results in two interesting effects that we will consider: an increase of the maximum linear entropy and altering its geometry. Increasing $|\alpha|$ leads to larger maximum linear entropies, this is especially pronounced for large detunings as shown in Fig. 2. Increasing $|\alpha|$ also results in decoherence occurring more rapidly for times significantly smaller than the linear entropies period of oscillation. In particular, for large $|\alpha|$ the linear entropy increases dramatically over a small period of time (after the start of each period) until $\zeta(t) > 0.5$ at which time the behavior of the linear entropy changes dramatically as it increases much more gradually. This behavior can be explained by the fact that for large $|\alpha|$ exponential terms in the linear entropy essentially cancel out except for times near the period $T$ where the $\csc^2(\Gamma t)$ term in $\chi(t)$ dominates, tending towards infinity, forcing the linear entropy to be zero. Mathematically we have that the linear entropy can be approximated as

$$\zeta_{R,\text{Cat-Th}}(t) \approx \zeta(t) = 1 - \frac{\Gamma^2}{2 \left[ \Gamma^2 + 2 \gamma^2 \pi \sin^2(\Gamma t) \right]}$$

(34)

for $|2 (t/T \mod 1) - 1| \ll 1$, as illustrated in Fig 2.

Note that $\xi(t)$ is periodic by $T$ and has minima of $\xi = 1/2$ at integral multiples of $T$. Thus, the observations that as Eq. (33) is a good approximation for times far away from integral multiples of the period, $T$, and that $\zeta_{R,\text{Cat-Th}}(nT) = 0 \forall n \in \{0, 1, \cdots\}$ account for the dramatic changes in linear entropy that occurs for large $|\alpha|$ described above.

FIG. 2: These plots illustrate that under the RWA the maximum value of $\zeta_{R,\text{Cat-Th}}(t)$ increases as $|\alpha|$ increases. Here $\omega_a = 10$, $\omega_b = \gamma = 1$, $\phi = 0$, and $\pi = 5$.

FIG. 3: These plots demonstrate the validity of Eq. (34). Here $\omega_a = \omega_b = \gamma = 1$, $\phi = 0$, and $\pi = 5$.

FIG. 4: These plots demonstrate the validity of Eq. (35). Here $\omega_a = \omega_b = \gamma = 1$, $\phi = 0$, and $\pi = 5$.

Under the ARWA the linear entropy of the system is given by

$$\zeta_{A,\text{Cat-Th}}(t) = 1 - \frac{\kappa^2 \csc^2(\kappa t) \Upsilon(t)}{2 (e^{2|\alpha|^2} + \cos \phi)^2 \left[ \omega^2 + \mu^2 (1 + 2\pi) + \kappa^2 \cot^2(\kappa t) \right]}$$

(35)
where

\[ \Upsilon(t) = e^{4|\alpha|^2} + 4e^{2|\alpha|^2}\cos(2\phi) + e^{4|\alpha|^2| \alpha |^2 \cos^2 (\phi)} + e^{4|\alpha|^2| \alpha |^2 \cos^2 (\phi)} + e^{4|\alpha|^2| \alpha |^2 \cos^2 (\phi)}. \] (36)

This linear entropy gives rise to results phenomena as found for cat-thermal under the RWA given in Eq. (32). In particular, it exhibits similar behavior for large |α| as illustrated in Fig. 4. Similar to the coherent-thermal case under the ARWA we have that \( \zeta_{\lambda, \text{Cat-Th}}(t) \) is periodic by \( \pi/\kappa \) for \( \mu < \varpi \) and is hyperbolic for \( \mu > \varpi \). Just as for the RWA case of cat-thermal increasing the phase, \( \phi \), of the initial cat state from 0 to \( \pi \) results in decoherence occurring more rapidly. The main difference being that under the ARWA the linear entropy is increased significantly everywhere whereas under the RWA there are two peaks form and the linear entropy undergoes little change at half periods (as shown in Fig. 5).

![Fig. 4](image4.png)

**FIG. 4:** These plots illustrate that under the ARWA the maximum value of \( \zeta_{\lambda, \text{Cat-Th}}(t) \) increases as |α| increases. Here \( \omega_a = \omega_b = 5 \), \( \gamma = 1 \), \( \phi = 0 \), and \( \pi = 5 \). Note that \( \Delta = 0 \) here yet \( \pi \) is large (relative to \( \gamma \)).

To obtain a better intuition for what is happening to the state of the system due to our dynamics we use Husimi function which is a quasiprobability distribution in phase-space (see Appendix B). In Fig. 6 we show the Husimi function for cat-thermal under the RWA, note that the Husimi function is periodic with the same period as the linear entropy, \( \pi/\Gamma \), so it suffices for us to focus on the first half-period. The relationship between the linear entropy and Husimi function is illustrated using Fig. 7 where we have highlighted the times corresponding to the displayed Husimi functions. The Husimi function for cat-thermal under the ARWA is shown in Fig. 8, the most significant difference between RWA and ARWA Husimi functions of cat-thermal is that under the RWA the two cats merge together at the origin whereas under the ARWA the two cats remain separated traversing phase-space in ellipses.

![Fig. 5](image5.png)

**FIG. 5:** Plots illustrating the linear entropies dependence on the phase, \( \phi \), of the initial cat state system in a thermal environment under the RWA and ARWA. A continuum of phases are considered \( 0 \leq \phi \leq \pi \), as indicated in the legend, to more easily display the effect of \( \phi \). Here \( \omega_a = \omega_b = 1 \), \( \gamma = 3/4 \), and \( \alpha = \pi = 1 \). Small \( \alpha \) and \( \pi \) values were used to make the effect of varying \( \phi \) more pronounced.

### C. Coherent-Coherent

Here we consider the initial state

\[ \rho_0 = |\alpha\rangle \langle \alpha| \otimes |\beta\rangle \langle \beta|, \] (37)

corresponding to a coherent system in a coherent environment. For this case we have found that no decoherence occurs as

\[ \zeta_{\text{R,Co-Co}}(t) = 0 \] (38)

for all times. This case has yet to be interpreted but we presume it comes down to a combination of the initial states of the system and environment being pure along with the dynamics of RWA. Under the ARWA for the coherent-coherent case the linear entropy does not vanish yet the explicit form of the linear entropy is quite long and will be omitted for brevity yet it is useful to note that it is a function of time \( t \), the coupling \( \mu \) and the average frequency \( \varpi \) not on the initial coherent states \( \alpha \) and \( \beta \). As for the previous ARWA cases the linear entropy is periodic with period \( \pi/\kappa \) for low coupling \( (|\mu| < \varpi) \) and hyperbolic for large coupling \( (|\mu| > \varpi) \) (see Fig. ??). We find the amplitude and period of the linear entropy decrease as \( \varpi \) is increased and that for periodic cases there is a slight dip in linear entropy at half-periods.
FIG. 6: Husimi function at several times for the cat-thermal case under the RWA. Here $\omega_a = \omega_b = \gamma = 1$, $\alpha = \pi = 5$, and $\phi = 0$.

IV. JAYNES-CUMMINGS RESULTS

In this section we consider several initial states of the qubit system and SHO environment under the Jaynes-Cummings approximation. Note, in Mathematica we have created animations for the evolution of the Bloch vector in the Bloch ball for the following systems and environments yet hitherto we have not created the corresponding tables of figures for this paper.

A. Superposition-Number

Here we consider the initial pure state

$$|\varphi\rangle = |\tau\rangle_a |n\rangle_b \quad (39)$$

where the qubit state is given by the superposition

$$|\tau\rangle = \sqrt{1 - \nu^2} |0\rangle + \nu |1\rangle. \quad (40)$$

This is the simplest case to evaluate due to the relations shown in Eq. (20) and we can use it to build up to more interesting environments. Here we have the linear entropy analytically yet it is too long to reasonably show here, we will show the special case where $\Delta = 0$ and $\nu = 1$ corresponding to starting in the excited state of the qubit

$$\zeta(t)_{JC,Ex-Num} = \frac{1}{2} \sin^2 \left(2\gamma t \sqrt{1 + n}\right), \quad \text{with } \Delta = 0, \nu = 1. \quad (41)$$

The presence of $\gamma t \sqrt{1 + n}$ in the argument is a result of letting $\nu = 1$ for $\nu = 0$ we obtain arguments of the form $\gamma t \sqrt{n}$, this comes from how $\hat{M}$ acts on these states. Increasing $n$ for the initial number state results in decoherence occurring more rapidly as well as larger amplitude and frequency of oscillation in the linear entropy (see Fig. 10). The linear entropy for superposition-number under Jaynes-Cummings are oscillatory yet not periodic due to the $\sqrt{n}$ and $\sqrt{n + 1}$ that are present in the arguments of trigonometric functions, however, the linear entropy does behave pseudo-periodically.

B. Superposition-Coherent

Here we consider the initial pure state

$$|\varphi\rangle = |\tau\rangle_a |\alpha\rangle_b \quad (42)$$

that is qubit in a superposition as the system in a coherent SHO state environment. My knowing how the time evolution operator acts on a number state we can find the time evolved state in this case by expanding the coherent environment in terms of number states as in Eq. (25). That is, we can write the time evolved state of the system and environment as

$$\rho(t) = \hat{U} |\varphi\rangle \langle \varphi| \hat{U}^\dagger, \quad (43)$$

which is an infinite double sum due to the coherent ket and bra. By partial tracing over the environment we obtain the state of the system to be a two-by-two reduced density matrix the elements of which are infinite...
FIG. 8: Husimi function at several times for the cat-thermal case under the ARWA. Here $\omega_a = \omega_b = \gamma = 1$, $\alpha = 0$, $n = 5$, and $\phi = 0$.

sums. Thus, in order to compute the linear entropy we truncate the sum going from $n = 0$ to some $n_{\text{max}}$, we illustrate that these sums converge in Fig. 11. As for the RWA cases for oscillator-oscillator couplings under Jaynes-Cummings we find that the maximum linear entropy decreases as detuning is increased (see Fig. 12).

C. Superposition-Thermal

Here we consider the initial state
\[
\rho_0 = |\tau\rangle_a \langle \tau|_a \rho_{\text{Th,b}}, \tag{44}
\]

of a superposition qubit system in a thermal oscillator environment. We obtain linear entropy results that are similar to those of superposition-coherent. On interesting result for superposition-thermal is that “beats” form in the linear entropy. That is, the linear entropy oscillates about a mean value and there are regions in which the amplitude of the oscillations are very large relative to other times. As detuning increases these beats become more distinct and the average linear entropy decreases.

V. CONCLUSIONS

We have considered the dynamics of minimal open systems of coupled oscillators and qubits using an algebraic method known as Wei-Norman. We have solved for the evolution of various initial states finding the corresponding linear entropy and Husimi function or Bloch sphere depending on whether the system is an oscillator or a qubit, respectively. Broadly, we have demonstrated that the interaction of an open systems with its environment gives rise to decoherence. For amplitude coupled SHOs and qubits decoherence occurs more rapidly when the
FIG. 11: The convergence of the linear entropy as the number of terms kept in the numerical approximations for the elements of the reduced density matrix is increased. Here $\omega_a = \omega_b = 1$, $\gamma = 1$, $\alpha = 1$, and $\nu = 0.5$.

FIG. 12: Jaynes-Cummings linear entropy while varying detuning. Here $\gamma = 1$, $\alpha = 1$, and $\nu = 0$.

coupling strength between the system and environment is increased or $\overline{n}$, the average photon number, is increased, for the case of a thermal environment. Under the RWA linear entropy linear entropy increases when $\omega_a, \omega_b$ are increased. For all cases investigated we have found that for a SHO system coupled to a SHO environment the linear entropy loses periodicity (for large couplings) under the ARWA. For Jaynes-Cummings with a qubit system coupled to a SHO environment we have found that the linear entropy is not periodic but oscillatory.

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Appendix A: Wei-Norman

In quantum dynamics, the Wei-Norman method uses the ansatz

$$\hat{U}(t) = \prod_{j=0}^{n} e^{\alpha_j(t) \hat{A}_j}, \quad (A1)$$

for the time evolution operator, where the $\alpha_i(t)$s are yet unknown functions and the $\hat{A}_i$s are time-independent basis elements for a Lie algebra $\mathfrak{a} = \{\hat{A}_1, \ldots, \hat{A}_n\}$. This basis is chosen such that the Hamiltonian can be expressed in terms of it,

$$\hat{H} = \sum_{i=1}^{n} c_i(t) \hat{A}_i, \quad (A2)$$

and to be closed under commutation. The $\alpha_i(t)$s can be found by substituting this $\hat{U}$ into Eq. (1) yielding

$$i\hbar \sum_{i=1}^{n} g_i(\alpha_1(t), \hat{\alpha}_1(t), \ldots, \alpha_n(t), \hat{\alpha}_n(t)) \hat{A}_i = \sum_{i=1}^{n} c_i(t) \hat{A}_i, \quad (A3)$$

where the $\{g_i\}$ functions depend on the algebra under consideration. Therefore, as $\{\hat{A}_i\}$ is linearly independent, the coefficients of Eq. (A3) must match and hence, by using the Wei-Norman method, one can separate the algebra from the time evolution of the operators. The dynamics of a system can therefore be reduced from handling an algebra of time-dependent operators to a system of at most $n$ coupled ordinary differential equations (involving the $\alpha_j(t)$s) which can be solved analytically for simple cases and numerically otherwise. One limitation of the Wei-Norman is that one must construct a Lie algebra closed under commutation which is not always possible.

Appendix B: Husimi Function

Husimi functions are quasi-probabilistic distributions which are useful in visualizing the evolution of quantum states in phase space. They are given by the expectation value of a state $\hat{\rho}$ with respect to the coherent states $|\alpha\rangle$ as

$$Q(\alpha) = \frac{\langle \alpha | \hat{\rho} | \alpha \rangle}{\pi}, \quad (B1)$$

where $\alpha = x + ip$ for $x$ and $p$ the expectation values of the position and momentum, respectively. Note that
a Husimi function does not give a simple trajectory in phase space as in classical mechanics as there are uncertainties involved as required by the Heisenberg uncertainty principle.

As an example we consider the time-dependent Husimi function for a coherent state subject the the SHO Hamiltonian of Eq. (6) given by $|\beta(t)\rangle = e^{-i\omega t/2} |\beta e^{-i\omega t}\rangle$ corresponding to the density matrix $ho = |\beta(t)\rangle \langle \beta(t)| = |\beta e^{-i\omega t}\rangle \langle \beta e^{-i\omega t}|$. Thus, we can find the corresponding Husimi function, first noting that the inner product of two coherent states is $\langle \alpha | \beta \rangle = e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2) + \alpha^* \beta}$, as $Q(\alpha, t) = \frac{\langle \alpha | \beta e^{-i\omega t} \rangle \langle \beta e^{-i\omega t} | \alpha \rangle}{\pi} = \frac{1}{\pi} e^{-|\alpha|^2 - |\beta|^2 + 2 \text{Re}(\epsilon^t \alpha \beta^*)}$.

From $Q(\alpha, t)$ one can observe that the coherent state $|\beta(t)\rangle$ traverses phase space by rotating in a circle about the origin without dispersion.

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