2020-07-30

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A Derivation of Scattered Intensity (Radiance) for Use in Optical Particle Counters

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ARTICLE INFO

Keywords:
Mie Scattering Theory
Intensity
Radiance
Optical Particle Counter

ABSTRACT

Mie scattering theory is often used to calculate the flux of light scattered off of small particles whose diameter is approximately the same order of magnitude as the wavelength of incident light. In many equations for the calculation of this flux, it is assumed that the area of interest is perpendicular to the ray of scattered light. This assumption does not apply to all cases such as low-cost (<$50) Optical Particle Counters (OPCs). Therefore, a conserved value, known as intensity (radiance), may be beneficial. By intensity it shall be meant the quantity containing the units of W/(m^2sr). On comparison with the International System of Units (SI) it is seen that this definition of intensity is also known as radiance. Due to the benefits of this quantity, this article presents a derivation of intensity (radiance) using Electromagnetic wave theory and Mie scattering theory. This quantity is simplified according to approximations relating to OPC modeling.

Nomenclature

Variables

$\Delta\Omega_{d-p}$	Finite Solid angle from pixel to particle
λ	Light source wavelength
$\langle S \rangle$	Time averaged Poynting vector
$\mathbf{B}$	Magnetic field
$\mathbf{E}$	Electric Field
$\mathbf{S}$	Poynting vector
μ_o	Permeability
ε_o	Permittivity constant
A	Stokes Parameter - Unpolarized
A_p	Projected area of particle
A_s	Total source area at distance d
d	Distance from source to particle
$E_{\parallel}$	Parallel Electric field
$E_{\perp}$	Perpendicular Electric Field
I_s	Scattered Intensity
P	Power
r	Distance from particle to detector
r_p	Particle radius
u_e	Total energy of a wave
x	Size Parameter
G	Irradiation
I	Intensity

Acronyms

OPC	Optical Particle Counter

Units

m	Meters
sr	Steradians
W	Watts

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1. Introduction

Light scattering from particles is an important problem in many engineering applications. An example of one such application is a sensor known as an Optical Particle Counter (OPC). In an OPC a collimated light source produces a beam that intersects with a gas stream containing particulates. These particulates scatter a portion of the incident light onto a photodetector which is used to produce a measurable response proportional to the power of incident light. This response is then correlated to a Particulate Matter concentration.

Mie scattering theory is a general development for light scattered off of any particle size (Mishchenko, Travis, & Lacis, 2002, p. 206). Rayleigh and Geometric Scattering are simplified cases of Mie Scattering for particle diameters much smaller and much larger than the source wavelength respectively (Hahn, 2009, p. 2). In order to properly calculate the incident power on the photodetector, it is required to know the scattered intensity.

Notation with regards to radiation transfer is often confusing and chaotic. This is distinctly evident in the understanding of intensity. This term, as defined by the International System of Units (SI), has units of candelas which is a photometric value (weighted to the spectral response of the human eye). In some disciplines, such as Mechanical Engineering, intensity is a conserved radiometric value such that its units are $W/m^2 sr$. Several books and articles exist that define intensity as a flux W/m^2, whereas in Mechanical Engineering this flux is an irradiation denoted as G. It is critical that one be clear in which value is meant and how they differ.

A better understanding of the difference of irradiation and intensity, as described in this article, may be understood by a consideration of the Sun. If one compares the irradiation incident on Mercury against the irradiation incident on Pluto due to the sun, it is easy to understand that these values are vastly different. However, the intensity of the sun incident on each planet is the same since as it is a conserved value. It is important to note that the International System of Units defines this intensity as radiance and this irradiation as irradiance (National Institute of Standards and Technology, 2019). In lieu of these various definitions it is important to note that this article will consider intensity as the conserved quantity radiance as defined by the SI.

2. Derivation

The following derivation and equations are performed using Mie scattering theory to derive a conserved value of intensity under the approximations used for OPCs. This article seeks to derive the relevant equations using terms of intensity rather than irradiation so that the scattered power may be properly calculated as in the case of an OPC. Although all cases of polarized light and particle shapes are not taken into account here-in, many of the general equations are given such that one should be able to continue their own derivation for a specific case that differs from the one presented in this article. In such an instance, it would be important to turn to the references for further understanding if needed.

It is known that Mie scattering theory provides the calculation of the Poynting vector ($\vec{S}$) which specifies the magnitude and direction of the rate of transfer of electromagnetic energy at all points of space (Bohren & Huffman, 1983, p. 23). Therefore, the Poynting vector is valid for any arbitrary area, angle, and distance away from the particle of interest. As such, the derivation contained in this article utilizes the Poynting vector along with Electromagnetic Wave theory in order to obtain an expression for the Intensity of light scattered off of a particle of arbitrary size. The mathematical definition of the Poynting vector is the electric field crossed with the magnetic field (Serway & Jewett, 2008) such that:

$$\vec{S} \equiv \frac{1}{\mu_o} \vec{E} \times \vec{B} \tag{1}$$

The tensors $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ represent the electric and magnetic fields respectively and μ_o represents the permeability of the scattered magnetic field. From Serway and Jewett (2008, p.961) it is known that "The magnitude of the Poynting vector represents the rate at which energy flows through a unit surface area perpendicular to the direction of wave propagation." Therefore, the magnitude of the Poynting vector (S) may be found by calculating the magnitude of the cross product described in Eq. 1.

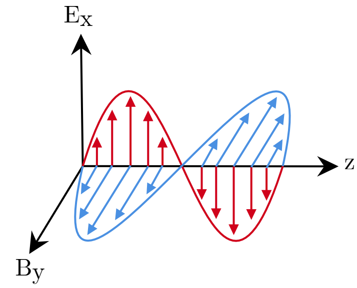

Figure 1: Shows a diagram of the propagation of an electromagnetic wave in the z direction. Note that the Electric field, E, in the x direction is always normal to the Magnetic field, B in the y direction (Serway & Jewett, 2008, p. 957).

$$S = \frac{1}{\mu_o} \left|\vec{E}\right| \left|\vec{B}\right| sin\left(\theta\right) \tag{2}$$

As can be seen in Fig. 1, $\vec{E}$ and $\vec{B}$ are perpendicular to each other such that θ from Eq. 2 is equal to 90°. As such, S travels in the direction of the propagation of the wave and can be simplified to Eq. 3:

$$S = \frac{EB}{\mu_o} \tag{3}$$

Serway and Jewett (2008, p. 961) state that the magnitude of the Magnetic field may also be defined in terms of the Electric field as in Eq. 4.

$$B = \frac{E}{c} \tag{4}$$

As such, a substitution may be made into Eq. 3 such that it becomes:

$$S = \frac{E^2}{\mu_o c} \tag{5}$$

The equations presented thus far represent instantaneous quantities; however, due to physical detection hardware limits, a time averaged Poynting vector is of practical use. This transforms Eq. 5 into Eq. 6 (Serway & Jewett, 2008, p. 961).

$$\langle S \rangle = \frac{E^2}{2\mu_o c} \tag{6}$$

It is necessary to calculate the squared Electric field, E^2. This may be done by evaluating different cases of polarization of the electric field which are known as Stokes Parameters. The Stokes parameters cover unpolarized light, horizontal and vertical polarizers, +45°and -45°polarizers, and

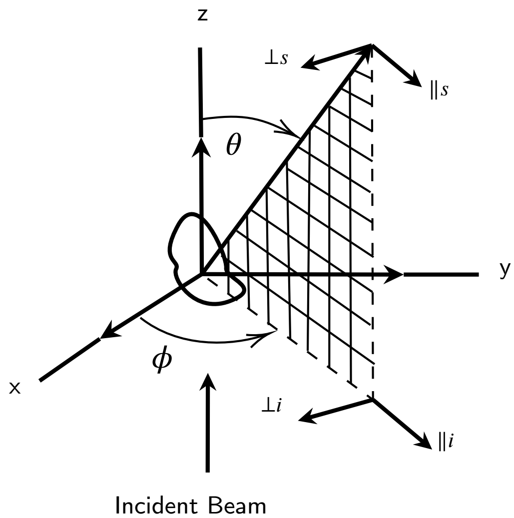

Figure 2: An arbitrary size/shape particulate scattering light into the parallel (||) and perpendicular (⊥) directions (Bohren & Huffman, 1983, p. 62)

circular polarizers. This article will focus on the case of unpolarized light since it also represents natural light. The other cases are given in more detail on p. 48 of Bohren and Huffman (1983). The Poynting vector may be defined in terms of the Stokes Parameter, A, for unpolarized light such that:

$$\langle S \rangle = \frac{1}{2\mu_o c} A \tag{7}$$

Where the Stokes Parameter A is:

$$A = E_{\|s} E^*_{\|s} + E_{\perp s} E^*_{\perp s} \tag{8}$$

The complex conjugate of the Electric field is denoted by the symbol *. The subscripts || and ⊥ represent the parallel and perpendicular directions respectively and the subscript s, denotes the field as scattered from the particle (see Fig. 2). As shown in Eq. 6 the magnitude of the Electric field is the quantity of interest. As such, the complex conjugate is used to calculate the magnitude of the Electric field.

The Scattered Electric field is defined by Bohren and Huffman (1983) as:

$$\begin{pmatrix} E_{\|s} \\ E_{\perp s} \end{pmatrix} = \frac{e^{ik[r-z]}}{-ikr} \begin{pmatrix} S_2 & S_3 \\ S_4 & S_1 \end{pmatrix} \begin{pmatrix} E_{\|i} \\ E_{\perp i} \end{pmatrix} \tag{9}$$

It is important to note that the S parameters used in this equation refer to the Amplitude scattering matrix which has to do with the Electric field and not the Poynting vector. As such, an S with a subscript is the Amplitude scattering matrix, whereas an S without the subscript refers to the Poynting Vector. The Amplitude scattering matrix parameters are functions of: direction (θ, ϕ), source wavelength (λ), size parameter (x) and refractive index $(m = n + ik)$.

S_2 is the fraction of $E_{\|i}$ scattered in the parallel direction. S_3 is the fraction of $E_{\perp i}$ scattered in the parallel direction. Likewise, S_4 is the fraction of $E_{\|i}$ scattered in the parallel direction. S_1 is the fraction of $E_{\perp i}$ scattered in the perpendicular direction.

Combining Eq.'s 8 and 9 yields the following:

$$\begin{aligned} A = &\left(S_2 E_{\|i} + S_3 E_{\perp i}\right)\left(S_2 E^*_{\|i} + S_3 E^*_{\perp i}\right) + \\ &\left(S_4 E_{\|i} + S_1 E_{\perp i}\right)\left(S_4 E^*_{\|i} + S_1 E^*_{\perp i}\right) \end{aligned} \tag{10}$$

Equation 10 is valid for cases where natural light is incident on a particle of arbitrary size and shape. In OPC modeling the particulates are approximated as homogeneous spheres, therefore due to the symmetry of a sphere, S_3 and S_4 are equal to 0 (van de Hulst, 1957, p. 35). Using this approximation leads to the simplification of Eq. 10 to:

$$A = S_1^2 E_{\perp i} E^*_{\perp i} + S_2^2 E_{\|i} E^*_{\|i} \tag{11}$$

It is now necessary to calculate the incident Electric field, $E_{\|i}$ and $E_{\perp i}$, must be found. When the Electric field is multiplied by its complex conjugate as in Eq. 11, Bohren and Huffman (1983, p. 49) show the following::

$$E_{\|i} E^*_{\|i} = a^2_{\|i} \tag{12}$$

$$E_{\perp i} E^*_{\perp i} = a^2_{\perp i} \tag{13}$$

Where the parameters $a_{\|i}$ and $a_{\perp i}$ are defined as the real non-negative amplitudes of the electric field incident on the particle (Mishchenko et al., 2002, p. 20). Therefore, Eq.'s 12 and 13 allow Eq. 11 to be simplified to:

$$A = \frac{1}{k^2 r^2}\left(S_2^2 a^2_{\|i} + S_1^2 a^2_{\perp i}\right) \tag{14}$$

In order to calculate the amplitude of the electric field incident on the particle ($a_{\|i}$ and $a_{\perp i}$), the Poynting vector incident on the particle may be considered. The Poynting vector is related to wave energy in the following way (Serway & Jewett, 2008, p. 962):

$$\langle S \rangle = u_e c \tag{15}$$

Where c is the speed of light and u_e is the average energy density of the electromagnetic wave both of which refer to the light incident on the particle. Also from Serway and Jewett (2008, p. 962), u_e may be defined as:

$$u_e = \frac{1}{2}\varepsilon_o E^2 \tag{16}$$

Where the variable ε_o is the permitivity of the incident electric field. Referring to the definition of the Poynting vector as stated for Eq. 1, it is known that the magnitude of the Poynting vector is the rate of energy that passes through an area such that:

$$\langle S \rangle = \frac{P}{A_s} \tag{17}$$

The variable A_s is the area of the light emitting source at the particle which is a distance, d, away from the initial emission of the source. This distinction is important if there is significant divergence in the light source. The variable P is the power of the source. Combining Eq.'s 15 - 17 and rearranging provide the following equation:

$$E^2 = \frac{2P}{A_s \varepsilon_o c} \tag{18}$$

From Bohren and Huffman (1983, p. 51) it is known that natural light has no preferred vibration ellipse or, in other words, it has an equal probablitity of traveling in both the parallel and perpendicular directions such that:

$$\frac{E}{2} = \left|E_{\parallel i}\right| = \left|E_{\perp i}\right| \tag{19}$$

Since Eq. 19 involves the magnitude of the Electric field in each direction respectively, the complex conjugate definitions from Eq.'s 12 and 13 provide suitable substitutions for Eq. 18 which leads to:

$$a_{\parallel i}^2 = \frac{P}{2A_s \varepsilon_o c} \tag{20}$$

$$a_{\perp i}^2 = \frac{P}{2A_s \varepsilon_o c} \tag{21}$$

Since the power term, P, for natural light in Eq.'s 20 and 21 is the same, these equations may be substituted into Eq. 14 to produce an equation to calculate the Stokes Parameter A.

$$A = \frac{1}{k^2 r^2}\left(S_1^2 + S_2^2\right)\frac{P}{2A_s \varepsilon_o c} \tag{22}$$

Equation 22 is then substituted into Eq. 6 to solve for the time averaged Poynting vector:

$$\left\langle S \right\rangle = \frac{1}{2\mu_o c k^2 r^2}\left(S_1^2 + S_2^2\right)\frac{P}{2A_s \varepsilon_o c} \tag{23}$$

Recalling that $\left\langle S \right\rangle$ is a flux, it is necessary to to turn this into an intensity or radiance such that, it becomes a conserved quantity as is the purpose of this article. Therefore $\left\langle S \right\rangle$ may also be written in terms of intensity:

$$\left\langle S \right\rangle = I_s cos\left(\theta\right)\Delta\Omega_{d-p} \tag{24}$$

Since this article is considering a spherical particle, the cosine term is equal to 1 and the definition of the solid angle for a sphere is:

$$\Delta\Omega_{d-p} = \frac{A_p}{r^2} \tag{25}$$

Where A_p is the projected area of the spherical particle. Therefore, $\left\langle S \right\rangle$ may also be written as:

$$\left\langle S \right\rangle = \frac{I_s A_p}{r^2} \tag{26}$$

Setting Eq. 26 equal to Eq. 23 and solving for the intensity gives:

$$I_s = \frac{P}{4\mu_o \varepsilon_o c^2 k^2 A_p A_s}\left(S_2^2 + S_1^2\right) \tag{27}$$

Equation 27 may be simplified by considering the following:

$$x = \frac{2\pi r_p}{\lambda} \tag{28}$$

$$k = \frac{2\pi}{\lambda} \tag{29}$$

$$\frac{1}{\mu_o} = c^2 \varepsilon_o \tag{30}$$

$$A_p = \pi r_p^2 \tag{31}$$

Combining Eq.'s 28 - 31 yields the following equation:

$$I_S = \frac{P}{4\pi x^2 A_s}\left(S_2^2 + S_1^2\right) \tag{32}$$

This derivation has produced an equation to calculate the scattered intensity in terms of known parameters. If a dimensional analysis is performed on Eq. 32 alone, it appears the units are W/m^2 and is therefore irradiation and not intensity. However, it is important to note that the units of Eq. 25 are steradians (sr). Therefore, this unit is carried through into the denominator of Eq. 32 which shows that dimensionally this is the intensity value defined in the beginning of this article.

As stated in the paragraph following Eq. 9, the Amplitude Scattering Parameters (S_1 and S_2) are functions of: direction (θ, ϕ), source wavelength (λ), size parameter (x) and refractive index ($m = n - ik$). In an OPC performance analysis the following approximations may apply: source wavelength is non-varying, particulates are homogeneous spheres ($\theta = \phi$), and particulates have a constant refractive index. Using these approximations Eq. 32 may also be written as:

$$I_s = \frac{P}{4\pi x^2 A_s}\left(\left|S_1^2\left(\theta, x\right)\right| + \left|S_2^2\left(\theta, x\right)\right|\right) \tag{33}$$

Equation 33 is the final product of this article, resulting in an equation that can calculate the scattered intensity off of a particulate in an OPC device.

3. Summary and Conclusion

This article has presented a logical derivation of radiometric intensity or radiance to be used in engineering applications. Although this article finished with a derivation for the case of natural light on a spherical particle, many of the general equations along with references for other specific cases were provided. This should enable one to derive an equation specific to applications other than that considered in this article. This work has provided an essential foundation to further articles on theoretical performance assessments of Optical Particle Counters.

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