Optimal noise matching for mutually-coupled arrays

Michael A. Jensen
jensen@byu.edu

Karl F. Warnick
warnick@byu.edu

Original Publication Citation

Warnick, K. F., and M. A. Jensen. "Optimal Noise Matching for Mutually Coupled Arrays." Antennas and Propagation, IEEE Transactions on 55.6 (27): 1726-31

Optimal Noise Matching for Mutually Coupled Arrays

Karl F. Warnick, *Senior Member, IEEE*, and Michael A. Jensen, *Senior Member, IEEE*

***Abstract*—From classical two-port noise theory, the noise figure of an amplifier is minimized when a source is matched to a particular optimal reflection coefficient at the amplifier input. In this paper, we show that this result extends in a natural way to the multiport case, with a coupled N-port source network such as an array antenna connected by a multiport matching network to the inputs of N low-noise amplifiers. For optimal noise performance, the matching network must decouple the array and present isolated, individually noise-matched ports to the amplifier inputs.**

***Index Terms*—Amplifier noise, antenna array mutual coupling, multiport circuits, signal-to-noise ratio.**

I. Introduction

Mutual coupling between elements of an array antenna impacts system performance in a variety of applications, including phased array radars [1], multiple-input–multiple-output (MIMO) communications [2], and array feeds [3], [4]. Most studies of mutual coupling have focused on the effects of mutual coupling on radiation patterns and the active element reflection coefficient [5]–[7]. More recent work has focused on developing a network theory framework for treating mutual coupling [8]–[11].

A network theory framework for mutually coupled arrays allows the effect of amplifier noise parameters on overall system noise performance to be modeled. In order to obtain the best possible system sensitivity, a matching network can be inserted between the array and amplifiers, in which case the goal is to specify the matching network such that noise contributed by the amplifiers is minimized. In [9], a simplified amplifier noise model is assumed for which optimal performance is obtained when the matching network maximizes the signal power delivered to the loads.

For a more realistic bidirectional amplifier noise model, a natural conjecture for the optimal matching network is one that presents to the amplifiers a diagonal matrix of reflection coefficients with the diagonal elements equal to the single-amplifier optimal reflection coefficient for minimum noise figure (i.e., the matching network minimizes the noise figure for each amplifier individually). Since the array is mutually coupled, however, reverse noise exiting the input of one amplifier scatters between array elements and is presented as forward waves at the inputs of all of the amplifiers. Because of this noise coupling, it is not obvious that a matching network which minimizes the noise figure of each amplifier individually is optimal. In [11], it was shown that this choice of matching network provides superior signal-to-noise ratio (SNR) performance relative to one that maximizes power transfer. While this latter work speculates that such a matching network is optimal, it does not provide a proof of this concept. A numerical demonstration of optimality was given for a particular array geometry and set of amplifier parameters in [4]. In this paper, we prove that the conjectured matching network criterion is optimal in the general case.

Manuscript received June 29, 2006; revised January 2, 2007. This work was supported in part by the National Science Foundation under Advanced Technology and Instrumentation under Grant 0352705, by the Information Technology Grants CCR-0313056 and CCF-0428004, and by the U.S. Army Research Office under the Multi-University Research Initiative (MURI) Grant W911NF-04-1-0224.

The authors are with the Department of Electrical and Computer Engineering, Brigham Young University, Provo, UT 84602 USA (e-mail: warnick@byu.edu).

Digital Object Identifier 10.1109/TAP.2007.898596

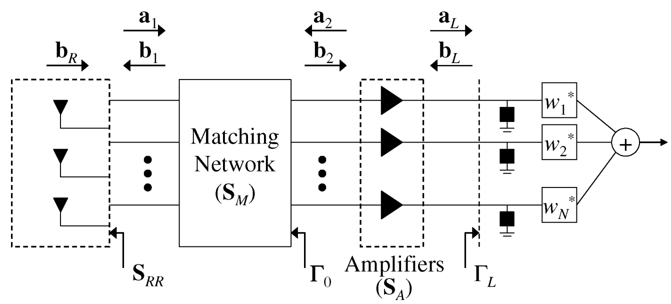

Fig. 1. Block diagram for the multiantenna receiver considered in this paper. The system includes a mutually coupled array, matching network, LNAs, and load terminations. After amplification, the voltages across the loads are either combined using a hardware beamformer or sampled and combined in digital signal processing. All connections are modeled as transmission lines with forward and reverse wave amplitudes labeled as indicated in the diagram.

II. Array and Receiver Model

The model receiver architecture to be considered in this paper is depicted in Fig. 1. The output ports of N array antenna elements are connected to the N input ports of a $2N$-port matching network. The output ports of the matching network are in turn connected to low-noise amplifiers (LNAs). The amplifiers drive loads which represent the inputs of a hardware beamformer or receivers that convert the signal to baseband and provide sampled complex voltages for digital beamforming. External noise is received by the array, and internal noise is introduced by the amplifiers and by conductor losses in the elements, transmission lines, and the matching network. In this analysis, we neglect noise generated by losses in the matching network and system elements after the LNAs. Signal and noise voltages are represented as phasors ($e^{j\omega t}$) with the frequency dependence suppressed.

A. Array Antenna Representation

An array antenna is characterized by an $N \times N$ mutual scattering matrix $\mathbf{S}_{RR}$ (referenced to a real impedance Z_0) and an

N element column vector of phasor voltage responses $\hat{\mathbf{v}}(\Omega)$ to a plane wave with unit amplitude arriving from the direction corresponding to the spherical angle Ω with all of the elements terminated by an open-circuit load. The array response is polarization dependent, so $\hat{\mathbf{v}}(\Omega)$ presumes a given polarization for the incoming wave. From the open-circuit loaded array responses, we can obtain the signal and external noise correlation matrices, which are, respectively

$$\begin{aligned}\hat{\mathbf{R}}_s &= \mathrm{E}\left\{\hat{\mathbf{v}}_s\hat{\mathbf{v}}_s^\dagger\right\}\\ \hat{\mathbf{R}}_n &= \mathrm{E}\left\{\hat{\mathbf{v}}_n\hat{\mathbf{v}}_n^\dagger\right\}\end{aligned}$$

where $\mathrm{E}\{\cdot\}$ denotes an expectation and the superscript $\dagger$ indicates a conjugate transpose. If the signal is stationary, then $\hat{\mathbf{R}}_s = \hat{\mathbf{v}}_s\hat{\mathbf{v}}_s^\dagger$. Here, $\hat{\mathbf{v}}_n$ represents noise from external sources (e.g., thermal radiation or interference received by the array) referred to the open-circuited array element ports.

B. Noisy Output Signal

In this section, we briefly review the framework of [4], [9], and [11], which allows us to determine the receiver outputs from the open-circuit loaded array response $\hat{\mathbf{v}}$. Forward and reverse voltage waves are denoted by $\mathbf{a}$ and $\mathbf{b}$, respectively. The receiver system consists of the coupled antenna elements, an impedance matching network, noisy amplifiers, and terminations, as shown in Fig. 1. The matching network can be described by a block $\mathbf{S}$-parameter matrix of the form

$$\mathbf{S}_M = \begin{bmatrix}\mathbf{S}_{11} & \mathbf{S}_{12}\\ \mathbf{S}_{21} & \mathbf{S}_{22}\end{bmatrix} \tag{1}$$

where 1 and 2 refer to input and output ports, respectively. The amplifiers have a similar $\mathbf{S}$-parameter description, with block elements denoted as $\mathbf{S}_{A,ij}$.

We assume that the LNAs are identical and that their noise contribution can be characterized using a classical two-port thermal noise model [12], wherein the mth amplifier injects forward and reverse traveling noise waves $a_{\eta,m}$ and $b_{\eta,m}$, respectively, at the amplifier input. With this model, the analysis of [4] can be used to construct the voltages across the amplifier terminations as

$$\mathbf{v}_L = \mathbf{Q}[\mathbf{G}_0\hat{\mathbf{v}} + \mathbf{\Gamma}_0\mathbf{b}_\eta - \mathbf{a}_\eta] \tag{2}$$

where

$$\mathbf{G}_0 = \frac{1}{2Z_0^{1/2}}\mathbf{S}_{21}(\mathbf{I}-\mathbf{S}_{RR}\mathbf{S}_{11})^{-1}(\mathbf{I}-\mathbf{S}_{RR}) \tag{3}$$

$$\begin{aligned}\mathbf{Q} = {}& Z_0^{1/2}(\mathbf{I}+\mathbf{\Gamma}_L)\\ &\times\left[(\mathbf{I}-\mathbf{\Gamma}_0\mathbf{S}_{A,11})\mathbf{S}_{A,21}^{-1}(\mathbf{I}-\mathbf{S}_{A,22}\mathbf{\Gamma}_L)-\mathbf{\Gamma}_0\mathbf{S}_{A,12}\mathbf{\Gamma}_L\right]^{-1}.\end{aligned} \tag{4}$$

The intent of (2) is to express the voltages at the amplifier outputs in terms of the antenna element open-circuit voltages and the network parameters of the array, matching network, and amplifiers. $\mathbf{G}_0$ relates open-circuit voltages at the antenna terminals to voltages at the amplifier inputs, $\mathbf{\Gamma}_0$ is the scattering matrix seen by the amplifier input ports looking into the matching network as shown in Fig. 1, and $\mathbf{Q}$ relates voltages at the amplifier inputs to voltages at the outputs.

If we assume that the noise generated by each amplifier is uncorrelated with that of all other amplifiers, the signal voltage, signal covariance, external noise covariance, and thermal noise covariance are given, respectively, as [4]

$$\mathbf{v}_s = \mathbf{Q}\mathbf{G}_0\hat{\mathbf{v}}_s \tag{5}$$

$$\mathbf{R}_s = \mathbf{Q}\mathbf{G}_0\hat{\mathbf{R}}_s\mathbf{G}_0^\dagger\mathbf{Q}^\dagger \tag{6}$$

$$\mathbf{R}_n = \mathbf{Q}\mathbf{G}_0\hat{\mathbf{R}}_n\mathbf{G}_0^\dagger\mathbf{Q}^\dagger \tag{7}$$

$$\mathbf{R}_{\mathrm{LNA}} = k_b B\mathbf{Q}\mathbf{R}_\eta\mathbf{Q}^\dagger \tag{8}$$

with

$$\mathbf{R}_\eta = T_\alpha\mathbf{I} + T_\beta\mathbf{\Gamma}_0\mathbf{\Gamma}_0^\dagger - T_\gamma\mathbf{\Gamma}_0 - T_\gamma^*\mathbf{\Gamma}_0^\dagger \tag{9}$$

where T_α, T_β, and T_γ are amplifier noise parameters [12] in units of Kelvin, B is the system bandwidth, and k_b is Boltzman's constant.

C. Matching Network Specification

Amplifier noise performance is determined by the scattering matrix $\mathbf{\Gamma}_0$ presented by the matching network to the amplifier input ports. Accordingly, we will parameterize the matching network in terms of $\mathbf{\Gamma}_0$. Each subblock of the matching network scattering matrix can be represented by a singular value decomposition (SVD) of the form $\mathbf{S}_{ij} = \mathbf{U}_{ij}\mathbf{\Lambda}_{ij}^{1/2}\mathbf{V}_{ij}^\dagger$. If we represent the subblocks of $\mathbf{S}_M$ in (1) using SVDs, the full matrix can be represented by eight $N\times N$ unitary matrices and $4N$ real singular values. For a matching network that is lossless and reciprocal, the scattering matrix must be unitary ($\mathbf{S}_M^\dagger\mathbf{S}_M = \mathbf{I}$) and symmetric ($\mathbf{S}_M = \mathbf{S}_M^T$, where ^{T} denotes the matrix transpose). With these constraints, it is straightforward to show that the off-diagonal blocks are determined by the diagonal blocks, and the $2N\times 2N$ degrees of freedom in $\mathbf{S}_M$ reduce to two unitary matrices $\mathbf{U}_{11}$ and $\mathbf{U}_{22}$ and N real singular values between zero and one arranged into the diagonal matrix $\mathbf{\Lambda}_{11}^{1/2}$. The subblocks of $\mathbf{S}_M$ are, then, given by

$$\begin{aligned}\mathbf{S}_{11} &= \mathbf{U}_{11}\mathbf{\Lambda}_{11}^{1/2}\mathbf{U}_{11}^T\\ \mathbf{S}_{12} &= j\mathbf{U}_{11}\mathbf{\Lambda}_{12}^{1/2}\mathbf{U}_{22}^T\\ \mathbf{S}_{21} &= j\mathbf{U}_{22}\mathbf{\Lambda}_{12}^{1/2}\mathbf{U}_{11}^T\\ \mathbf{S}_{22} &= \mathbf{U}_{22}\mathbf{\Lambda}_{11}^{1/2}\mathbf{U}_{22}^T\end{aligned} \tag{10}$$

where $\mathbf{\Lambda}_{12}^{1/2} = (\mathbf{I}-\mathbf{\Lambda}_{11})^{1/2}$.

The goal here is to solve for values of $\mathbf{\Lambda}_{11}$, $\mathbf{U}_{11}$, and $\mathbf{U}_{22}$ that achieve a specified $\mathbf{\Gamma}_0$. From the system topology in Fig. 1, we have

$$\mathbf{\Gamma}_0 = \mathbf{S}_{22} + \mathbf{S}_{21}(\mathbf{I}-\mathbf{S}_{RR}\mathbf{S}_{11})^{-1}\mathbf{S}_{RR}\mathbf{S}_{12}. \tag{11}$$

Using (10), this can be rewritten as

$$\tilde{\mathbf{\Gamma}}_0 = \mathbf{\Lambda}_{11}^{1/2} - \mathbf{\Lambda}_{12}^{1/2}\left(\mathbf{I}-\tilde{\mathbf{S}}_{RR}\mathbf{\Lambda}_{11}^{1/2}\right)^{-1}\tilde{\mathbf{S}}_{RR}\mathbf{\Lambda}_{12}^{1/2} \tag{12}$$

where

$$\tilde{\mathbf{\Gamma}}_0 = \mathbf{U}_{22}^\dagger\mathbf{\Gamma}_0\mathbf{U}_{22}^* \tag{13}$$

$$\tilde{\mathbf{S}}_{RR} = \mathbf{U}_{11}^T\mathbf{S}_{RR}\mathbf{U}_{11}. \tag{14}$$

One class of solutions arises if we choose $\mathbf{U}_{11}$ and $\mathbf{U}_{22}$ such that $\tilde{\mathbf{\Gamma}}_0$ and $\tilde{\mathbf{S}}_{RR}$ are diagonal. In this case, (12) reduces to N single-port matching problems. Each two-port matching network has three scalar degrees of freedom, of which the matching condition, being a complex equation, eliminates two, and there remains a one-parameter family of solutions for each port.

A particularly convenient member of this family of solutions results from choosing $\mathbf{U}_{11}$ and $\mathbf{U}_{22}$ such that $\tilde{\mathbf{\Gamma}}_0$ and $\tilde{\mathbf{S}}_{RR}$ are not only diagonal, but real and positive as well. If we denote the SVDs of the array and output scattering matrices as $\mathbf{\Gamma}_0 = \mathbf{U}_0\mathbf{\Lambda}_0^{1/2}\mathbf{U}_0^T$ and $\mathbf{S}_{RR} = \mathbf{U}_{RR}\mathbf{\Lambda}_{RR}^{1/2}\mathbf{U}_{RR}^T$, this is accomplished by setting $\mathbf{U}_{11} = \mathbf{U}_{RR}^*$ and $\mathbf{U}_{22} = \mathbf{U}_0$. In this case, (12) reduces to

$$\mathbf{\Lambda}_0 = \mathbf{\Lambda}_{11}^{1/2} - \mathbf{\Lambda}_{12}^{1/2}\left(\mathbf{I} - \mathbf{\Lambda}_{RR}\mathbf{\Lambda}_{11}^{1/2}\right)^{-1}\mathbf{\Lambda}_{RR}\mathbf{\Lambda}_{12}^{1/2}. \tag{15}$$

Solving for $\mathbf{\Lambda}_{11}^{1/2}$ leads to

$$\mathbf{\Lambda}_{11}^{1/2} = \left(\mathbf{\Lambda}_0^{1/2} + \mathbf{\Lambda}_{RR}^{1/2}\right)\left(\mathbf{I} + \mathbf{\Lambda}_0^{1/2}\mathbf{\Lambda}_{RR}^{1/2}\right)^{-1}. \tag{16}$$

With an explicit solution for the matching network, we can express (5)–(8) in terms of $\mathbf{\Gamma}_0$, in order to optimize noise performance over all possible choices for $\mathbf{\Gamma}_0$.

For an arbitrary $\mathbf{\Gamma}_0$, there may be other solutions for which $\tilde{\mathbf{\Gamma}}_0$ and $\tilde{\mathbf{S}}_{RR}$ are not diagonal, but these solutions are superfluous to the present analysis. As will be demonstrated later, optimizing the system noise performance requires that $\mathbf{\Gamma}_0$ is a scaled identity matrix, which by (12) and (13) implies that $\tilde{\mathbf{\Gamma}}_0$ and $\tilde{\mathbf{S}}_{RR}$ must be diagonal.

III. Optimal Matching Network

A. Classical Two-Port Noise Optimization

The amplifier noise figures are minimized when maximum cancellation of the forward and reflected reverse amplifier noise waveforms is achieved. For a single channel, the optimization procedure is a standard result in the noise theory of two-port devices [12]. As some formulas from the single-channel treatment are helpful in the multichannel extension, we review this case here.

The goal is to choose Γ_0, the reflection coefficient presented to the input of a two-port, such that the noise figure of the two-port is minimized. This requires minimization of the exchangeable noise temperature

$$T_{ee} = \frac{T_\alpha + |\Gamma_0|^2 T_\beta - T_\gamma\Gamma_0 - T_\gamma^*\Gamma_0^*}{1 - |\Gamma_0|^2}. \tag{17}$$

Taking the derivative of T_{ee} with respect to Γ_0^* leads to

$$\frac{\partial T_{ee}}{\partial \Gamma_0^*} = \frac{-T_\gamma\Gamma_0^2 + (T_\alpha + T_\beta)\Gamma_0 - T_\gamma^*}{\left(1 - |\Gamma_0|^2\right)^2}. \tag{18}$$

Forcing the derivative to vanish and solving for Γ_0 provides the optimal reflection coefficient for minimum amplifier noise figure

$$\Gamma_{\text{opt}} = \frac{T_\alpha + T_\beta \pm \sqrt{(T_\alpha + T_\beta)^2 - 4|T_\gamma|^2}}{2T_\gamma}. \tag{19}$$

The sign is chosen such that $|\Gamma_0| < 1$, corresponding to a passive matching network.

B. Multiport Noise Optimization

To characterize overall system noise performance in the multiport case, we combine the N receiver output channels using a beamformer and compute the SNR at the beamformer output. The received signals are combined using the complex beamformer weights $\mathbf{w}^\dagger = [w_1^* \; w_2^* \; \ldots \; w_N^*]$. Many algorithms for determining $\mathbf{w}$ are available, but the natural choice here is the one that maximizes the system output SNR.

The system output SNR after the beamformer is

$$\text{SNR} = \frac{\mathbf{w}^\dagger\mathbf{R}_s\mathbf{w}}{\mathbf{w}^\dagger\mathbf{R}_N\mathbf{w}} \tag{20}$$

where $\mathbf{R}_N = \mathbf{R}_n + \mathbf{R}_{\text{LNA}}$. Maximizing this expression with respect to $\mathbf{w}$ leads to the generalized eigenvalue problem [13]

$$\mathbf{R}_s\mathbf{w} = \lambda_{\max}\mathbf{R}_N\mathbf{w} \tag{21}$$

where $\lambda_{\max}$ is the generalized eigenvalue with largest magnitude. If the noise covariance matrix is nonsingular and $\mathbf{R}_s$ is of the form $\mathbf{v}_s\mathbf{v}_s^\dagger$, corresponding to a point signal source, the weights reduce to

$$\mathbf{w} = \mathbf{R}_N^{-1}\mathbf{v}_s. \tag{22}$$

From (20), the output SNR for this choice of beamformer is

$$\text{SNR}_{\max} = \mathbf{v}_s^\dagger\mathbf{R}_N^{-1}\mathbf{v}_s. \tag{23}$$

Using (5), (7), and (8), this can be written in terms of voltages measured at open-circuited array element ports as

$$\text{SNR}_{\max} = \hat{\mathbf{v}}_s^\dagger\hat{\mathbf{R}}_N^{-1}\hat{\mathbf{v}}_s \tag{24}$$

where

$$\hat{\mathbf{R}}_N = \hat{\mathbf{R}}_n + k_b B\underbrace{\mathbf{G}_0^{-1}\mathbf{R}_\eta\mathbf{G}_0^{\dagger^{-1}}}_{\hat{\mathbf{R}}_{\text{LNA}}} \tag{25}$$

is the noise correlation matrix with voltages referred to the open-circuited array. The goal now is to maximize this quantity with respect to the array-to-amplifier matching network.

Because signal correlation matrices are Hermitian and positive semidefinite, maximizing (24) implies minimizing the real, nonnegative eigenvalues of $\hat{\mathbf{R}}_N$. We will accomplish this by expanding $\hat{\mathbf{R}}_N(\mathbf{\Gamma}_0)$ with respect to a small matrix perturbation $\mathbf{X}$ around a fixed value of $\mathbf{\Gamma}_0$, and demonstrating that the linear term of this expansion vanishes at

$$\mathbf{\Gamma}_0 = \Gamma_{\text{opt}}\mathbf{I}. \tag{26}$$

The linear term of the matrix Taylor series expansion of $\hat{\mathbf{R}}_N(\mathbf{\Gamma}_0)$ is a combination of the partial derivatives of each element of the matrix $\hat{\mathbf{R}}_N$ with respect to each element of $\mathbf{\Gamma}_0$. If the linear term of the expansion vanishes for arbitrary $\mathbf{X}$ at some value of $\mathbf{\Gamma}_0$, then the derivatives vanish and that value of $\mathbf{\Gamma}_0$ represents a stationary point of $\hat{\mathbf{R}}_N$ as a function of $\mathbf{\Gamma}_0$ and, therefore, minimizes the noise contribution of the amplifiers according to (24).

It suffices to consider only the second term of (25), since the first term is independent of $\mathbf{\Gamma}_0$. From the definition of $\mathbf{G}_0$ and the matching network specification in Section II-C, it can be shown that

$$\mathbf{G}_0 = j\mathbf{U}_0(\mathbf{I}-\mathbf{\Lambda}_0)^{1/2}(\mathbf{I}-\mathbf{\Lambda}_{RR})^{-1/2}\mathbf{U}_{RR}^{\dagger}(\mathbf{I}-\mathbf{S}_{RR}). \quad (27)$$

Using this expression

$$\widehat{\mathbf{R}}_{\mathrm{LNA}} = \mathbf{A}\mathbf{U}_0^{\dagger}\mathbf{R}_{ee}\mathbf{U}_0\mathbf{A}^{\dagger} \quad (28)$$

where

$$\mathbf{A} = (\mathbf{I}-\mathbf{S}_{RR})^{-1}\mathbf{U}_{RR}(\mathbf{I}-\mathbf{\Lambda}_{RR})^{1/2} \quad (29)$$

is independent of $\mathbf{\Gamma}_0$, and

$$\mathbf{R}_{ee} = \left(\mathbf{I}-\mathbf{\Gamma}_0\mathbf{\Gamma}_0^{\dagger}\right)^{-1/2}\mathbf{R}_{\eta}\left(\mathbf{I}-\mathbf{\Gamma}_0\mathbf{\Gamma}_0^{\dagger}\right)^{-1/2}. \quad (30)$$

This expression is a matrix generalization of the exchangeable noise temperature T_{ee} in (17). In the single-port case, $(\mathbf{I}-\mathbf{\Gamma}_0\mathbf{\Gamma}_0^{\dagger})^{-1/2}(\mathbf{I}-\mathbf{\Gamma}_0\mathbf{\Gamma}_0^{\dagger})^{-1/2} = (1-|\Gamma_0|^2)^{-1}$, which gives rise to the denominator of (17), and from (9), it can be seen that $\mathbf{R}_{\eta}$ reduces to the numerator of (17).

Because $\mathbf{R}_{ee}$ has the same eigenvalues as the factor $\mathbf{U}_0^{\dagger}\mathbf{R}_{ee}\mathbf{U}_0$ in (28), we need only to minimize $\mathbf{R}_{ee}$. Let $\mathbf{\Gamma}_0 = \Gamma_{\mathrm{opt}}\mathbf{I}+\mathbf{X}$, where $\mathbf{X}$ is an arbitrary matrix. To first order in $\mathbf{X}$, we have

$$\begin{aligned}\mathbf{I}-\mathbf{\Gamma}_0\mathbf{\Gamma}_0^{\dagger} &= \mathbf{I}-(\Gamma_{\mathrm{opt}}\mathbf{I}+\mathbf{X})\left(\Gamma_{\mathrm{opt}}^{*}\mathbf{I}+\mathbf{X}^{\dagger}\right)\\ &\simeq \underbrace{\left(1-|\Gamma_{\mathrm{opt}}|^2\right)}_{T_0}\mathbf{I}-\underbrace{\left(\Gamma_{\mathrm{opt}}^{*}\mathbf{X}+\Gamma_{\mathrm{opt}}\mathbf{X}^{\dagger}\right)}_{\mathbf{T}_1}.\end{aligned}$$

Similarly

$$\mathbf{R}_{\eta} \simeq R_0\mathbf{I}+\mathbf{R}_1$$

where

$$\begin{aligned}R_0 &= T_{\alpha}+|\Gamma_{\mathrm{opt}}|^2T_{\beta}-T_{\gamma}\Gamma_{\mathrm{opt}}-T_{\gamma}^{*}\Gamma_{\mathrm{opt}}^{*}\\ \mathbf{R}_1 &= T_{\beta}\mathbf{T}_1-T_{\gamma}\mathbf{X}-T_{\gamma}^{*}\mathbf{X}^{\dagger}.\end{aligned}$$

Using these results in (30) leads to

$$\begin{aligned}\mathbf{R}_{ee} &= (T_0\mathbf{I}-\mathbf{T}_1)^{-1/2}(R_0\mathbf{I}+\mathbf{R}_1)(T_0\mathbf{I}-\mathbf{T}_1)^{-1/2}\\ &\simeq T_{ee}\mathbf{I}+\underbrace{R_0T_0^{-2}\mathbf{T}_1+T_0^{-1}\mathbf{R}_1}_{\mathbf{Y}} \quad (31)\end{aligned}$$

where $\mathbf{Y}$ is the linear term in the expansion of $\mathbf{R}_{ee}$ with respect to $\mathbf{\Gamma}_0$ at $\mathbf{\Gamma}_0 = \Gamma_{\mathrm{opt}}\mathbf{I}$.

Using (18) and its complex conjugate, together with the definitions of T_0 and R_0, the first-order term in (31) can be written as

$$\mathbf{Y} = \mathbf{X}\left.\frac{\partial T_{ee}}{\partial \Gamma_0}\right|_{\Gamma_0^*=\Gamma_{\mathrm{opt}}^*}+\mathbf{X}^{\dagger}\left.\frac{\partial T_{ee}}{\partial \Gamma_0^*}\right|_{\Gamma_0=\Gamma_{\mathrm{opt}}}. \quad (32)$$

From the definition of Γ_{opt}, both derivatives of T_{ee} vanish at $\Gamma_0 = \Gamma_{\mathrm{opt}}$, and we have $\mathbf{Y}=0$. This demonstrates that $\mathbf{R}_{ee}$ is stationary at $\mathbf{\Gamma}_0 = \Gamma_{\mathrm{opt}}\mathbf{I}$. To see that this represents an extremal point for the output SNR, consider the first-order Taylor series expansion

$$\mathrm{SNR}(\Gamma_{\mathrm{opt}}\mathbf{I}+\mathbf{X}) \simeq \mathrm{SNR}(\Gamma_{\mathrm{opt}}\mathbf{I})+\widehat{\mathbf{v}}_s^{\dagger}\mathbf{Z}\widehat{\mathbf{v}}_s \quad (33)$$

of (23) with respect to $\mathbf{\Gamma}_0$ about (26). From (28) and (31), the matrix $\mathbf{Z}$ in the linear term contains $\mathbf{Y}$ as a factor, and so the linear term must vanish for any signal $\widehat{\mathbf{v}}_s$.

Although (23) was derived for a stationary point source, the optimality result holds for other signal types as well. From (6) and (7), the signal and external noises couple through the matching network in the same way, so the matching network has no influence on the SNR without amplifier noise. Since the matching network improves SNR only by reducing noise added by the amplifiers, the optimal matching network is independent of the signal.

C. Uniqueness of the Optimal Solution

The specification (26) represents a family of possible optimal matching networks. There are internal degrees of freedom in the matching network itself, because a prescription for the output scattering matrix of the matching network does not uniquely specify the matching network, and other solutions for $\mathbf{S}_{ij}$ than that constructed in (16) exist. Moreover, for any given signal $\widehat{\mathbf{v}}_s$, there are other matching networks with $\mathbf{\Gamma}_0$ not equal to (26) which realize the optimal SNR. If one of the eigenvalues of $\mathbf{\Gamma}_0$ is equal to Γ_{opt}, for example, then the first-order term $\mathbf{Y}$ in (31) is not identically zero but is singular. In this case, the matrix $\mathbf{Z}$ in (33) is also singular. If $\widehat{\mathbf{v}}_s$ is in the null space of $\mathbf{Z}$, the first-order term in (33) vanishes, and the maximum possible SNR is obtained for that particular signal with a matching network that is nontrivially different from the optimal solution. For such suboptimal matching network, however, at least one of the eigenvalues of $\widehat{\mathbf{R}}_N$ must be larger than the eigenvalues realized with the optimal match, and hence, there must exist some other signal for which the SNR is suboptimal.

Numerically, the optimal matching network solution can be found by minimizing $\det(\widehat{\mathbf{R}}_N)$ or $\mathrm{tr}(\widehat{\mathbf{R}}_N)$ with respect to $\mathbf{\Gamma}_0$. Since these quantities are given by the product and sum, respectively, of the eigenvalues of the noise correlation matrix $\widehat{\mathbf{R}}_N$, neither will be minimized unless all of the eigenvalues are simultaneously minimized. This observation may be helpful in searching for the best suboptimal matching network over more restricted classes of network realizations, such as uncoupled matching networks with diagonal $\mathbf{S}_{ij}$. Because the optimal matching network that realizes $\mathbf{\Gamma}_0 = \Gamma_{\mathrm{opt}}\mathbf{I}$ in general is fully coupled, its realization will require N^2 connections between the array output ports and the amplifier inputs, so, in practice, it will be desirable to implement suboptimal networks.

We have modeled the matching network as lossless, but in practice, the network will have a finite loss. This will introduce additional system noise and lower the output SNR. In view of this, there is a tradeoff between complex networks which attempt to realize the optimal lossless solution and simpler suboptimal network with less loss.

D. Discussion and Physical Interpretation

Because $\mathbf{\Gamma}_0 = \Gamma_{\rm opt}\mathbf{I}$ is diagonal, the matching network that realizes this scattering matrix effectively decouples the array element ports. This provides a simple explanation of the main result of this paper. For a given amplifier, forward and reverse noise waves are partially correlated, and the output noise can be minimized by reflecting the reverse noise with proper phase such that the correlated parts of the forward and reverse noise waves cancel. The reverse noise is uncorrelated with the noise produced by all other amplifiers, however, so if reverse noise were to couple through the array back into the input ports of any other amplifier the overall available output noise increases. Optimal noise performance, therefore, can only be obtained with a matching network that uncouples the array ports. The matching network can be viewed as first decoupling the array elements to obtain a diagonal scattering matrix, after which each decoupled port is matched to the optimal single-port impedance that minimizes the LNA noise figures individually.

It is noteworthy that although mutual coupling is a near-field effect, a diagonal scattering matrix constrains the far-field radiation patterns of the array. Stein used conservation of energy to show that a diagonal scattering matrix implies that the embedded element radiation patterns are orthogonal [14]. The array and matching network together can be considered as a new N-port antenna array with scattering matrix $\mathbf{\Gamma}_0$. Each port has associated with it some effective radiation pattern formed by a linear combination of the element patterns. Because the scattering matrix is diagonal, the overlap integrals of these effective radiation patterns must vanish. This pattern decoupling is also closely related to decoupling of the signal correlation matrix for an antenna array in a strongly scattering multipath environment [15], [16].

As an example, consider the simple case of two parallel z-directed half-wave dipoles spaced one-half wavelength apart along the x-axis. In this case, the radiation pattern of one dipole with the other open-circuited is well approximated by the isolated radiation pattern, which is known in closed form, and the mutual scattering matrix can be approximated analytically [17]. If LNAs represented by a diagonal impedance matrix $\mathbf{Z}_A$ are attached directly to the output ports of the dipoles, the terminal voltages can be found in terms of open-circuit voltages using a multiport voltage divider as

$$\mathbf{v}_A = \mathbf{Z}_A(\mathbf{Z}_A + \mathbf{Z}_{RR})^{-1}\hat{\mathbf{v}} \tag{34}$$

where $\mathbf{Z}_{RR}$ is the mutual input impedance matrix of the array. Looking out of the LNA inputs, the array scattering matrix is not diagonal due to mutual coupling. The first two curves in Fig. 2 show the resulting receiving patterns. Mutual coupling perturbs the open-circuit radiation patterns so that the two dipoles have different radiation patterns, but the patterns are not orthogonal. The second pair of curves are the receiving patterns taken at the output of a matching network which decouples the array. In this case, the receiving patterns are orthogonal as must be the case

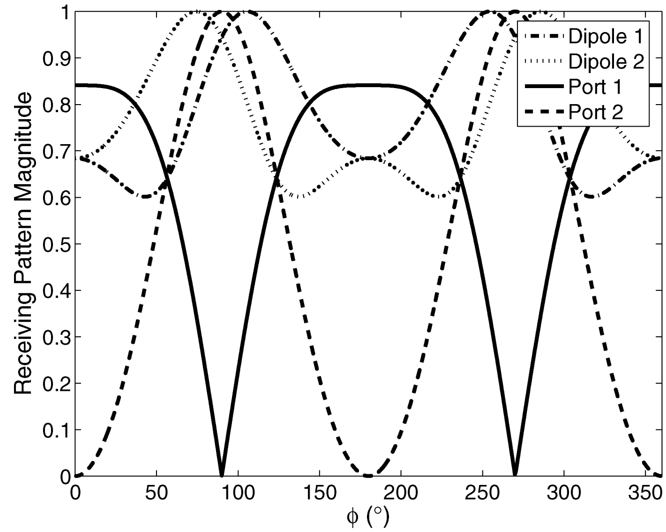

Fig. 2. Azimuthal receiving patterns for two parallel half-wave dipoles. The curves labeled Dipole 1 and Dipole 2 are the receiving patterns with a self-impedance matched load at the terminals of each dipole. The patterns are different due to mutual coupling but are not orthogonal. The second pair of curves is the receiving patterns taken at the output of a matching network that decouples the antennas, so that the receiving patterns are orthogonal.

for an uncoupled array. Optimal system noise performance can then be obtained by individually noise-matching each decoupled array port to an LNA using the classical two-port approach.

IV. Conclusion

We have generalized the classical two-port optimal noise matching condition to the multiport case. For a mutually coupled array, optimal LNA noise performance requires a matching network that uncouples the array elements, which in turn implies orthogonal effective receiving patterns at the matching network output ports.

The noise performance of optimal and suboptimal matching networks has been investigated in previous papers using simulations, but it remains for future work to build and test realizations of these matching networks and to investigate the required element values, tolerances, losses, and achievable bandwidth. Approaches to realizing the matching networks are available in the literature [15], [16], [18]. We conjecture that the optimal matching network considered in this paper provides the best possible channel capacity for an array on the receive side of a MIMO communication link.

Another outstanding problem is the determination of fundamental limits on coupling between antenna elements, especially in the case of an electrically small, closely spaced array. If the conducting structures that form the elements are shaped in such a way that mutual coupling is minimized over a given bandwidth, what is the lower bound on the amount of mutual coupling for an array of a given electrical size?

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Karl F. Warnick (S'95–M'98–SM'04) received the B.S. degree (*magna cum laude*) with University Honors and the Ph.D. degree from Brigham Young University (BYU), Provo, UT, in 1994 and 1997, respectively.

From 1998 to 2000, he was a Postdoctoral Research Associate and Visiting Assistant Professor at the Center for Computational Electromagnetics at the University of Illinois at Urbana-Champaign, Urbana. Since 2000, he has been a faculty member at the Department of Electrical and Computer Engineering at BYU, where he is currently an Associate Professor. He has coauthored a book chapter and over 85 conference presentations and scientific journal papers, as well as a book, *Problem Solving in Electromagnetics, Microwave Circuits, and Antenna Design for Communications Engineering* (Norwood, MA: Artech House, 2006). His research interests include array antenna systems, computational electromagnetics, rough surface scattering, remote sensing, and inverse scattering.

Dr. Warnick was a recipient of the National Science Foundation Graduate Research Fellowship and the Outstanding Faculty Member award for Electrical and Computer Engineering at BYU in 2005. He is a Technical Program Co-Chair for the 2007 IEEE International Symposium on Antennas and Propagation.

Michael A. Jensen (S'93–M'95–SM'01) received the B.S. (*summa cum laude*) and M.S. degrees from Brigham Young University (BYU), Provo, UT, in 1990 and 1991, respectively, and the Ph.D. degree from the University of California Los Angeles (UCLA), Los Angeles, in 1994, all in electrical engineering.

From 1989 to 1991, he was a Graduate Research Assistant in the Lasers and Optics Laboratory at BYU. From 1991 to 1994, he was a graduate student researcher in the Antenna Laboratory at UCLA. Since 1994, he has been at the Electrical and Computer Engineering Department at BYU where he is currently a Professor and Department Chair. His main research interests include antennas and propagation for personal communications, microwave circuit design, radar remote sensing, numerical electromagnetics, and optical fiber communications.

Dr. Jensen currently serves as a member of the Administrative Committee and the Joint Meetings Committee for the IEEE Antennas and Propagation Society as an Associate Editor for the IEEE TRANSACTIONS ON ANTENNAS AND PROPAGATION. He has also served the society as Vice-Chair and Technical Program Chair for several symposia. He was awarded the H. A. Wheeler paper award in the IEEE TRANSACTIONS ON ANTENNAS AND PROPAGATION in 2002, and the best student paper award at the 1994 IEEE International Symposium on Antennas and Propagation. He received a National Science Foundation Graduate Fellowship in 1990. He is a member of Eta Kappa Nu and Tau Beta Pi.