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Exploring the Solvability of the Jaynes-Cummings and Jaynes-Cummings-like Models: Implementing Quantum Control

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Abstract

In this paper we aim to explore the dynamics and overall solvability of the Jaynes-Cummings & Jaynes-Cummings-Like models. As a lens to understand these dynamics, we focused on cases where the parameters of the system were made time-dependent. All previous work on solving the dynamics of the Jaynes-Cummings models has relied heavily on the use of differential methods and setting the parameters as time-independent constants which were zero or one. To account for this, we utilized the Wei-Norman method which allowed us to analytical solve the time-dependent Hamiltonian. Through the use of this method, we can understand the more general characteristics of the dynamics associated with the Jaynes-Cummings model. This research related back to the field of Quantum information and control as the model highlights interactions with qubits and our time-dependent parameters can force certain transitions.

1 Introduction

This work will focus primarily on quantum control and how it relates back to quantum information. In our sense, quantum control involves the use of external factors in an experiment that affect the system at a quantum level to dictate how the states evolve in time. Understanding how quantum systems can evolve in time and be manipulated in experiment is interesting in both a general sense as well as an interest to the field of quantum information. In general, quantum system evolution has only recently been understood with respect to realistic systems. Furthermore, some of these realistic systems are only analyzed through special conditions. With respect to quantum information, when analyzing the evolution of two-state, or even three-state, systems these can be related back to the qubits, and qutrits, that underly the entire field. Specifically manipulating how these qubits and qutrits interact as they involve in time can allow for different degrees of entanglement which is another cornerstone of quantum information. To study these quantum states, we will an experimental models related to quantum optics. Quantum optics is an ideal field for Quantum Information because experimentation is performed in vacuum where many assumptions in models are realistic and achievable. Furthermore being related to Optics the field focuses on Light-Matter interactions where the matter is normally quantum states while the light is simple electromagnetic fields in cavities. This allows us to manipulate qubits directly and ask the following question:
To what extent can quantum control, in the form of time-dependent field frequency and field-atom interaction, be imposed in Jaynes-Cummings-like Models to produce quantum effects such as entanglement?
One can note that this question resolves how this study will model our theoretical system. By utilizing the Jaynes-Cummings Model, which focuses on one atom and one single mode field, we aim to create an equivalent two atom model where the atoms are non-directly interacting. To analyze these models and create important operators, such as the time-evolution operator, we will be utilizing the Wei Norman Method. The chosen system and method will be described in the next section. Lastly, although the Jaynes-Cummings Model has been analytically solved on numerous accounts the results found are not immediately relevant to my questions beyond the aid they provide to check my Method's results against known results.

2 Jaynes-Cummings & Wei-Norman

In this section, the Model and Method analyzed in this paper will described shortly. As well as general overview, the advantages and disadvantages will be highlighted.

2.1 Jaynes-Cummings:

The Jaynes-Cummings Model is basic model used in Quantum Optics to represent a system that is comprised of a two-state system and a single-mode field in a cavity. It is important to note that this model is an approximation of the Quantum Rabi Model, both models are described by the equation below:

$$H_{Rabi} = 1/2\hbar\omega_0\sigma_3 + \hbar\omega(N + 1/2) + \hbar\lambda(a + a^\dagger)(\sigma_+ + \sigma_-) \tag{1}$$

$$H_{JC} = (1/2)\hbar\omega_0\sigma_3 + \hbar\omega(N + 1/2) + \hbar\lambda(a\sigma_+ + a^\dagger\sigma_-) \tag{2}$$

As can bee seen in both Hamiltonian, there is a term dedicated to the particle's energy (σ_3), a term for the field's energy ($N + 1/2$), and a term for the interaction ($(a + a^\dagger)(\sigma_+ + \sigma_-)$ and $a^\dagger\sigma_- + a\sigma_+$ for Rabi and Jaynes-Cummings respectively) The approximation made, the Rotating Wave Approximation, results in the deletion of two terms: $a^\dagger\sigma_+$ and $a\sigma_-$. These two terms can be ignored because once evolved in time they vary rapidly as they correspond to the summation of the two frequencies rather than the different. Beside being a model that captures the dynamics we aim to study, the Jaynes-Cummings Model is fairly popular and thus has many different methods that produce complete analytic solutions.[GK08] Since there has yet to be any work applying the Wei-Norman method to the Jaynes-Cummings Model this allows us to check the validity of our method. As far as Quantum Control is concerned, this model has two main means of control. These controls are present in the interaction factor, λ, and the frequencies of the field and atom, $\omega\&\omega_0$, which in this simple Hamiltonian have been taken to be constants. In this work, we will attempt to add time dependence in these parameters to enforce stronger quantum control. Lastly, we will explore other Jaynes-Cummings like models enforce restrictions on allowed interactions.

2.2 Wei-Norman

The Wei-Norman Method is an algebraic technique used to study they dynamics of quantum mechanical systems. This method has been studied in great depth, so a brief overview will be given.[WN63, Say12] This is accomplished by forming an algebra from the operators present in the Hamiltonian. In quantum mechanics the Hamiltonian describes the energy of the system. Once the algebra is formed it has to be proven to close under the relations specified by Lie. In this case, the commutators of all elements in the algebra must produce elements in the algebra. If a new operator is produced by a commutator relation, it is added to the algebra and it must be commuted with every element.

Once the algebra closes and can be defined Lie Algebra, you can define a time-evolution operator. The time-evolution operator, $U(t)$ is defined below and is a unitary operator that can evolve a state to a desired time.

$$U(t) = \prod_i = 0^N e^{\alpha_i(t)A_i} \tag{3}$$

In this definition the A_i correspond to elements of your defined Lie Algebra. It is important to note that not A_i will be present in your Hamiltonian. The $\alpha_i(t)$ are time-dependent functions that will dictate how the state evolves in time. Although initially unknown, they can be solved by inputting the time-evolution operator into the Schroedinger Equation.

$$i\hbar\frac{\partial U}{\partial t}U^{-1} = H \tag{4}$$

As you can see, if you have a time-independent Hamiltonian the results are identical to evolution of energy eigenstates seen in introductory quantum mechanics. However, once you add time-dependence this becomes a series of non-linear coupled differential equations that determine the α_i. The main benefit of this method is that we can write our Hamiltonians in terms of time-independent operators, and only have to solve differential equations for the αs. Once the complex

coefficients, αs, are known, the system can be studied in time. Then the dynamics can be fully understood.

It is important to note that no literature could be found showcasing the results of the Wei-Norman method on the Jaynes-Cummings Model. Thus, the first step toward studying the creation of entanglement relies heavily on verifying the effectiveness of this method. In the next section we aim to confirm our Wei-Norman analytic results to those found in literature.

3 Verification of Wei-Norman on Jaynes-Cummings

In this section we will outline the efforts made to find analytic solutions to the Jaynes-Cummings Model utilizing the Wei-Norman method. After direct solutions have been found, each Lie algebra ordering will be used to evolve initial states and determine expectation values. These results will then be compared to analytic results determined for zero detuning in the literature.[GK08]

3.1 Closing the Algebra

When applying the Wei-Norman method, the initial algebra from the Hamiltonian is described below

$$LieAlgebra = \{N, \sigma_3, a^\dagger\sigma_-, a\sigma_+\} \tag{5}$$

After evaluating the commutators between these four elements, the commutators including N and σ_3 add no new terms to the algebra. This is due to the fact that an element in the algebra also accounts for any scalar multiple of itself. The main issue of this algebra comes from the commutator of $a\sigma_+$ and $a^\dagger\sigma_-$. The commutator's final solution is shown below.

$$[a^\dagger\sigma_-, a\sigma_+] = N\sigma_3 + 1/2(1+\sigma_3) \tag{6}$$

This commutator relation adds one new terms to the algebra, $N\sigma_3$. This new term clearly commutes with N and σ_3. However, the commutator between $N\sigma_3$ and the terms that created it ($a\sigma_+$ and $a^\dagger\sigma_-$) produce two new terms. These terms are $Na\sigma_+$ and $Na^\dagger\sigma_-$. Taking the commutator of these new terms in the manner outline below, two new terms are acquired.

$$[a^\dagger\sigma_-, Na^\dagger\sigma_-] = N^2a^\dagger\sigma_- \tag{7}$$

$$[a\sigma_+, Na\sigma_+] = N^2a\sigma_+ \tag{8}$$

It is important to recognize that scalar multiples of $a^\dagger\sigma_-$ and $a\sigma_+$ are produced in the respective cases, yet they are ignored here because they add nothing to the algebra. This produces a cyclic pattern where the n-th commutator between $a\sigma_+$ and the previous commutator results in $N^na\sigma_+$. Similar results will occur when starting with $a^\dagger\sigma_-$. In short, this constant addition of terms prevents the algebra from closing. In order to lead the algebra to closure, new operators must be defined which in turn will reshape the Hamiltonian.

The correction of the algebra is found from the first commutator between $a^\dagger\sigma_-$ and $a\sigma_+$. Its current form is the linear combination of $N\sigma_3$ and σ_3. However, when we utilize the fact that $\sigma_3^2 = 1$ we find the following relation.

$$1/2(1+\sigma_3) = 1/2(1+\sigma_3)\sigma_3 \tag{9}$$

Due to this symmetry, it is easy to see that the commutator solution can be rewritten as such:

$$[a^\dagger\sigma_-, a\sigma_+] = (N + 1/2(1+\sigma_3))\sigma_3 \tag{10}$$

This new term $N + 1/2(1+\sigma_3)$ is nothing more than the total quanta of the system which we will define below:

$$M = N + 1/2(1+\sigma_3). \tag{11}$$

It is important to note, that this new M is also a constant of motion as it commutes with every other term in the Hamiltonian. Knowing this, we rescale $a\sigma_+$ and $a^\dagger\sigma_-$ with M so they commute. To this end, we define two operators b and $b^\dagger$ below:

$$b = a\sigma_+/\sqrt{M} \tag{12}$$

$$b^\dagger = a^\dagger\sigma_-/\sqrt{M}. \tag{13}$$

The square root of M in the denominator effectively acts as a normalization for any product of two-state ket with a number ket. Furthermore, since M commutes with both $a\sigma_+$ and $a^\dagger\sigma_-$, the commutator of b and $b^\dagger$ can be simplified as follows.

$$[b, b^\dagger] = 1/M[a\sigma_+, a^\dagger\sigma_-] = (1/M)M\sigma_3 = \sigma_3 \tag{14}$$

Utilizing these new operators, the following Lie Algebra closes with the corresponding commutator relations.

$$Lie_{JC} = \{M, b^\dagger, b, \sigma_3\} \tag{15}$$

$$[M, \sigma_3] = 0 \tag{16}$$

$$[M, b] = 0 \tag{17}$$

$$[M, b^\dagger] = 0 \tag{18}$$

$$[\sigma_3, b] = 2b \tag{19}$$

$$[\sigma_3, b^\dagger] = -2b^\dagger \tag{20}$$

$$[b, b^\dagger] = \sigma_3 \tag{21}$$

Lastly the Hamiltonian must be redefined in terms of these new operators as seen below.

$$H_{JC} = 1/2\hbar\Delta\sigma_3 + \hbar\omega_N M + \hbar\lambda\sqrt{M}(b + b^\dagger) \tag{22}$$

$$\Delta = \omega_\sigma - \omega_N \tag{23}$$

It is important to note that this substitution results in the addition of new term Δ defined below as the detuning. The detuning corresponds to the difference in frequencies between the radiation field and qubit. More advanced levels of quantum control assume this term to be time-dependent. Furthermore, since M is a constant of the motion, we will be considering $\sqrt{M}$ to be a constant during calculations. In the next section we will underline the steps taken to find the characteristic coefficients α_i for the Lie algebra. Lie_{JC}

3.2 Solving for Alphas

In order to determine the coefficients $\alpha_i (i = 1{-}4)$ for the Lie Algebra above, we used a Mathematica program created by BYU graduate student Ty Beus. In this section, key steps of the Wei-Norman method will be outlined as well as simplification made to recover the results found in the Gerry & Knight text.

Taking the general structure for the time-evolution operator from section 2.2, one can find $\frac{\partial U}{\partial t}$ and U^{-1} as

$$\begin{aligned}\frac{\partial U}{\partial t} &= M\dot{\alpha_1}e^{M\alpha_1(t)}e^{b^\dagger\alpha_2(t)}e^{b\alpha_3(t)}e^{\sigma_3\alpha_4(t)} \\ &+ e^{M\alpha_1(t)}b^\dagger\dot{\alpha_2}e^{b^\dagger\alpha_2(t)}e^{b\alpha_3(t)}e^{\sigma_3\alpha_4(t)} \\ &+ e^{M\alpha_1(t)}e^{b^\dagger\alpha_2(t)}b\dot{\alpha_3}e^{b\alpha_3(t)}e^{\sigma_3\alpha_4(t)} \\ &+ e^{M\alpha_1(t)}e^{b^\dagger\alpha_2(t)}e^{b\alpha_3(t)}\sigma_3\dot{\alpha_4}e^{\sigma_3\alpha_4(t)}\end{aligned} \tag{24}$$

and

$$U(t)^{-1} = e^{-\alpha_4\sigma_3}e^{-\alpha_3 b}e^{-\alpha_2 b^\dagger}e^{-\alpha_1 M}, \tag{25}$$

where the time-evolution operator is factorized according to the Lie algebra in the expanded Schroedinger equation. In this case, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ correspond to $M, b^\dagger, b$ and σ_3 respectively. Utilizing the Baker-Campbell-Hausdorff (BCH), one can obtain the following differential equations.

$$\dot{\alpha_1} = -iw \tag{26}$$

$$\dot{\alpha_2} = i(-\lambda\sqrt{M} + \Delta\alpha_2 + \lambda\sqrt{M}\alpha_2^2) \tag{27}$$

$$\dot{\alpha_3} = -i(\lambda\sqrt{M} + (\Delta + 2\lambda\sqrt{M}\alpha_2)\alpha_3) \tag{28}$$

$$\dot{\alpha_4} = -i/2(\Delta + 2\lambda\sqrt{M}\alpha_2) \tag{29}$$

Solving these equations is rigorous and the generic results are left out this paper. However, when observing zero detuning ($\Delta = 0$) the alpha solutions simplify to the following.

$$\alpha_1(t) = -iwt \tag{30}$$

$$\alpha_2(t) = -i\tan(\lambda\sqrt{M}t) \tag{31}$$

$$\alpha_3(t) = -i\sin(\lambda\sqrt{M}t)\cos(\lambda\sqrt{M}t) \tag{32}$$

$$\alpha_4(t) = ln[-\cos\lambda\sqrt{M}t] - i\pi \tag{33}$$

These alphas and their effect on the expectation value of σ_3 and the unitary operator are outlined in the next section.

3.3 Results: Graphs of Alphas

Using the results from the previous section, the functions for the alphas have been plotted below.

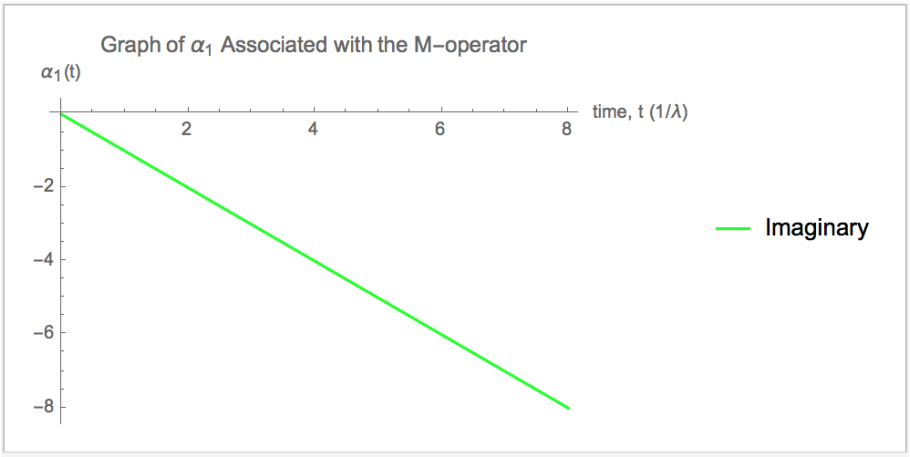

Figure 1: Coefficient α_1 is depicted as a function of time in units inversely proportional to the interaction factor.

As can be seen in Fig. 1, α_1 is a simple linear plot and purely imaginary. One can recall that α_1 is associated with the M operator which conserves the of quanta for the system. Knowing this association exists, these results show that constant of motion have no affect on the evolution of an initial state. This conclusion comes from the fact that the time-evolution operator is defined using the exponentials of the alphas multiplied by their respective operator. Thus, an exponential of a linear imaginary function is a phase typically seen in quantum mechanics. Thus when multiplying by complex conjugates, all potential effects of M will be removed. Lastly, the results obtained for this alpha are independent of detuning. In particular they are valid whether or not $\Delta = 0$ zero due to the relation of M in the time evolution operator to M in the Hamiltonian.

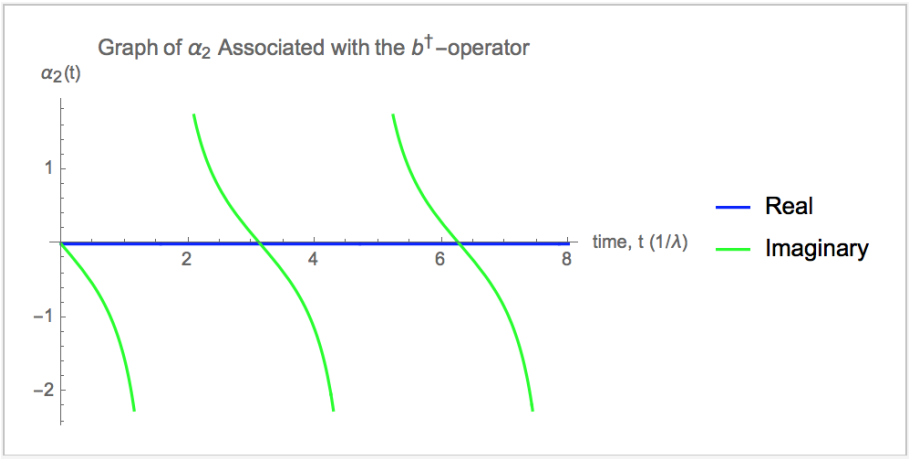

Figure 2: Coefficient α_2 as a function of time in units inversely proportional to the interaction factor. This plot is specific for zero detuning and matches the equation outlined in section 3.2.

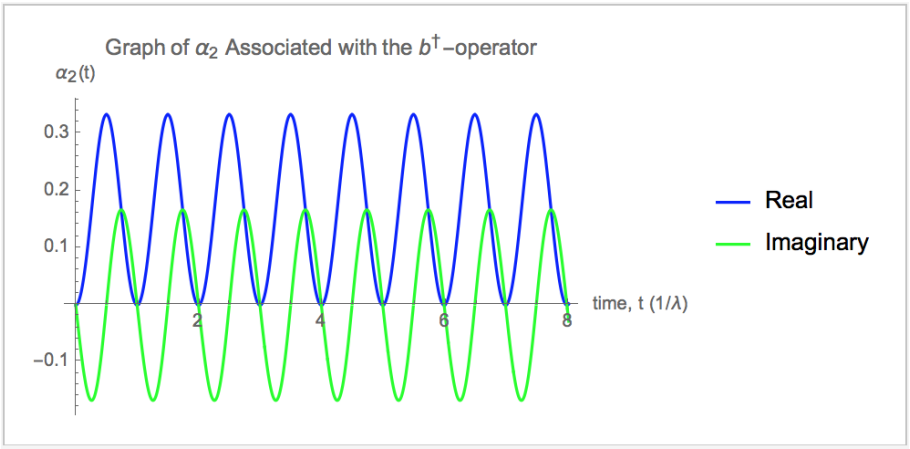

Figure 3: Coefficient α_2 as a function of time in units inversely proportional to the interaction factor. This plot is specific for a $\Delta = 5.98$.

For α_2 two plots have been given to depict the changes that occur when detuning is taken to be zero verse non-zero. For the zero detuning case (Fig. 2), it is a purely imaginary and periodic. Further, being a tangent function, the alpha has intermediate values that are positive and negative infinity. Physically, this can be interpreted as a time where this transition is highly allowed or impossible depending on the sign of the infinity. Fortunately, as will be seen in the final evolved state, the fact that α_2 increases toward infinity is compensated. In the case for a non-zero detuning, the graph changes to an imaginary cosine function and real cosine squared function. Similar to the zero-detuning case, this showcases that this alpha is periodic.

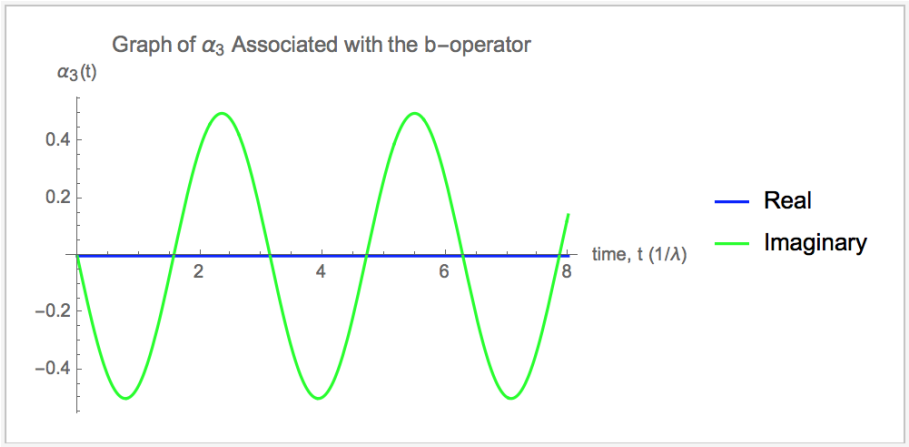

Figure 4: Coefficient α_3 as a function of time in units inversely proportional to the interaction factor. These results are for zero detuning and thus take the form of the equations in Section 3.2.

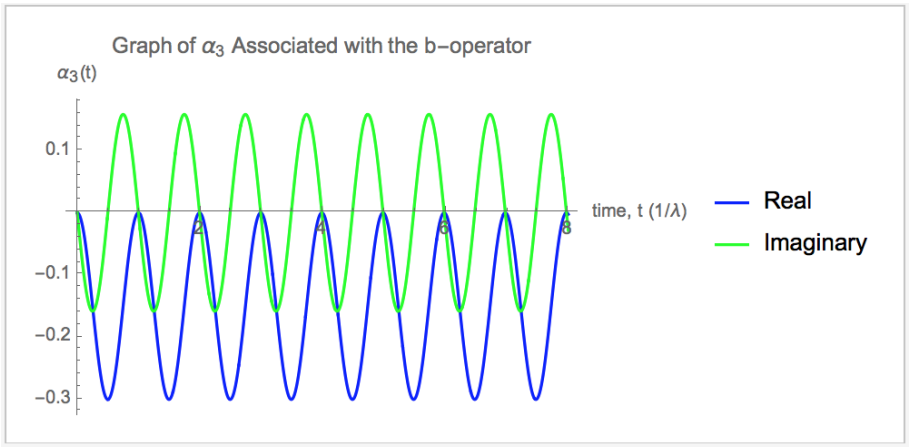

Figure 5: Coefficient α_3 as a function of time in units inversely proportional to the interaction factor. These results depict a $\Delta = 5.98$ and showcases transition from a purely imaginary function to a complex function.

As with α_2, α_3 has detuning dependence in the coefficient's most general form. Thus two plots have been made to highlight the differences. For zero-detuning the function is a purely imaginary negative cosine. However, once detuning is added it's real part is a cosine squared function and an imaginary cosine function. When compared to detuning results for α_2, the real part of the functions of alpha only differ by a sign. This can be interpreted as a correlation due to the similar effects that b and $b^\dagger$ have in the Hamiltonian. While b corresponds to photon field losing a photon and the atom becoming excited, $b^\dagger$ represented the reverse operation. This reversal of operations can be mathematical seen as the sign difference in their general solutions.Lastly, both results showcase that α_3 is periodic.

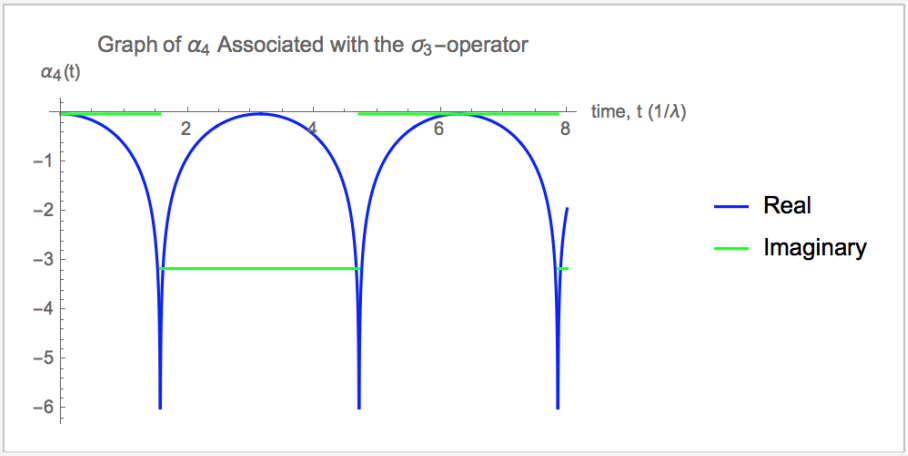

Figure 6: Coefficient α_4 as a function of time in units inversely proportional to the interaction factor. This is for the case of zero detuning and follows the results of section 3.2.

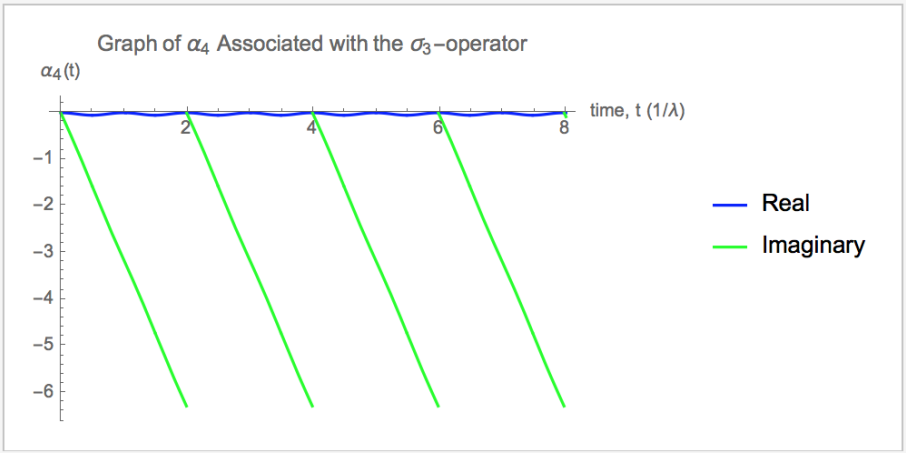

Figure 7: Coefficient α_4 as a function of time in units inversely proportional to the interaction factor. This graph depicts the results of $\Delta = 5.98$. Notice the function remains piecewise yet it is now linear as opposed to horizontal.

Lastly, α_4 is associated with σ_3 and has detuning dependence in its most general form. For zero-detuning it's imaginary part becomes piece wise, while the real part is again periodic similar to the other alphas. When detuning is large enough, the real part of the general function becomes a cosine function with a small amplitude while the imaginary part becomes a periodic linear plot similar to α_1. This last alpha showcases the periodicity that this model causes in the evolution as only two states are allowed.

3.4 Results: Expectation Values of Atomic Transitions & Photon Field

With generic, analytic alphas, we checked our results. In order to ensure that the Wei-Norman method produces correct results, we compared our solutions to well known literature. [citation] The specific result used was the expectation of σ_3 in time. With σ_3 being an operator, $< \sigma_3(t) >$ describes which state the atom is in at any given time. Utilizing numerics to describe atomic states, a value of 1 corresponds to the excited state while 0 corresponds to the ground state. By evolving a state initially excited with n-photons, using differential methods, Gerry & Knight obtained the expectation value

$$< \sigma_3(t) >= cos[2\Omega * t], \tag{34}$$

where Ω is the Rabi frequency of the system. The Rabi frequency is the frequency of oscillation between atomic levels which is defined for this system as,

$$\Omega = \lambda\sqrt{(n+1)} \tag{35}$$

Utilizing Mathematica, one can plot this for a specific n and λ. For the rest of this comparison n will be set equal to 0 and λ will be set equal to 1. This results in Fig. 8

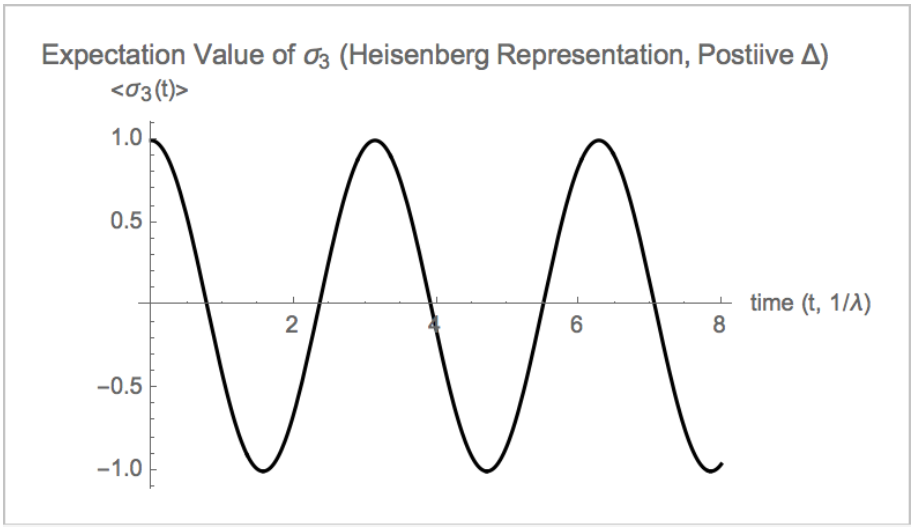

Figure 8: $<\sigma_3(t)>$ is depicted for time in units inversely proportional to the interaction factor. These results are taken directly from literature for the case of zero detuning. Notice that the frequency of this cosine graph is twice the Rabi Frequency.

As seen in the Fig. 8, the expectation value is sinusoidal in nature with a frequency equal to twice the Rabi frequency. In order to produce a plot for an expectation value we must first determine the Heisenberg representation of σ_3 for this particular Lie algebra. The analytic result is

$$\sigma_3(t) = \sigma_3(0) - 2\exp[2\alpha_4(t)]b^\dagger(0)\alpha_2(t) + 2\sigma_3(0)\alpha_2(t)\alpha_3(t) + 2\exp[-2\alpha_4(t)]b(0)\alpha_3(t)(1+2\alpha_2(t)\alpha_3(t)). \tag{36}$$

This new equation for $\sigma_3(t)$ can be sandwiched between a bra and ket for an excited state with zero photons in the field. This is equivalent to the time-dependent expectation of σ_3

$$< e, 0|\sigma_3(t)|e, 0 >= 1 + 2\alpha_2(t)\alpha_3(t) > \tag{37}$$

One can see that when evaluated for the zero-detuning alphas from the previous section $<\sigma_3(t)>$ simplifies to match the results from literature. To show this, the general results were plotted for zero-detuning against the results from literature.

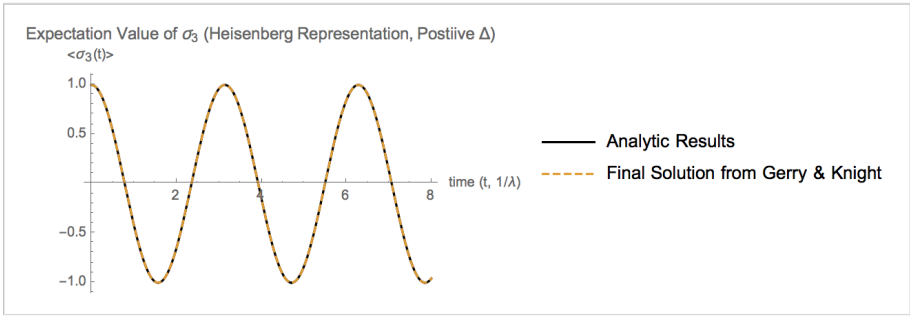

Figure 9: $<\sigma_3(t)>$ is depicted for time in units proportional to the interaction factor. In yellow the literatures results are highlighted while the analytical results are in black. The graphs directly overlap as the functions are identical.

As further verification, the expectation value of the number of photons in the system has been plotted. The method is exactly the same as for the expectation value of $< \sigma_3(t) >$, except that the graph should depict transitions that correspond to the atomic transitions.

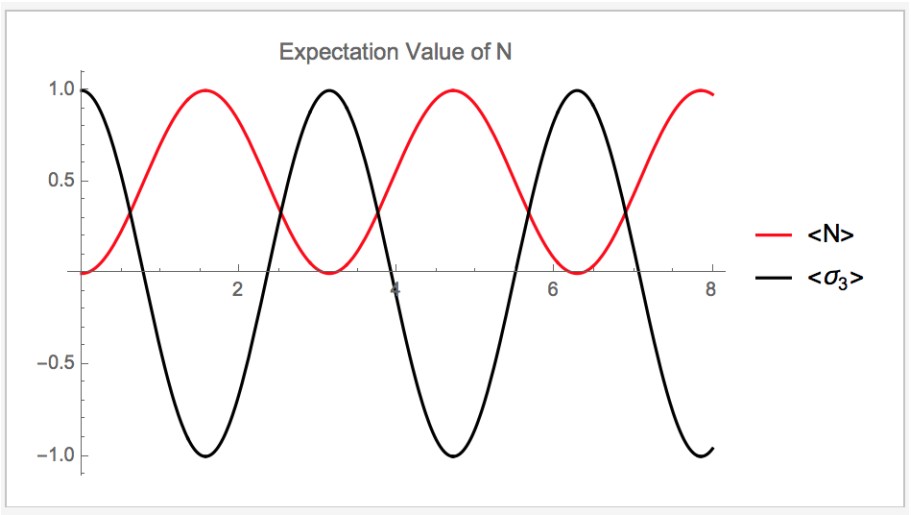

Figure 10: $<\sigma_3(t)>$ and <N(t)> as a function of time in units inversely proportional to the interaction factor. The graphs are related as the peaks and troughs are anti-correlated.

As can be seen in Fig. 10, the number of photons does oscillate between zero and one. When compared to the graphs from Gerry & Knight, the decrease in a photon corresponds to the excitation of the atom.

By comparing results with those well established in literature we were able to verify the results obtained from Wei-Norman. Although this is only shown for one Lie algebra ordering, the other orderings produce matching results and have been left out of this report. With this verification, we can move on to examine how the model is changed as detuning is applied to the system. We first do this analytically for time-independent parameters and then try to impose quantum control on the system through time-dependent parameters. For this particularly model, the changing parameters will be the frequencies of the atom and field as well as the interaction intensity factor.

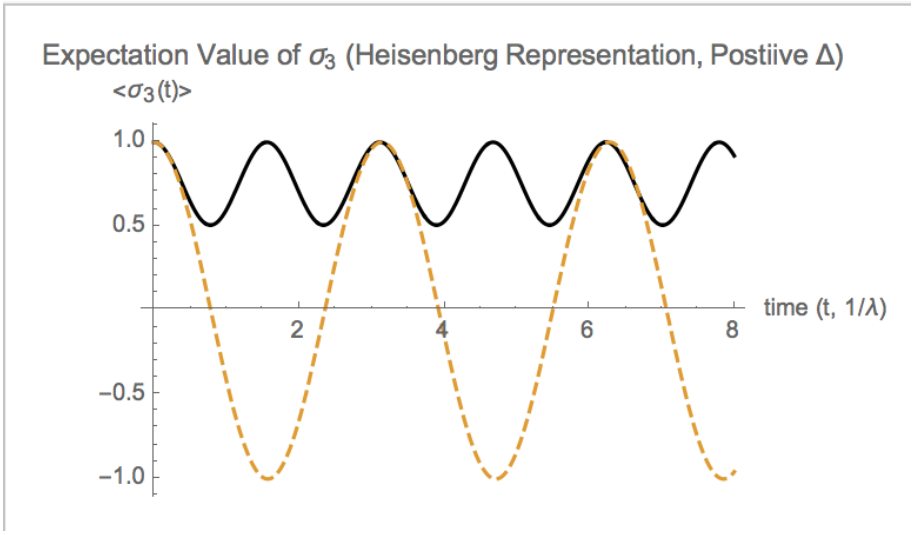

Figure 11: $<\sigma_3(t)>$ as a function of time in units inversely proportional to the interaction factor. The results of Gerry & Knight are in yellow while the positive detuning results are in black. Although the cosine function is maintained, the graph is shifted upward and the frequency increases.

In Fig. 11, one can see the original results from Gerry & Knight compared to analytic results with detuning. The two main differences are the increased frequency and the upward shift of the overall

graph. Looking back at the literature, one can find a general equation for the Rabi frequency in terms of detuning

$$\Omega(\Delta) = \sqrt{\Delta^2 + 4\lambda^2(n+1)} \tag{38}$$

This explains why the graph shows an increase in frequency. The amplitude decrease is mostly like due to the energy difference associated with the different frequencies. That difference now decreases the likelihood of atomic transition so the expectation value can no longer reach the full range. A way to understand this is to see the expectation value as a sum of the possible values multiplied by a time dependent probability of the value occurring. As the probability of being in the ground state decreases the minimum value is shifted upward; yet the general shape is maintained.

With results from this simple model understood and generalized to account for all of the expectation variables, the model can be expanded to allow for more complex interactions. In the next section we will present a model that adds a second two-state system to the Jayne-Cummings Model and then explore the behavior of the system in time through expectation values and numerical plots for time-dependent variables.

4 Jaynes-Cummings-like Models: Two Two-State Systems & Multi-Photon Transitions

4.1 Two Two-Level System Jaynes-Cummings: Determining The Dynamics

4.1.1 Equivalent Hamiltonian and New Operators

To fully analyze non-directly interacting qubit dynamics, the Jaynes-Cummings Model must be extrapolated to include a second two-state system. The way to ensure that the Hamiltonian contains no terms related to atom-atom interactions is to ignore all terms that combine σ_+ and σ_- with τ_+ or τ_-. From this point on, the τ terms correspond to operators that only act on the second atom. It is important to note that any τ operators commute with any σ operators and any operators associated with the photon field (N, a, and $a^\dagger$).

In order to avoid direct interaction terms, we followed the Jaynes-Cummings model closely and only added a new interaction term for the second atom

$$H_{JC2-atom} = (1/2)\hbar\omega_\sigma\sigma_3 + (1/2)\hbar\omega_\tau\tau_3 + \hbar\omega_N(N+1/2) + \hbar\lambda_\sigma(a\sigma_+ + a^\dagger\sigma_-) + \hbar\lambda_\tau(a\tau_+ + a^\dagger\tau_-). \tag{39}$$

Analyzing this equation in a similar fashion to the one atom case, the same Lie algebra non-closure occurs when taking the commutator between the interaction terms for each qubit separately. In order to avoid problem, a global constant of motion must be utilized. The constant of motion can then be used to remove terms that cause the algebra to expand. This model, being a sum of two individual atom models, allows us to utilize each atom's local constant of motion (C.O.M). These C.O.Ms are related to the M in the previous section but are written out below for clarity

$$M_\sigma = N + (1/2)(1+\sigma_3) \tag{40}$$

$$M_\tau = N + (1/2)(1+\tau_3). \tag{41}$$

Using the fact that both of these commute with their respective interaction terms (as well as with N, σ_3 and τ_3), we need only find a linear combination of the two and a function of N, σ_3 and τ_3 to make our global C.O.M. For this simple case the two local C.O.Ms can be added together and the function is simply minus N. This results in the following equation.

$$M_{2-atom} = N + 1/2(1+\sigma_3) + 1/2(1+\tau_3) \tag{42}$$

Identically to the one atom case, this C.O.M corresponds to the total quanta of the system. The main issue that should be immediately apparent is this new C.O.M doesn't cancel the additional terms from the interaction commutators. This can be offset by dividing by the square root of M_τ or M_σ and having M_{2-atom} occur in the Hamiltonian as an approximation of the interactions. The results for the new operators for the algebra are

$$b = \frac{a\sigma_+}{\sqrt{M_{2-atom}}} \tag{43}$$

$$b^\dagger = \frac{a^\dagger\sigma_-}{\sqrt{M_{2-atom}}} \tag{44}$$

$$c = \frac{a\tau_+}{\sqrt{M_{2-atom}}} \tag{45}$$

$$c^\dagger = \frac{a^\dagger\tau_-}{\sqrt{M_{2-atom}}}. \tag{46}$$

In this form the commutators between the same atom interaction terms now stay within the algebra. However, it should be noted that these terms do not commute between the atoms. More specifically, b doesn't commute with $c^\dagger$, nor $b^\dagger$ with c. To account for this, our model assumes that terms between different atoms immediately commute. Thus, the commutator relations and Lie Algebra of the system can be outline below.

$$LieAlgebra = \{M_{2-atom}, b^\dagger, b, c^\dagger, c, \sigma_3, \tau_3\} \tag{47}$$

$$[\sigma_3, b^\dagger] = -2b^\dagger \tag{48}$$

$$[\sigma_3, b] = 2b \tag{49}$$

$$[\tau_3, c^\dagger] = -2c^\dagger \tag{50}$$

$$[\tau_3, c] = 2c \tag{51}$$

$$[b, b^\dagger] = \sigma_3 \tag{52}$$

$$[c, c^\dagger] = \tau_3 \tag{53}$$

Now the Hamiltonian can be written in terms of these new operators similarly to what was done in the previous section.

$$H_{JC2-atom} = (1/2)\hbar\Delta_\sigma\sigma_3 + (1/2)\hbar\Delta_\tau\tau_3 + \hbar\omega_N(M_{2-atom} - 1/2) + \hbar\lambda_\sigma(b + b^\dagger) + \hbar\lambda_\tau(c + c^\dagger) \tag{54}$$

The two new terms that appear in the Hamiltonian are known from the previous section as detuning. The subscript denotes which atom the detuning describes. The definitions of these new symbols are

$$\Delta_\sigma = \omega_\sigma - \omega_N \tag{55}$$

$$\Delta_\tau = \omega_\tau - \omega_N. \tag{56}$$

Moving forward, this model is rich with terms that can lend the system to quantum control. The frequency of the photon field as well as the interaction factor strength (λ) between the atoms and field can be made time-dependent. Furthermore, through use of quantum effects, the frequency of the atom can be made time-dependent. To ensure every combination of variable types has been tested, analytic results for time-independent variables will be found followed by numerical results of time-dependent variables. In the next section the Wei-Norman method will be applied to the Hamiltonian and the results will be analyzed.

4.1.2 Solving for Alphas (Time-Independent Lambdas and Deltas

This section aims to show the process for finding the alphas with the following Lie Algebra.

$$LieAlgebra = \{M_{2-atom}, b^\dagger, c^\dagger, b, c, \sigma_3, \tau_3\} \tag{57}$$

We first used the Wei-Norman Mathematica program with the commutator relations outlined in the previous section. This produced the following list of differential equations for 7 alphas.

$$\dot{\alpha_1} = -i\omega_N \tag{58}$$

$$\dot{\alpha_2} = i(-\lambda_\sigma\sqrt{M} + \Delta_\sigma\alpha_2 + \lambda_\sigma\sqrt{M_{2-atom}}\alpha_2^2) \tag{59}$$

$$\dot{\alpha_4} = -i(\lambda_\sigma \sqrt{M_{2-atom}} + (\Delta_\sigma + 2\lambda_\sigma \sqrt{M_{2-atom}}\alpha_2)\alpha_4) \tag{60}$$

$$\dot{\alpha_6} = -i/2(\Delta_\sigma + 2\lambda_\sigma \sqrt{M_{2-atom}}\alpha_2) \tag{61}$$

$$\dot{\alpha_3} = i(-\lambda_\tau \sqrt{M_{2-atom}} + \Delta_\tau \alpha_3 + \lambda_\tau \sqrt{M_{2-atom}}\alpha_3^2) \tag{62}$$

$$\dot{\alpha_5} = -i(\lambda_\tau \sqrt{M_{2-atom}} + (\Delta_\tau + 2\lambda_\tau \sqrt{M_{2-atom}}\alpha_3)\alpha_5) \tag{63}$$

$$\dot{\alpha_7} = -i/2(\Delta_\tau + 2\lambda_\tau \sqrt{M_{2-atom}}\alpha_3) \tag{64}$$

We note that rather than one set of seven coupled equations, these results show two sets of coupled equations. For this ordering, the coupling separates the alphas associated with sigma from the tau alphas. Furthermore the coupled equations match the equations for the one atom model in all aspects besides the specific detuning. In this section, the specific results will be in terms of detuning for simplicity but the calculation will be based on the explicit form with frequencies, ωs. This is a precaution necessary for this model as each detuning is not independent of the other as through the mutual field frequency present in both equations. With these facts known, we can obtain analytic equations for the αs. Due to the symmetry in τ and σ αs, the results are identical to the one-atom case expect the τ αs have λ_τ and ω_τ instead of the corresponding σ terms.

Upon calculation of the differential equations, trigonometric functions and their inverses are used heavily. For that reason, alphas may vary based on the sign of the variables involved. In this model, the only variable that can change signs is the detuning. M_{2-atom} is always positive because the quanta can never be negative and the interaction variables are assumed positive.

In this section the alphas will not be plotted because they correspond directly to the alphas from the one atom section. The symmetry seen throughout this section will have strong effects on the expectation values that are plotted in the next section.

4.1.3 Results & Implications

Due to the fact that this system mirrors the single atom system, the derivations will be ignored and only the expectation values and their interpretations will be emphasized. The below figures depicts analytic plots of time-independent variables for $<\sigma_3(t)>$.

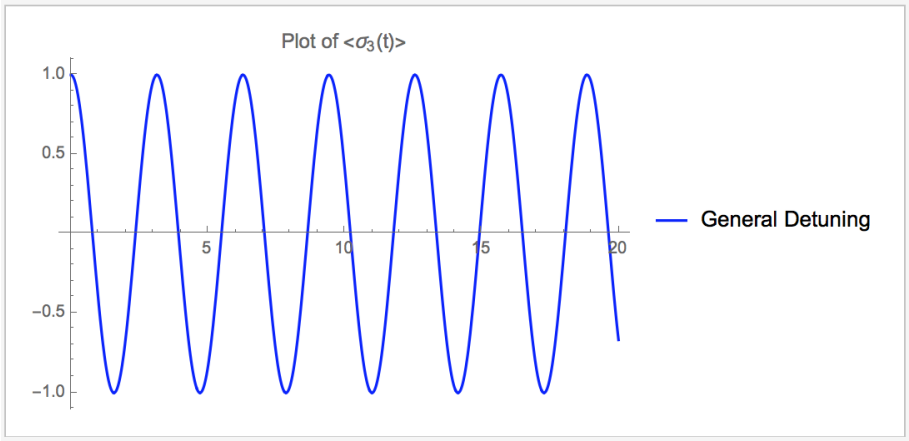

Figure 12: In this figure, the plot of $<\sigma_3(t)>$ is depicted for time in units proportional to the interaction factor. The results of this graph or specific for zero detunings for both atoms as well as all interaction factors equaling one. However, this graph remains unaffected by any changes in tau based parameter. Further, the same Jaynes-Cummings dynamics are maintained for changes in sigma parameters.

The top figure is set at zero detuning with a value of M_{2-atom} and λ_σ of 1. One can note that this results matches the results found in the one atom section. This is true down to the Rabi frequency of the system. These results are expected as the differential equations and initial conditions are identical.

The following changes were also explored: an increased Δ_τ and an increased λ_τ. The previous values for the sigma parameters were held constant. The most important find from these results is changing the factor associated with the tau atom have no effect on the expectation of sigma.

This is most likely due to the fact that the atoms are non-directly interacting and the field can be assumed sufficiently saturated with photons to allow for transitions in the one atom despite interacting with the second.

The results for just the expectation value of τ_3 are identical to the σ atom. Again, the most important result is the fact that the tau atom is also independent of the sigma atom.

For the purpose of this paper, we will move next to the sum of the two expectation values. The graph below showcase how varying detuning differently between the two atoms as well as their interaction factors may lead to quantum control.

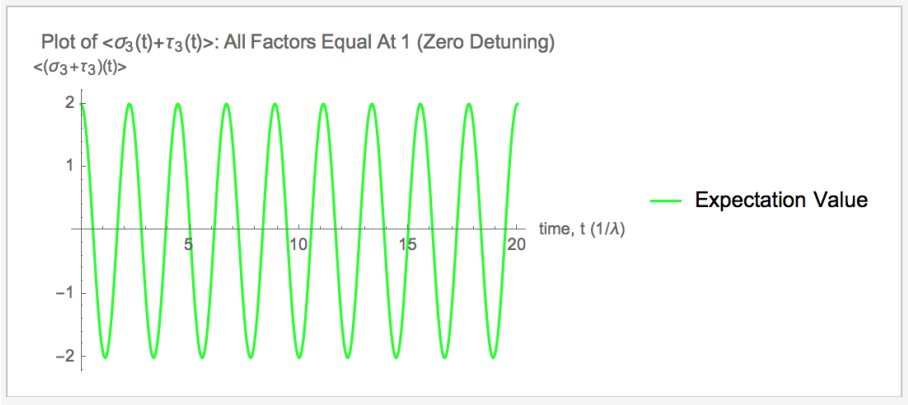

Figure 13: In this figure, the plot of $<\sigma_3(t) + \tau_3(t)>$ is depicted for time in units proportional to the interaction factors. The results of this graph are specific for zero detunings for both atoms as well as all interaction factors equaling one. You can see this graph looks identical to the results from the one atom model.The main difference is the bounds. Two corresponds to both excited and -2 to both decayed.

The first graph is the trivial case of zero detuning and comparable interaction factors. In this case the graph oscillates with the communal Rabi frequency and now goes between plus and minus two. For clarification, positive two is a double excited state and minus two is a double ground state.

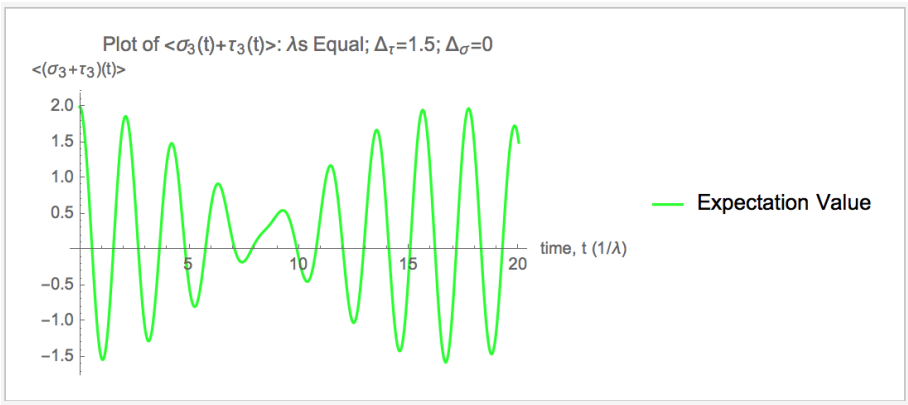

Figure 14: In this figure, the plot of $<\sigma_3(t) + \tau_3(t)>$ is depicted for time in units proportional to the interaction factor. The results of this graph or specific for one non-zero detuning for the tau atom as well as all interaction factors equaling one. As one can see, the creation of pulses can be seen at this detuning value.

The next graph showcases comparable interaction factors as well as detuning in one atom only. For low detuning, the production of pulses can be seen. This allows is simply a consequence of the simple oscillatory functions that make up each expectation value. However, relating back to quantum control, these results suggest that the mismatch can be varied such that the expectation may collapse toward a value of zero.

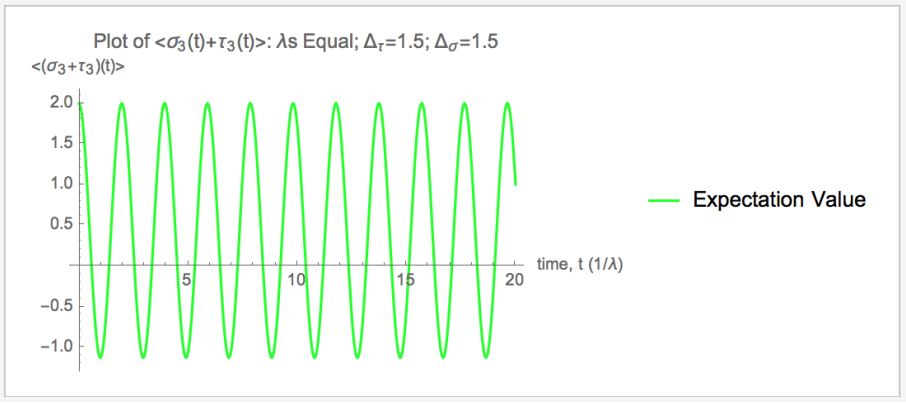

Figure 15: In this figure, the plot of $<\sigma_3(t) + \tau_3(t)>$ is depicted for time in units proportional to the interaction factor. The results of this graph or specific for non-zero detunings for both atoms that are equal as well as all interaction factors equaling one. Note that the graph looks identical to the double zero detuning except the the lower bound is raised to correspond to the decreased probability of transition.

The next graph, which has both atoms at comparable detuning, looks similar to the zero detuning case. However, the lower bound of the graph is shifted up which corresponds to the decrease in likelihood of transitions in both atoms.

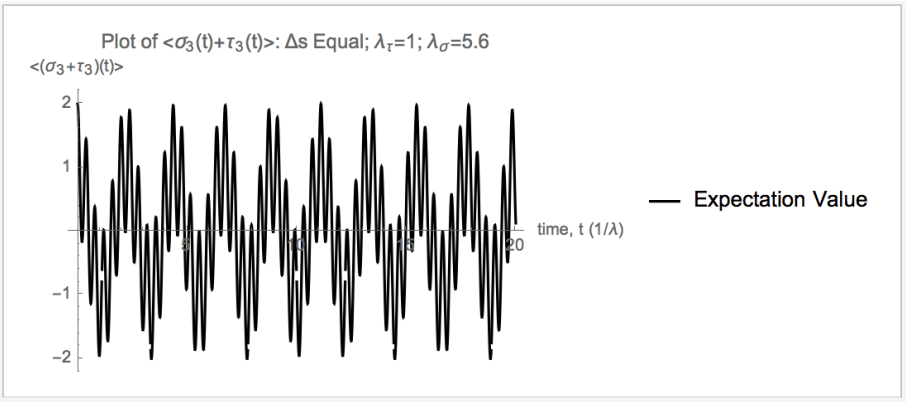

Figure 16: In this figure, the plot of $<\sigma_3(t) + \tau_3(t)>$ is depicted for time in units proportional to the interaction factor. The results of this graph or specific for zero detunings for both atoms as well as one interaction factor greater than one. In the one-atom model, an increase in the factor corresponds to an increase in oscillation frequency. seeing as this graph is a sum of two, you see this a oscillation with two frequencies.

The following graph depicts zero detuning with an increase in one atom's interaction factor. In the individual expectation values, this increase corresponds to a frequency increase. Thus, the sum of these will result in a an oscillating oscillation. This particular plot focuses on an increased factor for sigma, however, similar results occur for tau so it is not depicted.

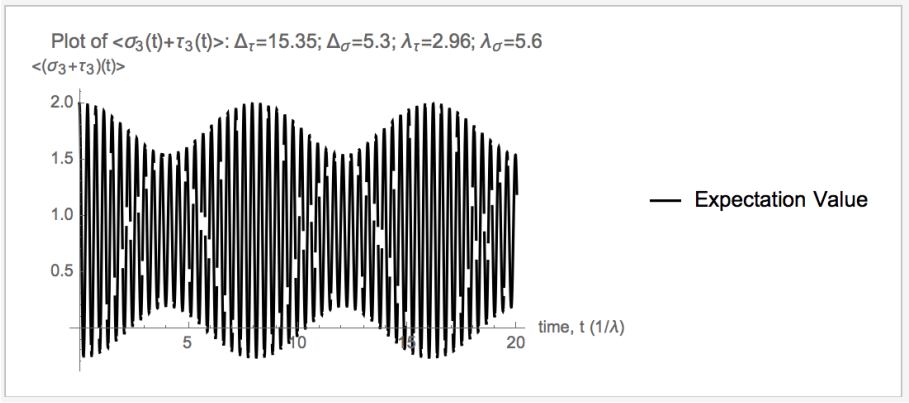

Figure 17: In this figure, the plot of $<\sigma_3(t)+\tau_3(t)>$ is depicted for time in units proportional to the interaction factor. This graph has unequal detunings such that the greater detuning corresponds to a lower interaction factor in that atom. The results are pulses shifted greatly up for the lowest value of -2. Unlike the previous pulses, these aren't as well defined because the non-one interaction factors balance the detuning to an extent.

Lastly, the most interesting case occurs when a mismatch occurs such that the detuning of one atom is large than the other and the larger detuning corresponds to a smaller interaction factor. In this graph, the sigma atom has the smaller detuning and larger interaction factor. The results become defined periodic pulses. These pulses do not oscillate but a fixed around a central value which can never be zero. This is due to the fact that both detuning being non-zero raise the overall graph and prevent symmetry around the center. Overall, these results speak very little to the idea of quantum control. This is due to the fact that both expectations are oscillatory in all cases. In short, this prevents the possibility of controlling the system toward staying in a particular state without time-dependence. Thus, the only "quantum control" would be the new information of the exact time to wait before the system has evolved to the desired results.

4.2 Two Photon Transition With Two Atoms

After getting physically expected results from the previous Jaynes-Cummings-like model, we next decided to affect how the qubits interact with the photon field. In this next model, we explore how the dynamics will change when the qubits interact with two photons at a time.

4.2.1 Definition an Equivalent JC & Closing the Algebra

We chose this next physical change because this corresponds to a^2 and $(a^\dagger)^2$ terms in the Hamiltonian. When the photon terms appear squared, this might result in effects related to non-linear optics. In this case,

$$H_{2\gamma} = (1/2)\hbar\omega_\sigma\sigma_3+(1/2)\hbar\omega_\tau\tau_3+\hbar\omega_N(N+1/2)+\hbar\lambda_\sigma(a^2)\sigma_+ + a^{\dagger 2}\sigma_-)+\hbar\lambda_\tau(a^2)\tau_+ + a^{\dagger 2}\tau_-). \quad (65)$$

It can be noted that the only difference compared to the 2-atom 1-photon model is substitution of the interaction terms. However, similar to the two atom with one photon interaction model, the commutator between the interaction terms prevent the algebra from closing. This can be further supported by the expanded version of the sigma interaction commutator below.

$$[a^2\sigma_+, a^{\dagger 2}\sigma_-] = (N^2+N)\sigma_3 + (2N+1)(1+\sigma_3) \quad (66)$$

Similar to the expansions seen in the 1-atom Jaynes-Cummings section, the $N^2\sigma_3$ and $N\sigma_3$ terms being added to the Lie Algebra will produce an infinite number of terms. We used the relation between $(1+\sigma_3)$, σ_3, and σ_3^2 to rewrite the following commutator relation in terms of an operator M_σ

$$M_\sigma = (N+\sigma_3)^2 + (N+\sigma_3) \quad (67)$$

$$[a^2\sigma_+, a^{\dagger 2}\sigma_-] = M_\sigma \sigma_3. \tag{68}$$

It is important to note that this new M_σ is a C.O.M. for the system as it commutes with all terms in the Hamiltonian. More specifically, it corresponds to the quanta of the σ-photon system squared plus itself. Thus, we can define a new b and $b^\dagger$ for the σ system using M_σ. Similarly, an M_τ can be defined for the τ interaction terms where the σ_3 terms are replaced by τ_3. Using M_τ, we can redefine the τ-photon interaction terms as new c and $c^\dagger$. These new operators are outlined below

$$b = \frac{a^2\sigma_+}{M_\sigma} \tag{69}$$

$$b^\dagger = \frac{a^{\dagger 2}\sigma_-}{M_\sigma} \tag{70}$$

$$M_\tau = (N + \tau_3)^2 + (N + \tau_3) \tag{71}$$

$$c = \frac{a^2\tau_+}{M_\tau} \tag{72}$$

$$c^\dagger = \frac{a^{\dagger 2}\tau_-}{M_\tau}. \tag{73}$$

It is important to note, that in this model we will be making the assumption the commutator between interaction term operators (b,c,$b^\dagger$,$c^\dagger$) of different atoms equal 0. Similar to the the 2-atom Jaynes-Cummings, we will need to use the local C.O.M.s (M_σ and M_τ) to create a global C.O.M. because in terms of the local C.O.M.s the Lie algebra will not close.

In order to accomplish this task, we started with a general structure for the global C.O.M. defined as $M_{2\gamma}$

$$M_{2\gamma} = M_\sigma + M_\tau + f(N, \sigma_3, \tau_3) \tag{74}$$

Where $f(N, \sigma_3, \tau_3)$ is a function of N, σ_3, and τ_3 used to conserve the quanta of the system. The reason we choose a function of these terms is these terms correspond to measuring the quanta of the system which should be conserved. Now the question becomes one of determining the function f. First, we defined f to cancel mutual terms between M_σ and M_τ. This will cancel out a N shared between the two. Lastly, through utilizing the the commutator relations between the interaction terms and the global $M_{2\gamma}$ we find

$$f(N, \sigma_3, \tau_3) = \sigma_3\tau_3 - N. \tag{75}$$

In this case the global C.O.M. can be simplified to clearly see the conserved quanta

$$M_{2\gamma} = (N + \sigma_3 + \tau_3 + 1)(N + \sigma_3 + \tau_3) - 2. \tag{76}$$

Now the global C.O.M. mirrors the form of the local C.O.M.s. This shows $M_{2\gamma}$ is the quanta of the system squared plus the quanta of the system with a scaling factor. With all operators for the system defined, we can define the 2-γ transition Lie algebra below.

$$LieAlgebra = \{\sqrt{M_{2\gamma}}, b^\dagger, c^\dagger, b, c, \sigma_3, \tau_3\} \tag{77}$$

In terms of the Lie algebra of the system, the effective Hamiltonian becomes

$$H_{2\gamma} = 1/2\hbar\Delta_\sigma\sigma_3 + 1/2\hbar\Delta_\tau\tau_3 + \hbar\omega_N\sqrt{M_{2\gamma} + 1/4} + \hbar\lambda_\sigma(b + b^\dagger) + \hbar\lambda_\tau(c + c^\dagger). \tag{78}$$

One can notice the $M_{2\gamma}$ which contains the non-linear effects of N^2 only appears underneath a square root. Thus the effect of M is similar to a N, negating any non-linear affects.

Furthermore, this Hamiltonian is interesting because it's commutator relations between operators mirror the results of the 2 atoms with 1 photon transitions. The main difference is seen in the new definitions for the Δ's which again are the effective detunings for the atoms with the photon field

$$\Delta_\sigma = \omega_\sigma - 2\omega_N \tag{79}$$

$$\Delta_\tau = \omega_\tau - 2\omega_N. \tag{80}$$

Compared to the 1-γ transition case, these new detunings are different from the 1-γ transition. This further showcases the similarities to the previous section's model. Knowing that these results mirror the previous sections, we will not repeat derivations of coefficients α_i and turn to the implications associated with this similarity.

4.2.2 Results and Implications

The results associated with this particular model mirror the results for the previous model. For this reason, we will avoid adding plots and focus more on the implications.

In this model, both atoms interacting with two-photons at a time is just equivalent to interacting with a photon at twice the frequency. This can be seen if you replace $2\omega_N$ with a new variable ω_{2N}. For this new variable, the field is twice as energetic and the Hamiltonian is now identical to the one-photon transition case. What this model shows is the fact that we must attempt to remove the symmetry of the interactions between the atoms and the field if we hope to affect the non-direct interactions between the atoms. Thus, in the next section we shall see what happens when one atom has a 1-γ transition and the other a 2-γ transition.

4.3 Mixed Photon Transitions Between Two Atoms

In a last attempt to impose quantum control in a Jaynes-Cummings-like system, we focused on an asymmetric case. In the case of two atoms where one has a 1-γ interaction and the other has a 2-γ interaction, we wanted to see whether this asymmetric photon requirement would result in new effects between the two atoms.

4.3.1 Defining the Hamiltonian & Closing the Algebra

The first step toward determining the dynamics of this new system is to define the Hamiltonian. To describe these field-atom interactions, one atom will have interaction operators with a and $a^\dagger$ while the other depends on a^2 and $a^{\dagger 2}$. Thus, the Hamiltonian is defined below

$$H_{mix} = (1/2)\hbar\omega_\sigma\sigma_3 + (1/2)\hbar\omega_\tau\tau_3 + \hbar\omega_N(N+1/2) + \hbar\lambda_\sigma(a^2\sigma_+ + a^{\dagger 2}\sigma_-) + \hbar\lambda_\tau(a\tau_+ + a^\dagger\tau_-). \tag{81}$$

The necessity of a C.O.M$>$ for the Hamiltonian is obvious. Even utilizing the assumption that the different atom interaction terms commute, we now have both commutator problems outlined previously. Specifically the commutator relation between the sigma interaction terms produces infinite terms. The same can be shown for the tau interaction terms. Seeing as this issue has occurred with most Jaynes-Cummings-like models, we will lay out a generic way to close Lie Algebras for these multi-atom models:

1. Determine the local constants of motion associated with the individual atom-field interactions.
2. Sum the local constants of motion and subtract off any terms mutual to both, such that it only appears once.
3. Add a generic $f(N, \sigma_3, \tau_3)$ which is defined as a sum of all combinations of N, σ_3 and τ_3. (Note: The highest order of N should be the same as the highest order associated with a and $a^\dagger$)
4. Next, apply the new global C.O.M to similar states for each atom (eg $a^2\sigma_+$ and $a\tau_+$)
5. Use the relations between the terms associated with the sum of the local C.O.Ms to make a system of linear equations for the coefficients associated with the operators in f.
6. Solve and verify the new C.O.M for the remaining interaction terms. Due to the symmetry of σ_3, τ_3, and N, this always commutes.

Applying this generic method to our varied photon-transitions Hamiltonian we can find a function, f, that created a C.O.M. that we will call M_{mixed}.

$$f(N, \sigma_3, \tau_3) = c_2 N + c_2\sigma_3 + \frac{c_2}{2}\tau_3 + \sigma_3\tau_3 + N\tau_3 \tag{82}$$

$$M_{mixed} = N^2 + 2N\sigma_3 + N\tau_3 + \sigma_3\tau_3 + (1+c_2)N + (1+c_2)\sigma_3 + \frac{(1+c_2)}{2}\tau_3 + 3/2 \tag{83}$$

These solutions are dependent on a scalar factor c_2. when writing our generic f to create our C.O.M, c_2 corresponded to the σ_3 operator. In this particular model, the conditions set by the commutator relations between $f(N, \sigma_3, \tau_3)$ dictate that the non-linear interaction term must commute with f while the linear term must account for extra terms. Thus, the τ interaction term commuted with the σ_3 so it's value had less restrictions than other terms. We set c_2 equal to zero.

With this new C.O.M., we define b's and c's as we normally do where they will use their local C.O.M.s. In this model, b's will correspond to σ_3 and c's will correspond to τ_3.

$$b = \frac{a^2\sigma_+}{\sqrt{M_\sigma}} \tag{84}$$

$$b^\dagger = \frac{a^{\dagger 2}\sigma_-}{\sqrt{M_\sigma}} \tag{85}$$

$$c = \frac{a\tau_+}{\sqrt{M_\tau}} \tag{86}$$

$$c^\dagger = \frac{a^\dagger\tau_-}{\sqrt{M_\tau}} \tag{87}$$

As with the previous model, we will assume that the commutator relations between b and c go to zero because the atoms aren't interacting. Lastly, we define our model such that M_{mixed} is comparable to the M_σ and M_τ. This allows us to write our final Hamiltonian in terms of M_{mixed} alone. It is important to note that these operators maintain the same relations as the other models.

$$H_{mixed} = (1/2)\hbar\Delta_\sigma\sigma_3 + (1/2)\hbar\Delta_\tau\tau_3 + \hbar\omega_N\sqrt{M_{mixed}} + \hbar\lambda_\sigma\sqrt{M_{mixed}}(b+b^\dagger) + \hbar\lambda_\tau\sqrt{M_{mixed}}(c+c^\dagger) \tag{88}$$

The difference between this and the other two Jaynes-Cummings-like Hamiltonians lies in the definition of the detuning. These definitions also showcase the fact that this model destroys any nonlinear effect for a generic state.

$$\Delta_\sigma = \omega_\sigma - 2\omega_N \tag{89}$$

$$\delta_\tau = \omega_\tau - \omega_N \tag{90}$$

With these definitions of the detunings, you can see the model is equivalent to one atom interacting with a photon that is twice the frequency of the other's photon. In the next section, we will talk about the general results as plots for this particular model are difficult to obtain with Mathematica.

4.3.2 Results & Implications

In this model we see that despite the fact that the atoms interact with a different number of photons, as long as the field is assumed sufficiently excited no interesting physics will occur. In this case, sufficiently excited refers to a field that has enough photons to create all possible states. All the alphas derived are oscillatory because they are functions of hyperbolic sines and cosines. Thus, the corresponding expectation values of atomic transitions are periodic and lend themselves to limited quantum control. Furthermore, the Jaynes-Cummings model is an approximation of the Quantum Rabi. This leads to some of the full physical description being lost. When we close the Lie algebra, our Hamiltonians in terms of a global C.O.M is a Jaynes-Cummings-like model approximation.

However, as mentioned previously, this originally expectation values assume a sufficiently excited photon field. In this model especially, there are implications that choosing states that correspond to insufficiently excited fields can lead to more interesting physics. Furthermore if the parameters such as frequencies or interaction factors were varied in non-periodic ways, more interesting physics can be obtained. In the next section we explore the effects that time-dependent factors can have on the Jaynes-Cummings Model. We specifically look at the original Jaynes-Cummings Models to begin to understand they time-dependent dynamics.

4.4 Time-Dependence In the 1-Atom 1-Photon Transition

The analytic Mathematica solver used to originally solved the differential alpha equations from the Wei-Norman method is not robust enough to tackle multiple time-dependent function. To counter this, we utilize numerical differential solvers for selective time ranges. In this study we utilized time dependence in the atom and field's frequencies (ω_σ and ω_N) as well as the interaction factor (λ). As a time-dependent Jaynes-Cummings Model is relatively unexplored we took a systematic approach to understanding the dynamics. We decided to plot expectation values of σ_3. This was a natural choice because the expectation value shows one how the state oscillates between it's allowed values. In short, if we can get this plot to get trapped in a specific value, we can verify that a degree of quantum control can be imposed.

We first by decided which time-dependence to use in our parameters vary. These included:

1. Constant
2. Linear t
3. Quadratic t^2
4. Periodic (Sine and Cosine)
5. Exponential e^t
6. Logarithmic. $ln[t]$

To decide how the function should be varied, we initially varied one parameters at a time. Then we varied two parameters at time while holding one at a constant. After primary plots had been made, we saw that the effects of changing one frequency while the other remained constant were constant. Thus, we decided to only focus on changes in the field frequency to obtain more results.

Below are specific numeric results that showcase a state transitioning from excited to the ground state and being completely trapped in the ground state after a certain time. This corresponds to the following parameter functions.

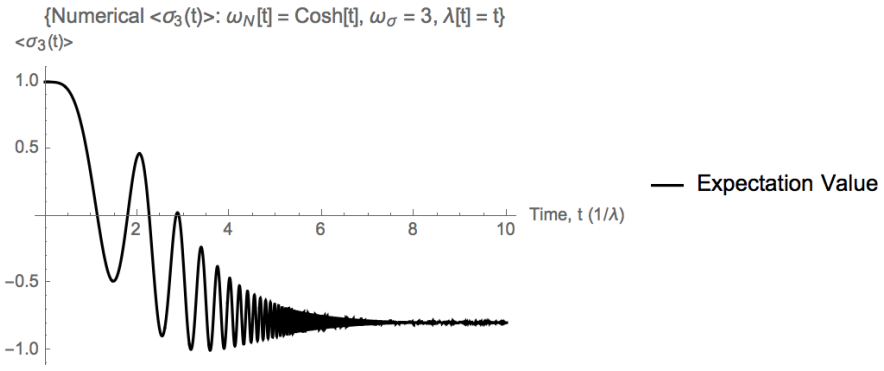

Figure 18: $<\sigma_3(t)>$ as a function time in units inversely proportional to the interaction factor. The results of this graph are specific for time-dependence in one frequency and the interaction factor. The graph collapse to a value close to -1. This showcases control over the transition of the atom.

$$\omega_\sigma[t] = 3 \tag{91}$$

$$\omega_N[t] = Cosh[t] \tag{92}$$

$$\lambda_\sigma[t] = t \tag{93}$$

One can note that Fig. 18 gets stuck at a value just above -1 with only slight oscillations. When examining the functions that made this graph you can see that the frequency of the field increases more rapidly then the interaction factor. So physically, the rate of interaction cannot make up for the mismatch in frequencies. Thus, after the time of transition, it as if the field and atom are no longer interaction.

In Fig. 19 below, we can see a graph where the value is stuck extremely close to zero. Although this value doesn't correspond to an pure state (excited or ground), it does correspond to a superposition. In this case the atom and field are maximally entangled. For this case, the functions are showcased below:

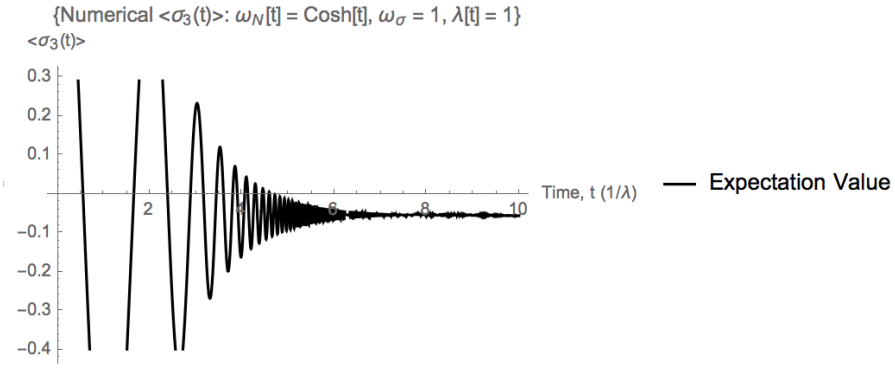

Figure 19: $<\sigma_3(t)>$ as a function time in units inversely proportional to the interaction factor. The results of this graph are specific for time-dependence in the field frequency alone. The graph collapse to a value close to $-.01$. This showcases control over the transition of the atom from excited to an almost completely entangled superposition.

$$\omega_\sigma[t] = 1 \tag{94}$$

$$\omega_N[t] = Cosh[t] \tag{95}$$

$$\lambda_\sigma = 1 \tag{96}$$

Fig. 19 again shows a situation where the detuning of the system becomes sufficiently greater than the interaction factor after a certain time. The time it takes for the system to reach this condition happens to correspond to when the system becomes maximally entangled.

From these two promising results we first see that quantum control can be enforced for systems once the parameters are allowed to vary in time. However, the mechanics behind this quantum control are rather simple:

1. The interaction factor tends to zero (not shown)

2. The frequency difference becomes comparable to the interaction factor.

In the first case, once the interaction factor tends to zero it is apparent that no interactions can take place and the state is frozen. In the second case, the interactions are again stopped. However, now it is due to the fact that the energy mismatch prevents transition due to the specific quanta needed. Unfortunately, in both cases, these results should be expected as they correspond to the two logical cases that would lead to a state being confined to a particular value. Thus, these graphs serve as verification of expected behavior. They are interesting in the fact that these chosen functions can be altered by scalars to affect the time of transition trapping or the specific value they get trapped in.

The next logical step, now that the dynamics of Jaynes-Cummings are understood, would be to apply time-dependence to a two-atom system. However, plotting the expectation values associated with those atomic transitions become harder to read as you can not distinguish between different ordering of states (eg $|e, g>$ and $|g, e>$) when plotting the sum. Thus we must introduce the notion of inter-particle purity to determine how entangled we can get a state in time.

4.5 Inter-particle Purity of Two-Atom One-Photon Transition

The term "Inter-Particle Purity" was coined in the 1980s as a method to quantize the entanglement of a system.[HS08] When first introduced, it was for system described in position representation and not abstract kets. Thus, the first thing to be done was to translate the integral based equation to a summation. From that point, we can then solve the equation analytically in terms of generic coefficients of our evolved two-state ket. With the Inter-Particle Purity being defined as "K(t)" the general equation is defined for a generic two spin state $|\Psi(t)>$.

$$K(t) = 1 - 2|C_1(t)C_4(t) - C_2(t)C_3(t)|^2, \tag{97}$$

$$|\Psi(t)>= C_1(t)|e,e,n> +C_2(t)|e,g,n+1> +C_3(t)|g,e,n+1> +C_4(t)|g,g,n+2> . \tag{98}$$

It is important to note that $K(t)$, as originally defined for wave functions, is bounded between 1 and 0. A value of 1 corresponds to no entanglement. A value of zero corresponds to a maximally entangled infinite system. This comes from its definition being defined for continuous variables. In our discrete two two-state system, the value of maximal entanglement corresponds to $K(t) = 1/2$.

To ensure all cases have been thoroughly checked, we started with three different states listed below. We then evolved them with our unitary operator for the following Lie Algebra and calculated $K(t)$.

$$LieAlgebra = \{M_{2-atom}, b^\dagger, c^\dagger, b, c, \sigma_3, \tau_3\} \tag{99}$$

1. Single State: $|e,e,n>$
2. Separable State: $\frac{1}{\sqrt{2}}(|g,e,n+1> +|g,g,n+2>)$
3. Entangled State: $\frac{1}{\sqrt{2}}(|g,e,n+1> +|e.g,n+1>)$

The mathematics of the result have been left out of the paper but the results bode poorly for the notion of creation of entanglement. For the two non-entangled initial states (single and separable), $K(t) = 1$. These results were found completely analytically so changing parameters should have no effect on this relation. This means that a state that starts non-entangled will remain non-entangled for all time. Similar results are found for the case of state starting entangled. After simplification, one finds $K(t) = 1/2$ for completely generic coefficients of the evolved state. This corresponds to entangled states maintaining entanglement for all time. The underlying reason for this is the fact that $C_1(t)C_4(t) - C_2(t)C_3(t)$ is either 0 (non-entangled) or $\frac{1}{\sqrt{2}}$ (entangled). We found this was due to the inherent symmetry that came from having a highly excited photon field.

When that photon-field is assumed, all states are allowed and the dependence on the coefficients runs the risk of canceling itself out. To counter this, we next went to a case of a non-entangled state that has a photon-field with an $M = 2$. This specific ket is defined as $\frac{1}{\sqrt{2}}(|g,e,0> +|g,g,1>)$. As you can note, the field is exhausted (set to 0) before a double excited state can be formed. Thus, when the state is evolved there will be only three coefficients, and $K(t) = 1$ will only occur when the final evolved state is in a pure state.

When analytically calculating this inter-particle purity, numerical errors occurred and this is currently being investigated. What we would expect is an oscillatory inter-particle purity. It would range and start at 1 and go as low was 1/2. This minimal value would correspond to $\frac{1}{\sqrt{2}}(|e,g> +|g,e>)|0>$, which is actually a Bell State.

5 Conclusion

Although the ten weeks has gone by rather fast, it is apparent that a lot of analytic work has been accomplished for the Jaynes-Cummings Model. We first verified that the Wei-Norman Method could be applied to this particular Quantum Optics Model. It was this fact that allowed us to later add time-dependence in the parameters.

After solving the original Jaynes-Cummings Model and recovering results for literature, we were able to solve several variations of the model. These included the addition of a second atom, changes in the interaction operators and a combination of both. In every case, we saw that

the dynamics between the atoms and field remained similar but for different parameter values. Despite this fact, we were able to determine a general system for solving any Jaynes-Cummings like model. This general method relies heavily on the dynamics and C.O.Ms found for the one and two photon transition models. We also found that true non-linearity is more complex than a change in interaction terms.

Once we had the models analytically solved, we set out to determine how quantum control could be imposed on the system. The first attempt revolved around varying the parameters in time. These results served a confirmation of the two ways to stop interactions between the field and photon. The first being a large detuning and the second being a low interaction factor. Despite being simplistic, full quantum control was achieved through the creation of superpositions and state transition in the original Jaynes-Cummings Model. Lastly, we moved away from checking the specific transitions of states and saw how we could impose quantum control to create entanglement. The general results were rather unexpected, but showcase the limitations of the model as far as the symmetry in the system with a sufficient photon field. Once that field is insufficiently excited for full transitions, results should reveal that entanglement can be created in time. Overall, these results showcase that quantum control can be imposed in Jaynes-Cummings like models for specific cases.

With regards to future work, one could explore numerical solutions for more complex time-dependent functions. Also, implementing time-dependence in a two-atom model could prove valuable as the detuning between the atoms can now vary in different ways. With regards to the further Jaynes-Cummings-like model, if a field is described by a N^2 in the Hamiltonian, this may result in stronger non-linear effects. Lastly, in our Models, we approximate the local C.O.M.s for the calculated global C.O.M. If the models were explored with their local C.O.M.s more interesting physics could be uncovered at the price of more rigerous analytics.

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