Thermally Developing Electro-Osmotic Convection in Circular Microchannels

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Brigham Young University - Provo
THERMALLY DEVELOPING ELECTRO-OsmOTIC CONVECTION
IN CIRCULAR MICROCHANNELS

by

Spencer L. Broderick

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GRADUATE COMMITTEE APPROVAL

of a thesis submitted by

Spencer L. Broderick

This thesis has been read by each member of the following graduate committee and by majority vote has been found to be satisfactory.

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Broderick in its final form and have found that (1) its format, citations, and bibli-
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requirements; (2) its illustrative materials including figures, tables, and charts are in 
place; and (3) the final manuscript is satisfactory to the graduate committee and is ready 
for submission to the university library.

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ABSTRACT

THERMALLY DEVELOPING ELECTRO-OSMOTIC CONVECTION
IN CIRCULAR MICROCHANNELS

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Masters of Science

Thermally developing, electro-osmotically generated flow has been analyzed for a circular microtube under imposed constant wall temperature (CWT) and constant wall heat flux (CHF) boundary conditions. Established by a voltage potential gradient along the length of the microtube, the hydrodynamics of such a flow dictate either a slug flow velocity profile (under conditions of large tube radius-to-Debye length ratio, $a/\lambda_d$) or a family of electro-osmotic flow (EOF) velocity profiles that depend on $a/\lambda_d$. The imposed voltage gradient results in Joule heating in the fluid with an associated volumetric source of energy. For this scenario coupled with a slug flow velocity profile, the analytical solution for the fluid temperature development has been determined for both thermal boundary conditions. The local Nusselt number for the CHF boundary condition is shown to reduce to the classical slug flow thermal development for imposed constant wall heat flux, and is independent of Joule heating source magnitude. For the CWT boundary cond-
ition, a local minimum in the streamwise variation in local Nusselt number for moderate positive dimensionless inlet temperature is predicted. For negative dimensionless inlet temperature, which arises if the fluid entrance temperature is below the tube wall temperature, the fluid is initially heated, then cooled, resulting in a singularity in the local Nusselt number at the axial location of the heating/cooling transition. The thermal development length is considerably larger than for traditional pressure-driven flow heat transfer, and is a function of the magnitudes of Peclet number and dimensionless inlet temperature.

For the EOF velocity profile scenario, numerical techniques were used to predict the fluid temperature development for both wall boundary conditions by utilizing a finite control volume approach. In addition to Joule heating as an energy source, viscous dissipation is also considered. The results predict that for decreasing $a/\lambda_d$, the local Nusselt number decreases for all axial positions and the thermal development shortens for both wall boundary conditions. Viscous dissipation has significant effect only at intermediate values of $a/\lambda_d$. Results predict local Nusselt numbers to increase for a CWT boundary condition and to decrease for an imposed constant wall heat flux with increasing viscous dissipation.
ACKNOWLEDGEMENTS

My educational goals, beginning as far back as high school, have always consisted of attending graduate school to earn a Masters Degree. I had long ago decided that a simple Bachelors Degree wasn’t all that I was made out to achieve in life. From an early age I have been keenly aware through my early academic success that I was capable of high academic achievement.

Almost exactly three years ago, when I was a senior in the Mechanical Engineering undergraduate program, my goal began to become a reality. I was personally and graciously invited by Dr. Brent Webb of the BYU Mechanical Engineering Department to pursue a graduate degree there through working on a particular thesis project that he had in mind. To be perfectly honest, the first time he described the project to me, I must have looked just as perplexed as many of my family, friends, and acquaintances do when I attempt to describe my project to them. I had absolutely no idea what electro-osmotic flow was, much less knew that it had ever existed. Yet my keen interest in heat transfer and its applications quickly led me to delving fully into the project and electro-osmotic flow quickly became a familiar and exciting concept to me. The project was fascinating and utilized many of my strengths to contribute fully to the research project.

I am forever grateful for the exciting opportunity I have had to work on this thesis project with the very capable support of Dr. Webb and Dr. Daniel Maynes. They always
provided sound and articulate advice to help me to produce excellent results. They were also very helpful and patient in giving me reviews of my thesis drafts that helped me to produce a sound and well-written thesis. They are true examples of excellence in all they do. I will forever be blessed by their examples and by the valuable lessons I learned from them through our association. I attribute any excellence within the content of this thesis to their able and freely-given support and guidance.

I would also like thank my parents for their valuable support, prayers, guidance, and advice. I never would have gotten this far without their patient and loving care that has been consistent and unconditional throughout my life.
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<th>Definition</th>
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<tr>
<td>2-D</td>
<td>Two-Dimensional</td>
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<tr>
<td>EDL</td>
<td>Electric Double Layer</td>
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<tr>
<td>EOF</td>
<td>Electro-Osmotic Flow</td>
</tr>
<tr>
<td>CHF</td>
<td>Constant Heat Flux</td>
</tr>
<tr>
<td>CWT</td>
<td>Constant Wall Temperature</td>
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<tr>
<td>TDMA</td>
<td>Tri-Diagonal Matrix Algorithm</td>
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**Roman Symbols**

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<td>m</td>
<td>capillary radius</td>
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<td>a_E</td>
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<td>a_N</td>
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<td>north neighbor coefficient of Equation (4.6)</td>
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<td>$I_i$</td>
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<td>$i^{th}$-order modified Bessel function of the first kind</td>
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<tr>
<td>$J$</td>
<td>——</td>
<td>variable representing sum of advection and diffusion fluxes</td>
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<td>dimensionless cylindrical coordinate in axial direction, $x/a \cdot Pe$</td>
</tr>
<tr>
<td>Symbol</td>
<td>Unit(s)</td>
<td>Definition</td>
</tr>
<tr>
<td>--------</td>
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</tr>
<tr>
<td>$X_e$</td>
<td>——</td>
<td>axial position at east control surface of control volume</td>
</tr>
<tr>
<td>$X_{fd}$</td>
<td>——</td>
<td>dimensionless thermal entrance length</td>
</tr>
<tr>
<td>$X_{max}$</td>
<td>——</td>
<td>maximum length of tube domain in the $X$ direction</td>
</tr>
<tr>
<td>$X_P$</td>
<td>——</td>
<td>center node axial position</td>
</tr>
<tr>
<td>$X_w$</td>
<td>——</td>
<td>axial position at west control surface of control volume</td>
</tr>
<tr>
<td>$w$</td>
<td>m</td>
<td>coordinate in direction perpendicular to tube wall</td>
</tr>
<tr>
<td>$z$</td>
<td>——</td>
<td>counterion charge number</td>
</tr>
<tr>
<td>$Z$</td>
<td>——</td>
<td>dimensionless radius-to-Debye length ratio, $a/\lambda_d$</td>
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### Greek Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Unit(s)</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>m²/s</td>
<td>thermal diffusivity</td>
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<tr>
<td>$\delta X_e$</td>
<td>——</td>
<td>axial distance between center and east nodes</td>
</tr>
<tr>
<td>$\delta X_w$</td>
<td>——</td>
<td>axial distance between west and center nodes</td>
</tr>
<tr>
<td>$\Delta Nu_{max}$</td>
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<td>maximum change in $Nu_e$ between two iterations for all $X$ locations</td>
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<tr>
<td>$\Delta Nu_e$</td>
<td>——</td>
<td>average change in $Nu_e$ between a test grid size and very fine grid</td>
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<tr>
<td>$\Delta R$</td>
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<td>radial width of control volume</td>
</tr>
<tr>
<td>$\Delta X$</td>
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<td>axial length of control volume</td>
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<td>dielectric constant</td>
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<td>C/N·m</td>
<td>fluid permittivity</td>
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<td>$\phi$</td>
<td>V</td>
<td>wall-normal electric potential</td>
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<tr>
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<td>$\Phi$</td>
<td>V</td>
<td>applied axial electric potential</td>
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<tr>
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<tr>
<td>$\lambda$</td>
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<td>dimensionless west node temperature</td>
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<tr>
<td>Symbol</td>
<td>Unit</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
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<tr>
<td>$\rho$</td>
<td>kg/m$^3$</td>
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<td>$\Omega \cdot m$</td>
<td>electrical resistivity</td>
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<tr>
<td>$\zeta$</td>
<td>V</td>
<td>zeta potential</td>
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1 - INTRODUCTION

1.1 INTRODUCTION TO ELECTRO-OSMOTIC FLOW

Fluid flow at the microscale (1–100 $\mu$m) has recently emerged as an area of
importance in many applications. Such applications include microscale cooling systems,
micropower generation, chemical separation processes, micropumping systems, etc.
Typically, such applications utilize standard pressure-induced flows. However, due to the
scales involved, significant pressures to drive the flow are generally required. Micro-
fluidic pumps are difficult to manufacture and maintain [1] and lack the precise control
that is often needed in microfluidic applications [2]. Electro-osmotic flow (EOF) may be
a viable alternative to pressure-induced flows due to its unique characteristics, precise
flow control, and lack of moving parts [1-4]. EOF has primarily seen use within the realm
of chemical separation processes, including their associated applications, and the reme-
diation of soils, but many other uses may be applicable [5]. One potential use is in
integrated microscale cooling systems [6].

As a result of these existing and possible applications, a need has arisen for a
fundamental study of the EOF convective heat transfer characteristics. Before such a
study can be performed, however, one must first understand the hydrodynamic charac-
teristics of EOF.

Electro-osmotic flow, first observed and reported by Reuss nearly two centuries
ago [7], can be induced in a channel of arbitrary cross-section filled with an ionic solution
by applying an external electric field. When such a solution is brought into contact with the surface of a soluble material, the material’s elements ionize into the solution, leaving either a positive or negative charge on the material surface depending upon the nature of the ion. For instance, in the case of fused-silica tubes, silanol groups ionize into the water, leaving a net negative charge on the tube surface. This small potential differential causes the formation of a region adjacent to the tube surface called the Electric Double Layer (EDL), as shown in Figure 1.1. The EDL is formed when positive ions, or cations, attach themselves to the negatively charged surface due to Coulomb forces, creating a single layer of immovable ions, called the Stern Layer [8]. However, the net charge of the cations is not sufficient to completely neutralize the charge on the material surface. As additional cations are attracted towards the tube surface, any negative ions, or anions, existent in the aqueous solution migrate to the positively charged immobile layer due to thermal motion and Coulomb forces and crowd out these ‘loose’ cations. This forms a partially mobile, primarily positive layer of anions and cations adjacent to the immobile layer called the Gouy-Chapman layer. The Stern and Gouy-Chapman layers together are what constitute the EDL [5,8].

Figure 1-1: Schematic illustration of Electric Double Layer (EDL) for a fused-silica tube.
The thickness of the EDL is often characterized by the Debye length, $\lambda_d$, which is the distance over which the wall-normal electric potential decreases to $\exp(-1)$ (37%) of the wall surface potential. This value is derived from the electric potential distribution obtained by solving the Poisson equation, which relates the spatial variation in electric field to the charge distribution. Utilizing the Boltzmann distribution to represent the ion concentration profile, which subsequently defines the charge density distribution in the fluid, the Poisson equation takes the form

$$\frac{d^2 \phi}{dw^2} = \frac{2zFc}{\varepsilon_p} \sinh \left( \frac{zF\phi}{R_uT} \right)$$

(1.1)

where the equation parameters are defined as follows: $\phi$—local electric potential, $w$—coordinate direction perpendicular to the tube wall, $z$—counterion (ion opposite in charge to the tube surface charge) charge number, $F$—Faraday's constant, $c$—average molar counterion concentration, $\varepsilon_p$—permittivity of the fluid medium, and $R_uT$—product of the universal gas constant and fluid temperature [5]. For small potentials, the Debye-Hückel linearization approximation assumes $zF\phi$ is much less than $R_uT$ in Equation (1.1), such that $\sinh(zF\phi/R_uT) \approx zF\phi/R_uT$. This approximation is used solely for mathematical purposes to allow Equation (1.1) to simplify to a form easily solvable by closed-form methods. The solution for the electric potential distribution then is $\phi = \phi_w \exp(-w/\lambda_d)$, where $\phi_w$ is the wall electric potential (the maximum for all radial locations). From Equation (1.1), the Debye length is approximated as $\lambda_d = \sqrt{\varepsilon_p R_uT / 2F^2z^2c}$, with typical values on the order of 1 nm to 1 $\mu$m [5]. For a circular tube of radius $a$ ($w = r$), the excess charge in the EDL decays exponentially to zero in the core fluid when $a/\lambda_d > 10$. For smaller $a/\lambda_d$
ratios, the EDL’s on opposite sides of the tube overlap each other, eliminating this core fluid area and creating an EDL potential distribution that varies with $a/\lambda_d$ [9].

The mobile layer of the EDL maintains a certain magnitude of electric potential depending upon the concentration of the solution and the solubility of the tube material. The zeta potential, or the electric potential at the interface of the two EDL semi-layers, is used to represent this electric potential and is defined as $\zeta = 4\pi\lambda_d e/\varepsilon$, where $e$ is the total excess charge in the solution per unit area and $\varepsilon$ is the dielectric constant of the fluid medium [8]. Note that $\zeta$ is only dependent upon fluid and tube material properties and independent of the tube radius. The use of the Debye-Hückel approximation, as defined in the previous paragraph, restricts $\zeta$ to be less than $3k_bT$ for simplified EOF studies ($k_b =$ Boltzmann’s constant), which is on the order of 10 to 26 mV [5].

The charged characteristics of the EDL described above are what allow electro-osmotic flow to occur. Even under the influence of an axial potential gradient, the positively charged cations and solvent molecules strongly adsorbed at the wall remain stationary. However, the positively charged and mobile annulus of the Gouy-Chapman layer is affected and begins to migrate towards the negative electrode. Due to viscosity effects inherent in the movement of the large cations relative to the electrons in the solution, this annulus of solution drags the core bulk solution with it, creating a uniform velocity profile (slug flow) in microtubes where $\lambda_d$ is very small ($a/\lambda_d > 500$) [8,9]. Typical velocities range from 0.1 to 20 mm/s for axial potential gradients on the order of 30 to 200 kV/m [10].

The theoretical fluid velocity distribution for all $a/\lambda_d$ ratios where $a/\lambda_d < 500$ is obtained by solving the general momentum equation with constant fluid properties. For
laminar, hydrodynamically fully-developed flow, the fluid velocity is one-dimensional (in the tube axial direction). Its profile is given in non-dimensional form for a circular tube as

\[ \frac{u}{u_{\text{max}}} = 1 - \frac{I_0(ZR)}{I_0(Z)} \]  

(1.2)

where \( R = r/a \), \( Z = a/\lambda_d \), and \( I_0 \) is the zero-order modified Bessel function of the first kind [9,11]. The variable \( u_{\text{max}} \) represents the maximum possible EOF velocity for a given applied axial potential. It is defined by the Helmholtz-Smoluchowski equation as \( u_{\text{max}} = (\varepsilon_p \zeta / \mu) d\Phi/dx \), where \( \mu \) is the fluid kinematic viscosity and \( d\Phi/dx \) is the applied axial potential gradient [5]. Figure 1-2 illustrates the velocity distribution given by Equation (1.2) for several \( Z \) ratios. As is evident in the figure, the velocity profile is almost completely uniform for \( Z > 500 \). For \( Z < 500 \) the velocity distribution deviates from a uniform flow profile since a significant portion of the tube diameter is occupied by the EDL. The

Figure 1-2: Non-dimensional EOF velocity profiles for \( Z = 1, 3, 10, 50, 100, \) and 500.
profile deviation becomes more pronounced as $Z$ decreases from 500 and the profile becomes nearly parabolic for small $Z (Z \cong 1)$ [9].

In addition to the unique velocity profiles described above, which are exhibited for all electro-osmotically driven flows, another factor that significantly affects EOF is the energy generated by the combined effects of Joule heating and viscous dissipation. For electro-osmotically driven flow in a channel, the total electrically-generated energy is equal to the power externally added to the system in the form of an applied voltage gradient and the corresponding induced electrical current. The total electrical current drawn in electro-osmotic flow consists of two components, the so-called conduction and convection currents. Conductive current occurs through the movement of electrons, as occurs in any standard electrical circuit. Convective current arises from the transfer of charge through the movement of ions from one electrode to the other. Only the conductive current causes Joule heating, which may be confidently modeled using Ohm’s law [12]. Although there is a net charge distribution in the near-wall region, the Joule heating is distributed uniformly across the tube cross-section for the case of low zeta potential or large $a/\lambda_d$ ratio [13]. Such energy generation will be represented per unit volume as $i_e^2 \sigma$, where $i_e$ is the conduction current density (amps/m$^2$) produced by the applied potential and $\sigma$ is the liquid electrical resistivity ($\Omega \cdot $m) of the aqueous solution. These two parameters are assumed constant under all EOF conditions studied in the present research.

The convection current also represents addition of energy to the flow, but volumetric energy generation due to this current is not described by Ohm’s law. It instead is characterized by viscous heating, which exists in the region of the channel cross-
section where \( \partial u / \partial r \neq 0 \). In the region close to the tube wall, the net excess charge of the EDL is acted on by the applied potential field, generating a force that moves ion particles through a distance, thus representing work done on the fluid. This work represents a “flow work” in the same manner pressure-driven flow exerts such work. In classical pressure-driven flow through a constant diameter horizontal duct, the difference in magnitude of the flow work between two points in the duct is proportional to the head loss, or the lost potential to do work due to viscous dissipation (i.e., the viscous forces absorbing energy to result in a pressure drop and ultimately a rise in fluid temperature). Likewise, the lost potential to do work (head loss) due to viscous dissipation in electro-osmotic flow is proportional to the difference in magnitude of the flow work between two points in the tube. Like for pressure-driven flow, it can also have a significant influence on the EOF flow behavior.

The sum of the energy generation due to Joule heating (caused by the conduction current) and the energy generation due to viscous dissipation (caused by the convection current) is the total volumetric heating that occurs. It is their combined effect that results in behavior quite different from the classical thermal entry problems. The volumetric generation \( s \) (W/m\(^3\)) at a specific radial location is thus equal to

\[
s = i_e^2 \sigma + \mu \left( \frac{\partial u}{\partial r} \right)^2
\]  

(1.3)

where \( \partial u / \partial r \) is the radial gradient of the streamwise velocity at that location.

For electro-osmotic flows with low zeta potentials, the conduction and convection currents and the radially dependent velocity distributions for arbitrary \( a/\lambda_d \) ratios are easily computed, as has been done previously [9]. Thus, the relative contributions to the
volumetric source from viscous dissipation and Joule heating can be determined. In the limiting case of large capillary radius-to-Debye length ratio \((a/\lambda_d > 500)\), the velocity across the tube cross-section is essentially uniform \((u = u_{\text{max}})\) and all of the viscous dissipation occurs in the very thin region near the tube wall where the velocity vanishes from its core magnitude to the no-slip wall condition. For capillaries of internal diameter greater than 100 \(\mu\)m an analysis reveals that the total energy added to the fluid due to Joule heating is at least two orders of magnitude larger than the total energy added due to viscous dissipation. Therefore, for all large \(a/\lambda_d\) ratios \((a/\lambda_d > 500)\) viscous dissipation can be neglected and will thus only be considered in the case where \(a/\lambda_d < 500\).

### 1.2 Motivation for and Definition of Problem

Due to possible significant applications that would utilize electro-osmotic flow, it is critical to first understand its theoretical heat transfer fundamentals. Previous studies have shown that higher heat transfer coefficients are exhibited in EOF compared to standard pressure-driven flows [11], yet very little has been done to completely characterize the heat transfer behavior for the range of relevant parameters. By providing a study of the thermally developing characteristics of EOF, the results of this research will serve as the basis for future studies, both theoretical and experimental and of possible industrial and commercial applications.

Thus, the aim of this thesis research is to analytically study the thermally developing heat transfer characteristics of electro-osmotic flow. Under the assumption of hydrodynamically fully-developed conditions, thermal transport is uncoupled from the
fluid momentum equation for constant fluid properties and is characterized by the steady flow differential energy equation

\[(\bar{u} \cdot \nabla)T - \alpha \nabla^2 T = \frac{i_e \sigma}{\rho C_P} + \frac{\mu}{\rho C_P} \nabla^2 \bar{u}\]  

(1.4)

where \(\bar{u}\) is the fully-developed velocity vector of the fluid moving through the tube (m/s) and \(T\) is the fluid temperature (K). The fluid properties \(\alpha\) (thermal diffusivity), \(\rho\) (mass density), \(C_P\) (specific heat), \(i_e\), \(\sigma\), and \(\mu\) are assumed to be constant. The terms on the left hand side of Equation (1.4) represent the heat advection and diffusion occurring in the fluid, respectively. The right hand side terms represent the spatially uniform volumetric energy generation arising from Joule heating and the generation of energy due to viscous dissipation. The temperature distribution and associated convective heat transfer coefficients for any EOF scenario will be obtained by solving Equation (1.4) using the associated assumptions and boundary conditions. An analytical solution will be obtained for the specific uniform velocity scenario defined by \(a/\lambda_d \to \infty\). Numerical solutions will be obtained for non-uniform EOF velocity flows given by finite \(a/\lambda_d\) [Equation (1.2)].
2 - LITERATURE REVIEW

2.1 FULLY-DEVELOPED ELECTRO-Osmotic Convection

Previous to the present study, research on electro-osmotic flow—it’s uses as well as its flow characteristics—have focused primarily on DNA extraction, ion separation, and other chemical analysis in improving chemical separation techniques. Relatively little prior work has appeared in the literature characterizing the convective heat transfer in electro-osmotically generated flows. Li et al. explored electrokinetic effects on the frictional and heat transfer characteristics of pressure-driven flow for round and rectangular microchannels [14-16]. The induced electrokinetic potential was found to result in a reduced flow rate, greater friction factor, and reduced Nusselt number from those of the classical laminar pressure-driven flow problem. Others have characterized the effect of thermal band broadening due to Joule heating in capillary electrophoresis [17-19].

Maynes and Webb performed a study on the thermally fully-developed characteristics of purely electro-osmotically driven flow for parallel plate and circular channels [11]. As the present study focuses on EOF in circular channels, only the results for this geometry are of interest. Their study obtained fully-developed solutions for the non-dimensional temperature distribution and Nusselt number for a specified range of $a/\lambda_d$ ($Z$) ratios and volumetric energy generation magnitudes. For $Z > 500$, the fully-developed Nusselt number, $N_{ufd}$, for the constant wall temperature and constant wall heat flux boundary conditions were found to be 5.783 and 8, respectively. A parametric study
revealed that as $Z \rightarrow 0.5$, where the EDL’s on opposite tube walls almost completely impinge upon each other, $Nu_{fd}$ falls to the classical pressure-driven parabolic flow values for each boundary condition with no energy generation.

Maynes and Webb also completed a similar study that included the effects of viscous dissipation on EOF heat transfer [20]. Their study found that viscous dissipation effects only contribute for $0.3 < Z < 100$. For increasing viscous dissipation, $Nu_{fd}$ decreased for the constant wall heat flux condition and increased for the constant wall temperature boundary condition.

### 2.2 Other Related Studies

No research has been reported in literature focusing on the thermally developing heat transfer characteristics of EOF. However, classical studies separate from EOF research have identified heat transfer solutions for hydrodynamically fully-developed flow characterized by either of the two unique factors that define EOF—slug flow and uniform volumetric energy generation. These studies involve solely pressure-driven flows and thus $a/\lambda_d$ ratios are not applicable.

Topper solved for the thermally developing temperature distribution in a circular tube with a constant wall temperature, slug flow, and volumetric energy generation while neglecting axial conduction [21]. Using this temperature distribution, the Nusselt number variation with axial position can be separately derived, yielding a fully-developed Nusselt number of $Nu_{fd} = 8$, which corresponds to the non-viscous dissipation heating results obtained by Maynes and Webb for the same wall boundary condition.

Slug flow studies without a generation source term have been presented by Burmeister [22], who solved the fundamental thermally developing slug flow problem in
circular tubes both for a constant wall temperature and for a constant wall heat flux while neglecting axial heat conduction. Results predict the local temperature distributions and the mean temperatures and Nusselt numbers along the tube length. Fully-developed Nusselt numbers were found to be 5.783 for the constant wall temperature scenario and 8 for the constant wall heat flux, coinciding with the non-viscous dissipation heating results obtained by Maynes and Webb for $Z > 500$.

Studies that address volumetric energy generation are categorized by the boundary condition applied at the duct wall. Using a constant wall temperature boundary condition, Topper solved the basic Graetz problem (parabolic fully-developed inlet velocity profile, negligible axial heat conduction, constant properties) in the thermal entry region of a circular tube with a constant energy generation source term [21]. Using this solution, the Nusselt number variation with axial position, as before, can be separately derived, yielding a fully-developed Nusselt number of $N_{ufd} = 6$.

Several studies have also appeared in the literature for the constant wall heat flux boundary condition. Sparrow and Siegel investigated both thermally developing and thermally developed laminar flow in a circular tube containing an arbitrarily varied energy generation source term while neglecting axial conduction [23]. Tao solved the problem in the thermally fully-developed region for arbitrary cross-sections, including circular tubes, and for uniform and arbitrary source term distributions [24,25]. For circular tubes, the resulting fully-developed Nusselt number for a zero source term coincided with that of classical laminar flow convection theory, where $N_{ufd} = 48/11 (4.364)$ [26]. Tyagi extended Tao’s work to include viscous dissipation as a source of heat generation [27].
Although no research has been reported in the literature investigating thermally developing EOF heat transfer, the above-mentioned studies will be used as comparative benchmarks for this thesis research. Note, however, that for the purpose of the present study neglecting axial conduction cannot be employed due to the low flow velocities, and therefore low Peclet numbers, inherent in EOF. Therefore, in the benchmarking studies, high Peclet number flows will be employed into the solutions for comparative purposes.

2.3 CONTRIBUTION OF CURRENT RESEARCH

As no previous studies have been reported for thermally developing EOF convective heat transfer, this thesis research will predict the non-dimensional temperature profiles and heat transfer coefficients for thermally developing, hydrodynamically fully-developed EOF in circular microchannels. The heat transfer behavior will be investigated by solving Equation (1.4) for various $a/\lambda_d$ ratios. The equation, in cylindrical coordinates, takes the form

$$\frac{u}{\alpha} \frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{i^2 \sigma}{k} + \mu \left( \frac{\partial u}{\partial r} \right)^2$$

(2.1)

where $x$ is measured in the axial direction and $k$ is the thermal conductivity of the fluid.

Equation (2.1) is a partial differential equation that can be solved using one of a variety of assumptions for the fully-developed velocity distribution, $u$, in the tube—uniform velocity flow (for infinitely thin Debye length, or $a/\lambda_d \rightarrow \infty$) or any of the family of profiles that depend on finite $a/\lambda_d$ [Equation (1.2)]. Each profile involves either of two wall boundary conditions—constant wall temperature (CWT) or constant wall heat flux (CHF). Chapter 3 addresses the infinitely thin Debye length case and presents the closed-form solutions along with parametric studies and discussions of their physical
interpretations. Chapter 4 outlines the numerical model used to solve Equation (2.1) for any EOF velocity profile and then presents and analyzes the resulting solutions. Finally, Chapter 5 summarizes the objectives and conclusions of this research and discusses possible areas of future studies.
3 - INFINITELY THIN DEBYE LENGTH STUDIES

The first velocity distribution explored assumes that the flow velocity is uniform across the whole tube radius, which applies to tubes where $a/\lambda_d > 500$ [9]. For such flow, viscous dissipation in the tube is negligible. After elimination of the viscous dissipation term and normalization of the remaining variables in Equation (2.1), the differential equation reduces for both wall boundary conditions to

$$
\frac{1}{2} \frac{\partial \theta}{\partial X} = \frac{1}{Pe^2} \frac{\partial^2 \theta}{\partial X^2} + \frac{1}{R} \frac{\partial \left( R \frac{\partial \theta}{\partial R} \right)}{\partial R} + S
$$

(3.1)

where $X = x/(aPe)$, $R = r/a$ and $Pe = 2au_{max}/\alpha$, with $u_{max}$ being given in the definition Equation (1.2). $\theta$ and $S$ are the dimensionless temperature and source term representing Joule heating in the fluid, respectively, and are defined differently for each wall boundary condition. Equation (3.1) can be solved analytically for both wall boundary conditions by using established partial differential equation solution techniques to yield the heat transfer characteristics of the flow.

3.1 CONSTANT WALL TEMPERATURE

Equation (3.1) will first be solved for a constant wall temperature, $T_w$, and uniform fluid entrance temperature, $T_e$. The definitions of $\theta$ and $S$ for the constant wall temperature boundary condition are
$$\theta = \frac{T - T_w}{a^2 \sigma^2 / k}$$  \hspace{1cm} (3.2a)  

$$S = 1$$ \hspace{1cm} (3.2b)

This CWT boundary condition scenario employs a different normalization approach for $\theta$ than normally used in obtaining solutions to energy equation problems. Traditional approaches for a CWT boundary condition use a given wall temperature difference, such as $T_e - T_w$, as the normalization factor, as exemplified by some of the cited literature [21,22,28,29]. However, this approach can result in some confusion in the resulting definition of the source term, $S = i_e^2 \sigma a^2 / k (T_e - T_w)$, since it could take on negative values as $T_e - T_w$ can be either positive or negative, which lacks physical significance. To avoid this problem, the energy generation is incorporated into the definition of $\theta$ (which can assume negative values) by specifying it as the local wall temperature difference normalized by the energy generation term given in Equation (2.1).

The thermal inlet and boundary conditions for the CWT scenario are, respectively,

$$\theta(X = 0, R) = \theta_e$$  \hspace{1cm} (3.3)

$$\frac{\partial \theta}{\partial R}_{|R=0} = 0$$  \hspace{1cm} (3.4a)

$$\theta(X, R = 1) = 0$$  \hspace{1cm} (3.4b)

where $\theta_e$ is the dimensionless inlet temperature, defined as $\theta_e = (T_e - T_w) / (i_e^2 \sigma a^2 / k)$. For the CWT boundary condition scenario, this inlet temperature is the parameter that reflects the strength of the volumetric heating in the fluid (not $S$), and may be viewed as an inlet fluid temperature excess for convection relative to the magnitude of the Joule heating.
Note that the normalized inlet temperature $\theta_e$ may take on magnitudes ranging from 0 (corresponding to $T_e = T_w$ or infinitely large Joule heating) to infinity (reflecting the limiting cases of large inlet temperature difference or no Joule heating in the fluid) and can be either positive or negative since the entrance temperature may lie above or below the wall temperature. Finally, as Equation (3.1) is second-order in $X$, a second boundary condition on $\theta$ in $X$ is needed. This condition requires that the solution be bounded as $X \to \infty$, which is defined by assuming a fully-developed condition at very large $X$.

Equation (3.1) is a linear, non-homogeneous partial differential equation that can be solved by one of several established methods. Separation of variables and partial solutions were used here, superimposing the thermally developing and fully-developed solutions [30]. The inlet condition, Equation (3.3), is satisfied using the orthogonality property of Bessel functions in an infinite series. Once the local dimensionless temperature solution is found, the heat transfer coefficients are then obtained in terms of the local Nusselt number, defined as $h_x 2a/k$ where $h_x$ is the local convection heat transfer coefficient. The Nusselt number is calculated by solving for the local mean temperature in the fluid and equating the heat flux at the wall to the internal convection ensuing from that mean temperature, which in dimensional form is expressed as $h_x (T_m - T_w) = -k \partial T / \partial r \bigg|_{r=a}$. The local mixed-mean temperature is evaluated in standard form from its definition

$$T_m = \frac{1}{\bar{u}A_c} \int\bar{u}TdA_c$$

(3.5)
which reduces in dimensionless form for the slug flow case to

\[ \theta_m(X) = 2 \int_0^1 R \theta(X,R) dR \quad (3.6) \]

where \( \theta(X,R) \) is the dimensionless temperature distribution solution for Equation (3.1).

### 3.1.1 Analytical Solution

The solution to Equation (3.1) for the above CWT-case inlet and boundary conditions is

\[ \theta(X,R) = (1-R^2)/4 + \sum_{k=0}^{\infty} A_k J_0(\lambda_k R) e^{i\sqrt{1+16\lambda_k^2/Pr^2} Pr^2 X/4} \quad (3.7) \]

where

\[ A_k = \frac{2}{J_1(\lambda_k)} \left( \frac{\theta_k}{\lambda_k^2} - 1 \right) \quad (3.8) \]

and \( J_0 \) and \( J_1 \) are the zero-order and first-order Bessel functions of the first kind, respectively. The eigenvalues, \( \lambda_k \), for this case are found from the eigencondition \( J_0(\lambda_k) = 0 \), which is derived from the wall boundary condition given in Equation (3.4b).

The local fluid mean temperature, \( \theta_m(X) \), is found from the above solution and Equation (3.6) to be

\[ \theta_m(X) = 1/8 + \sum_{k=0}^{\infty} A_k J_1(\lambda_k) e^{i\sqrt{1+16\lambda_k^2/Pr^2} Pr^2 X/4} \quad (3.9) \]

The local Nusselt number, defined as \( Nu_x = h_x 2a/k = -\frac{\partial \theta}{\partial R}|_{R=1}/\theta_m \), thus becomes

\[ Nu_x = \frac{1 + \sum_{k=0}^{\infty} A_k J_1(\lambda_k) \lambda_k e^{i\sqrt{1+16\lambda_k^2/Pr^2} Pr^2 X/4}}{1/8 + \sum_{k=0}^{\infty} A_k J_1(\lambda_k) e^{i\sqrt{1+16\lambda_k^2/Pr^2} Pr^2 X/4} / \lambda_k} \quad (3.10) \]
The resultant expressions for the local temperature, local mean temperature and local Nusselt number, given above with their infinite series summations, were each evaluated numerically. Established computer codes were used to evaluate the eigenvalues and Bessel functions [32]. Care was taken to ensure convergence of the series expansions over the full range of parameters \((\theta_e, Pe, X, \text{etc.})\) studied. Generally, the expressions converged very rapidly, using fewer than twenty terms, but for small values of \(X\) convergence of the series required considerably more terms. Thus, to ensure convergence for all scenarios studied, the data presented here were calculated using at least 5000 terms in the series summations.

### 3.1.2 Results

The development of the local dimensionless fluid temperature \(\theta(X,R)\) is illustrated in Figure 3-1 for \(Pe = 10\) and four different values of the dimensionless inlet temperature: \(\theta_e = 100, 0.18, 0.05, \text{ and } -0.05\). The significance of the \(\theta_e = 0.18\) case will be explained later. Recall that for \(\theta_e > 0\) the inlet temperature exceeds the tube wall temperature, while for \(\theta_e < 0\) the entrance temperature lies below the wall temperature. The inlet temperature value \(\theta_e = 100\) is representative of what might be encountered in a typical electro-osmotic flow scenario containing distilled water \((\sigma \approx 2273 \ \Omega \cdot m, \ k \approx 0.6 \ \text{W/m-K}, \ \alpha \approx 1.4 \times 10^{-7} \ \text{m}^2/\text{s})\) with an imposed electrical potential gradient, \(d\phi/dx\), in the microtube of magnitude 150 kV/m, a channel radius of \(a = 50 \ \mu m\), and an inlet temperature difference of \(|T_e - T_w| = 20^\circ C\). Recalling the definition of the fluid inlet temperature \(\theta_e = (T_e - T_w)/(i_e^2 \sigma a^2/k)\), for \(\theta_e = 100\), illustrated in Figure 3-1(a), the magnitude of \(\theta_e\) is such that the wall convection heat transfer is much greater than the energy generated by Joule heating, meaning
that the wall convection cooling influence will dominate over Joule heating for the temperatures at all radial positions. Thus, as the fluid moves with increasing streamwise position inside the microchannel, its local temperature decreases for all radial locations, causing the corresponding wall heat flux to decrease as the local mean temperature approaches the constant wall temperature. These trends continue until, far downstream ($X \rightarrow \infty$), the mean fluid temperature only slightly exceeds the wall temperature (this distribution can be seen in the last two plots of the figure for $X = 1.0$). At this point, the temperature gradient at the channel wall is just sufficient to completely dissipate the thermal energy generated by Joule heating, resulting in a fully-developed condition where
the temperature profile (centerline temperature of $\theta = 0.25$) and wall heat flux are constant.

For smaller values of $\theta_e$, energy generation plays a more significant part in the streamwise development of the temperature profiles. When $\theta_e$ is decreased to an approximate value of 0.18 [see Figure 3-1(b)], both convection from the tube and energy generation play significant roles in the thermal development region. Within the fluid core ($0 < R \leq 0.5$) energy generation is larger than the influence of wall convection cooling (Joule heating prevails), thus causing the temperature in that radial region to increase above the inlet temperature with streamwise position. However, near the channel wall, convection prevails over the Joule heating, causing the local temperatures near the wall to decrease with axial position below the entrance value. This convection results in a wall heat flux that decreases similarly to what was observed for $\theta_e = 100$. As before, the fully-developed condition is again characterized by a temperature profile whose wall gradient permits a balancing of the volumetric heating by convection at the wall.

As $\theta_e$ is further decreased, the energy generation prevails over the wall convection in an increasingly significant portion of the tube cross-section. This is exemplified in the temperature profile development for the $\theta_e = 0.05$ case, shown in Figure 3-1(c). Here the energy generation dominates the heat transfer over most of the entire tube radius, with temperatures only decreasing below the inlet value within a very small region next to the duct wall due to wall convection. This locally concentrated prevalence of wall convection vanishes for succeeding increasing streamwise position as the energy generation gains complete dominance, resulting in the subsequent increase of the temperature profile for

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all radial locations. Again, the same fully-developed condition as mentioned previously is obtained as $X \to \infty$.

Recalling the definition of the normalized inlet temperature, $\theta_e$, this temperature will assume negative values when $T_e$ is less than $T_w$. Figure 3-1(d) illustrates such a case, where the magnitude of the entrance temperature difference, $T_e - T_w$, and of the volumetric heating are identical to that of Figure 3-1(c), but the fluid entrance temperature is below that of the wall temperature, $(\theta_e = -0.05)$. In such a scenario, with the exception of a very small layer near the duct wall, energy generation will cause an increase in local temperature towards the wall temperature at all radial locations. At the small region near the wall, the temperature rises faster than in the fluid core due to the additional contribution of wall heating. This results in a temperature profile similar to that outlined for the $X = 0.021$ axial position in Figure 3-1(d), where temperatures nearer the wall are higher than those in the fluid core. At some point downstream, the fluid temperature in the vicinity of the wall reaches and exceeds the wall temperature. As a result, the wall temperature gradient and, hence, the wall heat flux reverse sign, causing the heat transfer at the wall to change from fluid heating to dissipation of the Joule heating. Note that this point, as exemplified by the $X = 0.021$ case, can be reached even if the fluid core temperature is below the wall temperature. For the particular dimensionless inlet temperature shown $(\theta_e = -0.05)$, the wall heat flux reversal occurs at a streamwise location of $X \approx 0.02$. From this location on, the wall temperature gradient increasingly dissipates the Joule heating until the same fully-developed condition as before is attained.

Similar overall thermal development behavior to that outlined above for $\theta_e = -0.05$ is exhibited for all $\theta_e < 0$ cases. The only difference lies in the inlet temperature and
in the portion of the tube cross-section in which energy generation prevails over wall convection, which is determined by the magnitude of $\theta_e$. Note also that the fully-developed normalized temperature profiles are always positive and identical regardless of $Pe$ and $\theta_e$. This is due to the nature of the CWT wall boundary condition, where at fully-developed conditions the wall convection from the tube completely dissipates the energy generation occurring within the fluid, yielding a temperature profile that will always be the same for fully-developed EOF slug flow with a CWT boundary condition. As a result, the normalized temperature profile is independent of the magnitude of the Joule heating within the fluid and is given by the first term on the right-hand side of Equation (3.7).

Figure 3-2 illustrates the variation of dimensionless local mean temperature with normalized streamwise position for $Pe = 5$ (the value of $Pe$ typically encountered under the same conditions mentioned earlier for $\theta_e = 100$) and both positive and negative values of the normalized inlet temperature $\theta_e$. The trends show the general fluid cooling and heating characteristics exhibited by the temperature profile developments described pre-

![Figure 3-2: $\theta_m$ vs. $X$ for $Pe = 5$ and values of $\theta_e$ ranging from -1 to 1.](image)
viously. For $\theta_e \to \infty$ (vanishing Joule heating), the thermal development of the classical slug flow, constant wall temperature case is reproduced [22]. For values of $\theta_e$ greater than 0.18, for which wall convection cooling dominates, the mean temperature decreases from its inlet value to an asymptotic fully-developed value of $\theta_m = 1/8$. When $\theta_e$ is less than 0.18, there is an increase in $\theta_m$ to the same value due to the prevalence of Joule heating, or its combination with wall convection heating. Note in Figure 3-2 the evident shift in heat transfer dominance from wall convection to Joule heating as $\theta_e$ decreases from 0.2 to 0.1, which straddles the critical 0.18 value. For all values of $\theta_e$ (both positive and negative), the mean temperature reaches its unchanging, fully-developed value at a dimensionless streamwise coordinate ($X$) between 1 and 5. This is well beyond the development length for the classical slug flow thermal entry length problem under constant wall temperature conditions with no Joule heating, where $X_{fd} \approx 0.1$ ($x/2aPe \approx 0.05$) [22].

Figure 3-3 illustrates the variation of mean temperature with Peclet number for a given inlet temperature ($\theta_e = 100$) and Peclet numbers ranging from $Pe = 0.1$ to 10. The high-Peclet-number limit, $Pe \to \infty$, is shown for comparison. As described previously, for $\theta_e = 100$ the wall convection prevails throughout the thermally developing region, causing the mean temperature to decay monotonically with streamwise coordinate $X$ for all $Pe$. Note that the thermal development terminates at smaller values of $X$ for increasing $Pe$. This is simply due to the increasing dominance of wall convection, which gradually nullifies the lengthening effect of axial conduction on the thermal development. Thus, as $Pe$ is increased to the large-Peclet number limit, where axial conduction becomes negligible, the dimensionless mean temperature’s variation with $X$ becomes independent.
In terms of the normalized coordinates the local Nusselt number can be expressed as

\[ Nu_x = \frac{2 \frac{\partial \theta}{\theta_m}}{R_{R=1}} = -\frac{\Phi}{\theta_m} \]  

(3.11)

where \( \Phi = 2q_w^*/(i_e^2 \sigma a) \) is the dimensionless local heat flux at the wall with positive values of \( \Phi \) reflecting fluid heating. The local heat flux may therefore be calculated from Equation (3.11) by using the relation \( \Phi = -Nu_x \theta_m \). In the thermally fully-developed regime, Figure 3-2 illustrates that the local mean temperature is unchanging. It is also subsequently found that \( Nu_x \) is unchanging in this regime. Far downstream of the inlet, the temperature profile ceases to change, and the physics require that all volumetric energy generation due to Joule heating be dissipated convectively at the walls, resulting of \( X \). This resulting variation is identical to the mean temperature distribution obtained by Topper for a circular tube with slug flow, constant wall temperature, and volumetric energy generation, while neglecting axial conduction [21].
in a uniform wall heat flux where the fluid is convectively cooled at the wall. For $X \to \infty$, an overall energy balance on the fluid yields $\Phi = -1$, and consequently, $Nu_{fd} = 1/\theta_e\big|_{X \to \infty} = 8$ [33]. This result is illustrated in Figures 3-4(a) and 3-4(b), which present the variation with $X$ of local Nusselt number and absolute value of the normalized wall heat flux, respectively, for $Pe = 5$ and positive values of $\theta_e$ ranging from 0 to $\infty$.

Figure 3-4(a) reveals two limiting Nusselt number cases corresponding to $\theta_e = 0$ and $\theta_e \to \infty$. For the $\theta_e \to \infty$ case, which corresponds to no volumetric heating, the local Nusselt number is identical to the classical behavior for slug flow in a tube with constant wall temperature and no volumetric generation, for which the fully-developed Nusselt number is $Nu_{fd} = 5.783$ [22,33]. For $\theta_e = 0$, the thermal development is dominated by volumetric generation, which must be dissipated by wall convection. The transport behavior for this case therefore approaches that of classical slug flow with a constant wall heat flux boundary condition as $X \to \infty$ [indicated by $\Phi = \text{constant}$ for large $X$ in Figure 3-4(b)], thus asymptoting to the corresponding fully-developed value of $Nu_{fd} = 8.0$ [22,33]. For finite, non-zero positive values of the fluid entrance temperature ($0 < \theta_e < \infty$), the local Nusselt number will exhibit distributions that lie between the two limiting cases. Their specific characteristics can be described using the normalized heat flux profiles in Figure 3-4(b). Initially, the heat flux profiles are characterized by a negative power-law relation in $\Phi$ with $X$ and follow very similar slopes on a log-log plot. This characteristic is due to conventional boundary layer growth associated with the convection-dominated transport, which occurs for all $\theta_e$ values at very small $X$. Deviation from the linear slope occurs when Joule heating begins to be significant, which location occurs further upstream for decreasing $\theta_e$ due to the increasingly dominant Joule heating.
Figure 3-4: (a) $N_u_e$ and (b) absolute value of the normalized heat flux, $|\Phi|$ vs. $X$ for $Pe = 5$ and over a positive range of $\theta_e$ (Note that for $\theta_e > 0$, $\Phi$ is always negative, indicating fluid cooling).

The above behavior is also evident in the Nusselt number distribution, where for small $X$ the local Nusselt number follows the $\theta_e \rightarrow \infty$ (convection dominated) behavior until the respective point of deviation.

For cases where $\theta_e \leq 0.18$ the point of deviation from convection-dominated heat transfer occurs where the local Nusselt number lies above $Nufd = 8.0$ and thus, the Nusselt
number will decrease monotonically to the fully-developed condition at $X \to \infty$. For $\theta_e \gtrsim 0.18$ the deviation point occurs far enough downstream that the Nusselt number has fallen below the critical value of 8. Thus, as Joule heating takes precedence, the Nusselt number begins to increase and deviates from the $\theta_e \to \infty$ case, creating a local minimum in the $N_u e$ distribution. This point of local minimum moves upstream for increasing $\theta_e$ just as the point of departure from the $\theta_e \to \infty$ solution does. $N_u e$ then asymptotes to the fully-developed magnitude of the $\theta_e = 0$ case downstream. Correspondingly, at $X \to \infty$ the normalized heat flux asymptotes to a constant value equal to $\Phi = -1$, as has been shown previously. The wall heat flux must asymptote to a value which balances the volumetric Joule heating. Thus, the transition from the constant wall temperature, large-$\theta_e$ behavior existent for all $\theta_e$ values early in the tube to constant wall heat flux character far downstream is confirmed.

In addition to the above analysis for $\theta_e > 0$, the local Nusselt number distributions will now be studied for negative $\theta_e$ values. Recall that such negative values of $\theta_e$ arise if the fluid entrance temperature is below the wall temperature. The local Nusselt number and normalized heat flux distributions for $Pe = 5$ for negative values of the dimensionless inlet temperature, $\theta_e$, are illustrated in Figure 3-5.

As observed previously relative to Figure 3-1(d), for $\theta_e < 0$ cases the fluid is heated both by Joule heating and wall convection early in the duct, where the mean temperature is below the wall temperature. As shown in Figure 3-5, the fluid-heating region is characterized by a decreasing local Nusselt number and positive $\Phi$ until the fluid mean temperature approaches the wall temperature ($\theta_m \to 0$). At this axial location, the normalized wall heat flux is identically 0, and the local Nusselt number (as tradition-
ally defined), which initially follows the $\theta_e \to \infty$ local Nusselt number behavior, drops dramatically ($\text{Nu}_e \to -\infty$). Here, the fluid is nominally at the wall temperature, as was seen for the $X = 0.021$ temperature profile in Figure 3-1(d) for $\theta_e = -0.05$. Downstream of this location, the wall heat flux reverses direction ($\Phi < 0$), the internally heated fluid is subsequently cooled by the walls, and the local Nusselt number falls from an infinitely high value characteristic of the formation of a new thermal boundary layer. Ultimately,
$Nu_x$ approaches the fully-developed magnitude ($Nu_{id} = 8.0$) along the $\theta_e = 0$ profile, corresponding again to a constant wall heat flux boundary condition. As $\theta_e$ approaches zero for negative values of $\theta_e$ (e.g., Joule heating increases), the location of wall heat flux reversal moves upstream. In the limiting case of $\theta_e \rightarrow 0$, the heat transfer reversal occurs at an infinitesimally small $X$ position, with no apparent singularity in the local Nusselt number behavior. As expected for flows dominated by volumetric heating, regardless of inlet temperature, the local Nusselt number versus $X$ profiles for both $\theta_e < 0$ (Figure 3-5) and $\theta_e > 0$ (Figure 3-4) become identical as $|\theta_e| \rightarrow 0$.

Figure 3-6 illustrates the variation in local Nusselt number as a function of $X$ for Peclet numbers ranging from 0.1 to 100 for $\theta_e = 100$. The individual curves generally follow the trends described previously for the variation of Nusselt number between the $\theta_e = 0$ and $\theta_e \rightarrow \infty$ limiting cases as expected for $\theta_e = 100$. Two observations may be made with regard to the dependence of the local Nusselt number on Peclet number. First, as noted similarly for $\theta_m$, the normalized streamwise location at which the thermal develop-
ment terminates decreases with increasing Peclet number, as anticipated given the more intense advection. Second, the decay in Nusselt number with $X$ is less rapid as Peclet number increases.

The solution for the thermally developing Nusselt number and its asymptotic fully-developed condition allows characterization of the thermal entrance length for isothermal electro-osmotic flow. The entrance length $X_{fd}$ may be defined as the $X$-location where the local Nusselt number is within 0.1% of its fully-developed value, $Nu_{fd} = 8.0$. As was seen in Figure 3-4(a), the local Nusselt number passes through $Nu_x = 8.0$ on its way to a local minimum for $\theta_e \gtrsim 0.18$ before reaching its fully-developed condition. Therefore, care was taken to determine the development length where the asymptotic value is truly reached. Figure 3-7 shows the thermal development length $X_{fd}$ thus determined as a function of $1/\theta_e$ for $Pe$ from 0.1 to 100, where the parameter $1/\theta_e$ is proportional to the magnitude of the Joule heating source term. The figure reveals that $X_{fd}$ decreases with increasing Peclet number for all values of $1/\theta_e$. The $X_{fd}$ behavior decreases with increasing Peclet number for all values of $1/\theta_e$. The $X_{fd}$ behavior

![Figure 3-7: $X_{fd}$ vs. $1/\theta_e$ for values of $Pe$ ranging from 0.1 – 100.](image-url)
eventually collapses to a single relationship in the large Peclet number limit. For large volumetric source ($1/\theta_e > 30$) the development length $X_{fd}$ is identical for both positive and negative inlet temperature. This is to be expected, given the constant wall heat flux limiting behavior for large volumetric energy generation. As $1/\theta_e \to \infty$, predictions reveal that the thermal entrance length becomes inversely proportional to $Pe$. Thus, $(x/aPe)_{fd}$ approaches a constant value ($= 2.8$ for $1/\theta_e \to \infty$ and $Pe = 1$). In the high Peclet number limit, the asymptotic $X_{fd}$ behavior for large Joule heating is independent of $Pe$, and can be expressed as $X_{fd} \approx 0.485$. By comparison, the development length for the classical pressure-driven flow with no volumetric heating is $X_{fd} \approx 0.1$ ($x/2aRePr \approx 0.05$) [29]. Thus, for all scenarios involving volumetric heating, the thermal entry region for electro-osmotically generated flow is significantly longer than for pressure-driven flow.

As $|1/\theta_e| \to 0$ (for both positive and negative values of $\theta_e$) the thermal development length increases in an unbounded manner. This behavior may be explained as follows. For large negative inlet temperature (small source) the Joule heating is so small as to require significant channel length before the reversal between fluid heating and fluid cooling occurs, as explained relative to Figure 3-5. For large positive $\theta_e$, the volumetric heating is so small as to require significant channel length before the departure from the constant wall temperature ($\theta_e \to \infty$) behavior to the fully-developed constant heat flux ($\theta_e \to 0$) behavior, as seen in Figure 3-4. Of course, in the limiting case of $\theta_e \to \infty$, the thermal entrance length is finite as given by the classical problem with no volumetric heating, and is considerably smaller than for finite fluid Joule heating.

Additionally, there is an interesting critical point in the thermal entry length dependence on $1/\theta_e$ near $1/\theta_e \approx 5.4$ ($\theta_e \approx 0.18$) where a local minimum occurs in the $X_{fd}$
behavior. The magnitude of the inlet temperature at this critical point is actually a weak function of $Pe$, ranging from $1/\theta_e \approx 5.7$ for $Pe \to \infty$ to $1/\theta_e \approx 5.2$ at small $Pe$. As $1/\theta_e$ increases from zero ($\theta_e$ decreasing from $\infty$), the point at which $Nu_e$ merges with the $\theta_e = 0$ case moves upstream until the critical point for $1/\theta_e$ is reached. At this point, the magnitude of $1/\theta_e$ is such that the local Nusselt number distribution ceases to fall below the fully-developed value $Nu_{fd} = 8.0$ to a local minimum like the $\theta_e \approx 0.18$ cases seen in Figure 3-4(a). For this nominal value of $1/\theta_e$, the influence of the wall convection and the volumetric generation on the overall transport is of the same order of magnitude and the development length is a minimum (for a given Peclet number), but for small $Pe$ is still considerably longer than the entry length for pressure-driven flow heat transfer. As $1/\theta_e$ increases further beyond the critical point, the point at which fully-developed conditions are reached moves downstream until $1/\theta_e \to \infty (\theta_e = 0)$ behavior is achieved.

### 3.2 Constant Wall Heat Flux

Equation (3.1) will now be solved for a constant wall heat flux, $q''_w$, and uniform fluid entrance temperature, $T_e$. The definitions of $\theta$ and $S$ for the CHF case are

$$\theta = \frac{T - T_e}{q''_w a/k} \quad (3.12a)$$

$$S = \frac{a l_e^2 \sigma}{q''_w} \quad (3.12b)$$

where $q''_w$ is the imposed wall heat flux with positive values of $q''_w$ reflecting fluid heating. Thus, from these definitions, the thermal inlet and boundary conditions for the CHF scenario are, respectively,
\[ \theta(X = 0, R) = 0 \]  
\[ \frac{\partial \theta}{\partial R} \bigg|_{R=0} = 0 \]  
\[ \frac{\partial \theta}{\partial R} \bigg|_{R=1} = 1 \]

In order to solve Equation (3.1) for the boundary conditions above, different partial differential equation solving methods were utilized than those used for the CWT scenario. Unlike the CWT case, Equation (3.1) is non-homogeneous both in the equation itself (through the existence of the constant source term, \( S \)) and in the boundary condition applied at the wall [Equation (3.14b)]. These two non-homogeneous conditions can be separated by splitting Equation (3.1) into two separate and solvable differential equations, each containing one of the two non-homogeneities, through the use of the superposition principle. Each equation was then solved by the variation of parameters method. Again, as for the CWT case, the solution for each equation was bounded by a fully-developed condition to provide the second boundary condition needed on \( \theta \) in \( X \). The fully-developed condition for each of the two equations was obtained by performing an energy balance at \( X \to \infty \).

The CHF-case solution to Equation (3.1) for the above inlet and boundary conditions was found to be

\[ \theta(X, R) = 2(S + 2)X + \sum_{k=1}^{\infty} A_k J_0 \left( \lambda_k R \right) \left( 1 - e^{\left[ 1 - \frac{16\lambda_k^2}{Pe^2} \right] Pe^2 X/4} \right) \]  
\[ A_k = \frac{2}{\lambda_k^2 J_0 \left( \lambda_k \right)} \]
The eigenvalues, $\lambda_k$, are generated from the eigencondition $J_i(\lambda_k) = 0$, which is derived, as before for the CWT case, from the wall boundary condition [Equation (3.14b)]. The local non-dimensional fluid mean temperature, $\theta_m(X)$, is obtained from the above solution by using Equation (3.6) to be

$$\theta_m(X) = 2(S + 2)X$$

(3.17)

and the local Nusselt number, defined as $Nu_x = h_x 2a/k = 2/\left(\theta_w - \theta_m\right)$, becomes

$$Nu_x = \frac{1}{\sum_{k=1}^{\infty} \left(1 - e^{-\sqrt{1+16\lambda_k^2/Pe^2}Pe^2X/4}\right)^2}$$

(3.18)

Although the local temperature distribution and local mean temperature depend on the magnitude of the volumetric Joule heating as shown in Equations (3.15) and (3.17), according to Equation (3.18) the Nusselt number does not. Thus the local Nusselt number solution for electro-osmotic flow and CHF boundary condition is identical to the classical slug flow solution for no volumetric heating ($S = 0$) with an imposed constant wall heat flux, for which the fully-developed Nusselt number asymptote is $Nu_{fd} = 8.0$ [22]. A further implication of this finding is that the thermally fully-developed length for EOF under constant heat flux conditions is identical to the $S = 0$ slug flow case, which is $X_{fd} \approx 0.21$ [22]. Therefore, all local heat transfer characteristics (excepting the temperature distribution and mean temperature) of the constant wall heat flux case reduce to the classical solution for no Joule heating, and previously published work for $S = 0$ with regard to local Nusselt number is applicable [15,22]. The only difference, therefore, between the electro-osmotic slug flow solution and the classical non-Joule heating solution is the inclusion of the dependence of the local temperature distribution and local
mean temperature on the magnitude of the energy source, which dependence is shown in Equations (3.15) and (3.17).
4 - ELECTRO-Osmotic FLOW VELOCITY
PROF ILE STUDIES

As mentioned in Chapter 1, electro-osmotic flow does not exhibit a completely uniform velocity profile for all \( a/\lambda_d \) ratios. Due to the no-slip condition located at the boundary of the Stern and Gouy-Chapman layers of the EDL, the fluid velocity distribution is a function of radial position, which becomes significant in the overall tube cross-section when \( a/\lambda_d < 500 \). Therefore, it is necessary to solve the governing differential energy equation taking this velocity distribution into account.

Renormalizing the governing energy equation, Equation (2.1), to include the velocity profile \( U = u/u_{\max} \) yields the normalized equation

\[
\frac{U(Z,R) \partial \theta}{2} + \frac{1}{\text{Pe}^2} \frac{\partial^2 \theta}{\partial X^2} + \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \theta}{\partial R} \right) + S'(S,S_v) = 0 \tag{4.1}
\]

where \( X = x/(a\text{Pe}) \), \( R = r/a \), \( Z = a/\lambda_d \) and \( \text{Pe} = 2a u_{\max}/\alpha \), with \( u_{\max} \) being given in the definition of Equation (1.2). As in Chapter 3, \( \theta \) and \( S \) are the dimensionless temperature and source term representing Joule heating in the fluid, respectively, and are defined differently for each wall boundary condition in the same manner as done previously for slug flow. The dimensionless parameter \( S_v \) is defined as

\[
S_v = \mu u_{\max}^2 / \iota_c^2 \sigma \lambda_d^2 \tag{4.2}
\]
and serves as a measure of the relative magnitudes of viscous dissipation and Joule heating. Note that this parameter can also be expressed as $S_v = \sigma \mu \mu_{eo}^2 / \kappa_\alpha^2$ ($\mu_{eo}$ is the electro-osmotic mobility of the fluid, defined as $\mu_{eo} = \varepsilon_\mu \zeta / \mu$) and is therefore based only on physical and electro-osmotic properties of the fluid. $S'(S, S_v)$ is a function that represents the dimensionless energy generation terms (Joule heating and viscous dissipation) derived from Equation (2.1) and depends upon the nature of the boundary condition applied at the wall. These functions will be presented later in their respective sections of the chapter. Finally, $U(Z,R)$ is the non-dimensional velocity profile, $u/u_{max}$ [Equation (1.2)], obtained from solving the general momentum equation. However, the inclusion of this non-uniform velocity distribution makes the general energy equation unsolvable in closed-form. Numerical methods will be utilized in this case.

4.1 NUMERICAL MODEL

Equation (4.1) can be solved numerically by employing a solution technique that predicts the temperatures at discrete points in the tube domain. The technique chosen here utilizes a control volume approach, where the domain is discretized into finite cells, and the differential equation is integrated around each control volume to generate a finite-difference equation for each [34]. Computational methods are then utilized to incorporate these equations with specified boundary conditions to obtain a solution. Each of these steps will be explained subsequently in further detail.

Taking advantage of symmetry within the tube, the fluid temperature distributions can be analyzed in a two-dimensional domain characterized by an axial cross-section of the tube bounded by the tube centerline and wall. This domain is discretized into finite
control volumes by deploying nodes and control surfaces to create a two-dimensional grid in $R$ and in $X$ within its interior, as illustrated by Figure 4-1. The control surfaces are deployed first parallel to both coordinate axes, including at the boundary surfaces, thus defining the geometric boundaries of each control volume. The nodes are then positioned at the geometric centers of each sequential pair of control surfaces and directly at the boundary surfaces and corners of the domain.

Equation (4.1) is now integrated around each finite control volume in both dimensions by employing the grid-point cluster shown in Figure 4-2 for each cell and by using the methodology of Patankar [34], yielding the integral equation:

$$
\int_{R_n}^{R_s} \int_{X_w}^{X_e} R \frac{\partial J}{\partial X} dRdX = \int_{X_w}^{X_e} \int_{R_n}^{R_s} \frac{\partial}{\partial R} \left( R \frac{\partial \theta}{\partial R} \right) dRdX + \int_{X_w}^{X_e} \int_{R_n}^{R_s} S'(S, S_x) RdRdX \quad (4.3)
$$

Here $X_e$ and $X_w$ represent the non-dimensional axial position values at the so-called east and west control surfaces bounding the control volume, respectively. $R_n$ and $R_s$ represent the non-dimensional radius values at the north and south boundaries, respectively. The $J$
Figure 4-2: (a) grid cluster for numerical integration; (b) definitions of finite integration distances

variable is defined as

\[ J = \frac{U(R)}{2} \theta - \frac{1}{Pe^2} \frac{\partial \theta}{\partial X} \]  \hspace{1cm} (4.4)

and represents the sum of advection and diffusion of thermal energy in the \( X \) direction. This variable is used due to the inseparable connection between these modes of heat transfer. Numerical schemes used for the discretization equations address this connection and will be given later.

The integrals in Equation (4.3) are performed for each cell by using simplifying assumptions for each parameter. The combined advection and diffusion flux, \( J \), is assumed to be only a function of \( X \) and \( R \frac{\partial \theta}{\partial R} \) is assumed to be only a function of \( R \). As well, the source term function \( S'(S, S_v) \) is assumed to be constant (independent of both \( R \) and \( X \)). Thus, after simple algebra, the definite integration illustrated in Equation (4.3) yields

\[ J_e - J_w = \frac{\Delta X}{R_e \Delta R} \left[ \left( R \frac{\partial \theta}{\partial R} \right)_n - \left( R \frac{\partial \theta}{\partial R} \right)_s \right] + S'(S, S_v) \Delta X \]  \hspace{1cm} (4.5)
where $R_p$ is the non-dimensional radius at the center node and the geometric parameters $\Delta X$ and $\Delta R$ are illustrated in Figure 4-2b.

For each interior cell, applying finite-difference methods to Equation (4.5) results in the following discretization equation:

$$a_p \theta_p = a_e \theta_E + a_w \theta_W + a_s \theta_N + a_s \theta_S + b$$  

(4.6)

where the coefficients are defined as

$$a_E = D_e \left[ 0, \left( 1 - \frac{0.1 \lvert F_e \rvert}{D_e} \right)^5 \right] + \left[ 0, -F_e \right]$$  

(4.7a)

$$a_W = D_w \left[ 0, \left( 1 - \frac{0.1 \lvert F_w \rvert}{D_w} \right)^5 \right] + \left[ F_w, 0 \right]$$  

(4.7b)

$$a_N = \frac{R_n}{R_p \delta R_n \Delta R} \Delta X$$  

(4.7c)

$$a_S = \frac{R_s}{R_p \delta R_s \Delta R} \Delta X$$  

(4.7d)

$$a_p = a_E + a_W + a_N + a_S$$  

(4.7e)

$$b = S'(S, S_e) \Delta X$$  

(4.7f)

The east and west coefficients, given by Equations (4.7a-b), are derived from Patankar’s power-law scheme, which is used to accurately represent the co-dependence of advection and diffusion in the $X$ direction [34]. Here $[\lvert A, B \rvert]$ denotes the greater of $A$ and $B$. The variables $D$ and $F$ represent the local dimensionless diffusion and flow conductance and
are defined as $1/\delta X$ and $U(R_p)Pe/2$, respectively. Equations (4.7a-f) are applicable to all interior cells except those adjacent to the boundary nodes.

Discretization equations for the boundary cells (i.e., inlet, outlet, wall, and centerline) are derived by using a cell integration technique similar to that outlined in Equation (4.3). Here the position of the boundary surface, whether in the $X$ or $R$ direction (depending upon the boundary condition involved), is one of the integration limits, as exemplified by Equation (4.8) for a wall boundary condition where $R_B$ represents the radial position of the wall boundary surface. Using the same integration assumptions made in performing the general integrals in Equation (4.3), one arrives at the result shown in Equation (4.9).

$$
\int_{X_w}^{X_R} \int_{R_{in}}^{R_{out}} R \frac{\partial J}{\partial X} dXdR = \int_{X_w}^{X_R} \int_{R_{in}}^{R_{out}} R \frac{\partial \theta}{\partial R} dRdX + \int_{X_w}^{X_R} \int_{R_{in}}^{R_{out}} S'(S, S_v) RdRdX \tag{4.8}
$$

$$
J_e - J_w = \frac{\Delta X}{R_p \Delta R} \left[ \frac{\partial \theta}{\partial R} \right]_{B} - \left[ \frac{\partial \theta}{\partial R} \right]_{s} + S'(S, S_v) \Delta X \tag{4.9}
$$

The centerline (or symmetry boundary) boundary condition, $\partial \theta / \partial R \big|_{R=0} = 0$, is the only common condition for the CWT and CHF wall boundary condition scenarios. This condition is applied between the centerline node and the first interior node in the $R$ direction, where the south control surface is now the boundary surface [$s = B$ in Equation (4.5)]. By setting $\partial \theta / \partial R \big|_{B}$ equal to zero, the south node temperature, where $\theta_s = \theta_B$, becomes nonexistent in Equation (4.6), resulting in the same discretization equation coefficients as those outlined in Equations (4.7a-f), except now with $a_s = 0$. The centerline boundary node temperature is evaluated equal to the temperature at the first interior node in the $R$ direction after the numerical solution process is complete.
Equation (4.6) can be generated for each interior \( m, n \) cell in the spatially discretized tube domain by using the corresponding coefficients from Equations (4.7a-f), where \( m \) and \( n \) represent the cell index in the \( R \) and \( X \) directions, respectively. The discretization equations for all cells in a single row \( m \) or column \( n \) in the 2-D grid make up an individual linear system of equations that can be solved numerically using the Tri-Diagonal Matrix Algorithm (TDMA). Each linear system of equations incorporates the node temperatures on either side of the row/column with their corresponding coefficients by treating them as known constants at each iteration and thus incorporating them into the \( b \) coefficient of Equation (4.6). The systems of equations for all rows and columns are then solved simultaneously in an iterative process (beginning with a temperature initialization) by using the line Gauss-Seidel method, as outlined by Patankar [34]. The resulting solution predicts the local temperatures at each node inside the discretized domain. Local mixed-mean temperatures may then be calculated by non-dimensionalizing Equation (3.5) and applying numerical integration techniques to the resulting equation:

\[
\theta_m(Z, X) = \frac{2}{\bar{U}_{\text{max}}(Z)} \int_0^1 R U(Z, R) \theta(X, R) dR
\]  

(4.10)

where \( \bar{U}_{\text{max}} = \bar{u}/u_{\text{max}} \) and is derived from Equation (1.2) to be

\[
\bar{U}_{\text{max}}(Z) = 1 - \frac{2l(Z)}{ZI_\theta(Z)}
\]  

(4.11)

For the present research, the Trapezoidal Rule was used to numerically evaluate Equation (4.10) since it only can be applied to a non-uniformly distributed grid (Simpson’s Rule requires that the two intervals over which it is applied be equally spaced). Nusselt numbers were then calculated from the resultant local mean temperatures using the definitions determined by the applied wall boundary condition. In the case of the CWT
boundary condition scenario, the derivative in the corresponding definition of $Nu_x$, given as $Nu_x = -2 \frac{\partial \theta}{\partial R_w} / \theta_m$, was evaluated using the temperature gradient between the wall boundary node and the adjacent interior node. For the CHF case, the local Nusselt number is defined as $Nu_x = 2/ (\theta_w - \theta_m)$.

Care was taken to ensure a converged solution for each parametric case in $\theta_c$ or $S$, $Pe$, $Z$, and $S_v$. Since the local Nusselt number is a significant parameter in illustrating convection heat transfer characteristics, the maximum difference in local Nusselt number between two iterations for all $X$ locations was adopted as the measure for convergence. The convergence criterion under these conditions was chosen by studying the change in average Nusselt number with the applied convergence criterion, where the average Nusselt number was calculated by averaging $Nu_x$ over the developing region (the most significant difference between convergence scenarios occurred in that region). Convergence studies for both wall boundary conditions are shown in Figure 4-3. The adopted criterion value was selected at where $\overline{Nu}$ became relatively level (ceased to change by

![Figure 4-3: Convergence studies for (a) constant wall temperature and (b) constant wall heat flux boundary conditions.](image-url)
more than 0.05%). As a result, the selected convergence criterion restricted the maximum difference in $N_{u_x}$ to be less than $10^{-5}$ for the CWT boundary condition and less than $10^{-2}$ for the CHF boundary condition.

Additionally, all solutions had to be independent of the grid size. In order to achieve this with the minimum number of cells required (which therefore minimizes the computational time to obtain a solution), the control surface locations were deployed using high-order polynomials in each coordinate direction to concentrate cells in areas of anticipated high temperature gradients (tube inlet and wall). Third-order polynomials were used because they moderately concentrate cells at the tube inlet and wall while at the same time maintain a nominally even distribution throughout the tube domain. The grid deployment algorithms used in the $X$ and $R$ directions, respectively, were

$$X_i = \left( \frac{i-1}{N_x} \right)^3 X_{\text{max}}$$

(4.12)

$$R_j = 1 - \left( 1 - \frac{j-1}{N_r} \right)^3$$

(4.13)

where $N_x$ and $N_r$ represent the number of cells deployed in the $X$ and $R$ directions, respectively, $i$ and $j$ represent the grid position in the $X$ and $R$ directions, ranging from 1 to $N_x + 1$ and from 1 to $N_r + 1$, respectively, and $X_{\text{max}}$ is the $X$ position at the tube outlet.

Under these conditions, tests for grid independence were conducted by first using a very course grid and then increasingly refining the grid with the same grid deployment algorithms and comparing the results. Comparisons were made based upon the percent difference in local Nusselt number at each $X$ position between the tested grid results and those of a very fine benchmarking grid (1200 and 500 cells in the $X$ and $R$ directions, respectively). Since each grid size has a different deployment of all $X$ positions, the local
Nusselt number differences between the test and benchmark results were calculated at matching \( X \) positions by using cubic spline interpolation methods in Mathcad\textsuperscript{®}. Figure 4-4 illustrates the change in the average percent difference in local Nusselt number with grid size for both wall boundary conditions. As shown, \( \Delta \text{Nu}_x \) changes significantly for course grid sizes but then converges to values that change by less than 0.1% for finer grids. The grid-independence was concluded to be attained at where \( \Delta \text{Nu}_x \) becomes less than 0.1%. As a result, 150,000 total cells (600 and 250 in the \( X \) and \( R \) directions, respectively) were found to be sufficient for grid-independence for both wall boundary conditions while exploiting a relatively minimum computation time.

The robustness of the numerical solutions was then evaluated by applying different classical flow theory conditions to the numerical model (such as pressure-driven flow with no energy generation, or EOF slug flow from Chapter 3) for comparison with the corresponding classical solutions. These model verifications will be presented later for each wall boundary condition scenario.
4.2 Constant Wall Temperature

Solutions to Equation (4.1) will first be presented for a constant wall temperature, \( T_w \), and uniform fluid entrance temperature, \( T_e \). The definitions of \( \theta \) and \( S \) for the constant wall temperature boundary condition are the same as those outlined in Equations (3.2a-b) and are given below.

\[
\theta = \frac{T - T_w}{a^2 \iota e^2 \sigma / k} \tag{4.14a}
\]

\[
S = 1 \tag{4.14b}
\]

Additionally, Equations (3.3) and (3.4a-b) apply as the definitions for the inlet and boundary conditions and are defined as

\[
\theta(X = 0, R) = \theta_e \tag{4.15}
\]

\[
\frac{\partial \theta}{\partial R} \bigg|_{R=0} = 0 \tag{4.16a}
\]

\[
\theta(X, R = 1) = 0 \tag{4.16b}
\]

Recall that \( \theta_e \) is the dimensionless inlet temperature, defined as \( \theta_e = (T_e - T_w) / (\iota e^2 \sigma a^2 / k) \).

For the CWT boundary condition scenario, the function \( S'(S, S_v) \), which represents the combination of Joule heating and viscous dissipation and is derived from non-dimensionalizing the energy generation terms in Equation (2.1), is defined to be

\[
S'(S, S_v) = S + S_v \frac{1}{Z^2} \left( \frac{\partial U}{\partial R} \right)^2 \tag{4.17}
\]

where \( S_v \) is given as \( S_v = \sigma \mu \nu_e^2 / \lambda_\alpha^2 \). Viscous dissipation will be included in the numerical model after non-viscous dissipation studies (\( S_v = 0 \)) are first presented and explored.

For the numerical model, the discretization equation coefficients outlined in Equations
(4.7a-f) are applicable except at the boundary cells, for which separate coefficients will be formulated. The centerline boundary condition formulation, however, was given in Section 4.1 and will not be repeated here. Once local temperature solutions are obtained, the local mean temperatures and Nusselt numbers are calculated using the methods outlined in Section 4.1.

### 4.2.1 Boundary Condition Formulations

The inlet, outlet, and wall boundary cell discretization equations are formulated by following the approach outlined in Section 4.1 for the centerline boundary condition formulation. For the inlet condition, \( \theta(X = 0, R) = \theta_e \), the temperature at the inlet boundary node is simply set equal to the desired inlet temperature, \( \theta_e \), and the discretization equation for the first interior node in the \( X \) coordinate direction [unchanged from Equations (4.6) and (4.7a-f)] directly incorporates that value. The same approach is used for the wall boundary condition, \( \theta(X, R = 1) = 0 \), where \( \theta \) is set equal to zero at the wall boundary node.

The outlet boundary condition is defined by thermally fully-developed conditions at the tube outlet. For the CWT boundary condition scenario, the local temperature distribution ceases to change when fully-developed conditions are attained. Thus, a zero gradient of \( \theta \) in \( X \) was forced at the outlet between the outlet boundary node and the adjacent interior node in the \( X \) direction, where the east control surface is now the boundary surface \([e = B \text{ in Equation (4.5)}]\). Due to the nature of the outlet boundary condition, \( \partial \theta / \partial X|_B = 0 \), which yields \( \theta_B = \theta_p \). Thus, the east node temperature, where \( \theta_E = \theta_B \), becomes non-existent in Equation (4.6), resulting in the same discretization
equation coefficients as those outlined in Equations (4.7a-f), except now with \( a_e = 0 \).

After the numerical solution process is complete, the outlet node temperature is evaluated equal to the temperature at the last interior node. It is important to note that the application of the outlet condition assumes that a sufficient length of tube has been used for fully-developed conditions to truly exist at the outlet. In order to satisfy this assumption, the fully-developed lengths from the analytical solutions for EOF slug flow were used to estimate the minimum tube lengths required, and \( X_{\text{max}} = x_{\text{max}}/\alpha_Pe = 100/Pe \) was found to be sufficient for all parametric cases.

### 4.2.2 Model Verification

Solutions for several flow scenarios with a CWT boundary condition were used to verify the numerical model outlined in Sections 4.1 and 4.2.1. These scenarios include the classical Graetz problem (parabolic velocity profile) with no energy generation [29], classical slug flow with no energy generation [22], and EOF slug flow [Section 3.1]. For the first two scenarios, the cited literature neglected axial conduction in obtaining results. Therefore, the numerical model was tested against such solutions in two stages—first, by applying a low Peclet number to verify the model for fully-developed conditions, and second, by applying a high Peclet number to determine accuracy in the small \( X \) region. Model verification was done in these two stages because a very large Peclet number (which represents more accurately for small \( X \) the zero axial conduction of the classical solutions) would require a very large \( X_{\text{max}} \) in order for results to reach a fully-developed condition. This would require a much larger grid size for grid-independence, thus resulting in an unacceptably high computational time. Additionally, comparisons against EOF slug flow will be presented for two Joule heating scenarios by using the limiting cases.
outlined previously for the dimensionless inlet temperature—\( \theta_e = 0 \) (corresponding to \( T_e = T_w \) or infinitely large Joule heating) and \( \theta_e = 100 \).

**Graetz Problem With no Energy Generation**

For the classical Graetz problem with no source term, numerical results were obtained by replacing the velocity term, \( U(Z,R) \), in Equation (4.1) with the fully-developed pressure-driven flow velocity profile, \( u/\bar{u} = 2(1 - R^2) \). The verification results for the low Peclet number case will be presented first by employing a Peclet number value of \( Pe = 100 \). Figure 4-5 illustrates for this Peclet number the comparison between the classical and numerical solutions accompanied by the magnitude of the percent error as a function of axial position. The local mean temperature exhibits less than 1.1% error between the numerical and classical results. Note that the error experienced for \( X > 0.4 \) is negligible since \( \theta_m \) in that region is so small that the difference between the numerical and classical solution is inconsequential.

Error results for the local Nusselt number are shown in Figure 4-5. With the given low Peclet number value, the numerical results will not match the Graetz solution in the thermally developing region since a low Peclet number results in significant axial con-

![Figure 4-5: CWT Graetz Problem model verification study for \( \theta_m \) and \( Nu_x \) with \( Pe = 100 \).](image_url)
duction. The numerical solution, which takes axial conduction into account, will predict higher local Nusselt numbers than those of the Graetz solution for small $X$. This will be discussed shortly in more detail for the high Peclet number case. For low Peclet number, the difference in the fully-developed regime is only of interest. In this region, the error is shown to be less than 0.2%.

Figure 4-6 illustrates for a high Peclet number ($Pe = 10^4$) the comparison between the classical and numerical solutions accompanied by the percent error as a function of axial position. The local mean temperature exhibits excellent matching between the numerical and classical results, with less than 0.075% error in the numerical solution. The Nusselt number comparison shows little error as well over the $X$ domain (less than 1%) with the exception of deviation for $X \leq 3 \cdot 10^{-5}$. The high deviation in this region is simply due to the finite value of the grid Peclet number, $P$, which is defined as $P = F/D = PeU(Z,R)\delta x/2$. Due to the third-order polynomial grid deployment in the $X$ direction (which gathers most of the cells toward the tube inlet), for very small axial positions, $\delta x$ is so small that $P$ will be less than 5 in that region, which according to Patankar predicts non-negligible axial conduction for the corresponding grid cells [34]. For the increase in $Pe$ from 100 to $10^4$, the value of the grid Peclet number for the range $3 \cdot 10^{-5} \leq X < X_{fd}$ in-

![Figure 4-6: CWT Graetz Problem model verification study for $\theta_m$ and $Nu_x$ with $Pe = 10^4$.](image-url)
creases above $P = 5$ such that axial conduction is predicted to be negligible in this region. Thus, the large difference between the numerical and Graetz solutions observed in Figure 4-5 vanishes. However, because the $\delta x$ values are so small for $X \lesssim 3 \cdot 10^{-5}$, even after significantly increasing the flow Peclet number (i.e., $Pe = 10^8$), the deviation in $Nu_x$ observed for the $Pe = 10^4$ case remains. Therefore, the observed deviation is not necessarily caused by significant error, but is simply a difference caused by the inclusion of axial conduction in the governing energy equation used by the numerical model.

Due to the error being less than 1.1% for both high and low $Pe$, it is concluded that the numerical model sufficiently predicts the classical solution for the Graetz problem with no energy generation and a constant wall temperature boundary condition.

**Slug Flow with no Energy Generation**

Numerical results were obtained for the conditions of classical slug flow with no energy generation by setting the velocity profile to be uniform ($U = 1$) and $1/\theta_e$ equal to zero (this forces zero Joule heating in the fluid). Figure 4-7 illustrates for low Peclet number ($Pe = 100$) the comparison between the classical and numerical solutions together with the percent error as a function of axial position for both $\theta_m$ and $Nu_x$. The local mean temperature exhibits good matching between the numerical and classical results, with less

![Figure 4-7: CWT Slug Flow model verification study for $\theta_m$ and $Nu_x$ with $Pe = 100$.](image)
than 1.1% error in the numerical solution. Note, as before, the error exhibited for large $X$ ($X > 0.25$) can be discarded since $\theta_m$ in that region is so small. Finally, error results are shown for the local Nusselt number, where the error in the fully-developed regime is shown to be less than 0.1%.

The comparison for high Peclet number between the classical and numerical solutions, together with the corresponding percent error as a function of axial position, is shown in Figure 4-8. The local mean temperature variations match for the numerical and classical results, exhibiting less than 0.04% error in the numerical solution. The Nusselt number solutions also show little error over the $X$ domain (less than 0.1%). Since the absolute value of the percent error is plotted in the figure, the downward spikes in the error show that the numerical results cross the classical local Nusselt number variation at the $X$ positions where the spikes exist ($X \approx 10^{-5}$ and $X \approx 0.0011$).

The error for both high and low $Pe$ values has been shown to be less than 1.1%. As a result, the numerical model sufficiently represents the classical solution for the slug flow problem with no energy generation and a CWT boundary condition.
**EOF Slug Flow**

Numerical results were obtained for the conditions of EOF slug flow by setting the velocity profile to be uniform \((U = 1)\) and by including Joule heating energy generation through a finite inlet temperature, \(\theta_e\). Figure 4-9 illustrates for \(\theta_e = 0\) and \(Pe = 5\) the comparison between the analytical (from Chapter 3) and numerical solutions accompanied by the percent error as a function of axial position. The local mean temperature exhibits excellent agreement between the numerical and analytical results, with less than 0.02% error in the numerical solution. Note that the error exhibited for small \(X (X < 0.005)\) can be neglected since \(\theta_m\) in that region is so small. Finally, the numerical error results are shown for the local Nusselt number, where the error in the fully-developed regime is shown to be less than 0.01%.

The comparison between the analytical solution from Section 3.1.1 and the numerical solution, accompanied by the percent error as a function of axial position for \(\theta_e = 100\) and \(Pe = 5\) is illustrated in Figure 4-10. Both the local mean temperature and Nusselt number results match between the analytical solution and numerical results with less than 0.4% error for both parameters.

![Figure 4-9: CWT EOF Slug Flow model verification study for \(\theta_m\) and \(Nu_x\) with \(\theta_e = 0\) and \(Pe = 5\).](image)

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Since the error is less than 0.4% for both high and low \( \theta_e \), it is has been shown that the numerical model reproduces the analytical solution for the EOF slug flow problem with a constant wall temperature boundary condition given in Section 3.1.

### 4.2.3 Constant Wall Temperature Results

Parametric studies with varying \( \theta_e \) and \( Pe \) for a given value of \( Z \) and a CWT boundary condition revealed that the dimensionless local temperature profile development, local mean temperature, and local Nusselt number follow trends very similar to those illustrated for EOF slug flow (\( Z \rightarrow \infty \)) in Section 3.1.2. Therefore, the results and subsequent physical interpretations will not be repeated here. Rather, this section will focus primarily on variations of the capillary radius-to-Debye length ratio, \( Z \), and the dimensionless viscous dissipation parameter, \( S_v \), for different values of \( \theta_e \). Viscous dissipation will be initially neglected for the \( Z \) parametric studies by setting \( S_v \) equal to zero.

#### Radius-to-Debye Length Ratio Studies

The development of the local dimensionless fluid temperature \( \theta(X,R) \) is illustrated in Figure 4-11 for \( \theta_e = 100 \), \( Pe = 5 \), \( S_v = 0 \) and four different values of the radius-to-Debye length ratio: \( Z = 100, 3, 1.5, \) and 0.5. Recall that \( \theta_e = 100 \) and \( Pe = 5 \) are typical
values of the dimensionless inlet temperature and Peclet number as mentioned in Section 3.1.2. Also recall that for $Z$ values near a magnitude of 500 the velocity field is nearly uniform, while for small $Z$ it is nearly parabolic (see Figure 1-2). The radius-to-Debye length ratio of $Z = 100$ represents the typical ratio value that might be encountered for distilled water ($\mu \approx 9 \times 10^{-4}$ N·s/m², $c \approx 0.0004$ mM) in a fused-silica tube with a radius of about 50 µm. The radius-to-Debye length ratio of $Z = 0.5$ represents the limiting case of small values where the Debye length encompasses the whole diameter of the capillary.

Figure 4-11 shows that the development of the local temperature profiles occurs more rapidly for decreasing $Z$. For $X = 0.001$, the temperature profiles for each $Z$ value are
nearly the same. At the other extreme, the fully-developed temperature profile is identical for all Z values. This result is expected since axial advection is equal to zero and no additional energy sources to Joule heating are present. Thus, the CWT fully-developed condition, where the temperature gradient at the channel wall is just sufficient to completely dissipate the thermal energy generated by Joule heating, applies to all Z cases.

Figure 4-12 illustrates the variation of dimensionless local mean temperature with normalized streamwise position for $\theta_e = 100$, $Pe = 5$, $S_v = 0$ and four different values of the radius-to-Debye length ratio: $Z = 100$, 10, 2, and 0.5. The EOF slug flow solution for $Z \rightarrow \infty$ from Equation (3.9) is shown for comparison. The mean temperature distributions—each decreasing monotonically with increasing $X$ from the inlet temperature to a fully-developed condition—follow unique behavior as Z is varied from 100 to 0.5. For $Z = 100$, as expected, $\theta_m$ follows a monotonic decrease with increasing streamwise position very similar to that of EOF slug flow for the specified inlet temperature and Peclet number. For smaller magnitudes of $Z$, the mean temperature initially develops at a slower

Figure 4-12: $\theta_m$ vs. $X$ for $\theta_e = 100$, $Pe = 5$, $S_v = 0$ and $Z = 100$, 10, 2, and 0.5.
rate than the $Z = 100$ scenario. Such is the case for $Z = 10$, where $\theta_m$ exhibits higher magnitudes for small $X$ and then asymptotes to a fully-developed value slightly greater than that of the $Z = 100$ case. However, as $Z$ is further decreased, the variation of $\theta_m$ in $X$ falls below that of the $Z = 100$ case towards the end of the thermal development, as seen for $Z = 2$. Note that the point where this occurs ($\theta_m$ falls below the $Z = 100$ case) moves upstream for decreasing $Z$. Thus, it can be concluded that for decreasing $Z$ (where $Z < 10$), the $\theta_m$ behavior as a function of $X$ will fall below the variations of $\theta_m$ for all higher $Z$ values at some point in the thermal development. However, when the $\theta_m$ variations for all $Z$ asymptote to fully-developed conditions, these trends are reversed, finally yielding fully-developed mean temperature values that vary monotonically in $Z$ between the two limiting cases—slug flow at $Z > 500$ ($\theta_m = 0.125$) and near-parabolic flow for very small $Z$ (derived independently to be $\theta_m = 0.167$) [21].

As mentioned at the beginning of this section, parametric studies in $Z$ were performed for several different values of the inlet temperature, $\theta_e$, thus providing insights into the effect of $\theta_e$ on the parametric variation of the radius-to-Debye length ratio. Recall that $\theta_e$ is an indirect measure of the strength of the Joule heating, where higher $\theta_e$ reflects lower volumetric energy generation. Figure 4-13 illustrates the effect of $\theta_e$ on the mean temperature variations for $Pe = 5$, $S_v = 0$, and three different values of the inlet temperature: $\theta_e = 10$, 1, and 0.1. For each $\theta_e$ value, $Z$ is varied from 100 to 0.5, with $Z \to \infty$ shown for comparison. For $\theta_e = 10$, the mean temperature distributions for all $Z$ follow roughly the same trends shown for the $\theta_e = 100$ case, where $\theta_m$ follows a monotonic decrease with increasing $X$ and variations in $\theta_m$ for $Z < 10$ cross from above to below the
mean temperature variations for all larger Z values. As the inlet temperature is decreased to $\theta_e = 0.1$, the figure shows that the crossing trends in the mean temperature variations vanish. Additionally, note that the fully-developed mean temperature is independent of the inlet temperature for a given value of Z, just as was concluded for EOF slug flow with an imposed constant wall temperature boundary condition in Chapter 3.

The local Nusselt number distribution is also a strong function of the radius-to-Debye length ratio, as shown in Figure 4-14 for $\theta_e = 100$, $Pe = 5$, $S_v = 0$ and Z ranging from 100 to 0.5. The variation in local Nusselt number for classical slug flow with Joule heating and no viscous dissipation is shown for comparison. As $X \rightarrow 0$ the Nusselt num-

Figure 4-13: $\theta_m$ vs. X for $Pe = 5$, $S_v = 0$, and $\theta_e = 10$, 1, and 0.1, with Z ranging from 0.5 to 100 for each $\theta_e$ value.
Figure 4-14: $\text{Nu}_x$ vs. $X$ for $\theta_e = 100$, $Pe = 5$, $S_v = 0$ and $Z$ ranging from 100 to 0.5.

Number variations converge for all $Z$ cases, thus $\text{Nu}_x$ is plotted for the axial range $0.02 \leq X \leq 4$ to more clearly illustrate its strong dependence on $Z$ in that region. Due to the given $\theta_e$ value, the Nusselt number distribution follows the general trend exhibited by EOF slug flow, where $\text{Nu}_x$ falls to a minimum and then rises to a fully-developed value. This fully-developed value is a function of $Z$ and decreases from that of classical EOF slug flow ($\text{Nu}_{fd} = 8$) for decreasing $Z$. For $Z = 100$, the Nusselt number distribution is slightly below that of the EOF slug flow solution for all $X$, asymptoting to a fully-developed value of $\text{Nu}_{fd} = 7.85$. As $Z$ is decreased to $Z = 5$, the local Nusselt number decreases from this distribution for all $X$ positions. However, at this radius-to-Debye length ratio ($Z = 5$), the thermal development length has begun to move upstream, resulting in earlier completion of the thermal development. This deviation becomes more significant as $Z$ becomes very small, with fully-developed values converging to that derived from the classical solution for parabolic flow with uniform internal energy generation and a constant wall temperature boundary condition, $\text{Nu}_{fd} = 6$ [21].
Figure 4-15 shows the effect of $\theta_e$ on the local Nusselt number variations for $Pe = 5$, $S_v = 0$, and three different values of the inlet temperature: $\theta_e = 1$, 0.3, and 0.2. These inlet temperature values were selected to illustrate the effect of $Z$ on the critical value $\theta_e = 0.18$ described in Section 3.1.2 for the CWT case of EOF slug flow. For each $\theta_e$ value, $Z$ is varied from 100 to 0.5, with $Z \to \infty$ shown for comparison. For $\theta_e = 1$, the local Nusselt number variations follow trends similar to those illustrated for $\theta_e = 100$. For each $Z$ value, $Nu_x$ falls below the respective fully-developed Nusselt number during the thermal development, then rises, asymptoting to the fully-developed condition. Note that as $\theta_e$ is decreased, there is a shift in heat transfer dominance from wall convection to Joule heat-
ing as $\theta_e$ decreases from 0.2 to 0.1, which straddles the critical 0.18 value. This characteristic, as shown by Figure 4-15, is found to be independent of $Z$.

The thermal entrance lengths will now be plotted as a function of radius-to-Debye length ratio. As done before in Chapter 3 for the EOF slug flow results, the entrance length, $X_{fd}$, is defined as the $X$ location where the local Nusselt number is within 0.1% of its fully-developed value, which value depends on the specified magnitude of $Z$. However, due to the discrete nature of the streamwise positions created by the discretized domain, it is difficult to obtain precise results, especially when the grid distribution is course compared to the range in which $X_{fd}$ varies for a given parametric study. Established interpolation methods can be used to increase the accuracy in calculating $X_{fd}$. For all data presented in this chapter, a parabolic spline curve was fitted between consecutive data points for interpolation purposes by using Mathcad®.

Figure 4-16 illustrates $X_{fd}$ as a function of $Z$ ($Z$ ranging from 100 to 0.5) for $Pe = 5, S_v = 0$ and four values of dimensionless inlet temperature: $\theta_e = 100, 10, 1, \text{ and } 0.1$. For

![Figure 4-16: $X_{fd}$ vs. $Z$ for $Pe = 5, S_v = 0$, and $\theta_e = 100, 10, 1, \text{ and } 0.1$.](image)
the parametric study in $Z$ where $\theta_e = 100$, the entrance length is nearly constant for $Z > 10$ at $X_{fd} = 1.626$, which is within 0.06\% of the EOF slug flow entrance length under the same parametric conditions. For $Z < 10$, which is the critical value where the EDL’s begin to overlap (see Section 1.1), $X_{fd}$ decreases as $Z$ is decreased, corresponding with the more rapid thermal development observed in the local temperature development plots of Figure 4-11. This trend yields an entrance length of $X_{fd} = 1.03$ for $Z = 0.5$, which is still significantly longer than for classical pressure-driven flow with no volumetric heating ($X_{fd} \approx 0.1$) [29]. The change in thermal development length with varied $Z$ for other $\theta_e$ values follow trends similar to that for $\theta_e = 100$, with the entrance length for $Z \to \infty$ in each $\theta_e$ case being within 0.1\% of the EOF slug flow entrance length under similar parametric conditions, which entrance length varies with $1/\theta_e$ as shown in Figure 3-7.

*Viscous Dissipation Studies*

Solutions for Equation (4.1) will now be presented for non-zero values of the dimensionless viscous dissipation parameter, $S_v$. Recall that $S_v$ can be defined as $S_v = \frac{\sigma \mu \mu^2}{\lambda \lambda_0^2}$ as mentioned at the beginning of the chapter. Realistic values of this parameter range from 0 (negligible viscous dissipation) to 30. The latter value represents the maximum typical value of $S_v$ that might be encountered for an aqueous solution ($\sigma \approx 10^5$ $\Omega \cdot m$, $\mu \approx 0.001$ N s/m$^2$) with a Debye length of 0.25 $\mu$m and electro-osmotic mobility of about $1.4 \cdot 10^{-7}$ m$^2$/V·s [10]. Predictions reveal that for large inlet temperature values (e.g., $\theta_e = 100$) realistic values of $S_v$ are not large enough to produce noticeable effects in the heat transfer characteristics. Therefore, a small inlet temperature value of $\theta_e = 0.1$, where viscous dissipation is more significant relative to wall convection, was used to obtain
most of the data presented in this section. Additionally, results provided for thermally fully-developed EOF studies with viscous dissipation and a CWT boundary condition show that viscous dissipation effects are most significant when $Z \approx 4 – 7$ [20]. As a result, $Z = 7$ was used here to obtain most of the presented data.

The development of the local dimensionless fluid temperature $\theta(X,R)$ is illustrated in Figure 4-17 for $\theta_e = 0.1$, $Pe = 5$, $Z = 7$ and four different values of the viscous dissipation parameter: $S_v = 1, 5, 10,$ and $30$. Recall that for a dimensionless inlet temperature less than 0.18, as described in Section 3.1.2., energy generation (Joule heating) prevails over wall convection over the majority of the tube radius, thus causing local temperatures

Figure 4-17: Dimensionless temperature profiles at various $X$ locations in the thermally developing region for $\theta_e = 0.1$, $Pe = 5$, $Z = 7$ and viscous dissipation parameter values of (a) $S_v = 1$, (b) $S_v = 5$, (c) $S_v = 10$, and (d) $S_v = 30$. 

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in that region to increase for increasing streamwise position. In the region near the tube wall, however, wall convection dominates, causing an initial decrease in local temperatures in that region. As noted before in Chapter 3, this locally concentrated prevalence of wall convection vanishes for succeeding increasing streamwise position as the energy generation gains complete dominance.

The presence of viscous dissipation as a source of energy generation, however, alters the trends described above significantly for increasing $S_v$. Figure 4-17 shows that as $S_v$ is increased from zero, the added contribution of viscous dissipation to the energy generation by Joule heating yields temperatures that are higher for all $X$. For $S_v = 1$ in Figure 4-17(a), the temperature profiles are slightly above that of the $S_v = 0$ case ($\theta_{R=0} = 0.25$). For $X < 0.01$, the difference in local temperature distributions between varied $S_v$ values is very slight as viscous dissipation has not had sufficient time to contribute significantly to the temperature development. For large $X (X > 0.01)$, local temperatures rise at increasingly higher rates for increasing $S_v$.

For large $S_v$, such as that seen in the $S_v = 30$ case in Figure 4-17(d), the significant influence of viscous dissipation near the wall due to high velocity gradients in that region will be large enough to completely eliminate any prevalence of wall convection except at very small values of $X$. This yields, for $X > 0.001$, increasing local temperatures for increasing streamwise position for all radial locations. As well, the concentration of viscous dissipation near the wall causes slightly higher temperatures in that region, as seen for $S_v = 30$ at $X = 0.01$. Regardless of parametric values, the thermal development will reach fully-developed conditions where the wall heat flux balances the energy generation within the fluid. Thus, the fully-developed temperature profile will be higher
for increasing $S_v$ as the wall temperature gradient must dissipate the higher total amount of energy generation. In addition, note that fully-developed conditions are reached at larger streamwise positions for increasing $S_v$, thus allowing the conclusion that increasing viscous dissipation has an overall lengthening effect on the thermal entrance region.

For specified parametric values of $\theta_e = 0.1$, $Pe = 5$, and $Z = 7$, the local mean temperature variations are shown in Figure 4-18 for the range of $S_v$ values from $S_v = 0$ to $S_v = 30$, where $S_v = 0$ represents the limiting case for zero viscous dissipation, or infinitely large Joule heating relative to viscous dissipation. The $\theta_m$ variation for $S_v = 1$ follows a monotonic increase in temperature that is slightly above the variation for $S_v = 0$ due to the higher amount of total energy generation. The same trend is observed for all $S_v$ cases, where the local mean temperature variations follow increasingly higher variations for increasing $S_v$. The figure also shows the slight variation in entrance length for varied $S_v$ values. As $S_v$ increases, fully-developed conditions are reached at increasingly longer entrance lengths.

![Figure 4-18: $\theta_m$ vs. $X$ for $\theta_e = 0.1$, $Pe = 5$, $Z = 7$, and $S_v$ ranging from 0 to 30.](image-url)
The variation in local Nusselt number is also a strong function of $S_v$. Figure 4-19 illustrates $Nu$ for $\theta_e = 0.1$, $Pe = 5$, $Z = 7$, and $S_v$ ranging from 0 to 30. Initially, as $X \to 0$, the variations in local Nusselt number coincide for all magnitudes of $S_v$ due to the dominance of wall convection in the very early stages of the thermal development. As noted previously for EOF slug flow under similar inlet temperature conditions ($\theta_e \approx 0.18$), the heat transfer deviates from wall convection-dominated conditions at small $X$ where the local Nusselt number lies above $Nud = 8.0$ and the Nusselt number decreases monotonically to the fully-developed condition at $X \to \infty$. However, the addition of viscous dissipation causes an increase in $Nud$ for all $X$. This is due to the additional presence of viscous dissipation coupled with Joule heating, thus requiring a higher wall heat flux to dissipate the increased amount of generated energy at fully-developed conditions [20]. Note that for a high value of the viscous dissipation parameter, such as for $S_v = 30$, the developing local Nusselt number variation does not follow the same curvature as for small $S_v$ values.

Figure 4-19: $Nu$ vs. $X$ for $\theta_e = 0.1$, $Pe = 5$, $Z = 7$, and $S_v$ ranging from 0 to 30.
The entrance length is also a function of the viscous dissipation parameter, as shown in Figure 4-20 with $S_v$ ranging from 0.01 to 30 for $Pe = 5$, $Z = 7$ and four values of dimensionless inlet temperature: $\theta_e = 10$, 1, 0.2, and 0.1. $X_{fd}$ is calculated by applying the 0.1% Nusselt number criterion used for previous entrance length calculations. For the parametric study in $S_v$ where $\theta_e = 0.1$, for very small values of $S_v$ the entrance length converges to that for $S_v = 0$ under the prescribed parametric conditions ($X_{fd} = 0.651$). As $S_v$ increases, $X_{fd}$ increases to a value of 0.815 for $S_v = 30$. This trend agrees with the thermal development behavior observed previously for $\theta_m$ and $Nu_x$, where the presence of viscous dissipation tends to lengthen the thermally developing region.

For larger inlet temperature values ($\theta_e \gtrsim 0.18$), however, the entrance length as a function of $S_v$ follows very different trends for $S_v > 1$. For $\theta_e = 0.2$, as shown in Figure 4-20, $X_{fd}$ decreases slightly from the case of no viscous dissipation under the same prescribed parametric conditions ($X_{fd} = 0.491$) until at $S_v \approx 1$ it begins to decrease rapidly to a minimum at $S_v \approx 5$. For $S_v > 5$, $X_{fd}$ increases to a value of 0.73 for $S_v = 30$. These trends

Figure 4-20: $X_{fd}$ vs. $S_v$ for $Pe = 5$, $Z = 7$, and $\theta_e = 10$, 1, 0.2, and 0.1.
suggest the existence of a critical $S_v$ value ($S_v \approx 5$ for $\theta_e = 0.2$), where the local Nusselt number distribution ceases to fall below its fully-developed value to a local minimum. A similar characteristic was observed for EOF slug flow in Section 3.1.2, where for increasing $\theta_e$ the entrance length reaches a minimum at $\theta_e \approx 0.18$. Recall that for EOF slug flow, increasing Joule heating resulted in the reduced influence of wall convection at earlier axial positions in the thermal development. However, for a specified inlet temperature and varied $S_v$, it is the increasing viscous dissipation that counteracts the dominance of wall convection. Thus, the local minimum in $X_{fd}$ occurs at higher $S_v$ values as $\theta_e$ is increased since more viscous dissipation would be required to counter the dominance of wall convection.

Once $\theta_e$ is high enough that the maximum relative magnitude of viscous dissipation ($S_v = 30$) is not sufficient to eliminate the local minimum in the Nusselt number distribution, no minimum entrance length is evident for the specified range of $S_v$ ($0 < S_v < 30$). This is exemplified by the $X_{fd}$ vs. $S_v$ plots for $\theta_e = 1$ and $\theta_e = 10$ in Figure 4-20. For $\theta_e = 1$ and small values of $S_v$, the entrance length converges to that for $S_v = 0$ ($X_{fd} = 1.297$). As $S_v$ increases, $X_{fd}$ moves downstream slightly due to the lengthening effect of viscous dissipation observed for $\theta_e = 0.1$ in Figure 4-20. However, at $S_v \approx 7$, $X_{fd}$ reaches a maximum and begins to move upstream. Eventually, it is expected that $X_{fd}$ would reach a minimum at an $S_v$ value above $S_v = 30$ where the viscous dissipation counters the wall convection dominance. However, since this value would be outside the applicable range of $S_v$, it is not of interest for the present studies.

For other $Z$ values where viscous dissipation does not have as significant of an effect, as exemplified in Figure 4-21 for $Z = 1$ and 100 (see Maynes and Webb [20]), the
same trends exhibited for $\theta_e \lesssim 0.18$ shown in Figure 4-20 are observed. Recall from Figure 4-15 that the critical point in the entrance length for varied inlet temperature ($\theta_e \approx 0.18$) holds for all $Z$ values. For the $Z = 1$ case, trends very similar to those of the $Z = 7$ case are observed for all $\theta_e$. However, the entrance length is considerably shorter than for $Z = 7$, showing that viscous dissipation has an increasing effect on the thermally developing region for small decreasing $Z$. For $Z = 100$, the entrance lengths follow the same relative trends as those exhibited by the $Z = 7$, $\theta_e = 1$ case for all $\theta_e \gtrsim 0.18$. This is due to the concentration of viscous dissipation near the tube wall where the only significant velocity gradients exist (see Figure 1-2). Though these gradients are high, the spatial area in which viscous dissipation has an influence is very small [20]. As a result, a significantly higher $S_v$ value ($S_v \gg 30$) would be required for all $\theta_e \gtrsim 0.18$ to counterbalance the wall convection and result in a local minimum in the entrance length distribution.

Beyond the parametric studies presented earlier both for $Z$ and $S_v$, it is of interest to explore the effects of $Z$ on non-zero values of the viscous dissipation parameter. For this reason, predicted results were obtained for given parametric values of $\theta_e = 0.1$, $Pe =$
Figure 4-22: Non-dimensional temperature profiles at various X locations in the thermally developing region for $\theta_e = 0.1$, $Pe = 5$, $S_v = 30$ and radius-to-Debye length ratios of (a) $Z = 100$, (b) $Z = 7$, (c) $Z = 2$, and (d) $Z = 0.5$.

5, $S_v = 30$, and $Z$ ranging from 0.5 to 100. Figure 4-22 illustrates the local temperature development for these parametric values, with $Z = 100$, 7, 2, and 0.5. For $Z = 100$, the temperature profiles are slightly above that of the $S_v = 0$ case ($\theta_{fd,R=0} = 0.25$) for all $X$, with similar regions of energy generation and wall convection dominance as exhibited previously for EOF slug flow in Section 3.1.2. As $Z$ decreases to a value of 7, the temperature profiles are higher in the fluid core region of the circular capillary compared to the $Z=100$ case. Recall that for $Z = 7$, viscous dissipation is significant, thus yielding a concentration of total energy generation near the wall that minimizes the direct effect of wall convection there. As $Z$ is further decreased to $Z = 2$, the fully-developed temperature
profile reaches a maximum compared to those of all magnitudes of $Z$. Note also that early in the thermal development ($0.01 \leq X \leq 0.1$), a maximum occurs in the local temperature profile at $R \approx 0.5 - 0.7$ rather than at the centerline axis ($R = 0$). The most significant of this characteristic occurs near $X \approx 0.05$. As $Z$ is decreased to a radius-to-Debye length ratio of 0.5, the local temperatures in the fluid core at any specified $X$ position decrease and the thermal development approaches that similar to the $Z = 100$ case.

The thermal development behavior observed above can be further explored by the local mean temperature variation for different values of $Z$. Figure 4-23 presents the $\theta_m$ variations for $\theta_e = 0.1$, $Pe = 5$, $S_v = 30$, and $Z$ ranging from 0.5 to 100. For $Z = 100$, the mean temperature variation varies very little with $X$, with the mean temperature initially decreasing before rising to a fully-developed value. As $Z$ is decreased, $\theta_m$ follows a monotonic increase for all $X$ to a fully-developed condition, which variation rises with decreasing $Z$ until $Z \approx 2$, where it reaches a maximum and then falls to a variation in $X$ for $Z = 0.5$ that is still greater than that observed for the $Z = 100$ case.

Figure 4-23: $\theta_m$ vs. $X$ for $\theta_e = 0.1$, $Pe = 5$, $S_v = 30$ and $Z$ ranging from 100 to 0.5.
The variations in local Nusselt number for the same conditions above, as shown in Figure 4-24, yield results that are significantly different from that observed for non-viscous dissipation studies at the beginning of this section. Initially, as $X \to 0$, the variations in local Nusselt number coincide for all magnitudes of $Z$ due to the dominance of wall convection in the very early stages of the thermal development. As noted previously for EOF slug flow under similar inlet temperature conditions ($\theta_e < \sim 0.18$), the heat transfer deviates from wall convection-dominated conditions at small $X$ where the local Nusselt number lies above $N_{ufd} = 8.0$ and the Nusselt number decreases monotonically to fully-developed conditions at $X \to \infty$. However, variations in $Z$ cause significant shifting of the Nusselt number development throughout the range of $Z$. For $Z > 10$ the $N_{ux}$ plots follow increasingly higher trends for all $X$ positions until $Z \approx 10$ where the $N_{ux}$ variation reaches a maximum. Then, for $Z < 10$ the magnitude of the Nusselt number decreases at all $X$ and the thermal development is more rapid for $Z > 10$, eventually falling below the $N_{ux}$ behavior for the $Z = 100$ case at $Z = 0.5$. 

**Figure 4-24: $N_{ux}$ vs. $X$ for $\theta_e = 0.1$, $Pe = 5$, $S_v = 30$ and $Z$ ranging from 100 to 0.5.**
Similar to what was observed for parametric studies in $S_v$ for $\theta_e = 10$, $Pe = 5$, and $Z = 7$, the entrance length is a unique function of the radius-to-Debye length ratio for non-zero values of the viscous dissipation parameter, as shown in Figure 4-25 with $\theta_e = 0.1$, $Pe = 5$, and $Z$ ranging from 0.5 to 100 for three values of $S_v$: $S_v = 30, 5, 1$, and 0.01. The $X_{fd}$ values were calculated by applying the 0.1% Nusselt number criterion used previously along with the parabolic spline interpolation method to obtain sub-grid accuracy. For $Z$ ranging from 0.5 to 5, $X_{fd}$ follows similar trends as those observed for non-viscous dissipation results as shown in Figure 4-16. As $Z$ increases above this range, the entrance length eventually reaches a maximum at $Z \approx 7$ and then begins to decrease. For $Z \geq 10$, $X_{fd}$ moves steadily upstream to $X_{fd} = 0.712$ for $Z = 100$ and $S_v = 30$. For the $S_v = 0.01$ case, however, only a slight maximum entrance length characteristic is exhibited at $Z \approx 10$.

![Figure 4-25: $X_{fd}$ vs. $Z$ for $\theta_e = 0.1$, $Pe = 5$ and $S_v = 30, 5, 1$, and 0.01.](image-url)
**CWT Results Summary**

The local mean temperatures, local Nusselt numbers, and thermal entrance lengths have been studied parametrically for varied $Z$, $\theta_\text{e}$, and $S_v$. Parametric studies in $Z$ reveal a significant dependence of $\theta_m$, $N_u$, and $X_{fd}$ on $Z$. For large $\theta_e$ values ($\theta_e \geq 1$), the mean temperature variation decreases monotonically for all $Z$ and asymptotes to a fully-developed value that varies with $Z$. For $Z < 10$, the $\theta_m$ variations fall below those for all higher $Z$ values. The local Nusselt number and thermal entrance lengths, however, decrease for all $X$ as $Z$ decreases for all applicable values of $\theta_e$.

With the inclusion of viscous dissipation, such that $S_v$ is non-zero, the local temperature profiles develop more rapidly to higher fully-developed profiles with increasing $S_v$ due to the additional energy generation contribution to the thermal development. For large $S_v$ and $Z = 7$, where viscous dissipation has the greatest influence, wall convection is dominant in a smaller region than that for small magnitudes of $S_v$ due to the more significant influence of viscous dissipation near the wall. For small $Z$ ($Z \approx 2$), the viscous dissipation causes a maximum in the temperature profile at intermediate $X$ positions that occurs in between the centerline axis and the wall. With regards to the variations in local mean temperature and Nusselt number, viscous dissipation simply increases the $\theta_m$ and $N_u$ magnitudes for all $X$ as $S_v$ increases due to the increased total amount of volumetric energy generation.

The change in thermal entrance length with varied $S_v$ is a strong function of the magnitudes of $Z$ and $\theta_e$. For $Z = 7$ and $\theta_e \leq 0.18$, $X_{fd}$ increases slightly with increasing $S_v$, which trend is observed for all inlet temperature magnitudes where $Z = 100$. For $\theta_e \geq 0.18$, the entrance length follows unique trends with varied $S_v$. For inlet temperatures near
the critical value of 0.18, the increasing presence of viscous dissipation relative to Joule heating results in a minimum \( X_{fd} \) magnitude. This characteristic is due to the shift in the Nusselt number distribution from a local minimum trend to a monotonic trend. This is observed for small \( Z \) where viscous dissipation effects are spread across the whole radial domain instead of being concentrated at the wall (e.g., \( Z = 100 \)) where it has little effect.

4.3 **CONSTANT WALL HEAT FLUX**

The solutions to Equation (4.1) are now presented for an imposed constant wall heat flux, \( q''_w \), and uniform fluid entrance temperature, \( T_e \). The definitions of \( \theta \) and \( S \) for the constant wall heat flux boundary condition are the same as those outlined in Equations (3.12a-b) and are repeated below.

\[
\theta = \frac{T - T_e}{q''_w a/k} \tag{4.18a}
\]

\[
S = \frac{a i_e^2 \sigma}{q''_w} \tag{4.18b}
\]

Additionally, Equations (3.13) and (3.14a-b) apply as the definitions for the inlet and boundary conditions and are defined as

\[
\theta(X = 0, R) = 0 \tag{4.19}
\]

\[
\left. \frac{\partial \theta}{\partial R} \right|_{R=0} = 0 \tag{4.20a}
\]

\[
\left. \frac{\partial \theta}{\partial R} \right|_{R=1} = 1 \tag{4.20b}
\]

For the CHF boundary condition scenario, the function \( S'(S, S_e) \) in Equation (4.1), which represents the combination of Joule heating and viscous dissipation and is derived from
non-dimensionalizing the energy generation terms in Equation (2.1), is defined as

\[ S'(S, S_v) = S \left( 1 + S_v \frac{1}{Z^2} \left( \frac{\partial U}{\partial R} \right)^2 \right) \]  

(4.21)

where \( S_v \) is given as \( S_v = \frac{\sigma \mu u^2}{\lambda_d^2} \). As for the CWT case studies, viscous dissipation will be included in the numerical model after non-viscous dissipation studies (\( S_v = 0 \)) are first presented and explored. For the numerical model, the discretization equation coefficients outlined in Equations (4.7a-f) are applicable except at the boundary cells, where separate coefficients will be formulated. The centerline boundary condition formulation given in Section 4.1 will not be repeated here. Once local temperature solutions are obtained, the local mean temperatures and Nusselt numbers are calculated using the methods outlined in Section 4.1.

### 4.3.1 Boundary Condition Formulations

The inlet, outlet, and wall boundary cell discretization equations are formulated by using approaches similar to those outlined in Section 4.2.1 for the CWT boundary condition scenario. For the inlet condition \( \theta(X = 0, R) = 0 \), the temperature at the inlet boundary node is simply set equal to zero and the discretization equation for the first interior node in the \( X \) coordinate direction [unchanged from Equations (4.6) and (4.7a-f)] directly incorporates that value.

The wall boundary condition \( \frac{\partial \theta}{\partial R}\big|_{R=1} = 1 \) follows the same numerical approach as used in the implementation of the centerline boundary condition, except in this case the boundary temperature gradient is equal to one instead of zero and is applied at the wall boundary cell instead of at the centerline. This condition is implemented by directly
setting $\frac{\partial \theta}{\partial R}\big|_B = 1$ in Equation (4.9). As a result, $\theta_N = \theta_B$ is nonexistent in the discretization equation, and after simple algebra the resulting coefficients differ from Equations (4.7a-f) only in $a_N$ and $b$, where

$$a_N = 0 \quad \text{(4.22a)}$$

$$b = S'(S,S_e)\Delta X + \frac{\Delta X}{R_p\Delta R} \quad \text{(4.22b)}$$

The last term of the $b$ coefficient results from the implementation of $\frac{\partial \theta}{\partial R}\big|_B = 1$ into Equation (4.9). The wall boundary temperature, $\theta_N$, is evaluated in terms of the first interior node temperature, $\theta_p$, by using the discrete form of the wall boundary condition, yielding

$$\theta_N = \theta_p + \frac{\Delta R}{2} \quad \text{(4.23)}$$

As stated previously, the outlet boundary condition is defined by the thermally fully-developed nature of the flow at the tube outlet. However, implementation of this condition is not as straightforward as with the CWT boundary condition scenario. At thermally fully-developed conditions, the gradient of $\theta$ in $X$ is constant (not zero as for the CWT case) and thus, the east coefficient, $a_E$, cannot simply be set to zero as before. Instead, fully-developed conditions require that $\frac{\partial \theta}{\partial X}$ match the mean temperature gradient, which from an energy balance is defined as

$$\frac{\partial \theta}{\partial X} = \frac{d\theta_m}{dX} = \frac{2(S'(S,S_e) + 2)}{\bar{U}_\text{max}(Z)} \quad \text{(4.24)}$$

The fully-developed condition in Equation (4.24) is forced at the outlet by applying finite-difference methods to define the boundary node temperature as a function of the
last interior node temperature, which yields the equation

\[
\theta_E = \theta_p + \frac{\Delta X (S'(S, S_v) + 2)}{U_{\text{max}}(Z)}
\]  \hspace{1cm} (4.25)

Substituting this definition into Equation (4.6) results in discretization equation coefficients that differ from those of Equations (4.7a-f) only in \(a_E\) and \(b\), defined as

\[
a_E = 0 \hspace{1cm} (4.26a)
\]

\[
b = a'_E \frac{\Delta X (S'(S, S_v) + 2)}{U_{\text{max}}(Z)} + S'(S, S_v)\Delta X \hspace{1cm} (4.26b)
\]

where \(a'_E\) is defined by the original power-law definition of \(a_E\) given in Equation (4.7a).

After the numerical solution process is complete, the outlet boundary temperature, \(\theta_E\), is evaluated in terms of the adjacent interior node temperature, \(\theta_p\), by using Equation (4.25). As noted previously for the CWT boundary condition scenario, it is important to note that the application of the outlet condition assumes that a sufficient length of tube has been used for fully-developed conditions to truly exist at the outlet. Again, the fully-developed lengths from the analytical solutions for EOF slug flow were used to estimate the minimum tube lengths required, and \(X_{\text{max}} = x_{\text{max}}/\alpha Pe = 100/Pe\) was found to be sufficient for all parametric cases.

### 4.3.2 Model Verification

Solutions for several flow scenarios with a CHF boundary condition were used to verify the numerical model outlined in Sections 4.1 and 4.3.1. These scenarios include the classical Graetz problem with no energy generation [29], classical slug flow with no energy generation [22], and EOF slug flow [Section 3.2]. As mentioned previously, the cited literature for the first two scenarios neglected axial conduction to obtain their...
results. Therefore, as before for the CWT boundary condition, the numerical model was tested against such solutions in two stages—first by applying a low Peclet number to verify the model for fully-developed conditions, and second by applying a high Peclet number to determine accuracy in the small $X$ region. Recall that the model verification was performed in these two stages because a very large Peclet number results in a longer $X$ domain and thus requires a larger grid size for grid-independence, which results in unrealistic computational times. Comparisons against EOF slug flow will be presented for two Joule heating scenarios with given values for the source term—$S = 0.01$ and $S = 1$.

**Graetz Problem With no Energy Generation**

For the classical Graetz problem with no source term, numerical results were obtained by replacing the velocity term, $U(Z,R)$, in Equation (4.1) with the fully-developed pressure-driven flow velocity profile, $u/\bar{u} = 2(1-R^2)$. The verification results for the low Peclet number case will be presented first by employing a Peclet number value of $Pe = 100$. Figure 4-26 illustrates for this Peclet number the comparison between the classical and numerical solutions accompanied by the magnitude of the percent error as a function of axial position. The local mean temperature exhibits less than 1.1% error between the numerical and classical results. Note that the error exhibited for small $X (X <$

![Figure 4-26: CHF Graetz Problem model verification study for $\theta_m$ and $Nu_x$ with $Pe = 100$.](image)

80
0.03) can be ignored since $\theta_m$ in that region is so small. Finally, error results are shown for the local Nusselt number. As before for the CWT boundary condition model verification, due to a low Peclet number value, the numerical results will not match the Graetz solution in the thermally developing region since the low Peclet number results in significant axial conduction. Thus, the numerical solution, which takes axial conduction into account, will predict higher local Nusselt numbers than those of the Graetz solution in this $X$ region. Therefore, the fully-developed regime is only of interest. In this region, the error is shown to be less than 0.1%.

Figure 4-27 illustrates for high Peclet number ($Pe = 10^4$) the comparison between the classical and numerical solutions accompanied by the percent error as a function of axial position. The $\theta_m$ variation exhibits very good matching between the classical and numerical results due to less than 0.6% error in the numerical solution. Note that the error experienced for $X < 4 \cdot 10^{-5}$ can be discarded since $\theta_m$ in that region is so small that the difference between the numerical and classical solution is negligible. The Nusselt number comparison shows little error as well over the $X$ domain (less than 1.2%) with the exception of significant deviation for $X \leq 0.0005$. This high deviation is simply due to the small value of the grid Peclet number in this region, which predicts non-negligible ax-

![Figure 4-27: CHF Graetz Problem model verification study for $\theta_m$ and $\nu_x$ with $Pe = 10^4$.]
ial conduction for very small values of $X$. Thus, just as was observed for the CWT boundary condition, the numerical solution will predict higher local Nusselt numbers than those of the classical solution in this $X$ region. Recall that the observed deviation is not necessarily caused by significant error, but is simply a difference caused by the inclusion of axial conduction in the governing energy equation used by the numerical model.

Due to the error being less than 1.2% for both high and low $Pe$, it is concluded that the numerical model sufficiently predicts the classical solution for the Graetz problem with no energy generation and an imposed CHF boundary condition.

**Slug Flow with no Energy Generation**

Numerical results were obtained for the conditions of classical slug flow with no energy generation by setting the velocity profile to be uniform ($U = 1$) and $S = 0$. Figure 4-28 illustrates for low Peclet number the comparison between the classical and numerical solutions with the corresponding percent error as a function of axial position. The local mean temperature exhibits excellent matching between the numerical and classical results due to less than 0.35% error in the numerical solution. As well, error results are shown for the local Nusselt number, where the error in the fully-developed regime is shown to be less than 0.06%.

![Figure 4-28: CHF Slug Flow model verification study for $\theta_m$ and $Nu_x$ with $Pe = 100$.](image-url)
Figure 4-29: CHF Slug Flow model verification study for $\theta_m$ and $Nu_x$ with $Pe = 10^4$.

The comparison between the classical and numerical solutions together with the percent error as a function of axial position is illustrated for a high Peclet number ($Pe = 10^4$) in Figure 4-29 for both $\theta_m$ and $Nu_x$. The local mean temperature variations match for the numerical and classical results due to less than 1% error in the numerical solution. Note that the error exhibited for small $X$ ($X < 0.0002$) can be discarded since $\theta_m$ in that region is so small. The Nusselt number comparison shows little error as well over the entire $X$ domain (less than 1%).

The error for both low and high $Pe$ values has been shown to be less than 1%. As a result, the numerical model sufficiently reproduces the classical solution for the slug flow problem with no energy generation and an imposed CHF boundary condition.

**EOF Slug Flow**

Numerical results were obtained for the conditions of EOF slug flow by setting the velocity profile to be uniform ($U = 1$) and by including Joule heating energy generation through a specified source term, $S$. Figure 4-30 illustrates for $S = 0.01$ and $Pe = 5$ the comparison between the analytical solution from Section 3.2 and the numerical solution, accompanied by the percent error as a function of axial position. The local mean temperature exhibits less than 0.06% error between the numerical and analytical results.
The local Nusselt number variation numerical results match the analytical solution with less than 0.8% error.

Figure 4-31 compares the analytical and numerical solutions accompanied by the percent error as a function of axial position for \( S = 1 \) and \( Pe = 5 \). The local mean temperature exhibits very good matching between the numerical and analytical results due to less than 0.06% error in the numerical solution. Finally, the numerical error results are shown for the local Nusselt number, where the error is shown to be less than 0.6%.

Since the error is less than 0.8% for both high and low \( \theta_e \), it is has been shown that the numerical model reproduces the analytical solution for the EOF slug flow problem in Section 3.2 with an imposed constant wall heat flux boundary condition.
4.3.3 Constant Wall Heat Flux Results

Parametric studies with varying $S$ and $Pe$ for a given value of $Z$ and a CHF boundary condition revealed that the dimensionless local temperature profile development, local mean temperature, and local Nusselt number follow trends very similar to the solutions for EOF slug flow in Section 3.2. Therefore, the results will not be repeated here. As in Section 4.2.3, this section will focus primarily on variations of the capillary radius-to-Debye length ratio, $Z$, and the dimensionless viscous dissipation parameter, $S_v$, for different values of $S$. As before, viscous dissipation will be initially neglected for the $Z$ parametric studies by setting $S_v$ equal to zero.

Prior to providing results for the studies performed on the applicable ranges of $Z$ and $S_v$, numerical inconsistencies evidenced in the results data must be discussed. Fundamental aspects of convective heat transfer dictate that developing Nusselt number variations will converge to a specific fully-developed value and remain constant from that point on in the fully-developed region. However, the results reveal a deviation of the local fully-developed Nusselt number by as much as 0.25% for large $Z$ and 4% for $Z = 3$ in the fully-developed region. At first, it was thought that the grid was not fine enough, but grid refinement studies proved this to not be a factor. However, several other causes can be proposed for the observed inconsistency. First of all, the most significant deviation occurs at the last three downstream node positions in the $X$ domain. This would suggest a discrepancy in the outlet boundary condition formulation used in the numerical model. As mentioned in Section 4.3.1, the formulation for the CHF scenario boundary conditions is not as straightforward as it is for the CWT case due to the non-zero value of the fully-developed temperature gradient, $\frac{\partial \theta}{\partial X}_{X \to \infty}$, at the outlet. The difficulty arises in the
fact that the fully-developed condition \( \frac{\partial \theta}{\partial X} \bigg|_{X \to \infty} = \frac{d \theta_m}{dX} \) cannot be directly incorporated into Patankar’s power-law coefficient scheme [34]. Several different attempts were made to accurately model this fully-developed outlet condition, and of these several models the method outlined in Section 4.3.1 was found to most significantly reduce the numerical anomaly at the outlet.

A second cause for the numerical inconsistencies in the fully-developed Nusselt number lies in the method used to calculate the local mean temperatures. Due to the discrete nature of the local temperature distributions, numerical integration must be used to evaluate the integral in Equation (4.10). The numerical integration was performed by using the Trapezoidal Rule [35]. When the numerical model predicts high temperatures in the fluid, such as for the \( Z = 0.5 \) case, slight error inherent in using the Trapezoidal Rule could contribute to slightly varying local mean temperatures. These variations then propagate into the calculation of the Nusselt number, which is very sensitive to the temperature difference, \( \theta_w - \theta_m \), shown in its definition (see Section 3.2), thus producing enough error that \( Nu_f \) varies in the fully-developed region. In order to minimize this inaccuracy, a finer scale was used for the numerical integration by separating each cell width into two intervals—between the west control surface and the cell node, \([X_w, X_P]\), and between the cell node and the east control surface \([X_P, X_e]\) (see Figure 4-2a). The Trapezoidal Rule was then individually applied over each new smaller interval, which decreased the numerical error in \( Nu_{fil} \) by as much as 55%. A side note to the above comparison is that no difference was noted in the Nusselt numbers over the entrance length between the separate methods of applying the Trapezoidal Rule. Therefore, in the thermally developing region of the flow, it can reasonably be concluded that the calcu-
lated local Nusselt numbers in that region are independent of the scale to which the Rule is applied for grid-independent solutions.

**Radius-to-Debye Length Ratio Studies**

The development of the local dimensionless fluid temperature \( \theta(X,R) \) is illustrated in Figure 4-32 for \( S = 0.01, \ Pe = 5, \ Sv = 0 \) and four different values of the radius-to-Debye length: \( Z = 100, 10, 5, \) and \( 3 \). Recall that \( Pe = 5 \) and \( Z = 100 \) are typical values of the Peclet number and radius-to-Debye length ratio as mentioned in Sections 3.1.2 and 4.2.3, respectively. Also recall that for \( Z \) values near a magnitude of 500 the velocity field is nearly uniform, while for small \( Z \) it approaches nearly parabolic conditions (see Figure 4-32).

**Figure 4-32:** Non-dimensional temperature profiles at various \( X \) locations in the thermally developing region for \( S = 0.01, \ Pe = 5, \ Sv = 0 \) and radius-to-Debye length ratios of (a) \( Z = 100 \), (b) \( Z = 10 \), (c) \( Z = 5 \), and (d) \( Z = 3 \).
1-2). The source term value of \( S = 0.01 \) is representative of what might be encountered in a typical electro-osmotic flow scenario containing distilled water (\( \sigma \approx 2273 \ \Omega \cdot m \)) with an imposed electrical potential gradient, \( d\Phi/dx \), in the microtube of magnitude 150 kV/m, a channel radius of \( a = 50 \ \mu m \), and an imposed wall heat flux to the tube of approximately 50 kW/m\(^2\). Figure 4-32(a) shows that for \( Z = 100 \) the local temperature increases with increasing streamwise position for all radial positions. Initially, for very small \( X \) (as seen for \( X \approx 0.01 \)) the temperatures near the wall increase more rapidly than in the fluid core region due to the direct effect of wall convection coupled with energy generation in that region. As the thermal development progresses, conduction in the radial direction distributes the thermal energy from the wall heat flux towards the centerline axis region of the tube. Thus, temperatures far from the wall region will increase relative to the wall temperature until a temperature distribution is reached such that the rate of increase of the local temperatures ceases to change. This condition is the classical definition of the fully-developed condition for an imposed wall heat flux scenario and is exemplified by the local temperature distributions at \( X = 0.5 \) and \( X = 1 \) in the Figure 4-32(a), where the relative temperature distribution is the same for both \( X \) positions. As \( Z \) is decreased from the \( Z = 100 \) case as shown in Figures 4-32(b), (c), and (d) for \( Z = 10, 5, \) and 3, respectively, the thermal development follows the same relative patterns as those described for \( Z = 100 \), but the local temperature distributions rise more rapidly for decreasing \( Z \).

Figure 4-33 illustrates the variations in the dimensionless local mean temperature for \( S = 0.01, Pe = 5, S_\nu = 0 \) and \( Z \) ranging from 100 to 3. The EOF slug flow solution for \( Z \rightarrow \infty \) from Equation (3.17) is shown for comparison. As expected for \( Z = 100 \), the mean temperature variation nearly equals the distribution for slug flow. As \( Z \) decreases,
the mean temperature develops at decreasing streamwise locations, yielding higher local temperatures for all $X$. This is similar to that observed for the local temperature profile development in Figure 4-32. Regardless of the radius-to-Debye length ratio, the mean temperature increases linearly with $X$ at a slope that varies with $Z$. This result agrees with the definition of $d\theta_m/dX$ given in Equation (4.24).

Figure 4-34 shows the local Nusselt number variations for $S = 0.01$, $Pe = 5$, $S_v = 0$ and $Z$ ranging from 100 to 3. The EOF slug flow solution for $Z \rightarrow \infty$ from Equation (3.18) is shown for comparison. Similar to that observed for the CWT case in Figure 4-14, the Nusselt number variation for $Z = 100$ falls slightly below that of the EOF slug flow solution for all $X$. The Nusselt number variation decreases for all $X$ as $Z$ is decreased to $Z = 3$. With careful observation, it can be seen that the entrance length is a slight function of $Z$ and moves upstream as $Z$ is decreased. This trend will be explained below.

Thermal entrance length values were calculated by finding the point where the local Nusselt number falls within 0.1% of the classical fully-developed Nusselt number.
as used in previous entrance length studies. The calculated values are plotted versus $Z$ in Figure 4-35 for three different source term values: $S = 1, 0.1, 0.01$. The figure shows that $X_{fd}$ is a slight function of the radius-to-Debye length ratio, and follows trends that change very little with varied $S$. At $Z = 300$, $X_{fd}$ for all $S$ values approaches the entrance length for EOF slug flow and a CHF boundary condition, which is $X_{fd} = 0.453$. This is expected

![Figure 4-34: $Nu_x$ vs. $X$ for $S = 0.01$, $Pe = 5$, $S_v = 0$ and $Z$ ranging from 100 to 3.](image)

![Figure 4-35: $X_{fd}$ vs. $Z$ for $S = 0.01$, $Pe = 5$ and $S_v = 0$.](image)
since as $Z \to \infty$, the fully-developed Nusselt number, as described in Section 3.2, is independent of the source term, $S$. As $Z$ is decreased from $Z = 300$ for the $S = 0.01$ case, the entrance length decreases at increasing rates until $X_{fd} = 0.413$ for $Z = 3$. For $S = 0.1$, the entrance length as a function of $Z$ deviates very slightly above the $S = 0.01$ scenario. For $S = 1$, the deviation is more significant although $X_{fd}$ is at most 1.5% higher than for the $S = 0.01$ condition. Additionally, note that all of the calculated entrance lengths are almost double the entrance length for classical slug flow with a CHF boundary condition and no energy generation, $X_{fd} \approx 0.21$ [22].

Viscous Dissipation Studies

Solutions for Equation (4.1) will now be presented for non-zero values of the dimensionless viscous dissipation term, $S_v$. Recall that $S_v$ is defined solely in terms of fluid properties as $S_v = \sigma \mu \mu_o^2 / \kappa_d^2$ as mentioned at the beginning of the chapter. Thus, the same range of parametric values used for the CWT boundary condition is applicable ($0 \leq S_v \leq 30$). The development of the local dimensionless fluid temperature $\theta(X,R)$ is illustrated in Figure 4-36 for $S = 1$, $Pe = 5$, $Z = 3$ and four different values of the viscous dissipation parameter: $S_v = 1, 5, 10,$ and $30$. Recall that $Pe = 5$ is the typical value of the Peclet number as mentioned in Section 3.1.2. Additionally, predictions reveal that for small source term values (e.g., $S = 0.01$) realistic values of $S_v$ are not large enough to produce noticeable effects in the heat transfer characteristics. Therefore, a larger source term value of $S = 1$ was used to obtain most of the data presented in this section. As well, results provided for thermally fully-developed EOF studies with viscous dissipation and a CHF boundary condition show that viscous dissipation effects are most significant when $Z \approx 3$ [20]. As a result, this $Z$ value was used to obtain most of the presented data.
Figure 4-36: Non-dimensional temperature profiles at various X locations in the thermally developing region for $S = 1$, $Pe = 5$, $Z = 3$ and viscous dissipation parameter values of (a) $S_v = 1$, (b) $S_v = 5$, (c) $S_v = 10$, and (d) $S_v = 30$.

Figure 4-36 shows that, for a given value of $S_v$, the temperature profile development follows trends very similar to that observed for parametric changes in $Z$. Thus, the fundamental physical interpretations of the flow presented for Figure 4-32(a) apply and are not repeated here. The figure also shows that $S_v$ has a significant effect on the development of the local temperature distributions. As $S_v$ is increased, the temperature profiles develop more rapidly, thus yielding temperature profiles with greater magnitude at corresponding axial positions. This is expected since the presence of viscous dissipation will contribute to higher temperatures at all radial positions for $Z = 3$ due to the existence of velocity gradients at all radial locations (see velocity profile in Figure 1-2).
Figure 4-37: $\theta_m$ vs. $X$ for $S = 1$, $Pe = 5$, $Z = 3$, and $S_v$ ranging from 0 to 30.

For specified values of $S = 1$, $Pe = 5$, and $Z = 3$, Figure 4-37 shows the local mean temperature variations as a function of $X$ for varied values of the viscous dissipation parameter. As shown, the local mean temperatures for $S_v = 1$ falls slightly above that of the non-viscous dissipation solution for all $X$ and increases for all $X$ as $S_v$ is increased. This results in higher linear temperature gradients in the fully-developed regime for increasing $S_v$. These increasing $\theta_m$ variations agree with the temperature profile developments observed in Figure 4-36.

Figure 4-38 illustrates the local Nusselt distribution for $S = 1$, $Pe = 5$, $Z = 3$, and $S_v$ ranging from 0 to 30. The local Nusselt number variation for $S_v = 1$ falls slightly below that of the non-viscous dissipation solution for all $X$ and decreases for all $X$ as $S_v$ is increased. This trend is the opposite of what was seen for the CWT boundary condition with viscous heating (Section 4.2), but does agree with the fully-developed results obtained by the analytical solution of Maynes and Webb [20]. Note that the entrance lengths are almost identical for each $S_v$ value. Using the methods used in previous $X_f$ studies, en-
entrance lengths were calculated for the above parametric conditions, as well as for \( S_v \) parametric studies over the applicable range of \( Z \) (\( 3 < Z < 100 \)) and different values of \( S \). Results show no noticeable variation in \( X_{fd} \) with \( S_v \) over the full range of parametric conditions, indicating that \( X_{fd} \) is independent of \( S_v \) for a CHF boundary condition.

It is also of interest to study the effects of \( Z \) on non-zero values of the viscous dissipation parameter. Figure 4-39 illustrates the local temperature development for \( S = 1, Pe = 5, S_v = 30 \), and four values of \( Z \): \( Z = 100, 10, 5, \) and \( 3 \). For \( Z = 100 \), the thermal development follows the same general trends as those observed both for non-viscous and viscous dissipation in Figures 4-32 and 4-36. As \( Z \) is decreased, Figure 4-39 shows that viscous dissipation has a significant increasing effect on the temperature development, where the magnitude of the temperature profiles increase more rapidly for smaller \( Z \) values. This is expected since for high \( Z \) values the viscous dissipation is concentrated in the wall region and has less of an effect on the overall temperature development than for small \( Z \) values where the velocity gradients are spread over the whole tube radius.
The above observed thermal development behavior can be further evidenced by the local mean temperature variation for different values of $Z$. Figure 4-40 presents the $\theta_m$ variations for $S = 1$, $Pe = 5$, $S_v = 30$, and $Z$ ranging from 3 to 100. For $Z = 100$, as expected, the mean temperature variation follows a monotonic increase for all $X$. As $Z$ is decreased, $\theta_m$ increases monotonically at higher rates with decreasing $Z$, thus coinciding with the results observed for the temperature profile development.

The variations in local Nusselt number for the same given conditions used above, as shown in Figure 4-41, yield results similar to those observed for non-viscous dissipation results earlier in this section (see Figure 4-34). However, in this case, with a high
viscous dissipation parameter value of $S_v = 30$, the change in $Nux$ with $Z$ for all $X$ is more significant. Note that fully-developed conditions are reached at further upstream $X$ positions for decreasing $Z$, just as was observed for non-viscous dissipation studies in Figure 4-35. This trend holds with no noticeable variation for different $S_v$ values, further illustrating the independence of the thermal entrance length on the magnitude of $S_v$.

Figure 4-41: $Nux$ vs. $X$ for $S = 1, Pe = 5, S_v = 30$ and $Z$ ranging from 100 to 3.
**CHF Results Summary**

The local mean temperatures, local Nusselt numbers, and thermal entrance lengths have been studied parametrically for varied $Z$, $S$, and $S_v$. Parametric studies in $Z$ reveal a significant dependence of $\theta_m$, $N_{ux}$, and $X_{fd}$ on $Z$. The mean temperature variations increase monotonically for all $Z$ and asymptote to the CHF fully-developed condition, where the gradient in the local temperature with $X$ is constant. The local Nusselt number decreases with decreasing $Z$ for all $X$. These $\theta_m$ and $N_{ux}$ trends are exhibited for all magnitudes of $S$. For thermal entrance length studies, results show that $X_{fd}$ follows trends very similar to that observed for a CWT boundary condition and varied $Z$. The magnitude of $S$ becomes significant in shifting these trends only when $S \geq 1$.

The inclusion of viscous dissipation produces predicted results for $\theta_m$ and $N_{ux}$ that differ little from the trends mentioned above. The only difference exists in the more significant effect of increasing $S_v$ than of increasing $Z$ on the thermal development. Increasing $S_v$ yields a more rapid increase in both the temperature profile development and local mean temperature. As well, the Nusselt number variation decreases more for all $X$ with increasing $S_v$ than for increasing $Z$. However, the thermal entrance length is independent of the viscous dissipation parameter.
5 - CONCLUSIONS AND RECOMMENDATIONS

5.1 OBJECTIVE SUMMARY

Solutions for the EOF convective heat transfer characteristics in circular micro-channels have been presented for the two prescribed velocity profile scenarios: i) uniform velocity flow (for infinitely thin Debye length, or \( a/\lambda_d \to \infty \)), and ii) the family of EOF velocity profiles that depend on finite \( a/\lambda_d \). Studies were performed for the imposed constant wall temperature and constant wall heat flux boundary conditions.

Closed-form solutions and physical analysis of the results were first presented for the infinitely thin Debye length condition. For an imposed constant wall temperature, it was concluded that the magnitude of the dimensionless inlet temperature, \( \theta_e \), determines the rate and nature of the thermal development by how it physically represents the strength of wall convection relative to that of Joule heating. Likewise, the Peclet number, \( Pe \), magnitude determines whether axial advection or diffusion controls the thermal development, resulting in a longer normalized developing region for decreasing \( Pe \). For all conditions, the thermal entrance length was found to be significantly longer than that for pressure-driven flow with no energy generation. For an imposed constant wall heat flux, the Nusselt number variation is equivalent to the classical non-energy source slug flow scenario for all \( S \) and \( Pe \). Only the local temperature and mean temperature developments are functions of the Joule heating energy source in this case.
Utilizing a finite volume solution method, numerical solutions were obtained and presented for the EOF velocity flow profile scenarios under both possible imposed wall boundary conditions. For the prescribed velocity profile scenarios, parametric studies show that $\theta_e$, $S$, and the Peclet number affect the thermal development similarly to that of the slug flow velocity scenario for both wall boundary conditions. The tube radius-to-Debye length ratio, $Z$, however, contributes significantly to the development trends exhibited in the local mean temperature and local Nusselt number variations for different values of $\theta_e$ and $S$. Generally speaking, the local Nusselt number and the entrance length decrease for decreasing $Z$ under both wall boundary conditions. Increasing the dimensionless viscous dissipation parameter, $S_v$, yields more rapid thermal development and reduced variations in local Nusselt number for both wall boundary condition scenarios. However, $S_v$ has a different effect on the thermal entrance length for the two applied wall boundary conditions. For a constant wall temperature condition, the thermal entrance length exhibits a minimum value with varied $S_v$ for intermediate and small magnitudes of $Z$ and inlet temperatures near the critical value of $\theta_e \approx 0.18$. For an imposed constant wall heat flux, the magnitude of the viscous dissipation parameter has no effect on the entrance length variation.

5.2 RECOMMENDATIONS

Many possible studies in EOF convective heat transfer, whether analytical or experimental, may be performed to build upon the present research. First of all, the present research can be extended to non-circular ducts to obtain solutions for each of the two general velocity profile scenarios that apply, as done here. Such a study is especially applicable for parallel plate channels, since research performed for that channel geometry
can serve as benchmarking results just as circular channel results were used in this research [11,20,22]. Before thermally developing EOF convection studies are performed for other channel geometries, however, studies similar to those performed by Maynes and Webb for circular and parallel channels should be completed in order to provide applicable benchmarking results. For all studies in thermally developing EOF, similar methods and precautions utilized in the present research should be used in obtaining any numerical solutions. In addition, analytical studies involving non-simplifying assumptions with regards to the flow characteristics and fluid properties may be performed, such as involving a non-uniform Joule heating energy source or fluids where the Debye-Hückel linearization approximation is inapplicable.

Experimental studies of EOF convective heat transfer could also be performed for different channel geometries. Results could be compared against existing or future theoretical studies and would provide ‘real world’ results for EOF application studies to utilize. Such studies would prove more difficult, though not impossible, as a concrete understanding of and experience with electro-osmotically controlled flows would be required.
REFERENCES


APPENDIX A– Infinitely Thin Debye Length Studies Source Code

CONSTANT WALL TEMPERATURE

Source code file: tconst.f

Compile instructions: use the following command in a UNIX command window in the directory where tconst.f is located

\texttt{f90 tconst.f -o desirednameofexecutable}

run executable by typing “./” then name of executable file, then press “Enter.”

User Inputs: values of $\theta_e$ and $Pe$ as prompted by program

Program Output File: Nudata.txt
Columnar data in order: $Nu_s$, $\theta_m$, $x/a$, $x/Pe$

Source Code:

C Program to evaluate temperature distributions in fully developing C
electro-osmotic flow

\begin{verbatim}
real*16 te, pe, lm(5000), cm(5000), j1m(5000), j1, j3
real*16 x, xx, nf, nf1, n, Nu, xplus

write(6,*) "Enter the magnitude of $T_e"
read (6,*) te
write(6,*) "Enter the magnitude of $Pe"
read (6,*) Pe
write(6,*) "Enter the upper value of xplus"
read (6,*) xplus

open(11, file='NUdata.txt')

C Values for first 20 eigenfunctions satisfying $Jo(lm)=0$
C
lm(1)=2.4048
lm(2)=5.5201
lm(3)=8.6537
lm(4)=11.7915
lm(5)=14.9309
lm(6)=18.071
lm(7)=21.213
lm(8)=24.351
\end{verbatim}
lm(9)=27.492
lm(10)=30.634
lm(11)=33.779
lm(12)=36.921
lm(13)=40.062
lm(14)=43.204
lm(15)=46.346
lm(16)=49.487
lm(17)=52.629
lm(18)=55.77
lm(19)=58.912
lm(20)=62.054

Compute the Nusselt number as a function of x
Start by computing the values for J1 and Cm

do 20 j=1,5000
nf=1
nf1=1
j1 = 0
j3 = 0
if(j.gt.20.)then
  lm(j)=lm(j-1)+3.1415927
endif

call bessj1(lm(j),j1,j3)
jlm(j)=j1
  cm(j)=2/j1*(te/lm(j)-1/lm(j)**3)
20  continue

xx=0.001
x=0.001
do while(x.lt.xplus)
  xx=x/8
  x=x+xx
  xpe=x/pe
  sumt=0
  sumb=0
  do 24 m=1,5000
    sumt=sumt+cm(m)*lm(m)*jlm(m)*exp(pe/4*(1-sqrt(1+16*lm(m)**2/pe**2))*x)
    sumb=sumb+cm(m)/lm(m)*jlm(m)*exp(pe/4*(1-sqrt(1+16*lm(m)**2/pe**2))*x)
  24 continue

  Nu = (1+2*sumt)/(1/8+2*sumb)
  thetm=(1/8+2*sumb)

  write(11,100) Nu,thetm, x,xpe
100  format(f14.6,x,f14.6,x,f14.6,x, f14.6)
Subroutine bessj1(lm, j1, j3)
c input: lm-eigenvalue
c output: j1-bessel function j1
c j3=j1/lm
real*16 lm, j1, j3, xx, y, ans, ans1, ans2, z
x=lm
if (x.lt.8) then
  y=x*x
  ans1=x*(72362614232.+y*(-7895059235.+y*(242396853.1
    +y*(-2972611.439+y*(15704.48260+y*(-30.16036606))))))
  ans2=144725229443.+y*(2300535178.+y*(18583304.74
    +y*(99447.43394+y*(376.9991397+y))))
  ans=ans1/ans2
else
  z=8/x
  y=z*z
  xx=x-2.356194491
  ans1=1+y*(.183105e-2+y*(-.3516396496e-4+y*(.2457520174e-5
    +y*(-.240337019e-6))))
  ans2=.04687499995+y*(-.2002690873e-3+y*(.8449199096e-5
    +y*(-.88228987e-6+y*.105787412e-6)))
  ans=sqrt(0.636619772/x)*cos(xx)*ans1-z*sin(xx)*ans2
endif
j1=ans
j3=ans/lm
return
end
APPENDIX B– EOF Velocity Profile Studies Source Code

B.1 CONSTANT WALL TEMPERATURE

Source code file: temptconst.c

Compile instructions: use the following command in a UNIX command window in the directory where temptconst.c is located:
gcc temptconst.c -o desirednameofexecutable -lm
run executable by typing “./” then name of executable file, then press “Enter.”

User Inputs: input file: input.txt
Input values given in first row of file with given order:
“$S_v \theta_e Pe Z X_{max} N_x N_r Rss$”
where Rss is the convergence criterion value
one space is placed in between each parametric value

Program Output Files: tempin.txt – regurgitates input data
tempout.txt – temperature distribution data matrix defined by $X$ and $R$ positions listed in first column
and first row, respectively.
tmean.txt – columnar data in order: $\theta_m, Nu_x, x/a$
terfile.txt – for program tracking purposes;
contains convergence parameter value and iteration number for each iteration as program runs

Source Code:

```c
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
define FILENAME "tempin.txt"
define FILENAME2 "tempout.txt"
define FILENAME3 "tmean.txt"
define FILENAME4 "iterfile.txt"
define FILENAME5 "input.txt"

double X[2001], R[2001], Th[2001][2001], Xp[2001], Rp[2001], Rpp[2001];
double delRn[2001], delRs[2001], U[2001], dU[2001], delX[2001];
double De[2001], Dw[2001], b[2001], a[2001], c[2001], r[2001];
```
double gam[2001], F[2001], ae[2001][2001], aw[2001][2001];
double an[2001][2001], as[2001][2001], ap[2001][2001], bb[2001][2001];
double Nuold[2001], Thfd[2001];

int main(void)
{
    int file, BC, NI, m, n, l, j, p, i, cont;
double Nx, Nr, N, t_e, s, pe, z, Xmax, Num, Nu_diff, Nu_diffold;
double delXe, delXw, delRp, ThB, bZ, bbZ, Rs, RsS, Rs_new, IOz;
double bet, Thm, Thmold, delTh, Numm, Nu, Nu_oldf, Svlam;
double Ilz, Fz, ubarmax
double besselIo(double Y);
double besselI1(double Yy);
double max(double A, double B);

printf("Program to evaluate temperature profiles in circular
duct\n");
printf("with constant temperature boundary condition in electro-

oslotic flow\n");

input = fopen(FILENAME5, "r");
itfile = fopen(FILENAME4, "w");
/* INPUTS */
    fscanf(input, "%lf %lf %lf %lf %lf %lf %lf %lf", &Svlam, &s, &pe,
    &z, &Xmax, &Nx, &Nr, &Rss);
    fprintf(iterfile, "\n");
fclose(iterfile);
fclose(input);

t_e=s;
s=1;
Fz=1/(z*z);

/* CELL FACE GRID DEPLOYMENT */
m=3;
/* i's and j's correspond to cell faces (for grid deployment) */
for(i=1;i<=(Nx+1);i++)
{    X[i]=pow(((i-1)/Nx), m)*Xmax;
}
for(j=1;j<=(Nr+1);j++)
{    R[j]=1-pow((1-(j-1)/Nr), m);
}

/* CALCULATE COEFF.'S AT EACH CELL */
/* Internal cells */
/* Cell Geometry */
/* X-direction */
for(i=1;i<=Nx;i++)
{    delX[i]=X[i+1]-X[i];
Xp[i]=X[i]+delX[i]/2;
    /* Inlet semi-BC */
    if(i==1)
    {    delXw=delX[i]/2;)
else
\[ \text{delXw} = \frac{X[i+1] - X[i-1]}{2}; \]

/* Outlet BC */
if (i == Nx)
    \{ delXe = delX[i]/2; \}
else
    \{ delXe = \frac{X[i+2] - X[i]}{2}; \}

/* Subcoeff.'s */
De[i] = 1/delXe;
Dw[i] = 1/delXw;

/* R-direction */
I0z = besselIo(z);
I1z = besselI1(z);
ubarmax = 1 - 2*I1z/(z*I0z);
for (j = 1; j <= Nr; j++)
    /* Nr is boundary since Nr+1 is a zero-volume boundary cell */
    \{ delRp = R[j+1] - R[j]; \}
    Rp[j] = R[j] + delRp/2;

/* Wall semi-BC */
if (j == Nr)
    \{ delRn[j] = delRp/2; \}
    ThB = delRn[j];
else
    \{ delRn[j] = \frac{R[j+2] - R[j]}{2}; \}

/* Center axis semi-BC */
if (j == 1)
    \{ delRs[j] = delRp/2; \}
else
    \{ delRs[j] = \frac{R[j+1] - R[j-1]}{2}; \}

/* Subcoeff.'s */
U[j] = 1 - besselIo(z*Rp[j])/I0z;
dU[j] = besselI1(z*Rp[j])/I0z;

if (j == 1)
    \{ U[j-1] = 1 - besselIo(z*Rp[j-1])/I0z; \}
    dU[j-1] = besselI1(z*Rp[j-1])/I0z;
}

/* for theta_mean calculations */
if (j == Nr)
    \{ Rp[j+1] = 1; \}
    U[j+1] = 0;
    dU[j+1] = 0;
}

F[j] = pe/2*U[j];
/ * INITIALIZE THETA */
/* i's and j's correspond to the cells (for node and coefficient deployment) */
/* i=0,Nx+1 and j=0,Nr+1 correspond to the zero-volume cells around domain perimeter */
/* Fully Developed Temperature Profile */
for(i=1;i<=(Nx+1);i++)
{ for(j=0;j<=(Nr+1);j++)
  { Th[i][j]=0;
  }
}
/* For introducing correct inlet temp for rightward sweep */
i=0;
for(j=0;j<=Nr;j++)
{ Th[i][j]=t_e;
}
/* Coeff.'s */
for(i=1;i<=Nx;i++)
{ for(j=0;j<=Nn;j++)
  { /* outlet BC */
    if(i==Nx)
    { ae[i][j]=0; }
    else
    { if(i==1)
        { ae[i-1][j]=0; }
        ae[i][j]=De[i]*max(0,pow((1-0.1*F[j]/De[i]),5))+max(0,-1*F[j]);
    }
    aw[i][j]=Dw[i]*max(0,pow((1-0.1*F[j]/Dw[i]),5))+max(0,F[j]);
    an[i][j]=R[j+1]*delX[i]/(Rpp[j]*delRn[j]);
    /* center axis BC */
    if(j==1)
    { as[i][j]=0; }
    else
    { /* wall BC */
      /* Set south boundary coeff. at zero for rightward sweep */
      if(j==Nr)
      { as[i][j+1]=0; }
      as[i][j]=R[j]*delX[i]/(Rpp[j]*delRs[j]);
    }
    ap[i][j]=as[i][j]+an[i][j]+ae[i][j]+aw[i][j];
    bb[i][j]=(s+Svlam*Fz*z*z*dU[j]*dU[j])*delX[i];
  }
}
/* GLOBAL ITERATION LOOP */
Rs=1000;
for(i=0;i<=Nx;i++)
{ Nuold[i]=-1;}

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NI=0;
cont=1;

while(cont==1) {
  tempin = fopen(FILENAME, "w");
tempout = fopen(FILENAME2, "w");
tmean = fopen(FILENAME3, "w");
   while(Rs>Rss)
   { iterfile = fopen(FILENAME4, "a");
     /* Reset Rs and Nu_diffold */
     Rs=0;
     Nu_diffold=0;
     /*Sweep right wall BC */
     NI=NI+1;
     
     j=Nr+1;
b[j]=b[j]+1;
     r[j]=0;

     N=Nr+1;
   
   /*Sweep right*/
   for(i=1;i<=Nx;i++)
   {
     /* Find coefficients*/
     for(j=1;j<=Nr;j++)
     { b[j]=ap[i][j];
       r[j]=ae[i][j]*Th[i+1][j]+aw[i][j]*Th[i-1][j]+bb[i][j];
     }
   
   /*Tridag algorithm*/
   if (b[1] == 0.0)
   { printf("Error 1%i in tridag", i);
     /*If this happens then you should rewrite your
     equations as a set of order N-1*/
     /*with u2 trivially eliminated*/
   }
   bet=b[1];
   Th[i][1]=r[1]/bet;

   /*Decomposition and forward substitution*/
   for (j=2;j<=N;j++)
   { gam[j]=-1*an[i][j-1]/bet;
     bet=b[j]-(-1*as[i][j]*gam[j]);
     if (bet == 0.0)
     { printf("Error 2%i in tridag", j);
       /*Algorithm fails*/
     }
     Th[i][j]=(r[j]-(-1*as[i][j]*Th[i][j-1]))/bet;
   }

   /*Backsubstitution*/
   for (j=(N-1);j>=1;j--)
   { Th[i][j] -= gam[j+1]*Th[i][j+1];
   }
b[0]=1;
r[0]=t_e;

for(j=2;j<=Nr;j++)
{
        /* Find coefficients */
        for(i=1;i<=Nx;i++)
        { 
                b[i]=ap[i][j];
                r[i]=an[i][j]*Th[i][j+1]+as[i][j]*Th[i][j-1]+bb[i][j];
        }
        /* Tridag algorithm */
        if (b[0] == 0.0)
        { printf("Error 3%i in tridag", j);
                /* If this happens then you should rewrite your
                equations as a set of order N-1*/
                /* with u2 trivially eliminated*/
        }
        bet=b[0];
        Th[0][j]=r[0]/bet;
/* Decomposition and forward substitution */
        for (i=1;i<=Nx;i++)
        { 
                gam[i]=-1*ae[i-1][j]/bet;
                bet=b[i]-(-1*aw[i][j]*gam[i]);
                if (bet == 0.0)
                { printf("Error 4%i in tridag", i);
                        /* Algorithm fails*/
                }
                Th[i][j]=(r[i]-(-1*aw[i][j]*Th[i-1][j]))/bet;
        }
/* Backsubstitution */
        for (i=(Nx-1);i>=1;i--)
        { Th[i][j] -= gam[i+1]*Th[i+1][j];
        }
}
/* Check for convergence using max change in Nusselt number */

/* CALCULATE UNKNOWN BOUNDARY TEMPERATURES */
for(i=1;i<=Nx;i++)
{ /* Center Axis */
        Th[i][0]=Th[i][1];
}
Thmn=0;
for(j=1;j<=Nr;j++)
{
    Thmn=Thmn+(Rp[j+1]-
Rp[j])/2*(Rp[j]*U[j]*Th[i][j]+Rp[j+1]*U[j+1]*Th[i][j+1]);
}
Thm=2*Thmn/ubarmax;

/*Calculate Nu */
j=Nr;
delTh=(Th[i][j]-Th[i][j-2])/(Rp[j]-Rp[j-2]);
Nu=2*delTh/(-1*Thm);

/* Compare new Nu to old Nu */
Nu_diff=Nu-Nuold[i];
if(Nu_diff<0)
{
    Nu_diff=-1*Nu_diff;
}Rs_new=max(Nu_diff,Nu_diffold);
Rs=max(Rs, Rs_new);
Nuold[i]=Nu;
  Nu_diffold=Nu_diff;

fprintf(iterfile, "\%i \%lf\n", NI, Rs);
fclose(iterfile);
}

printf("NI=%i\n", NI);

*/ CALCULATE UNKNOWN BOUNDARY TEMPERATURES */
for(i=1;i<=Nx;i++)
{ /* Center Axis */
    Th[i][0]=Th[i][1];
}
i=Nx+1;
for(j=0;j<=(Nr+1);j++)
{  Th[i][j]=Th[i-1][j];}

/*Print solution to file*/
s=t_e;
fprintf(tempin, " \%lf \%lf \%lf \%lf \%lf \%lf \%lf \%lf \%i \%i \%i \%lf",
Svlam, s, pe, z, Xmax, Nx, Nr, m, BC, NI, Rss);
fprintf(tempout, " 1")
for(j=0;j<=(Nr+1);j++)
{  fprintf(tempout, " \%lf", Rp[j]);
}
fprintf(tempout, "\n")
for(i=0;i<=Nx;i++)
{  fprintf(tempout, " \%lf", Xp[i]);
    for(j=0;j<=(Nr+1);j++)
    {  fprintf(tempout, " \%lf", Th[i][j]);
    }
    fprintf(tempout, "\n")
}

/*Calculate theta_mean and Nu */
Numm=0;
n=0;
for(i=0;i<=(Nx+1);i++)
{ /*Calculate theta_mean */
    Thmn=0;
    for(j=1;j<=Nr;j++)
    { Thmn=Thmn+(Rp[j+1]-Rp[j])/2*(Rp[j]*U[j]*Th[i][j]+Rp[j+1]*U[j+1]*Th[i][j+1]);
    }
    Thm=2*Thmn/ubarmax;
    /*Calculate Nu */
    j=Nr;
    delTh=(Th[i][j]-Th[i][j-2])/(Rp[j]-Rp[j-2]);
    Nu=2*delTh/(-1*Thm);
    fprintf(tmean, " %lf %lf %3.9f", Thm, Nu, Xp[i]);
    fprintf(tmean, " \n");
}
fclose(tempin);
fclose(tempout);
fclose(tmean);
/*printf("Continue with iterations?--Yes==1, No==2: ");
scanf("%i", &cont);
if(cont==1)
{ printf("Enter new convergence criterion (Rss): ");
  scanf("%lf", &Rss);
}
*/
return EXIT_SUCCESS;
}

double max(double A, double B)
{
    double mx;
    mx=A;
    if(mx<B)
    {mx=B;
    }
    return mx;
}

double besselIo(double Y)
{
    double ax, ans, w;
    
    ax=fabs(Y);
    if (ax < 3.75)
    {
        w=Y/3.75;
        w*=w;
        ans=1.0+w*(3.5156229+w*(3.0899424+w*(1.2067492+w*(0.2659732+w*(0.0360768+w*0.0045813)))))
    }
    else
    { w=3.75/ax;
        ans=(exp(ax)/sqrt(ax))*0.39894228+w*(0.01328592+w*(0.00225319+w*
double besselI1(double Yy)  
{
    double ax, ans, w;
    ax = fabs(Yy);
    if (ax < 3.75)
    {
        w = Yy / 3.75;
        w *= w;
        ans = ax * (0.5 + w * (0.87890594 + w * (0.51498869 + w * (0.15084934 + w * (0.02658733 + w * (0.00301532 + w * 0.00032411))))));
    }
    else
    {
        w = 3.75 / ax;
        ans = 0.02282967 + w * (-0.02895312 + w * (0.01787654 - w * 0.00420059));
        ans = (0.39894228 + w * (-0.03988024 + w * (-0.00362018 + w * (0.00163801 + w * (-0.01031555 + w * ans)))));
        ans *= (exp(ax) / sqrt(ax));
    }
    if (ans < 0)
    {
        ans = -1 * ans;
    }
    return ans;
}
B.2 Constant Wall Heat Flux

Source code file: Qflux.c

Compile instructions: use the following command in a UNIX command window in the directory where Qflux.c is located:
gcc Qflux.c -o desirednameofexecutable -lm
run executable by typing “./”, then name of executable file, then press “Enter.”

User Inputs: input file: input.txt
Input values given in first row of file with given order:
“S_y θ_e Pe Z X_{max} N_x N_r R_{ss}”
where R_{ss} is the convergence criterion value
one space is placed in between each parametric value

Program Output Files: tempin.txt – regurgitates input data
tempout.txt – temperature distribution data matrix defined by X and R positions listed in first column and first row, respectively.
tmean.txt – columnar data in order: \theta_m, \nu, x/a
iterfile.txt – for program tracking purposes;
contains convergence parameter value and iteration number for each iteration as program runs

Source Code:

```c
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#define FILENAME "tempin.txt"
#define FILENAME2 "tempout.txt"
#define FILENAME3 "tmean.txt"
#define FILENAME4 "iterfile.txt"
#define FILENAME5 "input.txt"
double X[2001], R[2001], Th[2001][2001], Rp[2001], Rpp[2001];
double delRn[2001], delRs[2001], U[2001], dU[2001], delX[2001];
double De[2001], Dw[2001], b[2001], a[2001], c[2001], r[2001];
double gam[2001], F[2001], ae[2001][2001], aw[2001][2001];
double an[2001][2001], as[2001][2001], ap[2001][2001], bb[2001][2001];
double Nuold[2001], Thfd[2001];

int main(void)
{
    int file, BC, NI, m, n, l, j, p, i, cont;
    double Nx, Nr, N, s, pe, z, Xmax, Num, Nu_diff, Nu_diffold;
    double delXe, delXw, delRp, ThB, bZ, bbZ, bp, Rs, Rss, R_{ss_{new}};
    double bet, Thmn, Thm, Thmold, delTh, Numm, Nu, Nu_{oldf}, Svlam;
    double I0z, I1z, Fz, ubarmax;
```
double besselIo(double Y);
double besselI1(double Yy);
double max(double A, double B);

printf("Program to evaluate temperature profiles in circular
duct\n");
printf("with given boundaries in electro-osmotic flow\n\n");

input = fopen(FILENAME5, "r");
iterfile = fopen(FILENAME4, "w");
/* INPUTS */
scanf(input, "%lf %lf %lf %lf %lf %lf %lf %lf", &Svlam, &s, &pe,
&z, &Xmax, &Nx, &Nr, &Rss);
fprintf(iterfile, "\n");
fclose(iterfile);
fclose(input);

Fz=1/(z*z);

/* CELL FACE GRID DEPLOYMENT */
m=3;
/* i's and j's correspond to cell faces (for grid deployment) */
for(i=1;i<=(Nx+1);i++)
{    X[i]=pow(((i-1)/Nx), m)*Xmax;
}

for(j=1;j<=(Nr+1);j++)
{    R[j]=1-pow((1-(j-1)/Nr), m);
}

/* CALCULATE COEFF.'S AT EACH CELL */
/* Internal cells */
/* Cell Geometry */
/* X-direction */
I0z=besselIo(z);
I1z=besselI1(z);
ubarmax=1-2*I1z/(z*I0z);
for(i=1;i<=Nx;i++)
{   delX[i]=X[i+1]-X[i];
    /* Inlet semi-BC */
    if(i==1)
    {   delXw=delX[i]/2;
    } else
    {   delXw=(X[i+1]-X[i-1])/2;
    }
    /* Outlet BC */
    if(i==Nx)
    {   delXe=delX[i]/2;
        delX[i+1]=0;
        bp=delX[i]*(s*(1+Svlam*Fz*z*z*dU[j]*dU[j])/(pe*ubarmax));
    } else
    {   delXe=(X[i+2]-X[i])/2;
    }
    /* Subcoeff.'s */
De[i]=1/delXi;
Dw[i]=1/delXw;
}

/* R-direction */
for(j=1;j<=Nr;j++) /* Nr is boundary since Nr+1 is a zero-volume boundary cell */
{
    delRp=R[j+1]-R[j];
    Rp[j]=R[j]+delRp/2;
    Rpp[j]=Rp[j]*delRp;

    /* Wall semi-BC */
    if(j==Nr)
    {    delRn[j]=delRp/2;
        ThB=delRn[j];}
    else
    {    delRn[j]=(R[j+2]-R[j])/2;}

    /* Center axis semi-BC */
    if(j==1)
    {    delRs[j]=delRp/2;
        Rp[j-1]=0;}
    else
    {    delRs[j]=(R[j+1]-R[j-1])/2;}

    /* Subcoeff.'s */
    U[j]=1-besselIo(z*Rp[j])/I0z;
    dU[j]=besselI1(z*Rp[j])/I0z;

    if(j==1)
    {    U[j-1]=1-besselIo(z*Rp[j-1])/I0z;
        dU[j-1]=besselI1(z*Rp[j-1])/I0z;}

    /* for theta_mean calculations */
    if(j==Nr)
    {    Rp[j+1]=1;
        U[j+1]=0;
        dU[j+1]=0;}

    F[j]=pe/2*U[j];
}

/* INITIALIZE THETA */
/* i's and j's correspond to the cells (for node and coefficient deployment)*/
/* i=0,Nx+1 and j=0,Nr+1 correspond to the zero-volume cells around domain perimeter */
/* Fully Developed Temperature Profile */
for(i=0;i<=(Nx+1);i++)
{    for(j=0;j<=(Nr+1);j++)
    {    if(i==0)
        {    Th[i][j]=0;}
    else
{Th[i][j]=2*(s*(1+Svlam*Fz*z*z*dU[j]*dU[j])+2)*(X[i]+delX[i]/2)/(pe*ubarmax);
}

/* Coeff.'s */
for(i=1;i<=Nx;i++)
{ for(j=1;j<=Nr;j++)
{ /* Inlet BC */
    if(i==1)
    {ae[i-1][j]=0;}
    ae[i][j]=De[i]*max(0,pow((1-
    0.1*F[j]/De[i]),5))+max(0,-1*F[j]);
    aw[i][j]=Dw[i]*max(0,pow((1-
    0.1*F[j]/Dw[i]),5))+max(0,F[j]);

    /* wall BC */
    if(j==Nr)
    { an[i][j]=0;}
    else
    { an[i][j]=R[j+1]*delX[i]/(Rpp[j]*delRn[j]);}

    /* center axis BC */
    if(j==1)
    { as[i][j]=0;}
    else
    { as[i][j]=R[j]*delX[i]/(Rpp[j]*delRs[j]);}

    /* outlet BC */
    if(i==Nx)
    { ap[i][j]=as[i][j]+an[i][j]+aw[i][j];}
    else{ ap[i][j]=as[i][j]+an[i][j]+ae[i][j]+aw[i][j];}

    /* wall BC */
    if(j==Nr)
    { /* outlet BC */
        if(i==Nx)
        {bb[i][j]=s*(1+Svlam*Fz*z*z*dU[j]*dU[j])*delX[i]+delX[i]/Rpp[j]+b
            p*ae[i][j];}
        else
        {bb[i][j]=s*(1+Svlam*Fz*z*z*dU[j]*dU[j])*delX[i]+delX[i]/Rpp[j];}
    }
    else
    { /* outlet BC */
        if(i==Nx)
        {bb[i][j]=bp*ae[i][j]+s*(1+Svlam*Fz*z*z*dU[j]*dU[j])*delX[i];}
        else{bb[i][j]=s*(1+Svlam*Fz*z*z*dU[j]*dU[j])*delX[i];}
    }
} }
i=Nx;
for(j=1;j<=Nr;j++)
{ae[i][j]=0;}

/* GLOBAL ITERATION LOOP */
Rs=1000;
for(i=0;i<=Nx;i++)
{Nuold[i]=-1;}
NI=0;
cont=1;

while(cont==1) { /*
tempin = fopen(FILENAME, "w");
tempout = fopen(FILENAME2, "w");
tmean = fopen(FILENAME3, "w");
while(Rs>Rss)
{ iterfile = fopen(FILENAME4, "a");
/* Reset Rs and Nu_diffold */
Rs=0;
Nu_diffold=0;
/*Sweep right wall BC */
NI=NI+1;

/*Sweep right*/
for(i=1;i<=Nx;i++)
{
    /* Find coefficients*/
    for(j=1;j<=Nr;j++)
    {  
b[j]=ap[i][j];
    r[j]=ae[i][j]*Th[i+1][j]+aw[i][j]*Th[i-1][j]+bb[i][j];
    }

    /*Tridag algorithm*/
    if (b[1] == 0.0)
    { printf("Error 1%i in tridag", i);
      /*If this happens then you should rewrite your
      equations as a set of order N-1*/
      /*with u2 trivially eliminated*/
    }

    bet=b[1];
    Th[i][1]=r[1]/bet;

    /*Decomposition and forward substitution*/
    for (j=2;j<=Nr;j++)
    { gam[j]=-1*an[i][j-1]/bet;
      bet=b[j]-(-1*as[i][j]*gam[j]);
      if (bet == 0.0)
      { printf("Error 2%i in tridag", j);
        /*Algorithm fails*/
      }
      Th[i][j]=(r[j]-(-1*as[i][j]*Th[i][j-1]))/bet;
    }

    /*Backsubstitution*/
    for (j=(Nr-1);j>=1;j--)
}
{ \text{Th}[i][j] = \text{gam}[j+1]*\text{Th}[i][j+1]; 
}

/* Sweep up */
/* Inlet BC */
b[0] = 1;
r[0] = 0;

for (j=1; j<=Nr; j++)
{
    /* Find coefficients */
    for (i=1; i<=Nx; i++)
    {
        b[i] = ap[i][j];
        r[i] = an[i][j]*\text{Th}[i][j+1]+as[i][j]*\text{Th}[i][j-1]+bb[i][j];
    }
    /* Tridag algorithm */
    if (b[0] == 0.0)
    {
        printf("Error 3%i in tridag", j);
        /* If this happens then you should rewrite your equations as a set of order N-1 */
        /* with u2 trivially eliminated */
    }
    bet = b[0];
    \text{Th}[0][j] = r[0]/bet;

    /* Decomposition and forward substitution */
    for (i=1; i<=Nx; i++)
    {
        \text{gam}[i] = -1*ae[i-1][j]/bet;
        bet = b[i] - (-1*aw[i][j]*\text{gam}[i]);
        if (bet == 0.0)
        {
            printf("Error 4%i in tridag", i);
            /* Algorithm fails */
        }
        \text{Th}[i][j] = (r[i] - (-1*aw[i][j]*\text{Th}[i-1][j]))/bet;
    }

    /* Backsubstitution */
    for (i=(Nx-1); i>=0; i--)
    {
        \text{Th}[i][j] -= \text{gam}[i+1]*\text{Th}[i+1][j];
    }
}

/* Check for convergence using max change in Nusselt number */
/* CALCULATE UNKNOWN BOUNDARY TEMPERATURES */
j = Nr+1;
for (i=1; i<=Nx; i++)
{
    /* Center Axis */
    \text{Th}[i][0] = \text{Th}[i][1];
    /* Wall */
    \text{Th}[i][j] = \text{Th}[i][j-1]+ThB;
}
/* Outlet BC */
i=Nx+1;
for(j=0;j<=(Nr+1);j++)
{    Th[i][j]=Th[i-1][j]+bp;
}

for(i=0;i<=(Nx+1);i++)
{    /*Calculate theta_mean */
        Thmn=Rp[1]*(Rp[1]*U[1]*Th[i][1])/2;
        for(j=1;j<=(Nr-1);j++)
{    Thmn=Thmn+(R[j+1]-Rp[j])/2*(Rp[j]*U[j]*Th[i][j]+R[j+1]*U[j]*(Th[i][j]+Th[i][j+1])/2)+(Rp[j+1]-
R[j+1])/2*(R[j+1]*U[j+1]*(Th[i][j]+Th[i][j+1])/2+Rp[j+1]*U[j+1]*Th[i][j+1]);
        }
    j=Nr;
    Thmn=Thmn+(Rp[j+1]-Rp[j])/2*(Rp[j]*U[j]*Th[i][j]);

        Thm=2*Thmn/ubarmax;

        /*Calculate Nu */
        j=Nr+1;
        Nu=2/(Th[i][j]-Thm);
        /* Compare new Nu to old Nu */
        Nu_diff=Nu-Nuold[i];
        if(Nu_diff<0)
{    Nu_diff=-1*Nu_diff;
        Rs_new=max(Nu_diff,Nu_diffold);
        Rs=max(Rs, Rs_new);
        Nuold[i]=Nu;
        Nu_diffold=Nu_diff;
 }

fprintf(iterfile, "%i %lf
", NI, Rs);
fclose(iterfile);
}

printf("NI=%i\n", NI);
/* CALCULATE UNKNOWN BOUNDARY TEMPERATURES */
j=Nr+1;
for(i=1;i<=Nx;i++)
{    /* Center Axis */
    Th[i][0]=Th[i][1];
    /* Wall */
    Th[i][j]=Th[i][j-1]+ThB;
    }

/* outlet BC */
for(j=0;j<=(Nr+1);j++)
{    i=Nx+1;
    Th[i][j]=Th[i-1][j]+bp;
    }

/*Print solution to file*/

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double max(double A, double B)
double besselIo(double Y)
{
    double ax, ans, w;

    ax=fabs(Y);
    if (ax < 3.75)
    {
        w=Y/3.75;
        w*=w;
        ans=1.0+w*(3.5156229+w*(3.0899424+w*(1.2067492 +w*(0.2659732+w*(0.0360768+w*0.0045813)))));
    }
    else
    {
        w=3.75/ax;

        ans=(exp(ax)/sqrt(ax))*(0.39894228+w*(0.01328592+w*(0.00225319+w*(-0.00157565+w*(0.00916281+w*(-0.02057706+w*(0.02635537+w*(- 0.01647633+w*0.00392377)))))));
    }
    return ans;
}

double besselI1(double Yy)
{
    double ax,ans, w;

    ax=fabs(Yy);
    if (ax < 3.75)
    {
        w=Yy/3.75;
        w*=w;
        ans=ax*(0.5+w*(0.87890594+w*(0.51498869+w*(0.15084934+w*(0.026587 33+w*(0.00301532+w*0.00032411))))));
    }
    else
    {
        w=3.75/ax;
        ans=0.02282967+w*(-0.02895312+w*(-0.01787654+w*(-0.00420059)));
        ans=(0.39894228+w*(-0.03988024+w*(-0.00362018+w*(0.00163801+w*(-0.01031555+w*ans))))) ;
        ans*=(exp(ax)/sqrt(ax));
    }
    if (ans<0)
    {
        ans=-1*ans;
    }
    return ans;
}