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Reasoning About Motion: A Case Study

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REASONING ABOUT MOTION: A CASE STUDY

by

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ABSTRACT

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Master of Arts

Several dance and industrial design students were given the opportunity to attend a non-traditional mathematics course. The nature of this course prompted student interaction and expected collaboration. My research focuses on one dance student, Sara, who did not consider herself a strong mathematics student, but who understood physical motion very well. This paper explores the evolution of Sara's representations for physical motion in a given task, and discusses her reasoning for keeping or dismissing various parts of her representations during the course of this task. I examine first how Sara learns mathematics with understanding in this task, and second how this class gave her the opportunity to learn significant mathematics by encouraging her to ask questions and reason about mathematics. The research presented in this paper shows that teaching mathematics can be successful if students are given the opportunity to investigate tasks designed to explore significant mathematics.

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CHAPTER I: INTRODUCTION AND BACKGROUND

There has been an ever-increasing concern about the teaching and learning of mathematics in the United States. According to the National Council of Teachers of Mathematics (NCTM), it is more important than ever for teachers and educators in mathematics to think differently about mathematics in the classroom in order to promote deeper student understanding of central mathematics concepts.   "All students should have the opportunity and the support necessary to learn significant mathematics with depth and understanding" (NCTM, 2000, p. 5). This research focuses on how students reason about and build their understanding of motion mathematically, and how that reasoning and understanding might be enriched and informed by their own personal experiences with motion. Deep understanding will be defined as what Skemp (1976) calls a relational understanding of mathematics. Skemp defines relational understanding as conceptual understanding or "knowing both what to do and *why*" (emphasis added).

Students can develop a relational understanding of mathematics by creating different *representations* and reasoning with them. In this paper, a representation will be defined as a presentation either to oneself, or to others (Speiser & Walter, 1997). Such representations can be used to convey ideas that aid in understanding mathematical concepts and situations, either individually or within a community of collaborative thinkers. Students build a plethora of representations, such as graphs, written symbols, and gestures, to develop and convey ideas and information.

Relatively few studies have examined how students actually *build* their own mathematical understanding of motion by creating and experimenting with their own representations (but see diSessa, et al., 1991; Sherin, 2000). Most research done with representing motion mathematically focuses on using dynamic software (such as SimCalc or Boxer) or graphical simulation devices (such as CBLs) to trace movement onto an already existing graph or to create a graph of the movement (Gonzalez-Lopez, 2001; Chien, 2000; Dixon, et al., 2000). This type of modeling simulates physical motion in order to help students understand standard ready-made graphs involving rate, distance, and time (Chien, 2000) and to investigate ideas relating to graphical representations of motion and speed. Other studies (Dixon, et al., 2000) use functions to model real world phenomena and analyze these phenomena by looking at standard graphs and tables of data.

DiSessa and others focus instead on the representations that students actually build, given appropriate conditions, as they work toward inventing graphing for themselves. These authors use the idea of *meta-representational competence* (MRC) to investigate how students use and think about their representations. They define MRC as the ability or knowledge (1) to *invent* novel representations, (2) to *critique* existing representations, (3) to *function* with representations and understand why they work, and (4) to *learn* how to use new representations (diSessa et al., 1991). They traced students' MRC by examining their work on a carefully designed and presented task that elicited the invention of representations that eventually evolved into standard graphs. They concluded that after such a task is thoroughly explored, students "are in a much better position to understand graphing and similar intellectual tools. For having engaged in the process of design, they are better prepared to invent replacements when graphs are inadequate" (p. 158). Furthermore, diSessa et al. (1991) emphasize that the students in their study “did genuine and important creative work and that their accomplishment warrants study as an exceptional example of student-directed learning” (p. 117).

Speiser, Walter, and Maher (2003) emphasize that “learners must adapt their understanding, in order to address new problems where their understanding proves inadequate” (p. 2). The NCTM reform documents suggest how understanding can be adapted and built; “students’ understanding of mathematical ideas can be built throughout their school years if they actively engage in tasks and experiences designed to *deepen* and *connect* their knowledge” (NCTM, 2000, emphasis added). These documents further explain that learning with understanding can be enhanced through social interactions in the classroom. This helps students to learn to propose mathematical ideas, make conjectures, evaluate their own and others’ thinking, and develop mathematical reasoning skills (NCTM, 2000). This becomes an important part in gaining a relational understanding of the properties of graphs and motion. While these ideas are important and certainly emphasized in the reform documents, there needs to be more research and progress beyond our current state in order to implement them effectively.

Thus we need to develop tasks and problems that can elicit deep mathematical conversation and thought, and we need to work intensively with students, using such tasks and problems as a basis. The research presented in this paper seeks to contribute to this effort. It is part of an investigation of the developing mathematical understanding of non-science major college students as they work intensively on rich problem tasks. It is based on student data from non-traditional mathematical modeling course taught Fall Semester 2002 at an academically selective, private western university. One purpose of this class was to understand how students outside the traditional science-based community might gain a relational understanding of some of the fundamental concepts of motion and calculus. My project focuses on one individual, Sara, who is a dance student. She took pre-calculus in high school, but put off taking another mathematics class until her senior year in college. Sara does not consider herself a strong mathematics student.

This study focuses on how Sara comes to understand how to model or represent movement mathematically, and the kinds of reasoning that she does with those representations. Motion is something that we all engage in, whether it's moving the mouse on the computer or running after a bus. How do our own experiences of motion help us to understand how to *represent* it? And from such representations, how do we *reason* about it? I am interested in seeing what students bring into the classroom that help them to develop a deeper understanding of motion, especially of their own motion? An important further goal of this research is to better understand conditions under which non-traditional mathematics students can succeed in mathematics, and where they can use their own experiences and prior knowledge to build meaningful mathematical connections, based on their own work and thinking.

The class in this study was structured by doing a number of tasks chosen or designed by the instructor to give students opportunities to build the basic concepts used for understanding motion. In this qualitative study, I will track Sara's involvement in the class, her interaction with the other students, and the contributions that she makes. This will be accomplished by analyzing classroom videotapes, transcripts, interviews, and Sara's notebook and journal. I will follow the evolution of Sara's representations in the course of one specific task as her changing understanding of standard velocity versus time graphs evolves with them.

“Today we realize the importance of understanding individual children in order to understand children in general” (Thompson, 1994). In this case study, I view Sara as a representative of her whole class. Further research, now in progress, may provide additional support for generalization. My central objective here is to look deeply at Sara’s understanding of motion as it relates to the choices she has made in representing motion mathematically. It will also be of interest to understand how Sara’s questions came to be answered in this class, and how Sara’s process of questioning and reflecting may have affected her use of representations and her ultimate understanding of investigating motion mathematically. Consequently, this research focuses on the following two specific research questions:

1. What choices about representations did Sara make in order to reason about representations for motion? What was her motivation for making those choices?
2. What evidence is there that Sara gained a deeper knowledge of the mathematics of change?

The following chapter will explore some basics of the didactic contract (Brousseau & Warfield, 1999) formed in Sara’s classroom, of interactions among students, and will include a more thorough look at how to analyze the use of representations and motion in classroom research. That discussion will be followed by a careful look at the methodology used in this study, and the analysis performed.

CHAPTER II: LITERATURE REVIEW

This research focuses on how students reason about representations and motion. This chapter focuses on prior research concerning student interactions in the classroom to facilitate rich dialogue and an enhanced learning environment by renegotiating a didactic contract (Brousseau & Warfield, 1999), student representations and how they are built, and student understanding of and modeling of motion and movement.

Didactic Contract

It has become widely accepted that to understand and explore significant mathematical concepts, students need to actively take part in creating a proper learning atmosphere. Brousseau and Warfield (1999) have done work involving didactical situations (or the theory of situations):

> The theory of situations is based on the idea that human knowledge is manifested in its role in the interactions between systems: actors, milieu and institutions. To each piece of knowledge it should be possible to associate a limited number of specific types of interaction whose proper development requires that knowledge, or even causes it to develop (p. 9).

The theory of didactical situations is an important tool in directing student learning. Brousseau and Warfield (1999) explore the theory of didactical situations through one case study of an eight-year-old boy in a mathematics classroom. This study shows some of the complexities of arranging didactical situations for this boy in order to help him gain a better understanding of mathematics. The teaching and learning environments were established to give the student an opportunity to feel confident with his abilities. This student was then motivated to master the subject through experimentation, searching, and decision-making. Brousseau explains that the teacher can then search for ways to promote this type of thinking in the student through questioning, forming new rules, or redefining the didactical contract with the student.

The theory of didactical situations is an important tool in directing student learning. Didactical contracts are made between the students and the teacher, and they are periodically adapted as new situations arise in the classroom. Students are given freedom to experiment, explore, and design solutions to problems given in class using different representations. How students perceive that contract influences the decisions they make in class and whether they experiment and ask questions to deepen their own understanding.

Representations and Motion

The representations that a student uses can help develop understanding of mathematics. In this paper, a representation will be defined as a presentation to either oneself as a reflection or thought, or to others as part of an evolving thought within a community of thinkers (Speiser & Walter, 1997). Students adapt their understanding of mathematics as a way to address new problems where their current understanding proves inadequate (Speiser, Walter, & Maher, 2003; Siegler, 1986; von Glasersfeld, 1987 and 1995; Confrey, 1990; Yackel, 2000). We come to know something through our own experiences and this often involves reflection and interpretation. The final result is of interest, but the process by which one arrives at the solution becomes a necessary component in student understanding.

Students build a plethora of representations, including graphs, written symbols, and even gestures and kinesthetic activities, to develop and convey ideas in the process of developing and understanding a problem and its solution. "Representation builds fundamentally toward social expression… Indeed, our only access to others' experiences is through interpreting, constructing meaning, in performances we share" (Speiser, Walter, & Maher, 1997, p. 41).

DiSessa and Sherin (2000) provide a good picture of how other researchers have viewed representations:

> Most of the literature on representation in science and mathematics learning has, by and large, concentrated on a small subset of competence related to representations. In particular, most studies have been about how students produce and interpret a small number of instructed scientific representations such as graphs and tables. Furthermore, even within this narrower context, the emphasis has been on the mistakes students make, rather than on *the capabilities that students possess* (pp. 385-386, italics added).

Most research about understanding and analyzing student representations has been focused on the difficulties that students encounter as they work with existing representations, such as line-graphs (Leinhardt, Zaslavsky, & Stein, 1990) and equations (Kieran, 1992). However, like diSessa and Sherin (2000), I will also look at "the capabilities that students possess."

Speiser, Walter, and Maher (2003) discuss what it means to experience working on a particular task within a community of thinkers. Enacting motion may be a useful tool or representation to more fully understand a given problem, as well as to begin to think about *how* to approach solving the problem. Both dynamic and written presentations were necessary for the students in Speiser, Walter, and Maher's (2003) study, in order for them to grapple with the deeper mathematical thinking needed for the given task. The students were given the Cat task developed by R. Speiser, in which they were asked to extrapolate information about motion from a set of 24 time-lapse photos (see also Speiser & Walter, 1994, 1996). As their research indicates, it is very difficult to make sense of instantaneous velocity from a discrete representation of actual motion, but complicated and rich ideas can come from the students as these issues are addressed (Speiser, Walter, & Maher, 2003). They state further that, “Our analysis, indeed, has shown how much such *dynamic presentations* help to convey how one may read the so-called static images. In keeping with our general approach to presentations, a reading, just like an inscription, can certainly present part of one’s thinking to oneself (perhaps even in a kinesthetic way), but also facilitate communication, especially to make important changes visible” (Speiser, Walter, & Maher, 2003, p. 24). Thus it becomes important to also view how representations are created within a community of thinkers, and how those representations and ideas are used within that community.

A study by Marguerite Etemad (1994) has indicated that some specific representations, such as acting out motion and kinesthetics, actually help somewhat to improve cognitive skills. In fact, Etemad states, “Teaching strategies need to incorporate art, movement, drama, music, a component that involves body motion rather than relying solely on visual and auditory means. When the child has the opportunity to become actively engaged by using one of these components in connection to the subject being studied, the child’s level of understanding will be increased” (p. 7). Similarly, a study by Maxine Sheets-Johnstone emphasizes that “modeled movement is no match for a real-life kinetics, which alone can provide detailed understandings of the…dimensions of movement itself and of the dynamics of kinetic relationships and contexts” (Sheets-Johnstone, 1999, p. 274). Creating an atmosphere where this type of learning can take place then becomes important.

An additional study on the importance of kinesthetic activity in the classroom comes from Reeve and Reynolds (2002). They claim that gesture and movement are helpful in the learning of mathematics for the following reasons; First, it helps to maintain joint attention, and secondly, it acts as a cognitive amplifier by giving students opportunities to participate in discussions in which the language of their problem domain proves insufficient. This is helpful for both the other students and the teacher in trying to understand different representations (especially gestures and other kinesthetic activities) that students present.

My study involves examining a variety of representations that students build in a mathematics classroom (such as graphs, journal entries, gestures, kinesthetic activities in the classroom, etc.) to analyze their lines of reasoning and how they both develop and convey their ideas through building these representations. Examining student representations will also be helpful in understanding what aspects of motion and rate might be represented in meaningful ways by students.

The previously stated literature has shown how didactic contracts can be renegotiated to help students construct ideas and meaningful connections, how those ideas are presented to their peers through representations, and how movement can aid in the learning process. The following literature will now ground the *invention* of graphical representations.

Graphical Representations of Motion

The research in this paper will focus on the student *construction* of mathematical representations without the aid of technology. An important connection must also be drawn between a student's work with representations and how that in turn helps them to apply the mathematics of motion to their own “worlds.” Students need to understand and work with motion from both the *inside* and the *outsid*e (Speiser, Walter, & Maher, 2003). To describe motion from the outside, students use measurements for speed or distance, and represent them graphically. Conversely, it is also possible to represent motion from the inside, which is carrying out a movement and reflecting on that experience. Looking at motion from the inside can lead to a more informal set of representations described by vernacular since they are typically built spontaneously.

Bruce Sherin (2000) focuses his research on how sixth grade and high school students *invent* and learn to use graphical representations of motion. The focus of Sherin’s work is on the broader context of meta-representational competence described in Chapter 1, but it will also be useful to understand his different classifications of representational forms and constructive resources. Those forms and resources include drawing, temporal sequence, features of the line segment, and graph-like representations. Students often represent their world in drawings. These can be adapted and changed, and the transition to formal mathematical models is not always easy to pinpoint. A temporal sequence is a form of a linear sequence of distinct elements; for example, “a list of items, written in a line, that tell the story of the motion one item at a time” (Sherin, 2000, p. 401). The features of a line segment, such as length, orientation, width, and color, are important in the construction of student representations. Finally, graphs or graph-like representations can be used to incorporate more detail in explaining motion. These are helpful classifications for understanding the types of representations that students may use to model motion.

A similar study by diSessa, et al. (1991) looks at the meta-representational competence of children (see also diSessa & Sherin, 2000). Their students were upper elementary students who had very little graphing knowledge or background, but they eventually invented graph-like representations to describe a problem about motion. Additionally, diSessa, et al. (1991) "believe the most significant result of this analysis is the discovery of substantial meta-representational expertise in children" (p. 152). The children knew what good representations were, how to critique them and ultimately, how to refine them.

The studies done by Sherin (2000) and diSessa, et al. (1991) will be useful to analyze how representations are built in the classroom and how they in turn help students to understand and model motion. They developed a strong conceptual framework for their study and this should be helpful in analyzing how representations are adapted over time, especially in the evolution of representations, for example, line segments and temporal sequences to a standard graph. Their studies also help to follow how representations are built within a community of thinkers.

This review of literature reveals strong agreement about the capabilities of students. In particular, when given the opportunity to explore and experiment with different representations, they are often able to solve complex problems.

CHAPTER III: DATA SOURCES AND METHODOLOGY

Background

Honors 250 was designed as a general education course in mathematical modeling. It was adapted to provide an environment for research in undergraduate mathematics education. It was taught fall semester 2002 in a laboratory classroom at a private western university. The main purpose of the research in this course was to more clearly understand how students build mathematical ideas about change and motion, both individually and collectively. The students engaged in a series of task-based mathematical explorations of movement, growth and change. Class was held once a week for one semester, and each class session was three hours long. Professors in both the Dance and Industrial Arts departments handpicked the six participants in this class: they included four dance and dance education majors (Sara, Cammie, Lainie, and Krista), and two industrial design majors (Ali and Brittany).

Throughout the semester, each class was videotaped by undergraduate and graduate research assistants. All six students sat at a large table and had easy access to each other's work and conversations. There was one camera on the group the entire time. The cameraperson was free to follow key conversations among the students in the classroom, or interesting individual work as they worked on the tasks. Data segments from these videotapes were then carefully selected, transcribed and analyzed. Video segments were chosen based on how the different representations were built and discussed in the classroom, especially those representations built by Sara.

The students were also required to keep on-going journals of their current thoughts and work. Each student kept a working notebook during class and also a reflective notebook that was used at home. Both Sara and Cammie recorded detailed notes in their journals that will also be used in the analysis. In addition to notes from the class, Sara recorded her own thoughts, notes from the whiteboards, drawings that were especially meaningful for her understanding of the problems, and any key events that she chose to record. All work done on whiteboards was also carefully recorded on videotape and any representations made in class (not in the notebook) were copied and preserved as well. In addition, the course instructor also took down field notes during each class session.

Because of the richness of data collected from Sara and her unusual and creative work in class, she will be the focus of this research paper. It must be noted, however, that each student contributed *significantly* to the class and the work of each was striking and informative in its own right. Indeed, much of the work done by other class members influenced Sara's thinking and thus contributed to and are a significant part of her emerging understanding and involvement in the class.

Data Analysis

The first task proposed to the class is called the Desert Motion Task. This task was adapted from the Motion Picture Task designed by Andrea diSessa and coworkers (diSessa, et al., 1991; Sherin, 2000). The Desert Motion task is the focus for analysis here because it raised several key issues for Sara which afforded her the opportunity to expose ideas during her participation in class that are important for her own and others understanding of the mathematics of motion. A framework for analysis was built simultaneously with the actual analysis as new information was discovered and used.

From the onset, a grounded theory approach seemed most reasonable for this type of analysis.

Video segments were chosen based on several straightforward criteria. All segments involving Sara and the Desert Motion task were captured for digital editing and saved for further study. After careful review of those data segments, it became clear that the building and analyzing of *representations* was very important for Sara. Hence, I began looking for segments that revealed how Sara built her various representations, where she suggests her motivations for choosing, discarding or redesigning them, and where she clarifies the choices that she made in making new representations. I made memos to document how Sara's representations supported the process of solution on one level, and how I made sense of Sara's choices on another level. Memos were useful in recording how Sara and her peers presented their thinking, how they reasoned about mathematics, and what choices they ultimately made with each representation.

Representations that Sara and others made will play a major role in this analysis. Some of her representations were made to help her grapple with an issue by herself, while others were used to present her ideas to peers or the instructor (Speiser, Walter, & Glaze, in press). As ideas about these representations and their possible interpretations were discussed within the group, they were often reconstructed, redefined, or reinterpreted to present further important aspects of Sara's thinking. For Sara, others' representations and their critiques of her representations are shown to help her to build more meaningful representations for herself.

As the analysis proceeded, it became clear that Sara made several pivotal choices about how to represent the task. Hence, I organize the analysis in episodes to distinguish between the representations that she chose as her own analysis evolved. When she chose to discard an representation or model, she tended to describe her motivation for the change. Each episode then contains a description 1) of Sara's choice of a particular representation, 2) of her motivation to keep certain features of that, and perhaps previous representations, and 3) of her motivation to discard certain features of those representations. In this way, Sara's understanding will emerge through an analysis of key choices or decisions that she makes along the way.

CHAPTER IV: RESULTS AND ANALYSIS

The results of this research will emerge through a case study. In particular, I will focus on one dance student, Sara. She not only documented her own learning in a journal, but she contributed in significant ways to building understanding in the classroom. The first section of this chapter focuses on Sara's background, as well as the classes' mathematical background and expressed feelings about mathematics. The next section describes the task that will be studied in this paper. The last section outlines key events surrounding Sara's representations and the choices that she makes in creating those representations. These key events are explored by examining Sara's journal, the representations she made in class (on paper and on the whiteboards), and the videotaped sessions surrounding those events. These events are organized as a sequence of different episodes.

The Case Study: Sara

Sara is senior studying dance education at a private western university. She studied dance from the age of five and has a special interest in ballet. She is an unusually talented dancer and choreographer with high recommendations from the Dance Department at this university. She always dreamed of being a professional dancer, but also felt that she needed a university education, so she attended this university instead of a conservatory. On the first day of class, Sara said, "It's an awesome education and I wouldn't trade it for anything."

Sara, like several other students in this group, seemed apprehensive about mathematics classes in general. Here is what Sara had to say about mathematics on the first day of class:

> The math side of things has been a challenge and I'm kind of sad because it's not that I don't want to learn them, and it's not that I don't think I can't do it, I just don't understand it… I mean I know it's sad to say, but I've just been so prone to do what comes naturally to me [dance].

Sara also indicated that the last math class she had was pre-calculus in high school. She mentioned that she tried several times to take a statistics class in college, but each time she dismissed the idea after attending the first day of class. She commented that the classes were always too big and impersonal. She felt that she couldn't learn mathematics in that kind of atmosphere, so she put off taking a mathematics class until her senior year of college.

During an informal interview on the first day of class, Sara gave us a look into how she learns:

> Another thing that I really needed, especially with math, is that I always wanted to know why. I think I've always wanted that with every aspect of my life, and I'm probably going to drive everyone nuts 'cause I'll ask a million questions and … I'll just wait until I understand; which is kind of bad in some situations, but I want to know why it works the way it does and how it's actually going to apply to something I'll do in the future, and how I can use it. I mean I've, … there's been a million times where I've just memorized the equation and plugged numbers into it, so I could pass a test… But, I just don't like that because it doesn't have any meaning to me, you know? I just really want to understand it.

Sara has indicated that mathematics should apply to something that she will need or do in the future. In addition, she appears to have a drive to understand through asking questions. This makes Sara an excellent case study here since we are interested in following her reasoning about mathematics and representations. A part of this analysis will be tracking changes in how she represents motion and her motivations behind such changes.

Class began with an open discussion among the instructor and all four of the dance students (the industrial design students joined the class a couple of class sessions later). When asked about their mathematical ability and experiences with mathematics, the students gave very similar answers. They were, in a sense, explaining the didactic contract (see Brousseau, 1999) established for them in previous mathematics classes. Their comments indicated special concerns about mathematics, and the teaching and learning of mathematics at the college level. The following comments were taken from a transcript of several videotape segments of the first day of class.

> *Cammie*: Um, math [shakes her head]… It's hard not to be apprehensive about this class because it's a math class.
>
> *Sara*: I could always understand things if I could make it work for me. Maybe I felt selfish or that I couldn't take the teacher's time to do that or, I don't know, but I always just had to turn it around somehow so I could understand it… I had to learn arithmetic with things. I had to use things and objects and ways of perceiving.
>
> *Krista*: I've just understood that once you get into a university, professors supposedly don't really 'care' about their students and they just teach the only way they know how and you have to try and pick it up. And they'll just do it and you just gotta learn it. They don't try and help everyone and apply it to everyone, and don't necessarily make it fun and enjoyable.

Before any tasks were posed on the first day of class, the students were asked to answer a few questions on a small index card. These questions asked for information about their previous mathematics courses and their current feelings or attitudes towards the learning of mathematics. The students commented freely about some of their difficulties with mathematics. The responses included, "I'm not a math person. This could be because I'm simply not good at it or because I have been misunderstood." "I'm a very right-brained person and a very touch-muscle, kinesthetic, and a bit of a visual learner. In past math classes, I've had to get up and run around the building while doing my work."

The students were free to use any equipment or resources in the mathematics education laboratory. In this new setting, they immediately dismissed the traditional didactic contract described above, and with the instructor's willing consent, helped to create an atmosphere of exploration, collaboration, and experimentation. The new working conditions that the students helped to establish gave them freedom to explore the following task in any way that seemed appropriate and made sense to them. For Sara, that meant asking thoughtful questions that would address central mathematical ideas. The students were expected to help each other to decide when new information might be necessary and in particular, when their representations might not be adequate enough to represent the entire problem.

The Task: Desert Motion

The Desert Motion Task was adapted from the Motion Picture Task designed by Andrea diSessa and coworkers (diSessa, et al., 1991; Sherin, 2000). The task, as posed by diSessa et al., read as follows: "A motorist is speeding across the desert, and he's very thirsty. When he sees a cactus, he stops short to get a drink from it. Then he gets back in his car and drives slowly away."

There were some important additions to diSessa's task given in the class studied in my research. The task was never formally written down by the instructor, but was presented verbally so that the students could interpret the problem individually and collectively, making it open to negotiation and reformulation throughout the solution process. Figure 1 is the task written in the words of Sara, as recorded in her journal.

Math
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#1 - Desert Motion

A woman was driving on a long road in a desert. She sees a cactus with a water faucet poking out. [When she realizes this] she slams on the breaks, backs up, stops at the cactus. Then she jumps out of the car & gets a glass of H^2O, jumps back into the car & takes off down the same road (same way).

- Analize this problem -

Figure 4.1. Sara's description of the Desert Motion on page 1 in her journal.

After posing the Desert Motion scenario, the instructor asked the students to represent the motion of the car. Initially, no constraints were given on the type of representations to be created or developed. Later on, however, the students were encouraged to develop a static non-moving representation of the Desert Motion. An excerpt from Sara's journal reveals some anxiety that she had about this task (see Figure 4.2).

At first I was concerned with solving this problem - I never really understand problems like (~~space~~ rate × time = distance). It was hard for me to think of the problem in terms of mathematical concepts. But when I thought about it in terms of movement, and the sequence of events - movement & change, it became easier. We started by drawing pictures.

Figure 4.2. Part of page 1 of Sara's journal from the first day of working on the task.

The class initially discussed the problem together, then turned to more individual efforts for an hour, working to make their own representations of the task. They collaborated freely with each other and ideas flowed around the class as they formed their own representations. Later, they shared their representations and continued to work both collectively and individually for another three-hour class period. On the third day, there was much more collaboration and working together until a final representation emerged and seemed to be generally accepted.

Sara Tackles the Task

Sara seemed very aware of the physical aspect of the Desert Motion task. In her journal, she said, "One interesting idea that came up as we were starting to formulate and analyze movement was sensing the way we would feel/react in a car, doing the movement." At the same time, she was very aware of the limitations of representing physical motion because she realized that one can never perform something the same way twice. This is shown in the following dialogue in the classroom.

> Sara: When you're dealing with movement, very rarely, well never, it's never going to be the same way twice. So, it's hard to do a movement problem for me mathematically because mathematically it's always going to equal the same thing if you have … If you design a problem and have a solution, that's always going to be the solution. But if it's movement… if we were to move it, it would be different every time. If the lady was actually driving and stopping at a cactus, it would be different.
>
> Krista: Ya, because she'd…
>
> Sara: She'd take a different number of steps, or she'd stumble or she'd get an extra drink, or…
>
> I: That's true.
>
> Sara: Like in real life.

This problem seems quite real for Sara. The students are invited to rely on their previous experience with change and motion in order to understand how they might choose to represent this task mathematically. Hence it seems natural that Sara and others place themselves, imaginatively, in the car the task describes.

Since this was a math class, the students expected the problem to involve numbers and calculations. In fact, after the dance students temporarily got stuck on how to represent more in a picture (i.e. speed, distance, *and* time), Sara said, "I guess that's where numbers come in or something." However, as it became evident that numerical information was not to be provided as a part of the Desert Motion task, new issues began to surface for these students. How can you represent something mathematically if there are no numbers in the problem? How do you work with the ideas of continuity or discontinuity in a static representation? From my perspective, the students began to question their prior understanding of mathematics, and how it might be used to enrich their own understanding of the world around them. At one point on the first day of class, Sara said, "I like to be able to understand why things are the way they are, and how things work. I understand things better if they relate to things I can visualize or relate to." For Sara, this sense later emerged as a central issue as she sought to apply her rich experience of dance to her developing mathematical enterprise.

After reviewing the classroom videotapes and journal entries from Sara, there were several issues that Sara addressed and then incorporated in building representations for the Desert Motion task. An analysis of the following episodes helps us understand how Sara thought about continuity, and how she coupled this idea with her own experiences to eventually understand a standard position versus time graph of the motion of the car. My discussion of these episodes will focus on what Sara gained from the different representations, her choices for using particular representations, and her motivations for using certain properties in new representations or discarding properties from previous representations. Sara used her own understanding of motion derived from her experiences and observations of physical movement to help her develop and understand different models of representing motion. These episodes show how Sara used color to help her understand continuous change, how she used Cuisenaire rod representations to give a more complete picture of the task, and how she used cone-like representations to describe her way of thinking about continuity.

Episode 1 – Color Progression

For Sara, color seems to be a very real way of understanding and representing motion. She said, "I always think in colors though. I'm a big color person… I can relate like feelings or movements with colors." She used this idea of color to understand the limitations of her different representations and to try to understand how to represent the Desert Motion more completely.

Sara's involvement with this task began by adding to a representation started by Krista on the whiteboard (see Figure 4.3). This type of representation was coined a temporal sequence by Bruce Sherin (2000) because it shows the different phases of the motion one frame at a time. Each frame represents a significant event in the motion. For example, in Figure 4.3, the fourth frame shows the car stopped and the woman getting a drink, and the fifth frame shows her speeding away to get to her deadline.

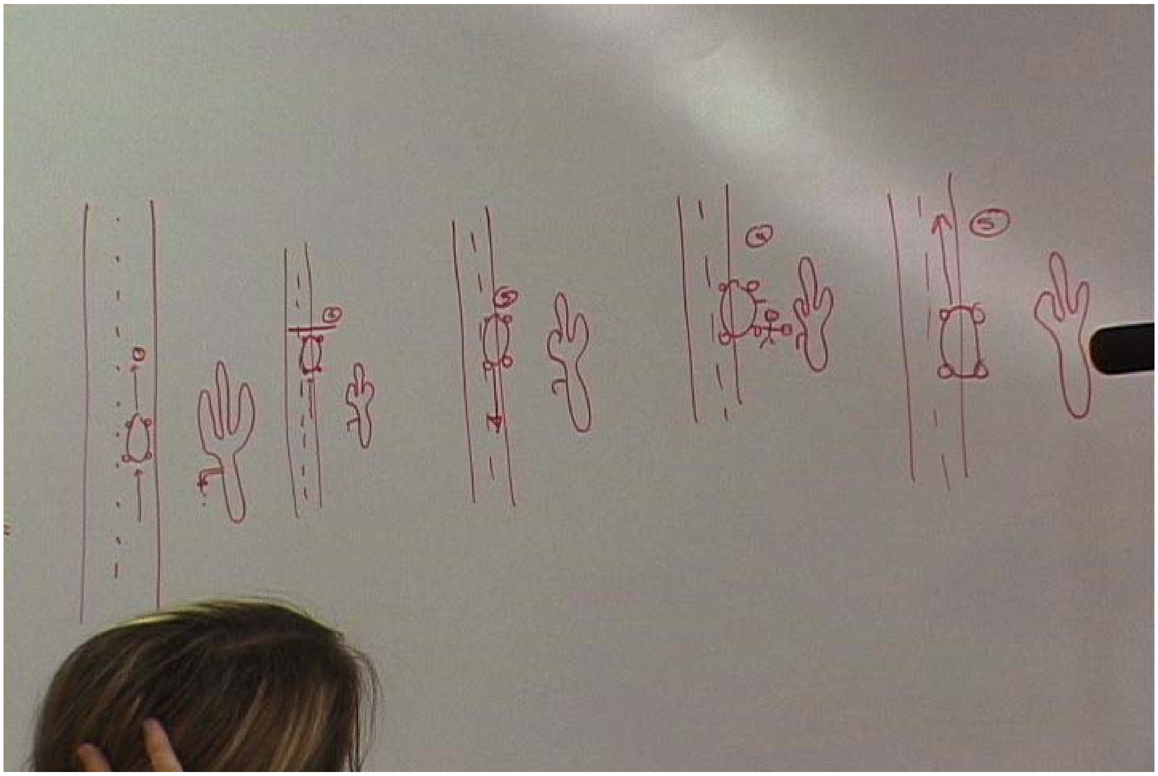

Figure 4.3. Sara's first representation of the Desert Motion task on the whiteboard.

After building a representation of the motion using a temporal sequence, Sara was prompted by the instructor (I) to think about some information that might not yet be present in her representation.

I: How do you get at how fast the car is going?

Sara: I didn't even think about that with this, but…

I: That might be another thing to think about.

Sara: Maybe, like a different color progression, or a different… I guess that's where numbers come in or something.

Of particular interest is Sara's mention of a color progression to describe how fast the car is moving. A dictionary[1] defines a progression to be a continuous series or movement from one member of a continuous series to the next. It can be thought of as a

[1] Definition of progression in http://dictionary.reference.com/search?q=progression

process of progressing. This is the first *suggestion* of any idea of continuity for Sara, although she does not explicitly use the term. In later segments, Sara uses this idea to explore continuity of physical movement and seeks to incorporate that into her representations. Sara continued to talk about this idea of a color progression to represent motion. The following excerpt happened about a minute after the dialogue above.

> Sara: But see this is interesting to me because when you choose something to represent the speed, like I said a color progression. But, this is kind of where I get stuck in math because I think mathematicians have… use numbers to form the way things are in the world…the way, like speed, motion, equations, to relate to that. And if I used color, I would need the people understanding the problem to understand my method of using…you know, my method of representation. What each color represents. And that's where I get stuck with math because I don't necessarily understand what equation represents what.

Here, Sara seems to be grappling with making her own understanding of motion compatible in some way with what she thinks a mathematician would do. She also seems concerned with mathematical equations and how they might relate to this problem. In other words, Sara seems fundamentally concerned with how her representations will communicate her ideas to others, while still incorporating important aspects of the representation for herself.

After the students where given time to work on the task individually, they shared their new representations. Sara modified her temporal sequence from the whiteboard slightly to incorporate some key elements, such as using her previous idea of a color sequence (see Figure 4.4). She drew colored lines in each frame of the temporal sequence: red is the fastest, orange is slowing down, and yellow is slowed down to almost a stop. She represented the process of slowing down by drawing a series of red dashes followed by orange dashes and ending with yellow dashes as the car came to a stop. The color order reversed as the driver sped away. In the third frame, Sara drew a series of yellow dashes followed by orange, and ending with two yellow dashes as the car sped up backwards and slowed down to a stop. She was beginning to address the issue of representing speed in her picture.

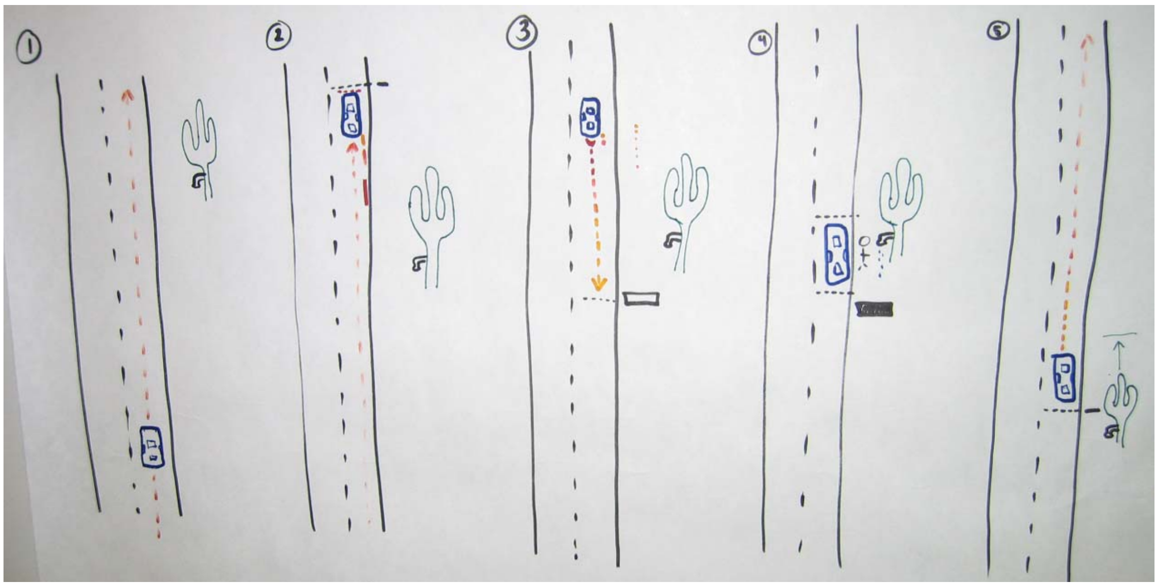

Figure 4.4. Sara's second rendition of the temporal sequence.

Sara is going back to something that is experientially real to her—color. She knows and has had experiences with color before, but she is not confident in her ability to use numbers and equations. Her experiences with color have made sense to her, but she is now grappling with the issue of representing motion with something that she sees as being non-standard mathematics. Sara wants to represent the speed of the car, but she also sees a need to bridge the gap between her world and a mathematician's world. Using colors serves Sara's purposes of conveying speed, but she seems to feel the restrictions placed on her from what she views as a mathematician's point of view.

Episode 2 – Rod Representations

After the initial attempt to represent the motion of the car, the instructor then asks the class to represent the motion using one static, non-moving representation of the Desert Motion. The idea of color resurfaces as Sara makes a new representation for the motion of the car. Sara commented in her journal about representing different aspects of the problem when she said, "We used colors to represent different speeds—drawing the lines with different colors, and found ways to represent different amounts of time stopped."

Knowing that she was not meeting all of the recently negotiated requirements for the problem (i.e. one static non-moving representation), Sara along with her classmates began to build other models, this time with Cuisenaire rods (Cuisenaire rods were a readily available resource to the class). They were introduced by the instructor at this time as an additional resource in representing the problem. All of the students produced unique representations using these rods.

Sara built the representation as shown in Figure 4.5. Her description of the different parts of this model reveals something about how she thinks about color and continuity and their importance in her new representation. Sara's rod model begins in the upper left-hand corner with four orange Cuisenaire rods. On top of the last rod is a red rod, which represents the car. The orange rods are followed by one yellow rod, and a purple rod, with a black rod perpendicular to them. The rods continue with a purple, two yellow rods, a purple and another black rod perpendicular to those mentioned. There is a green "cactus" at the end of the black rod and the rods continue with a purple, yellow, and two orange rods.

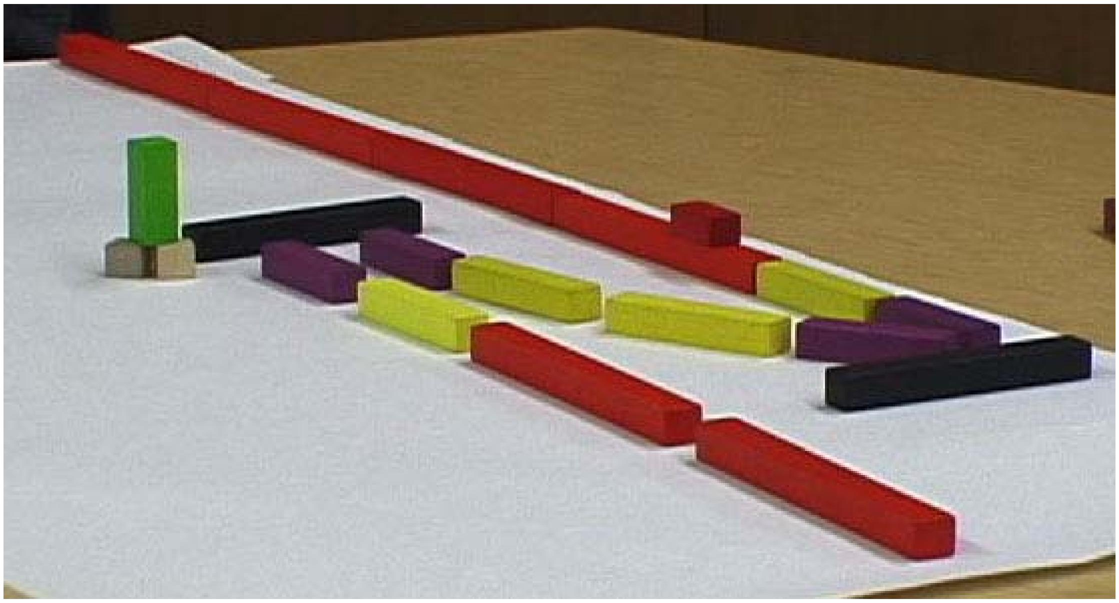

Figure 4.5. Sara's first Cuisenaire rod model.

Sara described her rod representation (Figure 4.5) to an observer: "Orange is a constant speed, yellow is the first, … I mean technically you'd need like fifty million colors to represent slowing down. Yellow is the first sequence of slowing down and purple is slowing down almost to a stop. Black is stopped…" Here we can see that the idea of a color progression is an important component of Sara's model and her understanding of motion. She is limited in her model because she does not have "fifty million colors" to represent this continuous process of slowing down. She adapts to the rod model then by allowing the yellow and purple rods to represent sequences of slowing down. It is also interesting to note that the rods do not follow the traditional color spectrum that she used in her description of the temporal sequence. There has been a shift in her focus. The length of the rod now seems to play the role of showing various stages of slowing down (differing speeds).

For Sara, this new rod model incorporated more information about the desert motion than the temporal sequence had. She was able to use the lengths and colors of the rods to represent different speeds, she was able to show the stops of the car, and all of this was shown on one representation.

However, Sara realizes that some important features are still not present in her representation. She has a difficult time seeing how to represent time in her model even though it is present. She also reasons about the continuity of the problem. She said, "I was thinking movement isn't like that [her rod model in Figure 4.5]. Movement goes from, if you're at fifty miles an hour, you're going to go, you know something like this [traces a smooth curve in the air with her fingers]. So then I just drew this… but it's not a good representation 'cause the way that I have it is kind of like this, and a step down and a step down, so it would be almost like the car went from fifty miles an hour to thirty and then to twenty and then to zero. You know? And cars don't do that."

Sara sees the importance of modeling the continuity of the motion, and she has a very clear picture of how that should be done, but she still seems at a loss to incorporate it into one single representation. In fact, on two separate occasions Sara asked, "Is there any *one* way to show everything?" Sara also sees this idea of trying to represent continuous motion in other student's models as well. She said, "So then when I was looking at hers [Krista's representation in Figure 4.6], that's exactly what I was thinking, with her continuum like this [traces Figure 4.6 in the air]. You know, and she just broke it down into all the little parts, but could you represent it like that [Figure 4.5] and still have it equal out to be the same?"

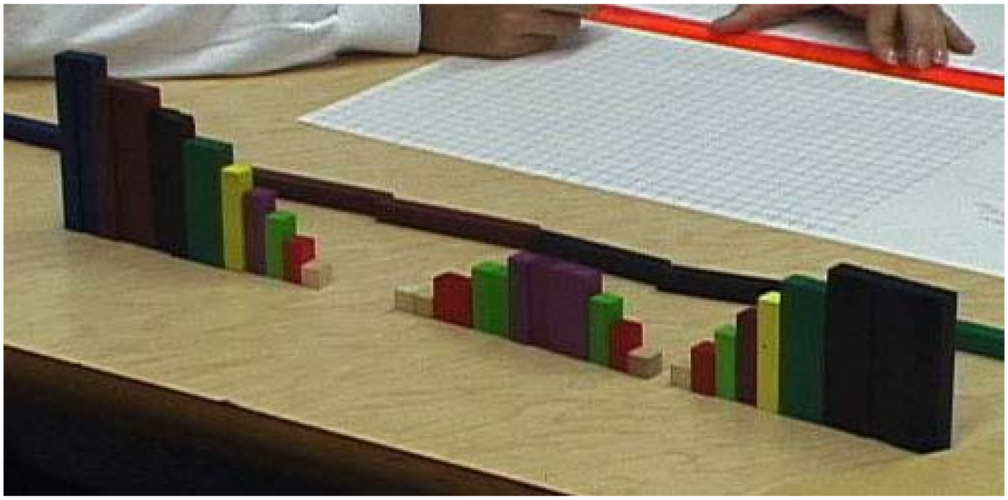

Figure 4.6. Krista's Cuisenaire rod representation of the Desert Motion task.

In Figure 4.6, Krista has Cuisenaire rods standing vertically on the table to represent different speeds the car travels (height of the rod) over time (the width of the rods and width of the holes). The two holes between the rods in Krista's representation show the time that the car was stopped. After Sara posed the previous question, she reproduced Krista's rod representation with some interesting modifications (see Figure 4.7). Even though Sara and Krista used the same rods in their representations, Sara chose to lay the rods flat on the table and to stagger them differently from Krista. We will see a possible motivation for this layout in the next episode.

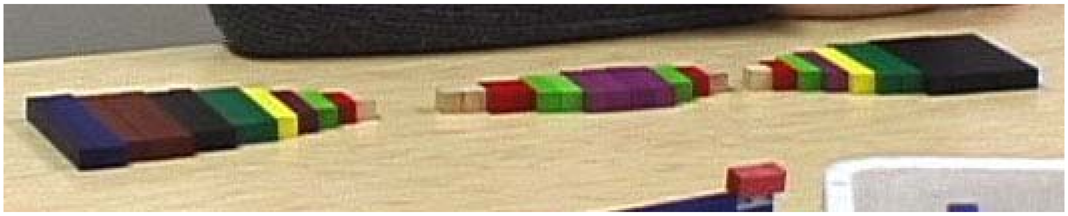

Figure 4.7. Sara's adaptation of Krista's rod representation.

Sara is concerned about representing all components of the motion, including distance, rate, and time. In a dialogue among the students and the instructor, Sara asked, "So wouldn't you have to take that [all components of motion] into consideration?" The students agreed with Sara and she concluded that, "I think Krista's the closest on that, but I think we've all hit the concept." This may indicate that Sara wanted to adopt Krista's model, with modifications, because it described the motion of the car better than her rod model. However, Sara also posed the question earlier about whether the representations actually did reveal the same information and whether they could "equal out to be the same."

At this point, Sara's motivations for adapting and discarding information seems to stem from the fact that she feels the need to represent the continuity of the motion of the car, and the distance, time and velocity of the car along the journey. She adopted a modified version of Krista's model seemingly because it showed the "continuum" or continuous nature of movement a little bit better than her representation had. Krista's representation also showed time explicitly (in the horizontal direction) whereas in Sara's previous representations, she could not readily identify how to see or even to show the time the car was stopped or as it was traveling. However, Sara is still not satisfied with the representations as is shown in the next two episodes.

Episode 3 – Cone Representations

An observer tries to understand Sara's first rod representation (Figure 4.3) by saying, "There is constant speed and then there are kind of slowing…" Sara interrupted by saying, "But slowing's a process, and I don't know how you can actually calculate it." She recognizes the connected nature of slowing, but does not yet seem to make the bridge to representing this idea *mathematically*.

The observer then asked Sara to follow up on her previous statement about the slowing process. Her response below indicates a deep familiarity with the concept of continuity, without having the formal mathematical language to describe it.

Sara: Well, depending on the car that she had, it would be a constant, … like, almost like a cone. Like she has a constant speed, but then it will narrow down like this [draws Figure 4.8] until it stops [Sara draws the line at the point perpendicular to the legs of the cone].

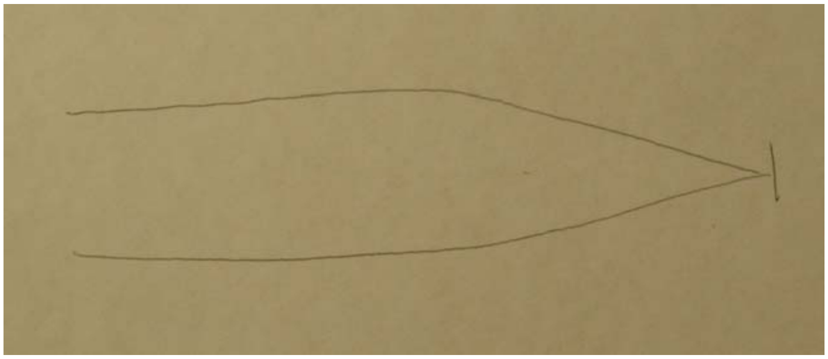

Figure 4.8. Sara's cone representation.

Sara uses parts of all her previous inscriptions and representations to try to make a more complete representation. This cone idea is very similar to the one in which she tried to describe the slowing process with the rods in Figure 4.5. Not only is this representation also visually similar to her second rod representation (Figure 4.7), but the stop is also represented perpendicular to the motion as in Sara's first rod representation (Figure 4.5). At this point, Sara strove to include as much as she could in her representations, but she saw the limitations of the discrete rod models. She seems to understand the need to represent the continuity of motion, and subsequently tells us why the rod model does not adequately represent that. The following conversation between Sara and an observer (O) help to illustrate this point.

Sara: But with colors, it's like I have one color and then another color and then another color and I can say that this is the speed, and this is a speed, then it would turn into, like, this, then like this, and like that [see Figure 4.9 (left)].

O: Ya, I see what you're saying. So this … so, instead of this nice continuous kind of flow [retraces Figure 4.8 with fingers], what you're seeing is… It's like taking snapshots here and here.

Sara: Ya, with this process of representation. Because I don't know if it would, if this [Figure 4.9 (left)], like, mathematically equals this [Figure 4.8]. But to me, they're different… Like, would these steps actually even out theoretically?

O: Say more about that, what do you mean? I think you've got something in mind.

Sara: I mean, if slowing down is actually a thing like this [Figure 4.9 (right) before she drew the steps on top of the smooth curve], but instead I have like steps where… I don't know how to draw it to scale [draws the step-like line over the straight lines in Figure 4.9 (right)], but do you know what I mean? Would like, these little pieces automatically even out? Like, what if you pulled that line [the step-like line], would it equal? Like, can you actually present it that way and have it without having a change of orange into yellow until almost like a white color or something?

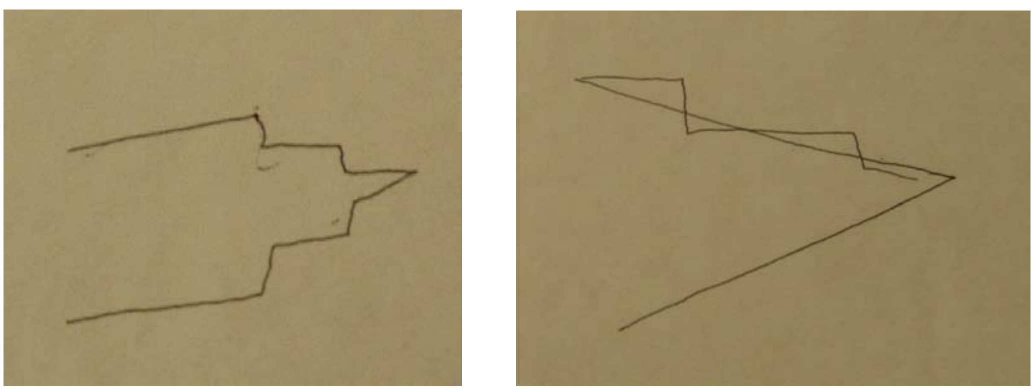

Figure 4.9. (Left) Jagged cone representation; (Right) Cone with overlaid jagged steps.

Sara's discrete and continuous models are different. She is still grappling with how to make the discrete model become a continuous one, or at least to show that aspect of the task into her representation. Even though Sara does not represent the entire Desert

Motion using these curved lines, they are used mainly for reasoning about the need to incorporate continuity into hers and other's representations.

Sara had multiple conversations with classmates, the observers, and the instructor about this cone idea to represent continuous slowing down and speeding up. She seems to understand the limitations of her rod model when she asked if the jagged line could somehow equal the curved line. At this point, she seems to be at a loss to create a new representation to incorporate this key idea. A fellow classmate, Ali, then begins to describe her representation and the following dialogue occurred:

Ali: My question is… 'cause like with the sticks [Cuisenaire rods]. So say like each one is one second right [referring to Krista's model, Figure 4.6]? But say point five… fifth of a second, it's still at the same speed and you don't kind of, [traces a jagged line in the air], you know? Like there's not…

Sara: That's exactly what I was trying to do last time.

Ali: … it's not the average because you have to um… If you had to plot it out with like points, then you could take any point anywhere along that line instead of like an average speed for that second.

Krista: Well, ya, I'm doing that…

I: Is that why your representation had the slanting parts?

Ali: Well, 'cause see…

I: To get closer to this?

Ali: Ya, like it…

Sara: Do you remember the drawing I drew? Like…

I: Please put it on the board [to Sara].

Krista: Like if you had a graph, this is just showing you if they accelerate at exactly this, or like double, or adding one more at exactly one second each. But you know if they are accelerating really fast in a shorter amount of time, I mean, you know…

Sara: Like this is just how I'm seeing it my mind. And with the Cuisenaire rods, this is how it shows. No matter how small you break it up you're still at those, you know, point blah blah blah seconds. But this is how real acceleration happens [Figure 4.10 (left)]. You know, if you're de-accelerating here, it's a gradual process. And this [Figure 4.10 (right)]. I guess you could draw a line [adds the lines in Figure 4.11 (right)] and maybe with numbers it would factor to be the same, like maybe that [makes the dot below the last step in Figure 4.11 (right)] equals that [makes the small dot above the step from the previous dot in Figure 4.11 (right)], but…

Ali: You couldn't take any point…

Sara: You couldn't take, like ya…

Ali: …like any second, and have it be…

I: So if we took the rods literally, then…

Sara: And just stretched them out.

I: …it would distort our mental impression of the motion.

Ali: Well, not only that, but if you like took… like if we were just going for like time, like a certain amount of time, like maybe wasn't rounded to the whole, then you would get an improper number just because it's the average and not the …

Sara: Right.

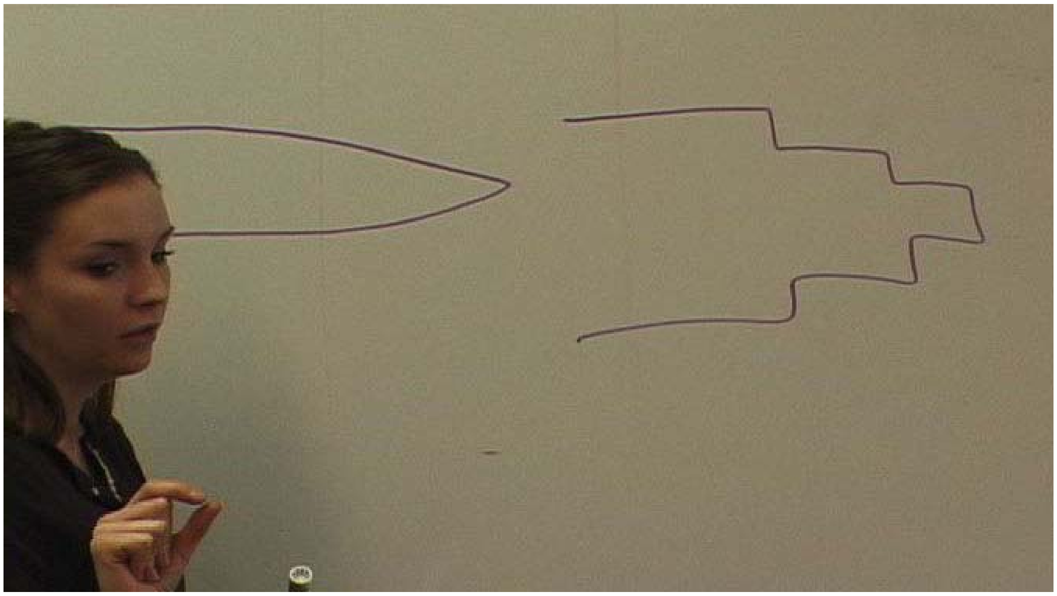

Figure 4.10. Sara’s smooth and jagged cone representations.

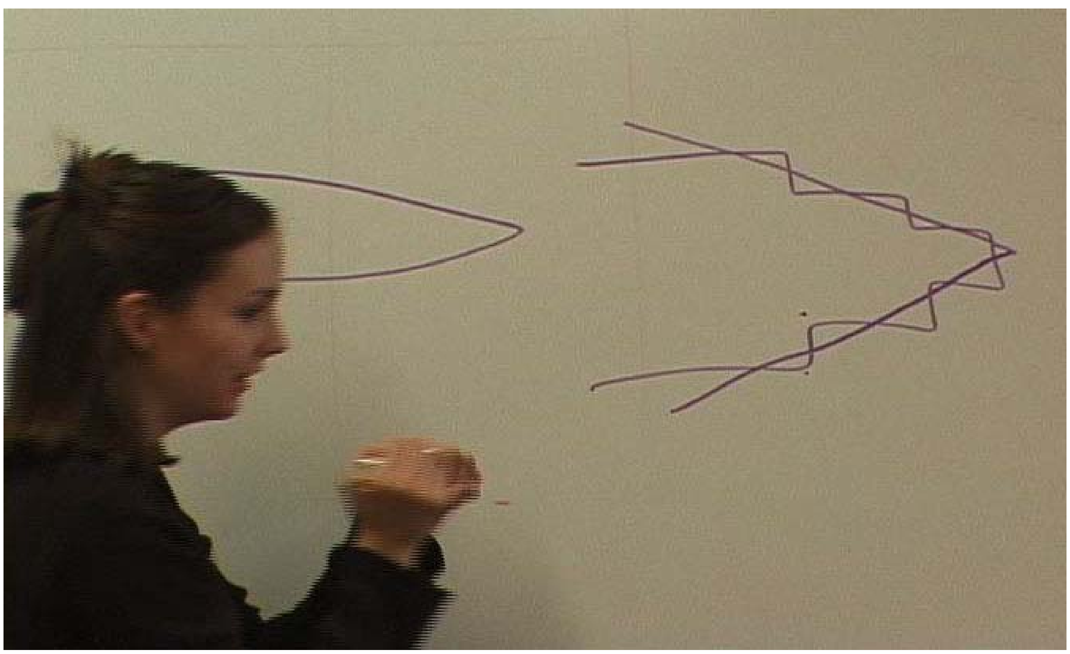

Figure 4.11. Sara drawing a line on the jagged cone representation.

Sara returned to her previous idea of representing continuity. She tried to fit her existing way of thinking about continuity to fit the different rod models, especially Krista’s model in Figure 4.6. It is clear that she is wondering if she can change the rod model in some way to make it continuous. Once again she is motivated to try to incorporate this idea of continuous physical motion into her static representation. Krista tried to explain how her rod representation might explain that, but Sara was unsatisfied with Krista’s discrete model (Figure 4.6) and turned to Ali for help. This gave Ali the opportunity to share what she had been working on and will be described in Episode 4.

Episode 4 – Ali’s Representation

Shortly after the above dialogue between Sara and Ali, Ali put her representation on the whiteboard (see Figure 4.12). Rate or velocity is shown on the vertical axis and time is shown on the horizontal axis. Sara seemed to connect with this representation in a way that she was not able to do with any other representation.

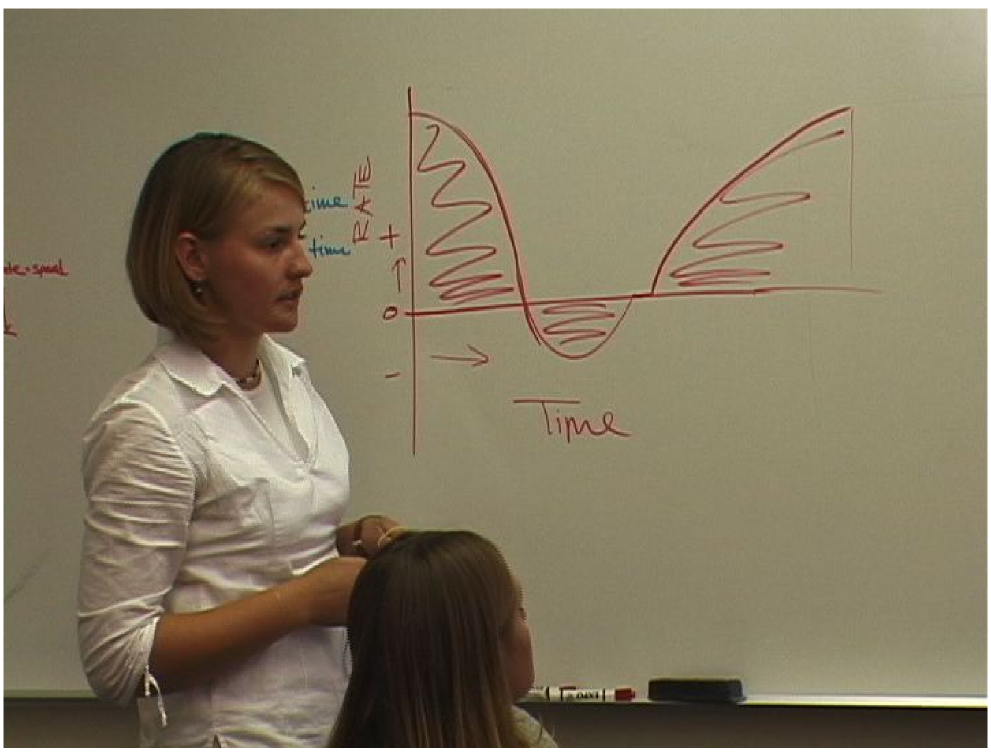

Figure 4.12. Ali’s representation of the Desert Motion task.

Sara: But if you used the rods, like if you drew that graph [Figure 4.12] to scale on graph paper, you could see it.

Ali: Well, if you even used [Krista's] graph [see Figure 4.13] and plotted the points and then drew…

Sara: Right.

Krista: You could see where it, like, faster deceleration and you could see…

Sara: You could take [Krista's] graph… Ya! Take that exact thing and plot the points and figure it out.

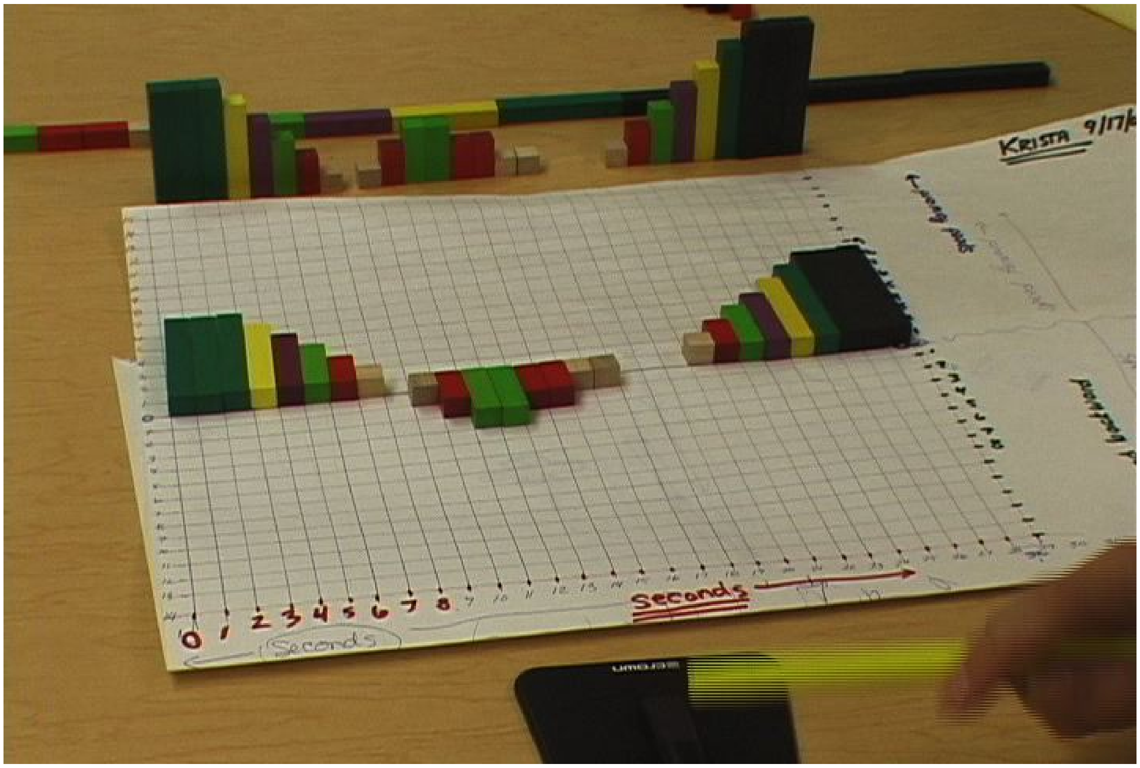

Figure 4.13. Krista's rod representation on graph paper.

Sara and Ali's excitement and emotion is hard to describe as they realized what Ali's representation meant. They both saw the direct correlation between Ali's representation and the one proposed by Krista, but Sara seemed to be more satisfied with Ali's representation because it solved her problem of presenting continuity, while preserving velocity, distance and time. There were no jagged lines or breaks in the graph that would suggest inconsistencies in movement. The following excerpt from Sara helps to clarify that point:

> Well, this kind of happens in movement too because something has to… If you have a movement where you put a lot of energy into it initially, you have to wind up that energy somehow. And it's not going to be in steps. It's going to be in a smooth motion. … Like a car isn't just going to go from thirty miles an hour to twenty miles an hour to ten to zero…

Sara found a representation that fit into her way of thinking. Figure 4.14 shows Sara's rendition of Ali's representation. Notice the important idea of area under the curve equating to distance!

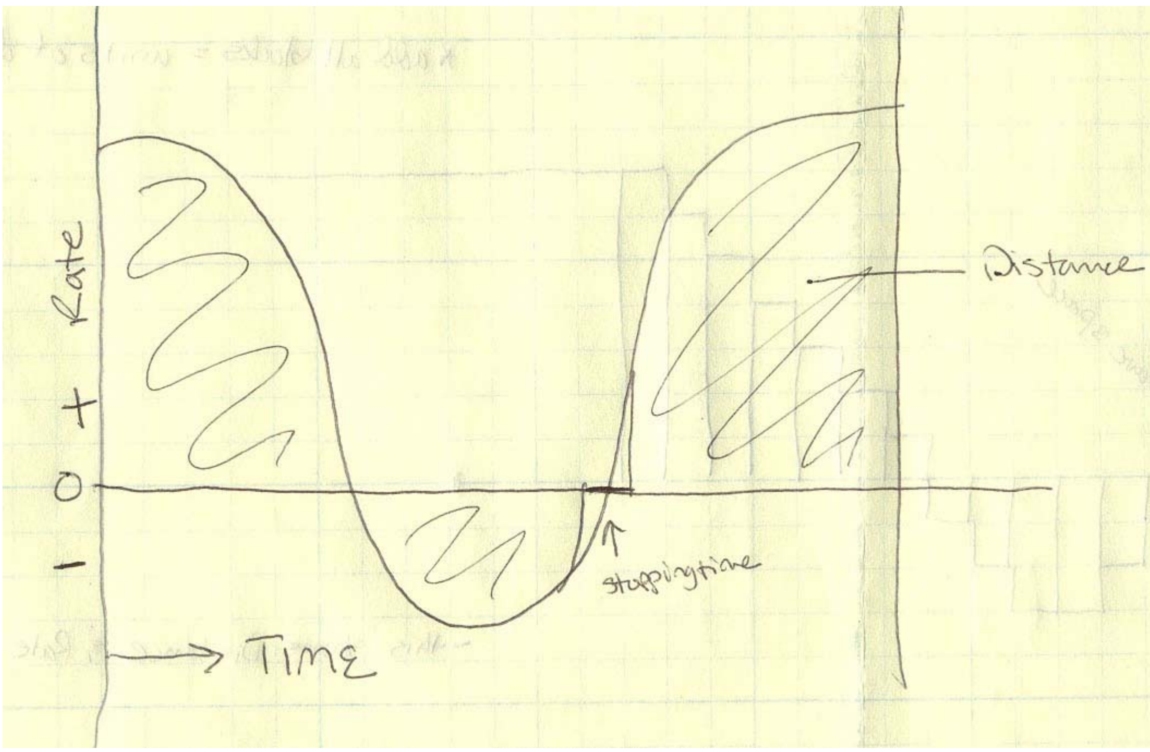

Figure 4.14. Sara's journal version of Ali's representation for motion.

CHAPTER V: DISCUSSION AND IMPLICATIONS

The research in this study focuses on the following two research questions:

1. What choices about representations did Sara make in order to reason about those representations for motion? What was her motivation for making those choices?
2. What evidence is there that Sara gained a deeper knowledge of the mathematics of change?

Discussion

The data and analysis provided in Chapter IV will lay the foundation to answer the research questions above. It will be of interest to examine first how Sara learns mathematics with understanding, and second how this class gave her the opportunity to learn significant mathematics by encouraging her to ask questions and reason about representations.

On the first day of class, Sara explained the conditions under which she understands mathematics:

> Another thing that I really needed, especially with math, is that *I always wanted to know why*. I think I've always wanted that with every aspect of my life, and I'm probably going to drive everyone nuts 'cause *I'll ask a million questions* and … *I'll just wait until I understand*; which is kind of bad in some situations, but I want to know why it works the way it does and how it's actually going to apply to something I'll do in the future, and how I can use it. I mean I've, … *there's been a million times where I've just memorized the equation and plugged numbers into it, so I could pass a test… But, I just don't like that because it doesn't have any meaning to me, you know? I just really want to understand it*.

This quote indicates that Sara felt strongly that in previous mathematics classes, her questions were not always answered, yet for her, building what she sees as answers to her questions is an essential component to her learning and understanding. Her understanding of the didactical contract that framed her classroom experiences was that she was expected to memorize equations and plug numbers into them without necessarily understanding why they work. Another quote from the first day of class indicates perhaps why her questions were not answered in previous mathematics classes:

> *I could always understand things if I could make it work for me. Maybe I felt selfish or that I couldn't take the teacher's time to do that* or, I don't know, but I always just had to turn it around somehow so I could understand it… I had to learn arithmetic with things. I had to use things and objects and ways of perceiving.

Throughout the course described in this research however, there is evidence that Sara's questions began to be answered by herself as she reasoned about the different representations in the group, and by the group directly. The focus of this research is not on the types of questions that Sara asked, but rather on what happened in this classroom that enabled her to ask her questions, and in turn to construct meaningful answers for those questions. For Sara, this happened as she was given the opportunity to reason about different representations describing the Desert Motion task.

Being able to ask questions and getting the help to answer them gave Sara a springboard to gain a deeper understanding of the mathematics of change. Many of the questions that she posed were not answered right away by her peers; in fact, these questions often stimulated conversations that raised central mathematical issues about speed and continuity, and about how to represent these in the Desert Motion task. For example, when Sara was comparing her rod representation with Krista's, she said, "[Krista] just broke it down into all the little parts, but could you represent it like that [Figure 4.5] and still have it equal out to be the same?" She is interpreting the different representations in meaningful ways to see how they might be similar, and what might be found useful in them. Sara and Krista then had a conversation about their different representations, and subsequently, Sara changed her rod model (see Figure 4.7) to reflect several features of Krista's representation that were not clear to Sara in her own representations. However, there are significant modifications to Krista's model that reflect Sara's desire to incorporate continuity into her model.

I see a delicate interplay between Sara's increased understanding of the mathematics of change and the choices that she made in keeping or discarding information in her representations for motion. As her awareness and understanding of motion increased, her representations became more detailed and she tried to find ways to incorporate more information into them.

I will now examine what I believe facilitated this interplay of ideas. This class had dismissed the traditional didactic contract on the very first day of the course. Sara and the others in the group described their experiences with what they perceived to be the traditional contract. In their view, this contract functioned as one which stipulated that they could not ask questions, that they should memorize formulas and plug numbers into them, and that the teachers did not need to care about their understanding of the mathematics. However, in this class, the students perceived the expectations placed on them to be different: the students could explore central mathematical ideas and they were given the opportunity to ask questions, and to create, with the instructor, an atmosphere of genuine learning. Indeed, Sara and her peers began to function as a group of gifted teachers for each other. They posed questions in order to genuinely understand what each other's representations showed, and why they were constructed as they were.

In Episode 1 of the analysis, Sara grappled with understanding how to represent different speeds in her first representation (Figure 4.3) so that anyone looking at her solution would understand how she represented the motion. In this way, she used the representation as a presentation of her own ideas to others. As Sara worked to understand how to represent speed, it occurred to her how colors would help her to achieve that goal, so she modified a previous representation to include that information (Figure 4.4). For Sara, using the spectrum or continuous sequence of colors to represent speed helped her to understand the continuity of the movement of the car. However, she also realized the limitations of that model because it was not clear to everyone who might look at her representation what the colors meant and how they represented different speeds. In other words, she was concerned with bridging the gap between her world and what she perceived as a mathematician's world.

She was then asked to show the motion of the car in a static non-moving representation, and that led to her rod representations presented in Episode 2. As others posed questions to Sara and as she in turn asked them questions, Sara began to find both limitations and benefits of her own representations. The rods were useful to show the speed the car traveled, but it was more difficult for Sara to show the continuously slowing motion and the stops of the car. She reasoned about the information to be represented in her models from her experiences with motion. She said, "I was thinking movement isn't like that [her rod model in Figure 4.5]. Movement goes from, if you're at fifty miles an hour, you're going to go, you know something like this [traces a smooth curve in the air with her fingers]. So then I just drew this… but it's not a good representation 'cause the way that I have it is kind of like this, and a step down and a step down, so it would be almost like the car went from fifty miles an hour to thirty and then to twenty and then to zero. You know? And cars don't do that." Her own experiences with motion let her discover an inconsistency in her representation—continuity.

In Episode 3, Sara explained her ideas about continuity and discontinuity in terms of her cone and jagged representations (see Figures 4.8 and 4.9); however, she did not seem to be able to connect these ideas right away to her other rod representations in order to make them continuous. So, Sara asked questions to try to understand the relationship between her rod and cone representations: "Can you actually present it that way [Figure 4.9 (right)] and have it without having a change of orange into yellow until almost like a white color or something?" She saw the merits of her rod representations, but was not satisfied that they did not articulate the continuous nature of movement. Sara was reasoning about what was important to represent about motion and what might be left out in a static representation. She took great care when deciding what to keep and what to discard from each of her representations. Her choices were based on her own experiences with motion.

When Ali began to pose similar concerns about the discontinuity of rod representations, Sara said, "That's exactly what I was trying to do last time [with the cone representations]." She presented her ideas to the class and that lead to Ali's final representation described in Episode 4. It is not difficult to see that all of the reasoning that Sara made about the representations for motion in the classroom gave her the foundation to understand the standard velocity versus time graph presented by Ali. In fact, it made the most sense to Sara because it incorporated time, velocity, distance (area under the curve), and continuity.

The shift of the didactical contract allowed Sara to ask questions and eventually find ways of getting them answered. She made choices about her different representations based on her own experiences with motion and what made sense to her from the other representations made in the class. This exploratory and inquisitive environment gave Sara the impetus that she needed to make meaningful connections in this math class. On a written questionnaire at the conclusion of the course, Sara explained her experiences with the class:

> I think that it was important to use objects and pictures first in order to break down the movement in a way that was very visual… I think it's important to be able to create tools as well for representations (making graphs, models, etc.)… It helped me to be asked open-ended *questions that made me think and discover solutions myself*. I think that *discovering the answers in ways that my natural thinking could relate to helped me internalize the concepts better*. I appreciated being helped to *understand why the problems work the way they do* instead of just being told to memorize how to do them. The class really helped me appreciate the value of mathematics.

Without much aid from the instructor, the class built analogues of Riemann sums (see Figure 4.13) to clarify how distance relates to velocity and time. Sara showed this understanding very clearly in her version of Ali's representation in Figure 4.14. She also commented on it in her concluding remarks about the course when she said, "One thing that was especially interesting to me was when we discovered the velocity. I think we had the colored rods and when we put the rods that represented the lady backing up down below the line of the others (see Figure 4.13) it made sense to me that the part of the trip where she backed up did not affect her overall distance."

Sara also began to understand the usefulness of graphs and how they can be used to represent motion. Over the next couple of weeks after doing one more task after the Desert Motion task, Sara wrote the following excerpt in her journal:

(this type)

- graphing is still a little dificult for me, but I finally feel that I can look at a graph & use it as a tool. They actually mean something. I guess this is a step in the right direction. Instead of just logically graphing the numbers in an equation, I am using a graph to discover unknown information, I am using it for a real purpose.

Implications for Future Teachers

There are populations of students from many disciplines who find it difficult to study mathematics. The research presented in this paper shows that teaching mathematics can be successful if students are given the opportunity to investigate tasks designed to explore significant mathematics. “All students should have the opportunity and the support necessary to learn significant mathematics with depth and understanding” (NCTM, 2000, p. 5). This can be accomplished when the didactic contract is negotiated and the students are given the opportunity to contribute to the overall learning environment of the class.

Students can contribute in a class when they are given freedom to ask questions. The research presented here suggests that whether those questions are answered directly right away or not is insignificant compared with the impact that those questions can have on the students themselves as well as on other students. Questions can provide the impetus for students to discuss important mathematical ideas, which may help them to internalize those ideas as they develop solutions to rich problem tasks both individually and collectively.

Future Research Opportunities

One possible avenue for future research would be to explore how students apply the mathematics of motion to their own lives. For Sara, it was important that mathematics be applied to something that she could use in the future. I do not think that Sara is unique in this view. It would be worth investigating what is useful for students to learn about motion and representing it mathematically, and what they actually apply in their lives. Further research may provide us with valuable insights into what students need in order to apply mathematics to their lives.

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