Rigorous Verification of Stability of Ideal Gas Layers

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ABSTRACT

Toward Rigorous Verification of Stability of Ideal Gas Layers

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In this thesis we develop tools for carrying out computer assisted proof of the stability of traveling wave solutions of the spatially one-dimensional compressible Navier-Stokes equations with an ideal gas equation of state. In particular, we obtain rigorous, tight error bounds on a high-accuracy numerical approximation of the traveling wave profile for parameters corresponding to air, and we obtain rigorous representations in a neighborhood of $\pm\infty$ of the solution to the first order ODE associated with linearizing the PDE equations about the traveling wave solution. We also develop supporting tools for rigorous verification of wave stability.

Keywords: Evans function, rigorous computation, stability of waves, Navier-Stokes model
ACKNOWLEDGEMENTS

I am grateful for Dr. Barker for supporting me and guiding me through my research. He was patient with me and gave me ideas. This thesis would not have been possible without him. I am grateful for my teachers for giving me the knowledge I have. I am grateful for my committee for approving my coursework and giving me this opportunity to do the thesis defence I need to do to graduate. I thank my family for supporting me throughout my education.
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Chapter 1. Introduction

In this thesis, we use computer assisted methods of proof (CAMP) to work toward proving the stability of traveling wave solutions of the viscous compressible Navier-Stokes equations in one spatial dimension with an ideal gas equation of state. In this context, a wave is nonlinearly orbitally stable if the solution to the Cauchy problem tends to a spatial translate of the traveling wave solution as time tends to infinity. To prove that a traveling wave solution of this systems is stable, it suffices to prove that the wave is spectrally stable; see [9, 12, 13, 14, 19, 25, 26, 29, 33, 34]. Spectral stability means that the eigenvalue problem formed by linearizing the PDE about the traveling wave profile has no eigenvalues with non-negative real part, except an eigenvalue at the origin which corresponds to translational invariance of the traveling wave solution. To prove a wave is spectrally stable, one may verify absence of zeros of the Evans function, a complex analytic function whose zeros correspond in location and multiplicity to the eigenvalues of the system. Numerics with careful attention to error estimates were carried out in [17] to approximate the Evans function to indicate that the waves are spectrally stable, hence nonlinearly orbitally stable. In this thesis, we work toward extending the results in [17] to rigorously verify spectral stability using computer assisted methods of proof. This requires complete control over all error involved in numerical approximations and computations.

Rigorous verification of stability of traveling waves using computer assisted methods of proof has been carried out in some other settings; see for example [1, 3, 5, 7, 28]. This work is particularly challenging because of the dimension of the system one must solve and the inherent stiffness involved in the ODE equations one solves to compute the Evans function.

To obtain rigorous error bounds on the numerical approximation of the Evans function, we bound all errors involved in the computation, including machine rounding errors, via use of the interval arithmetic package provided by the Python package mpmath. The main theoretical machinery used in this rigorous verification involves the Newton-Kantorovich
Theorem, the construction and proof of convergence of series solutions of ODEs, and numerical methods that are designed to deal with the challenges associated with the inherently stiff ODEs one must solve to compute the Evans function. To deal with the stiffness, we use the compound matrix method described in Section 2.3.

We describe our work toward rigorous verification of stability of traveling waves by first providing the mathematical background needed to prove stability of waves. We then describe how we obtain a rigorous bound on the error in approximating the traveling wave profile solution to the Navier-Stokes equations. This method also proves the existence and uniqueness of the solution. Finally, we describe our work toward rigorous error bounds on approximating the Evans function for the Navier-Stokes equations.

**Chapter 2. Mathematical background**

In this chapter, we describe the theoretical tools we will use to prove stability of traveling wave solutions of the Navier-Stokes equations. We begin with some general theory related to the stability of traveling waves. Next, we introduce the Evans function, which we use to show the traveling waves are spectrally stable, hence nonlinearly orbitally stable. We describe the compound matrix method, and describe the Newton-Kantorovich contraction mapping principle.

### 2.1 Stability of Traveling Waves

The Navier-Stokes equations belong to the class of PDEs considered in the nonlinear stability theory, which are of the form

\[
 u_t + f(u)_x = (B(u)u)_x + (C(u)u_{xx})_x, \tag{2.1}
\]

where \( t \geq 0, x \in \mathbb{R}, u : \mathbb{R}^2 \to \mathbb{R}^n \) is the dependent variable, \( f : \mathbb{R}^n \to \mathbb{R}^n \), and

\[
 B : \mathbb{R}^n \to \mathbb{R}^{n \times n}, \quad C : \mathbb{R}^n \to \mathbb{R}^{n \times n}.
\]
For the Navier-Stokes equations, $C(u) = 0$. A traveling wave solution of (2.1) is a solution that maintains the same spatial shape or profile, but moves at constant speed, as described below.

**Definition 1.** A solution of (2.1) that takes the form

$$u(x, t) = \tilde{u}(x - ct)$$

is called a traveling wave solution, where $c \in \mathbb{R}$ is the wave speed. We refer to $\tilde{u}$ as the traveling wave profile.

Rather than look for a traveling wave solution with speed $c$, we can make the transformation $x \to x - ct$ and look for stationary solutions of

$$u_t + (f'(u) - c)u_x = (B(u)u_x)_x + (C(u)u_{xx})_x.$$  

The traveling wave profile and its derivatives decay to the end-state at exponential rate as $x \to \pm \infty$,

$$\lim_{x \to \pm \infty} \tilde{u}(x) = u_{\pm}, \quad \lim_{x \to \pm \infty} \tilde{u}^{(n)}(x) = 0, \quad n \geq 1.$$  

In the above, the superscript $(n)$ indicates the $n$-th spatial derivative of $u$.

We are interested in proving stability, described below, of traveling wave solutions of the Navier-Stokes equations.

**Definition 2.** An acceptable perturbation is one such that the perturbation added to the traveling wave profile belongs to the domain of the initial value problem under consideration. A solution $U$ is asymptotically orbitally stable with respect to $\mathcal{P}$, the set of acceptable perturbations, if for all $V \in \mathcal{P}$,

$$U(x, t) = \tilde{U}(x) + V(x, t) \to \tilde{U}(x + \epsilon)$$  

(2.2)

for some $\epsilon \in \mathbb{R}$ as $t \to \infty$.

To prove that a traveling wave solution of (2.1) is stable, it suffices to prove that a wave is spectrally stable due to results outlined in [9, 12, 13, 14, 19, 25, 26, 29, 33, 34].
2.1.1 Spectral Stability. Spectral stability of a wave has to do with the eigenvalues of the ODE eigenvalue problem one obtains by linearizing the PDE about the traveling wave solution and looking for separated solutions of the resulting linear PDE. For (2.1), the eigenvalue problem takes the form

\[ \lambda V = LV := -\left(\alpha(\tilde{U}) \cdot V\right)_x + \left(\beta(\tilde{U}) \cdot V_x\right)_x - \left(\gamma(\tilde{U}) \cdot V_{xx}\right)_x \]  

(2.3)

where \(\alpha(\tilde{U}) \cdot V = dg(\tilde{U}) \cdot V - d\beta(\tilde{U}) \cdot V \cdot \tilde{U}_x - d\gamma(\tilde{U}) \cdot V \cdot \tilde{U}_{xx}\).

There are various types of eigenvalues of (2.3), as defined below.

Definition 3. We define:

- The spectrum \(\sigma(L)\) of \(L\) is the set of all \(\lambda \in \mathbb{C}\) such that \(L - \lambda I\) is not invertible.

- The point spectrum \(\sigma_p(L)\) of \(L\) is the set of all isolated eigenvalues of \(L\) with finite multiplicity.

- The essential spectrum \(\sigma_e(L)\) of \(L\) is the entire spectrum not including the point spectrum, i.e. \(\sigma_e(L) = \sigma(L) \setminus \sigma_p(L)\).

Spectral stability requires that none of the types of eigenvalues described above lie in the right half complex plane, as defined below.

Definition 4. A wave is spectrally stable if there is no spectrum of the eigenvalue problem (2.3) in the deleted right half plane: \(\mathbb{C}_+ := \{\lambda \in \mathbb{C} \setminus \{0\}|Re(\lambda) \geq 0\}\).

Spectral stability implies nonlinear orbital stability for a large class of PDEs; see [11, 15, 22, 23, 25, 24, 27, 31, 34].

To be spectrally stable, a wave can still have an eigenvalue at zero. In particular \(\sigma_p(L) \neq \emptyset\) because 0 is an eigenvalue whose corresponding eigenvector is the derivative of the traveling wave profile as given in the result of Sattinger.

Lemma 5. (Sattinger [30]) The derivative of the profile \(\tilde{U}'\) is an eigenfunction of \(L\) with eigenvalue 0.
Proof. Define
\[ F(u) := -(f'(u) - c)u_x + (B(u)u_x)_x + (C(u)u_{xx})_x \]
and note that \( \forall \delta \in \mathbb{R} \) that
\[ F(\tilde{U}(x + \delta)) = 0 \]
because the PDE is autonomous. Taking a \( \delta \) derivative and evaluating it at \( \delta = 0 \) yields
\[ 0 = \frac{\partial}{\partial \delta} F(\tilde{U}(x + \delta))|_{\delta=0} = F'(\tilde{U}(x))\tilde{U}'(x) = L(\tilde{U}'(x)). \]
Hence, \( \tilde{U}' \) is an eigenfunction with eigenvalue \( \lambda = 0 \). \hfill \Box

2.1.2 Essential Spectrum. To be spectrally stable the essential spectrum must also be in the left half plane. Fortunately, it is relatively easy to find the essential spectrum due to the following result of Henry.

**Theorem 6.** *(Henry [14])*

The essential spectrum of \( L \) in (2.3) is sharply bounded to the right of \( \sigma_e(L_+) \cup \sigma_e(L_-) \), where \( L_\pm \) correspond to the operators obtained by linearizing about the constant solutions \( \tilde{U} = U_\pm \), respectively.

By the result of Henry, it suffices to find the eigenvalues of a matrix because we can use the Fourier transform to solve the following eigenvalue problem,
\[ \lambda V = LV := -\alpha_\pm V_x + \beta_\pm V_{xx} - \gamma_\pm V_{xxx}, \quad \text{(2.4)} \]
where \( \alpha_\pm := dg(U_\pm) \), \( \beta_\pm := \beta(U_\pm) \), and \( \gamma_\pm := \gamma(U_\pm) \).

The linear operators \( L_\pm \) have no point spectrum, so \( \sigma(L_\pm) = \sigma_e(L_\pm) \).

Taking the Fourier transform, we have
\[ (\hat{L} - \lambda I)^{-1}V = (-i\zeta \alpha_\pm - \zeta^2 \beta_\pm + i\zeta^3 \gamma_\pm - \lambda I)^{-1}V, \quad \zeta \in \mathbb{R}. \quad \text{(2.5)} \]

We note that \( 0 \in \sigma(-i\zeta \alpha_\pm - \zeta^2 \beta_\pm + i\zeta^3 \gamma_\pm - \lambda I) \) if and only if \( L - \lambda I \) is not invertible,
\[ \lambda \in \sigma(L_\pm) \iff \lambda \in \sigma(-i\zeta \alpha_\pm - \zeta^2 \beta_\pm + i\zeta^3 \gamma_\pm - \lambda I), \quad \text{(2.6)} \]
for some $\zeta \in \mathbb{R}$. Consequently, the essential spectrum consists of $2n$-curves $\lambda_\pm^n$, that is,

$$\sigma(-i\zeta \alpha_\pm - \zeta^2 \beta_\pm + i\zeta^3 \gamma_\pm - \lambda I).$$

### 2.2 Theoretical Background of the Evans Function

We now describe what we mean by a traveling wave solution and what it means for that traveling wave solution to be spectrally stable. Consider a PDE of the form $U_t = F(U)$

Where $F(U)$ consists of the part of the PDE with spatial derivatives. A traveling wave solution is a solution of the form

$$U(x, t) = \tilde{U}(x - ct),$$

which represents a traveling wave with speed $c$. We want the traveling wave solution to be stationary in the coordinate system, so we make the change of coordinates $x \to x - ct$.

The eigenvalue problem comes from linearizing the PDE about the traveling wave profile $\tilde{U}$. This yields

$$\lambda V = LV := (DF(\tilde{U}) + c\partial_x) V,$$

where the spectral parameter $\lambda$ is complex valued. Spectral stability of the wave has to do with the eigenvalues of this equation. Define $P = \{ \lambda \in \mathbb{C} | \Re(\lambda) \geq 0 \} \setminus \{0\}$. The traveling wave is spectrally stable if $\sigma(L) \cap P = \emptyset$ and the dimension of the eigenspace with eigenvalue $\lambda = 0$ matches the dimension of the manifold of nearby traveling wave solutions. Spectral stability implies nonlinear-orbital stability in the case of the Navier-Stokes equations we consider; see [9, 12, 13, 14, 19, 25, 26, 29, 33, 34].

To determine whether or not the eigenvalue problem has point spectrum, we use the Evans function. The Evans function is a complex analytic function whose zeros correspond to eigenvalues of the ODE eigenvalue problem (2.7). The Evans function is constructed by considering the ODE eigenvalue problem as a system of ODEs,
\[ W'(x) = A(x, \lambda)W(x), \quad A_{\pm}(\lambda) := \lim_{x \to \pm\infty} A(x; \lambda). \quad (2.8) \]

One forms a basis for the space of solutions to (2.8) that correspond to the unstable manifold of the fixed point zero corresponding to \( x = -\infty \), say \( W_1, ..., W_k \), and the stable manifold of the fixed point zero corresponding to \( x = +\infty \), \( W_{k+1}, ..., W_n \). Then one takes the determinant of the matrix formed from these bases at \( x = 0 \) to compute the Evans function \( D, \quad D(\lambda) := \det[W_1, ..., W_k, W_{k+1}, ..., W_n]_{x=0} \). If the determinant is zero, that is if the Evans function has a zero, it indicates that there is a linear combination of the unstable manifold coming from the left with the stable manifold going to the right. That is, there exists an eigenfunction, which is a solution of (2.8) that decays at exponential rate to zero as \( x \to \pm\infty \).

If the basis elements for the unstable and stable manifolds vary analytically in the spectral parameter, then the Evans function is complex analytic. Thus we can compute the winding number of the image of a contour to determine if the Evans function has zeros inside the contour.

The Evans function is rather simple in definition, but requires careful numerics to deal with stiffness of the associated ODEs. There are several methods to deal with this stiffness; for a few examples see [4, 18, 20]. To deal with the stiffness of the ODE, we will use the compound matrix method, also known as the exterior product method, described below.

### 2.3 The Compound Matrix Method

If we simply numerically solve for a basis of solutions of (2.8), our result will not be representative of the actual solution due to extreme numerical error. The issue is that for \( |x| \gg 1 \), the solution behaves like that of the limiting system \( W' = A_{\pm}W \), which has solutions of an exponential form. Any numerical error will grow quickly in the direction of the largest exponential growth. Thus, what starts as linearly independent solutions will soon become nearly
linearly dependent due to blow up of error. To get around this, we may raise the system using exterior products and instead track a basis of solutions encoded in a single dimensional manifold corresponding to the solution with the greatest growth mode. For details about this method, see [8, 6].

For the Navier-Stokes equations we consider, the ODE system has dimension five and the manifold has dimension two. That is, \( n = 5 \) and \( k = 2 \). We let \( \{e_1, \ldots, e_5\} \) be the standard basis on \( \mathbb{C}^5 \). The basis we choose for \( \Lambda^2(\mathbb{C}^5) \) is given by

\[
(e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_1 \wedge e_5, e_2 \wedge e_3, e_2 \wedge e_4, e_2 \wedge e_5, e_3 \wedge e_4, e_3 \wedge e_5, e_4 \wedge e_5).
\]

Then the lift of

\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55}
\end{bmatrix}
\]  \hspace{1cm} (2.9)

to the space \( \Lambda^2(\mathbb{C}^5) \) is done by defining \( A^{(2)} \) as follows: \( A^{(2)} \circ e_i \wedge e_j = (Ae_i) \wedge e_j + e_i \wedge (Ae_j) \).

This yields

\[
A^{(2)} = \begin{bmatrix}
a_{11} + a_{22} & a_{23} & a_{24} & a_{25} & -a_{13} & -a_{14} & -a_{15} & 0 & 0 & 0 \\
a_{32} & a_{11} + a_{33} & a_{34} & a_{35} & a_{12} & 0 & 0 & -a_{14} & -a_{15} & 0 \\
a_{42} & a_{11} + a_{44} & a_{45} & 0 & a_{12} & 0 & a_{15} & 0 & -a_{15} & 0 \\
a_{52} & a_{11} + a_{55} & a_{54} & a_{53} & 0 & 0 & a_{12} & 0 & a_{13} & a_{14} \\
-a_{31} & a_{21} & 0 & 0 & a_{22} + a_{33} & a_{34} & a_{35} & -a_{24} & -a_{25} & 0 \\
-a_{41} & 0 & a_{21} & 0 & a_{43} & a_{22} + a_{44} & a_{45} & a_{25} & 0 & -a_{25} \\
-a_{51} & 0 & 0 & a_{21} & a_{53} & a_{54} & a_{52} + a_{55} & 0 & a_{23} & a_{24} \\
0 & -a_{41} & a_{31} & 0 & -a_{42} & a_{32} & 0 & a_{33} + a_{44} & a_{45} & -a_{35} \\
0 & -a_{51} & 0 & a_{31} & -a_{52} & 0 & a_{32} & a_{54} & a_{33} + a_{55} & a_{34} \\
0 & 0 & -a_{51} & a_{41} & 0 & -a_{52} & a_{42} & -a_{53} & a_{43} & a_{44} + a_{55}
\end{bmatrix}
\]  \hspace{1cm} (2.10)

This results in the ODE

\[
W' = A^{(k)}W, \quad W \in \mathbb{C}^{10}.
\]  \hspace{1cm} (2.11)

We now only need solve for a one-dimensional solution that corresponds to the dominant mode at \( \pm \infty \).

Now that we have described the system that we must solve, we are ready to discuss the methods we use. We begin with the parameterization method, which is important for
obtaining a rigorous enclosure of the solutions to the profile equation, introduced in equation (3.3), and the Evans ODE system (2.11).

2.4 Parameterization Method for Vector Fields

In this section, we describe how we use the parameterization method. For details about this method, see [32].

Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a smooth vector field. Let \( Df \) be the Jacobian of \( f \) and let \( \lambda_1, \ldots, \lambda_n \) be the eigenvalues of \( Df \) evaluated at a fixed point. We will assume the eigenvalues are distinct. Let \( \zeta_1, \ldots, \zeta_n \) be the corresponding eigenvectors. Suppose the first \( m \) eigenvalues have positive real part and the rest have negative real part. We seek a solution to \( u_t = f(U) \) of the form

\[
U(t) = \sum_{k_1, \ldots, k_m=0}^{\infty} U_{k_1 \ldots k_m} e^{\lambda_1 t} \ldots e^{\lambda_k t}
\]

where \( U_{0, \ldots, 0} \) is given by the fixed point and \( U_{0,0,\ldots,k,0,\ldots,0} = \zeta_k \) for \( 1 \leq k \leq m \). Note that if a series of this form exists, then on its radius of convergence it is a solution to the problem that approaches the fixed point along the unstable manifold as \( x \to -\infty \).

We solve for a series solution of this form by obtaining a recursive formula and solving for a finite number of terms. We then use a proof by induction to show that the remaining terms are bounded by a geometric series. This proof by induction involves the use of rigorous computation to verify the base cases.

Once we have used the parameterization method to compute the unstable manifold of (2.8) at the fixed point zero corresponding to \( x = -\infty \) and the stable manifold of (2.8) at the fixed point zero corresponding to \( x = +\infty \), we use series solutions with the standard polynomial basis to obtain a solution of (2.8) in the region of the real line near the origin, and then the Newton-Kantorovich Theorem to show there is a unique, nearby solution to these pieces of solution. We describe the Newton-Kantorovich Theorem next.
2.5 Newton Kantorovich

The Newton-Kantorovich Theorem helps us prove that a function has a unique zero within a quantifiably small neighborhood of our numerical approximation of a function. The statement of the theorem is as follows.

**Theorem 7** (a-posteriori Newton-Kantorovich). *(see [21])*

Suppose that $F : X \to Y$ is continuously differentiable. Let $B(X,Y)$ be the space of bounded linear functionals from $X$ to $Y$. Let $\bar{x} \in X, A^\dagger \in B(X,Y), A \in B(X,Y)$ with $A$ one-to-one. Let $Y_0, Z_0, Z_1 > 0$ be positive constants, and $Z_2 : [0, \infty) \to [0, \infty)$ be a positive function, all satisfying the following conditions:

- $\|AF(\bar{x})\|_X \leq Y_0$
- $\|I - AA^\dagger\|_{B(X)} \leq Z_0$
- $\|A(A^\dagger - DF(\bar{x}))\|_{B(X)} \leq Z_1$
- $\sup_{x \in B_r(\bar{x})} A(DF(\bar{x}) - DF(x))\|_{B(X)} \leq Z_2(r)r$.

If $p(r) := Z_2(r)r^2 - (1 - Z_0 - Z_1)r + Y_0$ is negative for some $r \in (0, \infty)$, then there is a unique $\tilde{z} \in B_r(\bar{z})$ such that $F(\tilde{z}) = 0$.

**2.5.1 Example 1.** We provide an example of using interval arithmetic computations and the Newton-Kantorovich theorem to obtain a tight enclosure of the zero of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 - 2$.

$x := 1.4142$

$A_t := 2.8284$

$A := 0.35355$

We only consider values of $r$ less than $1e-2$. We record in Table 2.1 the bounds we obtain.
Since $1 - Z_0 - Z_1 - Z_2(r)r > 0$, then $p(r) < 0$ if $Y_0/(1 - Z_0 - Z_1 - Z_2(r)r) < r \leq 1e-2$. Thus there is a root $\tilde{x}$ such that $|\tilde{x} - x| < r = 1.3659122931369878e-5$. There must be a solution between 1.4141863408770685 and 1.4142136591229315.

<table>
<thead>
<tr>
<th>$Y_0$</th>
<th>0.000013562178000162330649</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_0$</td>
<td>0.00001917999999896502061</td>
</tr>
<tr>
<td>$Z_1$</td>
<td>1.5700774014248965167e-16</td>
</tr>
<tr>
<td>$Z_2(r)r$</td>
<td>0.007071000000001639597</td>
</tr>
</tbody>
</table>

Table 2.1: Bounds obtained to use in the Newton-Kantorovich Theorem for an example problem.

### 2.5.2 Example 2.

We provide an example of using interval arithmetic computations and the Newton-Kantorovich Theorem to obtain a tight enclosure of a zero of the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = [\sin(x + y), x + y^2 - \pi/2 - \pi^2/4]$. In our example, we use the infinity norm. We make the following choices:

\[
x := 1.5708
\]
\[
y := 1.5707
\]
\[
A := \begin{bmatrix} 0 & 1 \\ .3184 & .3184 \end{bmatrix}
\]
\[
A^\dagger := \begin{bmatrix} -1 & 3.14 \\ 1 & 0 \end{bmatrix}
\]

We only consider values of $r$ less than $1e-2$. The bounds we obtain are given in Table 2.2.

Since $1 - Z_0 - Z_1 - Z_2(r)r > 0$, then $p(r) < 0$ if $Y_0/(1 - Z_0 - Z_1 - Z_2(r) \ast r) < r \leq 1e-2$. Thus there is a root $\tilde{x}, \tilde{y}$ such that $|\tilde{x} - x|, |\tilde{y} - y| < r = 3.8631981913716246e-5$.

<table>
<thead>
<tr>
<th>$Y_0$</th>
<th>0.000013562178000162330649</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_0$</td>
<td>0.00022400000000003905337</td>
</tr>
<tr>
<td>$Z_1$</td>
<td>0.00044576000000009233454</td>
</tr>
<tr>
<td>$Z_2(r) \ast r$</td>
<td>0.006368000000000064678</td>
</tr>
</tbody>
</table>

Table 2.2: Bounds obtained to use in the Newton-Kantorovich Theorem for an example problem.
2.6 A POLYNOMIAL AND SERIES CLASS OF INTERVALS

In obtaining a traveling wave profile solution, we obtain rigorous error bounds on the coefficients of series solutions of the traveling wave profile ODE. In preparation for obtaining the coefficients of the matrix that is used in computing the Evans function, we build an object oriented class for carrying out operations on series. Specifically, to do series operations, we use a class for handling polynomials and series of the form \( f(x) = \sum_{k=0}^{\infty} a_k x^k \).

The polynomial class stores all coefficients up to a certain degree \( n \), and for each \( k > n \), \( a_k \leq c_0 c^k / (Q + k)^2 \) for some \( c_0, c, Q \in \mathbb{R} \) such that \( c_0, c > 0 \) and \( Q \geq 1 \).

In what follows, we provide details regarding the rigorous enclosure of various operations.

2.6.1 Addition.

**Proposition 8.** Let \( f(x) = \sum_{k=0}^{\infty} a_k x^k \) and \( g(x) = \sum_{k=0}^{\infty} b_k x^k \). Let \( c_0, d_0, c, Q \in \mathbb{R} \) such that \( c_0, d_0, c > 0 \) and \( Q \geq 1 \). Assume \( |a_k| \leq c_0 c^k / (Q + k)^2 \) and \( |b_k| \leq d_0 c^k / (Q + k)^2 \) for each \( k > n \). Then \( f(x) + g(x) = \sum_{k=0}^{\infty} l_k x^k \) where \( l_k = a_k + b_k \). For each \( k > n \), \( |l_k| \leq b_0 c^k / (Q + k)^2 \) where \( b_0 = c_0 + d_0 \).

The proof is trivial.

2.6.2 Multiplication by constant.

**Proposition 9.** Let \( f(x) = \sum_{k=0}^{\infty} a_k x^k \) and \( b \in \mathbb{C} \). Let \( c_0, c, Q \in \mathbb{R} \) such that \( c_0, c > 0 \) and \( Q \geq 1 \). Assume \( |a_k| \leq c_0 c^k / (Q + k)^2 \) for each \( k > n \). Then \( bf(x) = \sum_{k=0}^{\infty} l_k x^k \) where \( l_k = ba_k \). For each \( k > n \), \( |l_k| \leq d_0 c^k / (Q + k)^2 \) where \( d_0 = |b| c_0 \).

The proof is trivial.

2.6.3 Multiplication by series.

**Proposition 10.** Let \( f(x) = \sum_{k=0}^{\infty} a_k x^k \) and \( g(x) = \sum_{k=0}^{\infty} a_k x^k \). Let \( c_0, d_0, c, Q \in \mathbb{R} \) such that \( c_0, d_0, c > 0 \) and \( Q \geq 1 \). Assume \( |a_k| \leq c_0 c^k / (Q + k)^2 \) for each \( k > n_1 \) and \( |b_k| \leq d_0 c^k / (Q + k)^2 \) for each \( k > n_2 \). Then \( fg(x) = \sum_{k=0}^{\infty} m_k x^k \) where \( m_k = \sum_{j=0}^{k} a_j b_{k-j} \). For each \( k > n_1 + n_2 \), \( |m_k| \leq e_0 c^k / (Q + k)^2 \) where \( e_0 = |b| c_0 + d_0 c_0 \).

The proof is trivial.
\[ \frac{d_0 c^k}{(Q + k)^2} \] for each \( k > n_2 \). Then \( f(x)g(x) = \sum_{k=0}^{\infty} l_k x^k \) where \( l_k = \sum_{j=0}^{k} a_j b_{k-j} \). For each \( k > n_1 + n_2 \), \( |l_k| \leq \frac{b_0 c^k}{(R + k)^2} \) where

\[
\begin{align*}
  b_0 &= c_0 d_0 k_{Q,n1,n2} + d_0 \sum_{j=0}^{n_1} |a_j| c^j (n_1 + n_2 + 2Q)^2 \frac{(n_1 + n_2 + 2Q)^2}{(n_1 + n_2 + 1 - j + Q)^2} + c_0 \sum_{j=0}^{n_2} |b_j| c^j (n_1 + n_2 + 1 - j + Q)^2 \\
  \text{and} \quad R &= 2Q - 1.
\end{align*}
\]

**Proof.** Assume the hypothesis of the theorem.

It is trivial to show \( f(x)g(x) = \sum_{k=0}^{\infty} l_k x^k \).

Assume \( k > n_1 + n_2 \)

\[
|l_k| \leq \left| \sum_{j=0}^{k} a_j b_{k-j} \right| \leq \left| \sum_{j=n_1+1}^{k-n_2-1} a_j b_{k-j} \right| + \left| \sum_{j=0}^{n_1} a_j b_{k-j} \right| + \left| \sum_{j=k-n_2}^{k} a_j b_{k-j} \right| \leq \left| \sum_{j=n_1+1}^{k-n_2-1} \frac{c_0 d_0 c^k}{(Q + j)^2 (k-j + Q)^2} \right| + \left| \sum_{j=0}^{n_1} \frac{|a_j|}{c^j} \frac{d_0 c^k}{(k-j + Q)^2} \right| + \left| \sum_{j=0}^{n_2} \frac{|b_j|}{c^j} \frac{c_0 c^k}{(k-j + Q)^2} \right| \leq \frac{b_0 c^k}{(R + k)^2}.
\]

\[ \Box \]

2.6.4 Derivative.

**Proposition 11.** Let \( f(x) = \sum_{k=0}^{\infty} a_k x^k \). Let \( c_0, c, Q \in \mathbb{R} \) such that \( c_0, c > 0 \) and \( Q \geq 1 \).

Assume \( |a_k| \leq \frac{c_0 c^k}{(Q + k)^2} \) for each \( k > n \). Then \( f'(x) = \sum_{k=0}^{\infty} l_k x^k \) where \( l_k = (k+1)a_{k+1} \).

For each \( k > (n-1) \), \( |l_k| \leq \frac{d_0 c^k}{(R + k)^2} \) where \( d_0 = cc_0 \), \( d = c \sqrt{n+1} \) and \( R = Q + 1 \).

**Proof.** Assume all hypothesis of the proposition.

It is trivial to show \( f'(x) = \sum_{k=0}^{\infty} l_k x^k \).

We first show \( \sqrt{n+1}^k \geq (k+1) \) for \( k > (n-1) \) using induction.
The base case where $k=n$ is trivial.

For the inductive step, we assume $\sqrt[n+1]n^k \geq (k+1)$.

Then $\sqrt[n+1]n^{k+1} \geq \sqrt[n+1]n(k+1) \geq \sqrt\prod_{i=1}^n \frac{i+1}{i}(k+1) \geq \frac{k+2}{k+1}(k+1) = k+2$. Thus we have $\sqrt[n+1]n^k \geq (k+1)$.

Let $k > n-1$.

Then $|l_k| = |(k+1)a_{k+1}| \leq (k+1)\frac{c_0c_{k+1}}{(Q+k+1)^2} \leq \sqrt[n+1]n^{k} \frac{c_0c_k}{(R+k)^2} \leq \frac{d_0d_k}{(R+k)^2}$. 

\[ \Box \]

2.6.5 Derivative of Exponential.

**Proposition 12.** Let $f(x) = \sum_{k=0}^{\infty} a_k e^{\sigma k x}$ and $\sigma \in \mathbb{R}$. Let $\sigma \in \mathbb{R}$. Let $c_0, c, Q \in \mathbb{R}$ such that $c_0, c > 0$ and $Q \geq 1$. Assume $|a_k| \leq c_0 e^k/(Q + k)^2$ for each $k > n$. Then $f'(x) = \sum_{k=0}^{\infty} l_k e^{\sigma k x}$ where $l_k = k \sigma a_k$. For each $k > n$, $|l_k| \leq d_0 d^k/(Q + k)^2$ where $d_0 = |\sigma|c_0$ and $d = c(\min(3, n + 1))^{1/\min(3, n+1)}$.

**Proof.** Assume all hypothesis of the proposition.

It is trivial to show $f'(x) = \sum_{k=0}^{\infty} l_k e^{\sigma k x}$.

The function $g(k) = k^{1/k}$ is decreasing for $k > e$ and has a maximum at $k = 3$ when restricted to natural numbers.

Let $k > n$.

Then $|l_k| = |k\sigma a_k| \leq k|\sigma|\frac{c_0 c^{k}}{(Q+k)^2} \leq g(k)^k|\sigma|\frac{c_0 c^{k}}{(Q+k)^2} \leq g(\min(3, n + 1))^k|\sigma|\frac{c_0 c^{k}}{(Q+k)^2} \leq \frac{d_0 d^k}{(Q+k)^2}$.

\[ \Box \]

2.6.6 Truncation Error.

**Proposition 13.** Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$. Let $x \in \mathbb{C}$. Let $c_0, c, Q \in \mathbb{R}$ such that $c_0, c > 0$ and $Q \geq 1$. Assume $|a_k| \leq c_0 e^k/(Q + k)^2$ for each $k > n$. Let $c|x| < 0$. Then the absolute value of the truncation error, $|f(x) - \sum_{k=0}^{n} a_k x^k| \leq \frac{b_0(|c|x|)^{n+1}}{(n+Q)^2(1-c|x|)}$.

**Proof.** Assume all hypothesis of the proposition.
Then
\[
\left| f(x) - \sum_{k=0}^{n} a_k x^k \right| = \left| \sum_{k=n+1}^{\infty} a_k x^k \right| \\
\leq \frac{1}{(n + Q)^2} \sum_{k=n+1}^{\infty} b_0(c|x|)^k \\
\leq \frac{b_0(c|x|)^{n+1}}{(n + Q)^2(1 - c|x|)}.
\]

2.7 A 2D POLYNOMIAL AND SERIES CLASS OF INTERVALS

For the left manifold, we use a different class for handling polynomials and series of two variables. These take the form \( f(x) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} a_{k_1,k_2} x_1^{k_1} x_2^{k_2} \). The polynomial class stores all coefficients such that \( k_1 + k_2 r \leq n \) and for each \( k_1, k_2 \) such that \( k_1 + r k_2 > n \), \( a_{k_1,k_2} \leq c_0 c_1^{k_1} c_2^{k_2} / ((Q + k_1)^2(Q + k_2)^2) \) for some \( c_0, c, Q \in \mathbb{R} \) such that \( c_0, c_1, c_2 > 0 \) and \( Q > 1 \).

2.7.1 Addition.

Proposition 14. Let \( f(x) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} a_{k_1,k_2} x_1^{k_1} x_2^{k_2} \) and \( g(x) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} b_{k_1,k_2} x_1^{k_1} x_2^{k_2} \).

Let \( c_0, d_0, c_1, c_2, Q \in \mathbb{R} \) such that \( c_0, d_0, c_1, c_2 > 0 \) and \( Q > 1 \). Assume \( |a_{k_1,k_2}| \leq c_0 c_1^{k_1} c_2^{k_2} / ((Q + k_1)^2(Q + k_2)^2) \) and \( |b_{k_1,k_2}| \leq d_0 c_1^{k_1} c_2^{k_2} / ((Q + k_1)^2(Q + k_2)^2) \) for each \( k_1 + r k_2 > n \). Then \( f(x) + g(x) = \sum_{k_2=0}^{\infty} l_{k_1,k_2} x_1^{k_1} x_2^{k_2} \) where \( l_{k_1,k_2} = a_{k_1,k_2} + b_{k_1,k_2} \). For each \( k_1 + r k_2 > n \), \( |l_{k_1,k_2}| \leq b_0 c_1^{k_1} c_2^{k_2} / ((Q + k_1)^2(Q + k_2)^2) \) where \( b_0 = c_0 + d_0 \).

The proof is trivial.

2.7.2 Multiplication by Constant.

Proposition 15. Let \( f(x) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} a_{k_1,k_2} x_1^{k_1} x_2^{k_2} \) and \( b \in \mathbb{C} \). Let \( c_0, c_1, c_2, Q \in \mathbb{R} \) such that \( c_0, c_1, c_2 > 0 \) and \( Q > 1 \). Assume \( |a_{k_1,k_2}| \leq c_0 c_1^{k_1} c_2^{k_2} / ((Q + k_1)^2(Q + k_2)^2) \) for each \( k_1 + r k_2 > n \). Then \( b f(x) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} l_{k_1,k_2} x_1^{k_1} x_2^{k_2} \) where \( l_{k_1,k_2} = b a_{k_1,k_2} \). For each \( k_1 + r k_2 > n \), \( |l_{k_1,k_2}| \leq d_0 c_1^{k_1} c_2^{k_2} / ((Q + k_1)^2(Q + k_2)^2) \) where \( d_0 = |b| c_0 \).

The proof is trivial.
2.7.3 Multiplication by Series.

Proposition 16. \( f(x) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} a_{k_1,k_2} x_1^{k_1} x_2^{k_2} \) and \( g(x) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} b_{k_1,k_2} x_1^{k_1} x_2^{k_2} \).

Let \( c_0, d_0, c_1, c_2, Q \in \mathbb{R} \) such that \( c_0, d_0, c_1, c_2 > 0 \) and \( Q > 1 \). Assume \( |a_{k_1,k_2}| \leq c_0 c_1^{k_1} c_2^{k_2}/((Q + k_1)^2(Q + k_2)^2) \) for each \( k_1 + r k_2 > n_1 \) and \( |b_{k_1,k_2}| \leq d_0 c_1^{k_1} c_2^{k_2}/((Q + k_1)^2(Q + k_2)^2) \) for each \( k_1 + r k_2 > n_2 \). Define \( \tilde{a}_{k_1,k_2} = \max(0, a_{k_1,k_2} - c_0 c_1^{k_1} c_2^{k_2}/((Q + k_1)^2(Q + k_2)^2)) \), \( \tilde{b}_{k_1,k_2} = \max(0, b_{k_1,k_2} - d_0 c_1^{k_1} c_2^{k_2}/((Q + k_1)^2(Q + k_2)^2)) \). Then \( f(x)g(x) = \sum_{k_2=0}^{\infty} l_{k_1,k_2} x_1^{k_1} x_2^{k_2} \) where \( l_{k_1,k_2} = \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} a_{j_1,j_2} b_{k_1-j_1,k_2-j_2} \). For each \( k_1 + r k_2 > n_1 + n_2 \), \( |l_{k_1,k_2}| \leq b_0 c_1^{k_1} c_2^{k_2}/((R + k_1)^2(R + k_2)^2) \) where

\[
 b_0 = c_0 d_0 Q^2 + \frac{d_0}{Q^4} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \frac{\tilde{a}_{j_1,j_2}(j_1 + R)^2(j_2 + R)^2}{c_1^{j_1} c_2^{j_2}} + \frac{c_0}{Q^4} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \frac{\tilde{b}_{j_1,j_2}(j_1 + R)^2(j_2 + R)^2}{c_1^{j_1} c_2^{j_2}}
\]

and

\[ R = 2Q - 1. \]

Proof. Assume the hypothesis of the theorem.

It is trivial to show \( f(x)g(x) = \sum_{k_2=0}^{\infty} l_{k_1,k_2} x_1^{k_1} x_2^{k_2} \).

Assume \( k_1 + r k_2 > n_1 + n_2 \).
Then

\[
|l_{k_1,k_2}| \leq \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} a_{j_1,j_2} b_{k_1-j_1,k_2-j_2} \left| c_0 d_0 c_1^{k_1} c_2^{k_2} \right| + \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} \tilde{a}_{j_1,j_2} d_0 c_1^{k_1-j_1} c_2^{k_2-j_2} \left( k_1 - j_1 + Q \right)^2 (k_2 - j_2 + Q)^2 \left( k_1 - j_1 + Q \right)^2 (k_2 - j_2 + Q)^2 c_1^j c_2^j
\]

\[
+ \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} \tilde{b}_{j_1,j_2} c_0 c_1^{k_1-j_1} c_2^{k_2-j_2} \left( k_1 - j_1 + Q \right)^2 (k_2 - j_2 + Q)^2 c_1^j c_2^j
\]

\[
\leq \frac{c_0 d_0 N^2 Q c_1^{k_1} c_2^{k_2}}{(k_1 + R)^2 (k_2 + R)^2} + \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} \tilde{a}_{j_1,j_2} d_0 c_1^{k_1-j_1} c_2^{k_2-j_2} \left( k_1 - j_1 + Q \right)^2 (k_2 - j_2 + Q)^2 c_1^j c_2^j
\]

\[
+ \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} \tilde{b}_{j_1,j_2} c_0 c_1^{k_1-j_1} c_2^{k_2-j_2} \left( k_1 - j_1 + Q \right)^2 (k_2 - j_2 + Q)^2 c_1^j c_2^j
\]

\[
\leq \frac{c_0 d_0 N^2 Q c_1^{k_1} c_2^{k_2}}{(k_1 + R)^2 (k_2 + R)^2}
\]

\[
+ \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} \frac{(j_1 + R)^2 (j_2 + R)^2 \tilde{a}_{j_1,j_2} d_0 c_1^{k_1-j_1} c_2^{k_2-j_2}}{(k_1 + R)^2 (k_2 + R)^2 Q^4 c_1^j c_2^j}
\]

\[
+ \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} \frac{(j_1 + R)^2 (j_2 + R)^2 \tilde{b}_{j_1,j_2} c_0 c_1^{k_1-j_1} c_2^{k_2-j_2}}{(k_1 + R)^2 (k_2 + R)^2 Q^4 c_1^j c_2^j}
\]

\[
\leq \frac{b_0 c_1^{k_1} c_2^{k_2}}{(R + k_1)^2 (R + k_2)^2}.
\]

2.7.4 Derivative.

**Proposition 17.** Let \( f(x) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} a_{k_1,k_2} x_1^{k_1} x_2^{k_2} \). Let \( c_0, c_1, c_2, Q \in \mathbb{R} \) such that \( c_0, c_1, c_2 > 0 \) and \( Q > 1 \). Assume \( |a_{k_1,k_2}| \leq c_0 c_1^{k_1} c_2^{k_2} / ((Q + k_1)^2(Q + k_2)^2) \) for each \( k_1 + r k_2 > n \).

Then \( \frac{d}{dx_1} f(x) = \sum_{k=0}^{\infty} l_k x_1^{k_1} x_2^{k_2} \) where \( l_k = (k_1 + 1) a_{k_1+1,k_2} \). For each \( k_1 + r k_2 > (n - 1) \),
\[ |l_k| \leq d_0 d_1 d_2^2/(R + k)^2 \text{ where } d_0 = c_1 c_0, \; d_1 = 2c_1, \; d_2 = c_2 \text{ and } R = Q + 1. \]

**Proof.** Assume all hypothesis of the proposition.

It is trivial to show \( d/dx f(x) = \sum_{k=0}^{\infty} l_{k_1,k_2} x_1^{k_1} x_2^{k_2}. \)

Then
\[
|l_{k_1,k_2}| = |(k_1 + 1)a_{k_1+1,k_2}| \leq (k_1 + 1)(Q + k_1 + 1)(Q + k_2 + 1) \leq \frac{c_1 c_0 c_1^{k_1+1} c_2^{k_2}}{(R + k_1)^2(R + k_2)^2} \leq \frac{d_0 d^k}{(R + k_1)^2(R + k_2)^2}
\]

\[ \square \]

**2.7.5 Derivative of Exponential.**

**Proposition 18.** Let \( f(x) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} a_{k_1,k_2} e^{\sigma_1 k_1 x} e^{\sigma_2 k_2 x} \). Let \( \sigma_1, \sigma_2 \in \mathbb{C} \). Let \( c_0, c_1, c_2, Q \in \mathbb{R} \) such that \( c_0, c_1, c_2 > 0 \) and \( Q > 1 \). Assume \( |a_{k_1,k_2}| \leq c_0 c_1^{k_1} c_2^{k_2}/((Q + k_1)^2(Q + k_2)^2) \) for each \( k_1 + rk_2 > n \). Then \( f'(x) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} l_{k_1,k_2} e^{\sigma_1 k_1 x} e^{\sigma_2 k_2 x} \) where \( l_{k_1,k_2} = (k_1 \sigma_1 + k_2 \sigma_2)a_k \).

For each \( k_1 + rk_2 > n \), \( |l_{k_1,k_2}| \leq d_0 d_1 d_2^2/((Q + k_1)^2(Q + k_2)^2) \) where \( d_0 = (|\sigma_1| + |\sigma_2|)c_0, \; d_1 = c_1^{\sqrt{3}} \) and \( d_2 = c_2^{\sqrt{3}} \).

**Proof.** Assume all hypothesis of the proposition.

It is trivial to show \( f'(x) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} l_{k_1,k_2} e^{\sigma_1 k_1 x} e^{\sigma_2 k_2 x} \).

The function \( g(k) = k^{1/k} \) is decreasing for \( k > e \) and has a maximum at \( k = 3 \) when restricted to natural numbers.

Let \( k > n \).

Then
\[
|l_{k_1,k_2}| = |(k_1 \sigma_1 + k_2 \sigma_2)a_k| \leq |(k_1 \sigma_1 + k_2 \sigma_2)| \leq \frac{c_0 c_1^{k_1} c_2^{k_2}}{(Q + k_1)^2(Q + k_2)^2} \leq \frac{d_0 d_1 d_2^2}{(Q + k_1)^2(Q + k_2)^2}.
\]

\[ \square \]
2.7.6 Truncation Error.

Proposition 19. Let \( f(x_1, x_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} a_{k_1,k_2} x_1^{k_1} x_2^{k_2} \). Let \( x_1, x_2 \in \mathbb{C} \). Let \( c_0, c_1, c_2, Q \in \mathbb{R} \) such that \( c_0, c_1, c_2 > 0 \) and \( Q > 1 \). Assume \( |a_{k_1,k_2}| \leq c_0 c_1^{k_1} c_2^{k_2} / ((Q + k_1)^2(Q + k_2)^2) \) for each \( k_1 + rk_2 > n \). Assume \( c_1|x_1|, c_2|x_2| < 1 \). We let

\[
    n_2(k_1) = \begin{cases} 
    1 + [(n - k_1)/r] & k_1 \leq n \\
    0 & n_1 > n
    \end{cases}
\]

Then the absolute value of the truncation error,

\[
    \left| f(x_1, x_2) - \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \mathbb{I}_{k_1 + rk_2 < n} a_{k_1,k_2} x_1^{k_1} x_2^{k_2} \right| \leq g(x_1, x_2)
\]

where

\[
    g(x_1, x_2) = c_0 \sum_{k_1=0}^{n} \frac{(c_1|x_1|)^{n_1} (c_2|x_2|)^{n_2(k_1)}}{(k_1 + Q)^2(n_2(k_1) + Q)^2(1 - c_2|x_2|)} + c_0 \frac{(c_1|x_1|)^{n_1+1}}{(n + 1 + Q)^2Q^2(1 - c_1|x_1|)(1 - c_2|x_2|)}.
\]

Proof. Assume all hypothesis of the proposition.
Then

\[
|f(x_1, x_2) - \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \mathbb{1}_{k_1 + k_2 < n} a_{k_1, k_2} x_1^{k_1} x_2^{k_2}| \\
= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} |\mathbb{1}_{k_1 + k_2 > n} u_{k_1, k_2} x_1^{k_1} x_2^{k_2}| \\
\leq \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\mathbb{1}_{k_1 + k_2 > n} c_0 (c_1 |x_1|)^{k_1} (c_2 |x_2|)^{k_2}}{(k_1 + Q)^2 (k_2 + Q)^2} \\
\leq c_0 \sum_{k_1=0}^{\infty} \frac{(c_1 |x_1|)^{k_1}}{(k_1 + Q)^2} \sum_{k_2=n_2(k_1)}^{\infty} \frac{(c_2 |x_2|)^{k_2}}{(k_2 + Q)^2} \\
\leq c_0 \sum_{k_1=0}^{\infty} \frac{(c_1 |x_1|)^{k_1}}{(k_1 + Q)^2} \frac{1}{(n_2(k_1) + Q)^2 (1 - c_2 |x_2|)} \\
+ c_0 \sum_{k_1=n+1}^{\infty} \frac{(c_1 |x_1|)^{k_1}}{(k_1 + Q)^2} \frac{1}{Q^2 (1 - c_2 |x_2|)} \\
\leq c_0 \sum_{k_1=0}^{\infty} \frac{(c_1 |x_1|)^{k_1} (c_2 |x_2|)^{n_2(k_1)}}{(k_1 + Q)^2 (n_2(k_1) + Q)^2 (1 - c_2 |x_2|)} \\
+ c_0 \frac{(n + 1 + Q)^2 (1 - c_1 |x_1|)(1 - c_2 |x_2|)}{(n + 1 + Q)^2 (1 - c_1 |x_1|)(1 - c_2 |x_2|)} \\
\leq g(x_1, x_2).
\]

\[
\sum_{k=a}^{n-b} \frac{1}{(k + Q)^2 (n + Q - k)^2} \leq \frac{\mathbb{N}_{Q,a,b}}{(2Q + n - 1)^2}
\]

2.8 A SUPPORTING BOUND

We use the bound given in Proposition 20 extensively in what follows. The proof of the proposition is given in the appendix. For convenience, we define

\[
\mathbb{N}_Q = \mathbb{N}_{Q,0,0}, \quad \mathbb{N}_{Q,a} = \mathbb{N}_{Q,a,a}.
\]

**Proposition 20.** Let \( n \in \mathbb{N}, Q \in \mathbb{R}, a, b \in \mathbb{N} \cup \{0\} \) such that \( Q > 1 - a \) and \( Q > 1 - b \),

Then

\[
\sum_{k=a}^{n-b} \frac{1}{(k + Q)^2 (n + Q - k)^2} \leq \frac{\mathbb{N}_{Q,a,b}}{(2Q + n - 1)^2}
\]
where

\[ N_{Q,a,b} = \frac{1}{(a + Q - 1)} \left( \frac{2}{e} + 1 \right) + \frac{1}{(b + Q - 1)} \left( \frac{2}{e} + 1 \right) . \]

**Chapter 3. The Navier-Stokes equations**

In this chapter, we introduce the Navier-Stokes equations, describe our computer assisted proof of the existence of a locally unique traveling wave solution, and describe work toward rigorous verification of stability through rigorous computation of the Evans function.

### 3.1 Background

Following the notation of [17], the Navier-Stokes equations for compressible gas in Lagrangian coordinates with an ideal gas equation of state are given by

\[ \begin{align*}
    v_t - u_x &= 0, \\
    u_t + p_x &= \left( \frac{\mu u_x}{v} \right)_x, \\
    E_t + (pu)_x &= \left( \frac{\mu uu_x}{v} \right)_x + \left( \frac{\kappa T_x}{v} \right)_x,
\end{align*} \tag{3.1} \]

where the pressure function for ideal gas is given by \( p = \Gamma e/v \). The physical interpretation of the functions and coefficients are as follows:

- \( v \) - specific volume
- \( u \) - velocity
- \( p \) - pressure
- \( E \) - interval energy (\( E = e + u^2/2 \)) where \( e \) is internal energy and \( u^2/2 \) is kinetic energy
- \( \Gamma \) - Gruneisen constant
- \( \mu \) - coefficient of viscosity
• κ - coefficient of heat conductivity

• e = cνT for ideal gas, where cν is a constant the characterizes the gas

A traveling wave solution of (3.1) satisfies the system

\[ \begin{align*}
    v_t - sv_x - u_x &= 0, \\
    u_t - su_x + p_x &= \left( \frac{\mu u_x}{v} \right)_x, \\
    E_t - sE_x + (pu)_x &= \left( \frac{\mu uu_x}{v} \right)_x + \left( \frac{\kappa T_x}{v} \right)_x,
\end{align*} \]

where s is the speed of the traveling wave. Using the rescaling of [17],

\[ (x, t, v, u, T) \rightarrow (-\varepsilon sx, \varepsilon s^2 t, v/\varepsilon, -u/(\varepsilon s), T/(\varepsilon^2 s^2)) \],

yields the the system

\[ \begin{align*}
    v_t + v_x - u_x &= 0, \\
    u_t + u_x + p_x &= \left( \frac{\mu u_x}{v} \right)_x, \\
    E_t + E_x + (pu)_x &= \left( \frac{\mu uu_x}{v} \right)_x + \left( \frac{\kappa T_x}{v} \right)_x.
\end{align*} \] (3.2)

A traveling wave solution is a stationary solution of (3.2) that decays at exponential rate to the end-states that are specified by the Rankine-Hugoniot conditions. If we set time derivatives in (3.2) to be zero and write this as a first order system, we arrive at the profile ODE (see [16])

\[ \begin{align*}
    v' &= \frac{1}{\mu} [v(v - 1) + \Gamma (e - ve_-)] \\
    e' &= \frac{v}{\nu} [-(v - 1)^2/2 + e - e_- + \Gamma e_- (v - 1)]
\end{align*} \] (3.3)

where the endstate values of the solution are determined by the Rankine-Hugoniot conditions,

\[ e_- = \frac{(\Gamma + 2)(v_+ - v_+)}{2\Gamma(\Gamma + 1)}, \quad e_+ = \frac{v_+(\Gamma + 2 - \Gamma v_+)}{2\Gamma(\Gamma + 1)}, \quad v_+ = \frac{\Gamma}{\Gamma + 2}. \] (3.4)

In the above, the subscript ± indicates the asymptotic limit of the function; for example \( v_\pm := \lim_{x \rightarrow \pm \infty} v(x) \). Traveling wave solutions are connecting orbits between the fixed points \((1, e_-)\), corresponding to \( x = -\infty \), and \((v_+, e_+)\), corresponding to \( x = +\infty \). The fixed point \((1, e_-)\) is a source, and \((v_+, e_+)\) is a saddle.
The existence of a traveling wave profile solution to (3.3) was proved by Gilbarg in [10] by showing that the stable manifold, in one direction, of a saddle point of (3.3) is contained within nullclines that connect to another fixed point. The other fixed point is a source, and so one can conclude that a connecting orbit exists from one fixed point to the other since there are no other fixed points enclosed by the nullclines in this two-dimensional phase space. The existence result does not give a tight bound on the profile solution or how it varies spatially. A tight enclosure of the profile is needed in order to carry out Evans function computations to determine stability of the wave. Our main result in this thesis is providing a constructive proof of existence of the profile that provides a tight error bound on its spatial structure. We next describe how we obtain a numerical approximation of the profile and how we obtain these tight error bounds on the approximate profile.

3.2 Traveling wave solution

To obtain a rigorous enclosure of the traveling wave solution, we set up a zero finding problem so we can employ the Newton-Kantorovich Theorem to obtain rigorous error bounds on our numerical approximation of the profile. The function whose zeros we seek, call it $F : \mathbb{R}^n \to \mathbb{R}^n$, is defined so that a zero of $F$ corresponds to a global solution of (3.3) that connects the unstable manifold of the fixed point $(v_-, e_-)$ with the stable manifold of the fixed point $(v_+, e_+)$. That is, a zero of $F$ corresponds to a traveling wave solution.

To describe our formulation of $F$, we need to define a few terms. We will denote with $\Phi_L(\tilde{\theta}_1, \tilde{\theta}_2)$ a parameterization of the 2D unstable manifold of $(v_-, e_-)$ in a local neighborhood. We will denote with $\Phi_R$ a parameterization of the 1D stable manifold of $(v_+, e_+)$ in a local neighborhood. We will denote by $\Psi_{-\Delta x}(v_0, e_0)$ the solution of (3.3), initialized with $(v_0, e_0)$ at $x = 0$, evaluated at $-\Delta x$.

We first need to approximate the profile before obtaining a rigorous enclosure of it. Let $x_0 > x_1 > ... > x_N$ be evenly spaced with such that $x_{k+1} - x_k = \Delta x$. We find rigorous bounds on $(v_0, e_0)$ by evaluating $\Phi_R$ at $x_0$. We then use a built in shooting scheme to evolve...
the solution of (3.3) initialized at \((v_0, e_0)\) backward in \(x\) to obtain approximations of \((v_n, e_n)\) for each \(n \in \{0, 1, ..., N\}\). We then use a built in zero-finding algorithm to find the values of \(\tilde{\theta}_1\) and \(\tilde{\theta}_2\) such that \(\Phi_L(\tilde{\theta}_1, \tilde{\theta}_2) \approx (v_N, e_N)\).

We specify the function \(F\) to be a function of the variables \((\tilde{\theta}_1, \tilde{\theta}_2, v_N, e_N, ..., v_1, e_1)\). We define

\[
F(\tilde{\theta}_1, \tilde{\theta}_2, v_N, e_N, ..., v_1, e_1) := (\Phi_L(\tilde{\theta}_1, \tilde{\theta}_2) - (v_N, e_N), \Psi_{-\Delta x}(v_N, e_N) - (v_{N-1}, e_{N-1}) , ..., \\
\Psi_{-\Delta x}(v_1, e_1) - (v_0, e_0)).
\]

We will need the Jacobian of \(F\) in order to use the Newton-Kantorovich Theorem,

\[
DF(\tilde{\theta}_1, \tilde{\theta}_2, v_N, e_N, ..., v_1, e_1) = 
\begin{pmatrix}
D\Phi_L(\tilde{\theta}_1, \tilde{\theta}_2) & -I_2 & 0 & ... & 0 \\
0 & D\Psi_{-\Delta x}(v_N, e_N) & -I_2 & ... & 0 \\
... & ... & ... & ... & ...
\end{pmatrix}.
\]

where \(I_2\) is the two by two identity matrix. We have now defined a function \(F\) whose zeros correspond to traveling wave solutions of (3.3). We next describe how we obtain \(\Phi_L, \Phi_R,\) and \(\Psi_{\Delta x}\).

### 3.2.1 Right manifold.

In this section, we describe how we approximate \(\Phi_R\) and how we obtain a rigorous bound on the approximation error involved. The Jacobian of (3.3) evaluated at the fixed point \((v_+, e_+)\) has exactly one eigenvalue, call it \(\sigma\), with negative real part. Using \(\sigma\), we make the ansatz that a solution of (3.3) that belongs to the stable manifold of the fixed point \((v_+, e_+)\) takes the form

\[
v(x) = \sum_{n=0}^{\infty} v_n e^{\sigma nx}, \quad e(x) = \sum_{n=0}^{\infty} e_n e^{\sigma nx},
\]
for some coefficients $v_n, e_n \in \mathbb{R}$. Plugging the ansatz into (3.3), we arrive at the recursion formula

$$(\sigma n I - A) \begin{bmatrix} v_n \\ e_n \end{bmatrix} = w_n, \quad n \geq 2$$

where

$$w_n = \begin{bmatrix} \frac{1}{\mu} g_n \\ \frac{1}{\nu} \sum_{j=1}^{n-1} g_j v_{n-j} - \frac{3}{2} v_0 g_n + g_n + \sum_{k=1}^{n-1} c_k v_{n-k} + \Gamma e - g_n \end{bmatrix},$$

$$g_n = \sum_{k=0}^{n} v_{n-k} v_k,$$

$$A = \begin{bmatrix} \frac{2v_++1-\Gamma e_-}{\mu} & \frac{\Gamma}{\mu} \\ \frac{2\Gamma e_-v_+-\Gamma e_-+e_-e_+-e_--3v_+^2}{\nu} + 2v_+ - \frac{1}{2} \frac{e_-}{\nu} & v_+ \end{bmatrix}. $$

The zeroth order term is the fixed point $(v_0, e_0) = (v_+, e_+)$, and the first order term is an eigenfunction of $A$ corresponding to the eigenvector $\sigma$.

We solve the recursion formula with interval arithmetic to get a rigorous enclosure of the coefficients up to some order $N \in \mathbb{N}$. We then use a proof by induction to prove that the tail end of the series is bounded by a geometric series. The proof by induction involves many base cases, but these we verify using interval arithmetic computations. The statement of the relevant theorem follows.

**Theorem 21.** Let $N, Q \in \mathbb{N}$, $c_0, d_0, c > 0$, and let $A$ be the Jacobian of (3.3) evaluated at $(v_+, e_+)$. Assume that $v_0 = v_+, e_0 = e_+$, and that $(v_1, e_1)$ is an eigenvector of $A$ corresponding to the eigenvalue $\sigma$ of $A$ with negative real part. Let $v_n$ and $e_n$ satisfy the recursion formula (3.5) for $2 \leq n \leq N$. Assume that $|v_n|, |e_n| \leq c_0 c^n/(n + Q)^2$ and $|g_n| \leq d_0 c^n/(n + Q)^2$ for all $0 < n \leq N$. Assume that $(N + 1)|\sigma| > |A|_{\infty}$ and

$$\max(b_1, b_2) \leq c_0$$

where $b_1 := \kappa_{Q,1}/\mu$ and $b_2 = [\kappa_{Q,1}(c_0 d_0/2 + c_0^2) + 3v_+ d_0/2 + d_0 + \Gamma e - d_0]/\nu$ and $A$ is the Jacobian of (3.3) evaluated at $(v_+, e_+)$. Assume that $c_0^2 \kappa_{Q,1} \leq d_0$. Then $|v_n|, |e_n| \leq c_0 c^n/(n + Q)^2$ and $|g_n| \leq d_0 c^n/(n + Q)^2$ for all $n \in \mathbb{N}$.
Proof. Assume the hypothesis of the theorem. Let \( n > N \) and assume that \(|v_k|, |e_k| \leq c_0 c^n/(k + Q)^2\), and \(|g_k| \leq d_0 c^n/(k + Q)^2\) for all \(0 < k \leq n\). Then

\[
|g_{n+1}| = \left| \sum_{k=1}^{n} v_k v_{n+1-k} \right|
\leq \sum_{k=1}^{n} \frac{c_0^2 c^{n+1}}{(k + Q)^2(n + 1 - k + Q)^2}
\leq \frac{c_0^2 n Q c^{n+1}}{(n + Q + 1)^2}
\leq \frac{d_0 c^{n+1}}{(n + 1 + Q)^2}.
\]

Next, we obtain bounds on the first component of \(w_{n+1}\) given in (3.5),

\[
|w_{(n+1),1}| = \left| \frac{1}{\mu} g_{n+1} \right|
\leq \frac{d_0 c^{n+1}}{\mu(n + 1 + Q)^2}
\leq b_1 \frac{c^{n+1}}{(n + 1 + Q)^2}.
\]

Next we bound the second component of \(w_{n+1}\),

\[
|w_{(n+1),2}| = \left| \frac{1}{\nu} \left[ - \frac{1}{2} \sum_{j=1}^{n} g_j v_{n+1-j} - \frac{3}{2} g_0 g_{n+1} + g_{n+1} + \sum_{k=1}^{n} e_k v_{n+1-k} + \Gamma e_{-g_{n+1}} \right] \right|
\leq \frac{1}{\nu} \left[ \frac{1}{2} \sum_{k=1}^{n} \frac{c_0 d_0 c^{n+1}}{(k + Q)^2(n - k + 1 + Q)^2} + \frac{3}{2} d_0 \frac{d_0 c^{n+1}}{(n + 1 + Q)^2} \right]
\leq b_1 \frac{c^{n+1}}{(n + 1 + Q)^2}.
\]

Combining these two analyses, we have

\[
|w_{(n+1)}|_{\infty} \leq \max(b_1, b_2) \frac{c^{n+1}}{(n + 1 + Q)^2}.
\]

Using this bound and the formula (3.5), we have

\[
|w_{n+1}|_{\infty} = |(\sigma(n + 1) I - A) \begin{bmatrix} v_{n+1} \\ e_{n+1} \end{bmatrix} | \geq (n + 1)|\sigma| \begin{bmatrix} v_{n+1} \\ e_{n+1} \end{bmatrix}_{\infty} - |A| \begin{bmatrix} v_{n+1} \\ e_{n+1} \end{bmatrix}_{\infty}.
\]
This yields
\[ |w_{n+1}|_\infty \geq ((N + 1)|\sigma| - |A|_\infty) \begin{bmatrix} v_{n+1} \\ e_{n+1} \end{bmatrix}_\infty. \]

Recalling the assumption that \((N + 1)|\sigma| > |A|_\infty\), we have
\[ \begin{bmatrix} v_{n+1} \\ e_{n+1} \end{bmatrix}_\infty \leq \frac{|w_{n+1}|_\infty}{((N + 1)|\sigma| - |A|_\infty)}. \]

Hence, from the bounds above,
\[ \begin{bmatrix} v_{n+1} \\ e_{n+1} \end{bmatrix}_\infty \leq \frac{c_0 \max(b_1, b_2)}{((N + 1)|\sigma| - |A|_\infty)} c_0^n + 1 \frac{n + Q + 1}{2}. \]

By assumption,
\[ \frac{\max(b_1, b_2)}{((N + 1)|\sigma| - |A|_\infty)} \leq c_0. \]

Hence, by mathematical induction, the result holds for all \(n \in \mathbb{N}\).

To use this theorem, we use interval arithmetic to compute the coefficients up to some order \(N\) and verify that the assumptions of the theorem hold with rigorous computation. If the assumptions do not hold, we increase \(N\) until they do. We use the polynomial and series class with \(e^{\sigma x}\) substituted for \(x\) to account for truncation error.

### 3.2.2 Left manifold.

We describe how we approximate \(\Phi_L\) and how we obtain a rigorous bound on the approximation error involved. Similar to how we obtained an enclosure of the stable manifold of the fixed point \((v_+, e_+)\) of (3.3), we obtain a rigorous enclosure of the unstable manifold of the fixed point \((v_-, e_-)\) of (3.3). However, the dimension of the unstable manifold we seek is two, so we look for solutions of the following form,
\[
v(x) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} v_{n_1,n_2} \theta_1^{n_1} \sigma_1^{n_1} \theta_2^{n_2} \sigma_2^{n_2} e^{n_1 \sigma_1 x} + n_2 \sigma_2 x, \quad e(x) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} e_{n_1,n_2} \theta_1^{n_1} \sigma_1^{n_1} \theta_2^{n_2} \sigma_2^{n_2} e^{n_1 \sigma_1 x} + n_2 \sigma_2 x,
\]
where \(\sigma_1\) and \(\sigma_2\) are the eigenvalues of \(A\) with positive real part, where \(A\) is the Jacobian of (3.3) evaluated at the fixed point \((v_-, e_-)\).
We have that
\[ A = \begin{bmatrix}
\frac{2v_2 - 1 - \Gamma e_-}{\mu} \\
\frac{2\Gamma e_- v_2 - \Gamma e_- - \frac{3v_2}{\nu} + 2v_2 - \frac{1}{2}}{\nu}
\end{bmatrix}.\]

For convenience when evaluating at \( x_n \), we also have the formula
\[ v(x) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} v_{n_1,n_2} \tilde{\theta}_1^{n_1} e^{n_1 \sigma_1(x-x_n)} \tilde{\theta}_2^{n_2} e^{n_2 \sigma_2(x-x_n)}, \]
\[ e(x) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} e_{n_1,n_2} \tilde{\theta}_1^{n_1} e^{n_1 \sigma_1(x-x_n)} \tilde{\theta}_2^{n_2} e^{n_2 \sigma_2(x-x_n)}, \]
where \( \tilde{\theta}_1 = \theta_1 e^{x_n \sigma_1} \) and \( \tilde{\theta}_2 = \theta_2 e^{x_n \sigma_2} \). This ansatz is then of the form,
\[ v(x_n) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} v_{n_1,n_2} \tilde{\theta}_1^{n_1} \tilde{\theta}_2^{n_2}, \quad e(x_n) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} e_{n_1,n_2} \tilde{\theta}_1^{n_1} \tilde{\theta}_2^{n_2}. \]

To help us obtain a concise recursion formula, we define a few terms.

We define
\[ a(k_1, k_2) = \begin{cases} 1 & (k_1 > 0 \lor k_2 > 0) \land (k_1 < n_1 \lor k_2 < n_2) \\ 0 & \text{otherwise.} \end{cases} \]

We define
\[ b(k_1, k_2, j_1, j_2) = \begin{cases} 1 & (j_1 > 0 \lor j_2 > 0) \land (k_1 < n_1 \lor k_2 < n_2) \land (j_1 - k_1 < n_1 \lor j_2 - k_2 < n_2) \\ 0 & \text{otherwise.} \end{cases} \]

We have that
\[ (\sigma_1 n_1 + \sigma_2 n_2) v_{n_1,n_2} = \frac{1}{\mu} \left[ \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} v_{k_1,k_2} v_{n_1-k_1,n_2-k_2} - v_{n_1,n_2} + \Gamma e_{n_1,n_2} - \Gamma e_- v_{n_1,n_2} \right], \]
which is equivalent to
\[ (\sigma_1 n_1 + \sigma_2 n_2) v_{n_1,n_2} = \frac{1}{\mu} \left[ 2v_{0,0} v_{n_1,n_2} + \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} a(k_1,k_2) v_{k_1,k_2} v_{n_1-k_1,n_2-k_2} - v_{n_1,n_2} + \Gamma e_{n_1,n_2} - \Gamma e_- v_{n_1,n_2} \right]. \]

Gathering terms with \( v_{n_1,n_2} \) and \( e_{n_1,n_2} \) to the left hand side, we have
\[ \left( \sigma_1 n_1 + \sigma_2 n_2 + \frac{1 + \Gamma e_- - 2v_{0,0}}{\mu} \right) v_{n_1,n_2} - \frac{\Gamma}{\mu} e_{n_1,n_2} = \frac{1}{\mu} \left[ \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} a(k_1,k_2) v_{k_1,k_2} v_{n_1-k_1,n_2-k_2} \right]. \]
Similarly, we obtain the formula

\[
\frac{(\sigma_1 n_1 + \sigma_2 n_2) e_{n_1, n_2}}{\nu} = \frac{1}{\nu} \left[ -\frac{3}{2} v_{0,0} e_{n_1, n_2} \right. \\
-\frac{1}{2} \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} b(k_1, k_2, j_1, j_2) v_{k_1,k_2} v_{j_1-j_2-k_1-k_2 n_1-n_2-j_2} \\
+ \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} a(k_1, k_2) v_{k_1,k_2} v_{n_1-n_2-k_2} - \frac{v_{n_1,n_2}}{2} \\
+ \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \epsilon(k_1, k_2, j_1, j_2) v_{j_1-j_2-k_1-k_2 n_1-n_2-j_2} - \epsilon_{v_{n_1,n_2}} \\
+ \Gamma \epsilon v_{0,0} v_{n_1,n_2} + \frac{\epsilon_{v_{0,0} e_{n_1,n_2} + \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} a(k_1, k_2) e_{k_1,k_2} v_{n_1-n_2-k_2} - \epsilon_{v_{n_1,n_2}}}}{2} \\
+ 2 \epsilon_{v_{0,0} v_{n_1,n_2} + \epsilon_{v_{0,0} e_{n_1,n_2} + \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} a(k_1, k_2) e_{k_1,k_2} v_{n_1-n_2-k_2} - \epsilon_{v_{n_1,n_2}}}}. 
\]

Reorganizing, we obtain

\[
\frac{(\sigma_1 n_1 + \sigma_2 n_2) e_{n_1, n_2}}{\nu} = \frac{1}{\nu} \left[ -\frac{3}{2} v_{0,0} e_{n_1, n_2} \\
-\frac{1}{2} \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} b(k_1, k_2, j_1, j_2) v_{k_1,k_2} v_{j_1-j_2-k_1-k_2 n_1-n_2-j_2} \\
+ \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} a(k_1, k_2) v_{k_1,k_2} v_{n_1-n_2-k_2} - \frac{v_{n_1,n_2}}{2} \\
+ \epsilon_{v_{0,0} v_{n_1,n_2} + \epsilon_{v_{0,0} e_{n_1,n_2} + \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} a(k_1, k_2) e_{k_1,k_2} v_{n_1-n_2-k_2} - \epsilon_{v_{n_1,n_2}}}}. 
\]

\[
\left( \frac{3}{2} v_{0,0}^2 - 2 v_{0,0} + \frac{1}{2} - 2 \epsilon_{v_{0,0}^2} + \frac{\epsilon_{v_{0,0}^2}^2}{\nu} \right) v_{n_1,n_2} + \frac{(\sigma_1 n_1 + \sigma_2 n_2 - \frac{v_{0,0}}{\nu}) e_{n_1,n_2}}{\nu} = \frac{1}{\nu} \left[ -\frac{1}{2} \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} b(k_1, k_2, j_1, j_2) v_{k_1,k_2} v_{j_1-j_2-k_1-k_2 n_1-n_2-j_2} \\
+ \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} a(k_1, k_2) v_{k_1,k_2} v_{n_1-n_2-k_2} \\
+ \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} a(k_1, k_2) e_{k_1,k_2} v_{n_1-n_2-k_2} \\
+ \Gamma \epsilon_{v_{0,0} v_{n_1,n_2} + \epsilon_{v_{0,0} e_{n_1,n_2} + \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} a(k_1, k_2) e_{k_1,k_2} v_{n_1-n_2-k_2} - \epsilon_{v_{n_1,n_2}}}}. 
\]
We define \( g_{n_1,n_2} = \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} v_{k_1,k_2} v_{n_1-k_1,n_2-k_2} \) to help track notation. We have then that
\[
\left( \frac{3}{2} \nu^2 \theta - 2v_{0,0} + \frac{1}{2} - 2 \Gamma e_- v_{0,0} + \Gamma e_- \right) v_n + \left( \sigma_1 n_1 + \sigma_2 n_2 - \frac{v_{0,0}}{\nu} \right) e_n
\]
\[
= \frac{1}{\nu} \left\{ -\frac{1}{2} \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} a(j_1,j_2) v_{n_1-j_1,n_2-j_2} g_{j_1,j_2}
- \frac{v_{0,0}}{2} \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} a(j_1,j_2) v_{n_1-j_1,n_2-j_2} v_{j_1,j_2}
+ \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} a(k_1,k_2) v_{k_1,k_2} v_{n_1-k_1,n_2-k_2}
+ \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} a(k_1,k_2) e_{k_1,k_2} v_{n_1-k_1,n_2-k_2}
+ \Gamma e_- \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} a(k_1,k_2) v_{k_1,k_2} v_{n_1-k_1,n_2-k_2} \right\} .
\]

Thus the coefficients \( v_n \) and \( e_n \) can be computed by the recursion formula,
\[
((\sigma_1 n_1 + \sigma_2 n_2) I - A) \begin{bmatrix} v_{n_1,n_2} \\ e_{n_1,n_2} \end{bmatrix} = w_{n_1,n_2}
\]
(3.6)
where
\[
w_{n_1,n_2,1} = \frac{1}{\mu} \left[ \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} a(k_1,k_2) v_{k_1,k_2} v_{n_1-k_1,n_2-k_2} \right]
\]
and
\[
w_{n_1,n_2,2} = \frac{1}{\nu} \left\{ -\frac{1}{2} \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} a(j_1,j_2) v_{n_1-j_1,n_2-j_2} g_{j_1,j_2}
- \frac{v_{0,0}}{2} \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} a(j_1,j_2) v_{n_1-j_1,n_2-j_2} v_{j_1,j_2}
+ \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} a(k_1,k_2) v_{k_1,k_2} v_{n_1-k_1,n_2-k_2}
+ \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} a(k_1,k_2) e_{k_1,k_2} v_{n_1-k_1,n_2-k_2}
+ \Gamma e_- \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} a(k_1,k_2) v_{k_1,k_2} v_{n_1-k_1,n_2-k_2} \right\} .
\]
In the cases where \( n_1 = 1 \) and \( n_2 = 0 \) or \( n_1 = 0 \) and \( n_2 = 1 \), \( \begin{bmatrix} v_{1,0} \\ e_{1,0} \\ e_{0,1} \end{bmatrix} \) and \( \begin{bmatrix} v_{0,1} \\ \end{bmatrix} \) are eigenvectors corresponding to eigenvalues \( \sigma_1 \) and \( \sigma_2 \) respectively. For other combinations, we use the linear algebra solver in \textit{mpmath}.

### 3.2.3 Bound on Coefficients.

We solve the recursion formula with interval arithmetic to get a rigorous enclosure of the coefficients such that \( \sigma_1 n_1 + \sigma_2 n_2 \leq \sigma_1 N \) for \( N \in \mathbb{N} \) and \( \max(n_1, n_2) > 0 \). We then use a proof by induction to prove that the tail the series is dominated be a geometric series. The proof by induction involves many base cases, but these we verify using interval arithmetic computations. The statement of the relevant theorem follows.

**Theorem 22.** Let \( Q, n_1, n_2 \in \mathbb{N}, c_0, d_0, c_1, c_2 > 0 \), and let \( A \) be the Jacobian of \((3.3)\) evaluated at \((v_-, e_-)\). Assume that \( v_{0,0} = v_- \) and \( e_{0,0} = e_- \), and that \((v_{1,0}, e_{1,0})\) and \((v_{0,1}, e_{0,1})\) are eigenvectors corresponding to eigenvalues \( \sigma_1 \) and \( \sigma_2 \) of \( A \) with positive real part respectively.

Let \( \sigma_1 < \sigma_2 \). Let \( n_1, n_2 \in \mathbb{N} \). Let \( v_{n_1,n_2}, e_{n_1,n_2} \) satisfy the recursion formula \((3.6)\) when \( \sigma_1 n_1 + \sigma_2 n_2 \leq \sigma_1 N \). Let \( v_{n_1,n_2}, e_{n_1,n_2} \leq c_0 c_1^{n_1} c_2^{n_2} / ((n_1 + Q)^2(n_2 + Q)^2) \), \( g_{n_1,n_2} \leq d_0 c_1^{n_1} c_2^{n_2} / ((n_1 + Q)^2(n_2 + Q)^2) \) for \( n_1, n_2 \) such that \( \sigma_1 n_1 + \sigma_2 n_2 \leq \sigma_1 N \) and \( \max(n_1, n_2) > 0 \). Let \((N + 1)\sigma_1 > |A|_\infty \). Assume that

\[
\max(b_1, b_2) \leq \frac{c_0}{(N + 1)\sigma_1 - |A|_\infty} \leq c_0
\]

where \( b_1 := c_0^2 N_Q^2 / \mu \) and \( b_2 := (\frac{1}{2} c_0 d_0 + (\frac{\nu_0}{2} + 2 + \Gamma e_-) c_0^2) N_Q^2 / \nu \). Assume that

\[
c_0^2 N_Q^2 + 2c_0 v_{0,0} \leq d_0.
\]

Then \( |v_{n_1,n_2}|, |e_{n_1,n_2}| \leq c_0 c_1^{n_1} c_2^{n_2} / ((n_1 + Q)^2(n_2 + Q)^2) \) and \( |g_{n_1,n_2}| \leq d_0 c_1^{n_1} c_2^{n_2} / ((n_1 + Q)^2(n_2 + Q)^2) \) for all \( n_1, n_2 \in \mathbb{N} \cup \{0\} : \max(n_1, n_2) > 0 \).

**Proof.** Assume the hypothesis of the theorem. It is sufficient to prove the theorem holds when \( n_1 \sigma_1 + n_2 \sigma_2 \leq n \sigma_1 \) for each \( n \in \mathbb{N} \). We use induction on \( n \).
Let \( n \geq N \). Suppose \( v_{n_1,n_2}, e_{n_1,n_2} \leq c_0 e_1^n n_1^n e_2^n / ((n_1 + Q)^2(n_2 + Q)^2) \) and \( g_{n_1,n_2} \leq d_0 e_1^n e_2^n / ((n_1 + Q)^2(n_2 + Q)^2) \) for each \( n_1, n_2 \in \mathbb{N} \cup \{0\} \) such that \( \max(n_1, n_2) > 0 \) and \( \sigma_1 n_1 + \sigma_2 n_2 \leq n \). Let \( n \sigma_1 < n_1 \sigma_1 + n_2 \sigma_2 \leq (n + 1) \sigma_1 \). We denote \( v_{n_1,n_2} = w_{n_1,n_2,1} \) and we denote \( e_{n_1,n_2} = w_{n_1,n_2,2} \). We have that

\[
|w_{n_1,n_2,1}| = \left| \frac{1}{\mu} \left[ \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} a(k_1, k_2)v_{k_1,k_2}v_{n_1-k_1,n_2-k_2} \right] \right| \\
\leq \frac{2c_0 n_1 c_2^2}{\mu} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \frac{1}{(k_1 + Q)^2(n_1 - k_1 + Q)^2} \sum_{k_2=0}^{n_2} \frac{1}{(k_2 + Q)^2(n_2 - k_2 + Q)^2} \\
\leq \frac{c_0 n_1 c_2^2}{\mu(n_1 + Q)^2(n_2 + Q)^2} \\
\leq b_1 \frac{c_0 n_1 c_2^2}{(n_1 + Q)^2(n_2 + Q)^2}, \text{ and} \\
|w_{n_1,n_2,2}| = \left| \frac{1}{\nu} \left[ \frac{1}{2} \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} a(j_1, j_2)v_{n_1-j_1,n_2-j_2}g_{j_1,j_2} \right. \right. \\
- \frac{v_{0,0}}{2} \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} a(j_1, j_2)v_{n_1-j_1,n_2-j_2}v_{j_1,j_2} \\
+ \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} a(k_1, k_2)v_{k_1,k_2}v_{n_1-k_1,n_2-k_2} \\
+ \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} a(k_1, k_2)e_{k_1,k_2}v_{n_1-k_1,n_2-k_2} \\
+ \Gamma e_- \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} a(k_1, k_2)v_{k_1,k_2}v_{n_1-k_1,n_2-k_2} \left. \right] \right| \\
\leq \frac{1}{2} c_0 d_0 + \frac{v_{0,0}}{2} + 2 + \Gamma e_- \right] \frac{c_0^2}{\nu} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \frac{c_1^{n_1} c_2^{n_2}}{(k_1 + Q)^2(n_1 - k_1 + Q)^2} \sum_{k_2=0}^{n_2} \frac{c_1^{n_1} c_2^{n_2}}{(k_2 + Q)^2(n_2 - k_2 + Q)^2} \\
\leq \left( \frac{1}{2} c_0 d_0 + \frac{v_{0,0}}{2} + 2 + \Gamma e_- \right) \frac{c_0^2}{\nu(n_1 + Q)^2(n_2 + Q)^2} \frac{c_1^{n_1} c_2^{n_2}}{c_1^{n_1} c_2^{n_2}} \\
\leq b_2 \frac{(n_1 + Q)^2(n_2 + Q)^2}{(n_1 + Q)^2(n_2 + Q)^2}.
Thus

\[ |w_{n_1,n_2}|_\infty \leq \max(b_1, b_2) \frac{c_1^{n_1} c_2^{n_2}}{(n_1 + Q)^2(n_2 + Q)^2}; \]

\[ |w_{n_1,n_2}|_\infty = \left| (\sigma_1 n_1 + \sigma_2 n_2) I - A \right| \begin{bmatrix} v_{n_1,n_2} \\ e_{n_1,n_2} \end{bmatrix} \]

\[ \geq (\sigma_1 n_1 + \sigma_2 n_2) \begin{bmatrix} v_{n_1,n_2} \\ e_{n_1,n_2} \end{bmatrix}_\infty - A \begin{bmatrix} v_{n_1,n_2} \\ e_{n_1,n_2} \end{bmatrix}_\infty \]

\[ \geq ((N + 1) \sigma_1 - |A|_\infty) \begin{bmatrix} v_{n_1,n_2} \\ e_{n_1,n_2} \end{bmatrix}_\infty \]

\[ |v_{n_1,n_2}|, |e_{n_1,n_2}| \leq \begin{bmatrix} v_{n_1,n_2} \\ e_{n_1,n_2} \end{bmatrix}_\infty \]

\[ \leq \frac{|w_{n_1,n_2}|_\infty}{((N + 1) \sigma_1 - |A|_\infty) \max(b_1, b_2) c_1^{n_1} c_2^{n_2}} \]

\[ \leq \frac{c_0 c_1^{n_1} c_2^{n_2}}{(n_1 + Q)^2(n_2 + Q)^2}; \]

\[ |g_{n_1,n_2}| = \left| \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} v_{k_1,k_2} v_{n_1-k_1,n_2-k_2} \right| \]

\[ \leq 2v_{0,0} v_{n_1,n_2} + \sum_{k_1=1}^{n_1} \frac{c_0 c_1^{n_1}}{(k_1 + Q)^2(n_1 - k_1 + Q)^2} \sum_{k_2=0}^{n_2} \frac{c_0 c_2^{n_2}}{(k_2 + Q)^2(n_2 - k_2 + Q)^2}. \]

\[ \leq \frac{d_0 c_1^{n_1} c_2^{n_2}}{(n_1 + Q)^2(n_2 + Q)^2}. \]

Like earlier, to use this theorem, we use interval arithmetic to compute the coefficients up to some order \( N \) and verify that the assumptions of the theorem hold with rigorous computation. We can also increase \( N \) here as needed so that the assumptions hold. We use the 2d polynomial and series class with \( \tilde{\theta}_1 \) and \( \tilde{\theta}_2 \) substituted for \( x_1 \) and \( x_2 \) to account for truncation error.
3.2.4 Middle part. We describe how we approximate $\Psi_{\Delta x}$ and how we obtain a rigorous bound on the approximation error involved. We make the ansatz that a solution of (3.3) takes the form

$$v(x) = \sum_{n=0}^{\infty} v_n x^n, \quad e(x) = \sum_{n=0}^{\infty} e_n x^n$$

for some coefficients $v_n, e_n \in \mathbb{R}$. The zero order coefficients, $v_0, e_0$, correspond to the initial values for the initial value problem, and all other coefficients come from a recursion formula.

Plugging the ansatz into (3.3), we arrive at the formulas,

$$v_{n+1} = \frac{1}{\mu(n + 1)} \left[ \sum_{j=0}^{n} v_j v_{n-j} - (1 + \Gamma e_{-}) v_n + \Gamma e_n \right]$$

$$e_{n+1} = \frac{1}{\nu(n + 1)} \left[ -\frac{1}{2} \sum_{j=0}^{n} \sum_{k=0}^{j} v_k v_{j-k} v_{n-j} - \left( \frac{1}{2} + \Gamma e_{-} + e_{-} \right) v_n + \left( 1 + \Gamma e_{-} \right) \sum_{j=0}^{n} v_j v_{n-j} + \sum_{j=0}^{n} v_j e_{n-j} \right].$$

We define

$$g_n = \sum_{k=0}^{n} v_k v_{n-k},$$

and rewrite

$$v_{n+1} = \frac{1}{\mu(n + 1)} \left[ g_n - (1 + \Gamma e_{-}) v_n + \Gamma e_n \right]$$

$$e_{n+1} = \frac{1}{\nu(n + 1)} \left[ -\frac{1}{2} \sum_{j=0}^{n} g_j v_{n-j} - \left( \frac{1}{2} + \Gamma e_{-} + e_{-} \right) v_n + \left( 1 + \Gamma e_{-} \right) g_n + \sum_{j=0}^{n} v_j e_{n-j} \right]. \quad (3.7)$$

In order to use the Newton Kantorovich Theorem, we need the derivatives of the coefficients with respect to $v_0$ and $e_0$. These are computed using the recurrence formulas,

$$\frac{\partial v_0}{\partial v_0} = 1, \quad \frac{\partial v_0}{\partial e_0} = 0, \quad \frac{\partial e_0}{\partial v_0} = 0, \quad \frac{\partial e_0}{\partial e_0} = 1$$

$$\frac{\partial g_n}{\partial v_0} = 2 \sum_{k=0}^{n} \frac{\partial v_k}{\partial v_0} v_{n-k}$$

$$\frac{\partial g_n}{\partial e_0} = 2 \sum_{k=0}^{n} \frac{\partial v_k}{\partial e_0} v_{n-k}$$
\[
\frac{\partial v_{n+1}}{\partial v_0} = \frac{1}{\mu(n+1)} \left[ \frac{\partial g_n}{\partial v_0} - (1 + \Gamma e_-) \frac{\partial v_n}{\partial v_0} + \frac{\partial e_n}{\partial v_0} \right]
\]

\[
\frac{\partial v_{n+1}}{\partial e_0} = \frac{1}{\mu(n+1)} \left[ \frac{\partial g_n}{\partial e_0} - (1 + \Gamma e_-) \frac{\partial v_n}{\partial e_0} + \frac{\partial e_n}{\partial e_0} \right]
\]

\[
\frac{\partial e_{n+1}}{\partial v_0} = \frac{1}{\nu(n+1)} \left[ -\frac{1}{2} \sum_{j=0}^{n} \frac{\partial g_j}{\partial v_0} v_{n-j} - \frac{1}{2} \sum_{j=0}^{n} \frac{\partial g_j}{\partial e_0} v_{n-j} - \frac{1}{2} \sum_{j=0}^{n} \frac{\partial g_j}{\partial v_0} v_{n-j} - \frac{1}{2} \sum_{j=0}^{n} \frac{\partial g_j}{\partial e_0} v_{n-j} - \frac{1}{2} - (1 + \Gamma e_- + e_-) \frac{\partial v_n}{\partial v_0} - (1 + \Gamma e_- + e_-) \frac{\partial v_n}{\partial e_0} + \frac{\partial e_n}{\partial v_0} - (1 + \Gamma e_- + e_-) \frac{\partial e_n}{\partial e_0} \right]
\]

\[
\frac{\partial e_{n+1}}{\partial e_0} = \frac{1}{\nu(n+1)} \left[ -\frac{1}{2} \sum_{j=0}^{n} \frac{\partial g_j}{\partial v_0} v_{n-j} + \frac{1}{2} \sum_{j=0}^{n} \frac{\partial g_j}{\partial e_0} v_{n-j} + \frac{1}{2} \sum_{j=0}^{n} \frac{\partial g_j}{\partial v_0} v_{n-j} + \frac{1}{2} \sum_{j=0}^{n} \frac{\partial g_j}{\partial e_0} v_{n-j} + \frac{1}{2} + (1 + \Gamma e_- + e_-) \frac{\partial v_n}{\partial v_0} + (1 + \Gamma e_- + e_-) \frac{\partial v_n}{\partial e_0} + \frac{\partial e_n}{\partial v_0} + (1 + \Gamma e_- + e_-) \frac{\partial e_n}{\partial e_0} \right]
\]

\[
\frac{\partial^2 v_0}{\partial v_0^2} = \frac{\partial^2 v_0}{\partial v_0 \partial e_0} = \frac{\partial^2 v_0}{\partial e_0^2} = 0
\]

\[
\frac{\partial^2 e_0}{\partial v_0^2} = \frac{\partial^2 e_0}{\partial v_0 \partial e_0} = \frac{\partial^2 e_0}{\partial e_0^2} = 0
\]

\[
\frac{\partial^2 g_n}{\partial v_0^2} = 2 \sum_{k=0}^{n} \frac{\partial^2 v_k}{\partial v_0^2} v_{n-k} + 2 \sum_{k=0}^{n} \frac{\partial v_k}{\partial v_0} \frac{\partial v_{n-k}}{\partial v_0}
\]

\[
\frac{\partial^2 g_n}{\partial e_0^2} = 2 \sum_{k=0}^{n} \frac{\partial^2 v_k}{\partial e_0^2} v_{n-k} + 2 \sum_{k=0}^{n} \frac{\partial v_k}{\partial e_0} \frac{\partial v_{n-k}}{\partial e_0}
\]

\[
\frac{\partial^2 e_{n+1}}{\partial v_0^2} = \frac{1}{\mu(n+1)} \left[ \frac{\partial^2 g_n}{\partial v_0^2} - (1 + \Gamma e_-) \frac{\partial^2 v_n}{\partial v_0^2} + \frac{\partial^2 e_n}{\partial v_0^2} \right]
\]

\[
\frac{\partial^2 v_{n+1}}{\partial v_0 e_0} = \frac{1}{\mu(n+1)} \left[ \frac{\partial^2 g_n}{\partial v_0 e_0} - (1 + \Gamma e_-) \frac{\partial^2 v_n}{\partial v_0 e_0} + \frac{\partial^2 e_n}{\partial v_0 e_0} \right]
\]

\[
\frac{\partial^2 e_{n+1}}{\partial e_0^2} = \frac{1}{\mu(n+1)} \left[ \frac{\partial^2 g_n}{\partial e_0^2} - (1 + \Gamma e_-) \frac{\partial^2 v_n}{\partial e_0^2} + \frac{\partial^2 e_n}{\partial e_0^2} \right]
\]
Let statement of the relevant theorem follows. We solve the recursion formula with interval arithmetic to get a rigorous enclosure of the relation
\[
\frac{\partial^2 e_{n+1}}{\partial v_0^2} = \frac{1}{\nu(n+1)} \left[ -\frac{1}{2} \sum_{j=0}^{n} \frac{\partial^2 g_j}{\partial v_0^2} v_{n-j} - \sum_{j=0}^{n} \frac{\partial g_j}{\partial v_0} \frac{\partial v_{n-j}}{\partial v_0} - \frac{1}{2} \sum_{j=0}^{n} g_j \frac{\partial^2 v_{n-j}}{\partial v_0^2} + \left( \frac{1}{2} + \Gamma e_- + e_- \right) \frac{\partial^2 v_n}{\partial v_0^2} + (1 + \Gamma e_-) \frac{\partial^2 g_n}{\partial v_0^2} + \sum_{j=0}^{n} \frac{\partial^2 v_j}{\partial v_0^2} e_{n-j} + \sum_{j=0}^{n} \frac{\partial v_j}{\partial v_0} \frac{\partial e_{n-j}}{\partial v_0} + \sum_{j=0}^{n} v_j \frac{\partial^2 e_{n-j}}{\partial v_0^2} \right]
\]
\[
\frac{\partial^2 e_{n+1}}{\partial v_0 \partial e_0} = \frac{1}{\nu(n+1)} \left[ -\frac{1}{2} \sum_{j=0}^{n} \frac{\partial^2 g_j}{\partial v_0 \partial e_0} v_{n-j} - \sum_{j=0}^{n} \frac{\partial g_j}{\partial v_0} \frac{\partial v_{n-j}}{\partial e_0} - \frac{1}{2} \sum_{j=0}^{n} g_j \frac{\partial^2 v_{n-j}}{\partial v_0 \partial e_0} + \left( \frac{1}{2} + \Gamma e_- + e_- \right) \frac{\partial^2 v_n}{\partial v_0 \partial e_0} + (1 + \Gamma e_-) \frac{\partial^2 g_n}{\partial v_0 \partial e_0} + \sum_{j=0}^{n} \frac{\partial^2 v_j}{\partial v_0 \partial e_0} e_{n-j} + \sum_{j=0}^{n} \frac{\partial v_j}{\partial v_0} \frac{\partial e_{n-j}}{\partial e_0} + \sum_{j=0}^{n} v_j \frac{\partial^2 e_{n-j}}{\partial v_0 \partial e_0} \right]
\]
\[
\frac{\partial^2 e_{n+1}}{\partial e_0^2} = \frac{1}{\nu(n+1)} \left[ -\frac{1}{2} \sum_{j=0}^{n} \frac{\partial^2 g_j}{\partial e_0^2} v_{n-j} - \sum_{j=0}^{n} \frac{\partial g_j}{\partial e_0} \frac{\partial v_{n-j}}{\partial e_0} - \frac{1}{2} \sum_{j=0}^{n} g_j \frac{\partial^2 v_{n-j}}{\partial e_0^2} + \left( \frac{1}{2} + \Gamma e_- + e_- \right) \frac{\partial^2 v_n}{\partial e_0^2} + (1 + \Gamma e_-) \frac{\partial^2 g_n}{\partial e_0^2} + \sum_{j=0}^{n} \frac{\partial^2 v_j}{\partial e_0^2} e_{n-j} + \sum_{j=0}^{n} \frac{\partial v_j}{\partial e_0} \frac{\partial e_{n-j}}{\partial e_0} + \sum_{j=0}^{n} v_j \frac{\partial^2 e_{n-j}}{\partial e_0^2} \right]
\]

We solve the recursion formula with interval arithmetic to get a rigorous enclosure of the coefficients and derivatives up to some order \( N \in \mathbb{N} \). We then use a proof by induction to prove that the tail end of the series is bounded by a geometric series. The proof by induction involves many base cases, but these we verify using interval arithmetic computations. The statement of the relevant theorem follows.

**Theorem 23.** Let \( N, Q \in \mathbb{N} \). Let \( v_0, e_0, c_0, d_0, c \in \mathbb{R} \) and let \( v_n, e_n \) satisfy the recurrence relation 3.7. Assume that \( |v_n|, \frac{\partial v_n}{\partial v_0}, \frac{\partial v_n}{\partial e_0} \leq b_0 c^n/(n+Q)^2, |e_n|, \frac{\partial e_n}{\partial v_0}, \frac{\partial e_n}{\partial e_0} \leq c_0 e^n/(n+Q)^2 \) and \( |g_n|, \frac{\partial g_n}{\partial v_0}, \frac{\partial g_n}{\partial e_0} \leq d_0 c^n/(n+Q)^2 \) for all \( 0 < n \leq N \) Assume the following inequalities hold.

\[
c \geq \frac{1}{b_0 \mu (N+1)} \left[ (d_0 + (1 + \Gamma e_-) b_0 + \Gamma c_0) \frac{(N+Q+1)^2}{(N+Q)^2} \right]
\]
\[
c \geq \frac{1}{c_0 \nu (N + 1)} \left[ \left( \frac{1}{2} + \Gamma e_- + e_- \right) b_0 + (1 + \Gamma e_-) d_0 \right] \frac{(N + Q + 1)^2}{(N + Q)^2} \\
+ (b_0 d_0 + 2b_0 c_0) R_Q \]

and

\[
d_0 \geq 2b_0^2 N_Q^2
\]

Then

\[
|g_n|, \left| \frac{\partial g_n}{\partial v_0} \right|, \left| \frac{\partial g_n}{\partial e_0} \right| \leq \frac{d_0 c^n}{(n + Q)^2}
\]

\[
|v_n|, \left| \frac{\partial v_n}{\partial v_0} \right|, \left| \frac{\partial v_n}{\partial e_0} \right| \leq \frac{b_0 c^n}{(n + Q)^2}
\]

\[
|e_n|, \left| \frac{\partial e_n}{\partial v_0} \right|, \left| \frac{\partial e_n}{\partial e_0} \right| \leq \frac{c_0 c^n}{(n + Q)^2}
\]

for all \( n \in \mathbb{N} \cup \{0\} \).

**Proof.** Assume the hypothesis of the theorem.

Let \( n \geq N \) and assume that \( |v_k|, |e_k| \leq c_0 c^n / (k + Q)^2 \), and \( |g_k| \leq d_0 c^n / (k + Q)^2 \) for all \( 0 \leq k \leq n \). Then

\[
|v_{n+1}|, \left| \frac{\partial v_{n+1}}{\partial v_0} \right|, \left| \frac{\partial v_{n+1}}{\partial e_0} \right| \leq \frac{1}{\mu(n + 1)} \left[ \left( d_0 + (1 + \Gamma e_-) b_0 + \Gamma c_0 \right) \frac{c^n}{(n + Q)^2} \right]
\]

\[
\leq \frac{1}{\mu(n + 1)} \left[ \left( d_0 + (1 + \Gamma e_-) b_0 + \Gamma c_0 \right) \frac{(n + Q + 1)^2 c^n}{(n + Q)^2(n + Q + 1)^2} \right]
\]

\[
\leq \frac{1}{\mu(n + 1)} \left[ \left( d_0 + (1 + \Gamma e_-) b_0 + \Gamma c_0 \right) \frac{(N + Q + 1)^2 c^n}{(N + Q)^2(n + Q + 1)^2} \right]
\]

\[
\leq \frac{b_0 c^{n+1}}{(n + Q + 1)^2}
\]
\begin{align*}
|e_{n+1}|, \left| \frac{\partial e_{n+1}}{\partial v_0} \right|, \left| \frac{\partial e_{n+1}}{\partial e_0} \right| & \leq \frac{1}{\nu(n+1)} \left[ \left( \left( \frac{1}{2} + \Gamma e_- + e_- \right) b_0 + (1 + \Gamma e_-)d_0 \right) \frac{c^n}{(n+Q)^2} 
+ (b_0d_0 + 2b_0c_0) \sum_{j=0}^n \frac{c^n}{(j + Q)^2(n - j + Q)^2} \right] \\
& \leq \frac{1}{\nu(n+1)} \left[ \left( \left( \frac{1}{2} + \Gamma e_- + e_- \right) b_0 + (1 + \Gamma e_-)d_0 \right) \frac{c^n}{(n+Q)^2} 
+ (b_0d_0 + 2b_0c_0) \frac{\mathcal{N}_Q}{(n+1+Q)^2} \right] \\
& \leq \frac{1}{\nu(n+1)} \left[ \left( \left( \frac{1}{2} + \Gamma e_- + e_- \right) b_0 + (1 + \Gamma e_-)d_0 \right) \frac{(n+Q+1)^2c^n}{(n+Q+1)^2(n+Q)^2} 
+ (b_0d_0 + 2b_0c_0) \frac{\mathcal{N}_Q}{(n+1+Q)^2} \right] \\
& \leq \frac{c_0c^{n+1}}{(n + Q + 1)^2}
\end{align*}

\begin{align*}
|g_{n+1}|, \left| \frac{\partial g_{n+1}}{\partial g_0} \right|, \left| \frac{\partial g_{n+1}}{\partial e_0} \right| & \leq \sum_{j=0}^{n+1} \frac{2b_0^2c^{n+1}}{(j + Q)^2(n + 1 - j + Q)^2} \\
& \leq \frac{2b_0^2\mathcal{N}_Qc^{n+1}}{(n + Q + 1)^2} \\
& \leq \frac{d_0c^{n+1}}{(n + Q + 1)^2}.
\end{align*}

We use the polynomial and series class to account for truncation error.

\section*{3.2.5 Newton Kantorovich argument.} In this section, we describe the details of our rigorous computation to prove a tight error enclosure of the traveling wave solution, depicted in Figure 3.1, of the profile equation (3.3) for the parameters $\nu = 4/3$, $\Gamma = 2/3$, and $v_+ = 0.8$. The setup for the Newton-Kantorovich Theorem is described in Section 3.2. Here we record the details in Table 3.1.
In Table 3.1, $R_s$ is what we scale the eigenvector by in solving for the parameterization of the stable manifold as described in Section 3.2.1. Similarly, $L_s$ is what we scale the eigenvectors by in the process of solving for the unstable manifold as described in Section 3.2.2.

As described in Section 3.2, the right manifold is evaluated at $x = 0$. The value $x_N$ given in Table 3.1 corresponds to where the left manifold is evaluated. The number of spatial intervals between $x = x_N$ and $x = 0$ is given by $N$ and $\Delta x$ is the width of these intervals.

The values $N_\infty$ and $N_{-\infty}$ given in Table 3.1 correspond to the number of terms used in computing the series solution for the right and left manifolds respectively. The number of terms is the series computed for a solution on an interval in the midel of width $\Delta x$ is given by $N_m$.

The remaining items in Table 3.1 are the bounds and terms pertaining to the assumptions of the Newton-Kantorovich Theorem as described in Section 2.5.
<table>
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<th>Value</th>
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</tr>
<tr>
<td>$L_s$</td>
<td>-1/15</td>
</tr>
<tr>
<td>$x_N$</td>
<td>-15</td>
</tr>
<tr>
<td>$N$</td>
<td>60</td>
</tr>
<tr>
<td>$\Delta x$</td>
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<td>$N_{-\infty}$</td>
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<tr>
<td>$N_m$</td>
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</tr>
<tr>
<td>$Y_0$</td>
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</tr>
<tr>
<td>$Z_0$</td>
<td>$7.57306 \times 10^{-29}$</td>
</tr>
<tr>
<td>$Z_1$</td>
<td>$2.57851 \times 10^{-10}$</td>
</tr>
<tr>
<td>$Z_2(r)r$</td>
<td>0.0123434</td>
</tr>
<tr>
<td>$p_r(r)$</td>
<td>$Z_2(r)r^2 - (1 - Z_0 - Z_1)r + Y_0$</td>
</tr>
<tr>
<td>$r$</td>
<td>$1.16542 \times 10^{-13}$</td>
</tr>
</tbody>
</table>

Table 3.1: Bounds and parameter choices used for the application of the Newton-Kantorovich Theorem to obtaining a rigorous, tight enclosure of the profile.

3.3 Evans function

In this section, we describe work toward obtaining a rigorous enclosure of the Evans function. The main work is to approximate well the solutions of (2.8) and obtain concrete, rigorous error bounds on that approximation. For the Navier-Stokes system we consider, equation (2.8) takes the form below, as described in [17]. The system is given by

$$W'' = A(x, \lambda)W, \quad W = (\tilde{\epsilon}, \tilde{\epsilon}', \tilde{u}, \tilde{v}, \tilde{v}')^T,$$

where

$$A(x, \lambda) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
\lambda \nu^{-1} & \nu^{-1} & \frac{1}{2\nu} w_x - v_{xx} & \lambda g & g - h \\
0 & 0 & 0 & \lambda & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & \Gamma & \lambda v + \Gamma v_x & \lambda v & f - \lambda
\end{bmatrix},$$
where $w = v^2$ and $v$ and $e$ are the components of the traveling wave profile, and
\[
g = \nu^{-1} \Gamma e - \frac{\nu + 1}{\nu} (w - v + \Gamma (e - ve_-)),
\]
\[
h = -\nu^{-1} \left( -\frac{w}{2} + v - \frac{1}{2} + (e - e_-) + (v - 1)\Gamma e_- \right),
\]
\[
f = 2v - 1 - \Gamma e_-.
\]
The dimension of the unstable manifold of this system is two at $x = -\infty$ and three at $x = +\infty$. Integrating these manifolds is not numerically stable, so we use the compound matrix method.

3.3.1 Compound matrix method system. The lifted matrix needed for a five dimensional system with a two dimensional manifold is given in (2.10), which for this system is given by

\[
A^{(2)} = \begin{bmatrix}

\nu^{-1}v & \frac{1}{2}w_x - ve_x & \lambda g & g - h & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
\Gamma & \lambda v + \Gamma ve_x & \lambda v & f - \lambda & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & \lambda v^{-1}v & 0 & 0 & \nu^{-1}v & \lambda & 1 & -\lambda g & h - g & 0 \\
0 & 0 & \lambda v^{-1}v & 0 & 0 & \nu^{-1}v & 1 & \frac{1}{2}w_x - ve_x & 0 & h - g \\
0 & 0 & \lambda v^{-1}v & \lambda v + \Gamma ve_x & \lambda v & \nu^{-1}v + f - \lambda & 0 & \frac{1}{2}w_x - ve_x & \lambda g \\
0 & 0 & 0 & 0 & 0 & \lambda v & f - \lambda & \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\Gamma & 0 & \lambda v & f - \lambda & 1 \\
0 & 0 & 0 & 0 & 0 & -\Gamma & 0 & -\lambda v - \Gamma ve_x & 0 & f - \lambda
\end{bmatrix}.
\]

The equations to solve are then
\[
W'_L(x; \lambda) = (A^{(2)}(x; \lambda) - \mu_L(\lambda))W_L(x; \lambda), \quad x \in (-\infty, 0], \quad \lim_{x \to -\infty} W_L(x; \lambda) = W^0_L(\lambda)
\]
\[
W'_R(x; \lambda) = (-A^{(2)*}(x; \lambda) + \mu_R^*(\lambda))W_R(x; \lambda), \quad x \in [0, \infty), \quad \lim_{x \to +\infty} W_R(x; \lambda) = W^0_R(\lambda)
\]

where $\mu_L(\lambda) \in \mathbb{C}$ is the eigenvalue of $A^{(2)}_-$ with the greatest real part and $W^0_L(\lambda)$ is a corresponding eigenvector, and $\mu_R(\lambda) \in \mathbb{C}$ is the eigenvalue of $A^{(2)}_+$ with least real part and $W^0_R(\lambda)$ is a corresponding eigenvector. We note that the reason for using the negative of the conjugate transpose of $A^{(2)}$ in the ODE on the right half-plane is to use the adjoint method so that the we only need take exterior products between two vectors instead of three, since the stable manifold of $A_+$ at zero is three dimensional. For additional explanation, see [2, 6].

As certain eigenvalues and eigenvectors of $A^{(2)}_\pm$ are needed, we next describe how we obtain a rigorous enclosure of these.
3.3.2 Eigenvalues and eigenvectors at infinity. To obtain a rigorous enclosure of the eigenvalues and eigenvectors of $A^{(2)}_{\pm}(\lambda)$, we pose the eigenvalue solving problem as a zero-finding problem for the function $f : \mathbb{R}^{10} \to \mathbb{R}^{10}$ defined by

$$f(x_2, ..., x_{10}, \mu) = (A^{(2)}_{\pm} - \mu I)(1, x_2, ..., x_{10})^T.$$ 

In practice, we can choose any non-zero value for one of $x_j$, $j = 1, ..., 10$, rather than taking $x_1 = 1$. We use built in solvers to find approximations of the eigenvalue and eigenvector and then use the Newton-Kantorovich Theorem to obtain a tight, rigorous enclosure of the eigenvalue, eigenvector pair.

We can, in practice, obtain an enclosure of the eigenvalue and eigenvector as $\lambda$ varies by using polynomials in $\lambda$ with interval coefficients in place of entries of $A^{(2)}_{\pm}$ and the approximate eigenvalues and eigenvectors. In this case, the eigenvalues and eigenvectors are approximated using interpolation, and again the Newton-Kantorovich Theorem is used to obtain a uniform, rigorous bound over the interval of $\lambda$ values.

3.3.3 Right manifold. The dimension of the stable manifold of (2.3) at $x = +\infty$ is three, but by using the adjoint formulation, $W' = -A^*W$, where $A^*$ is the complex conjugate transpose of $A$, we arrive at a system where the relevant manifold is dimension two. Then the lifted matrix is of the same form. The equation to solve is

$$W'_R(x; \lambda) = (-A^{(2)*}(x; \lambda) + \mu^*_R(\lambda))W_R(x; \lambda), \quad x \in [0, \infty), \quad \lim_{x \to +\infty} W_R(x; \lambda) = W^0_R(\lambda),$$

but for ease of notation, in what follows we will drop the negative and the hermitian symbol, which equates to redefining the notation for $A^{(2)}$ and $\mu_R$. We make the same ansatz as we did for the profile. That is, we assume that the solution is of the form $W(x) = \sum_{n=0}^{\infty} e^{\sigma_n x} W_n$ where $\sigma$ is the negative eigenvalue of the Jacobian of (3.3) evaluated at $(v_+, e_+)$. In order to obtain a series solution of this form, we need $W_0 = 0$, and we need $W_1$ to be an eigenvector of the limiting marix $A^{(2)}_+ := \lim_{x \to +\infty} A^{(2)}(x)$. In order for this to be the case, we need to rescale the problem because the eigenvalue $\mu_+$ of $A^{(2)}_+$ with the smallest real part will not
in general be equal to $\sigma$. Thus, we make the coordinate change $W(x) = e^{(\mu - \sigma)x}Z(x)$, which yields the system

$$Z'(x; \lambda) = (A^{(2)}(x; \lambda) - (\mu - \sigma)I)Z(x).$$

The recursion formula is given by

$$Z_n = -(A^{(2)}_+ - (\mu - \sigma)I)^{-1}\sum_{k=1}^{n} A^{(2)}_k Z_{n-k}$$

for $n > 1$, with $Z_0 = 0$, and $Z_1$ being an eigenvector of $A^{(2)}_+$ corresponding to the eigenvalue $\mu_+$.

We use a proof by induction to prove convergence of the series and a bound on the truncation error of the series, similar to how we did in obtaining a rigorous, tight enclosure of the profile solution. We refer to Proposition 20 to recall the definition of $\aleph_Q$.

We are now ready to state the proposition.

**Proposition 24.** Let $N \in \mathbb{N}$ and let $C_0, C, R_0, R, M > 0$. Assume that

$$\|[(n-1)\sigma + \mu - A_0]^{-1}\|_2 \leq M$$

for all $n \geq N$. Assume that

$$\|A_n\|_2 \leq \frac{C_0C^n}{(n + Q)^2}$$

for all $n \in \mathbb{N}$ and that

$$\|Z_n\| \leq \frac{R_0R^n}{(n + Q)^2}$$

for all $0 \leq n \leq N$. Then if $R \geq MC_0\aleph_QC$, it holds that

$$\|Z_n\| \leq \frac{R_0R^n}{(n + Q)^2}$$

for all $n \in \mathbb{N}$.

**Proof.** Let $N \in \mathbb{N}$ and let $C_0, C, R_0, R, M > 0$. Assume that the hypotheses of the proposition hold. Let $n > N + 1$ and assume that

$$\|Z_k\| \leq \frac{R_0R^k}{(k + Q)^2}$$

for all $k \leq n$.
for all $0 \leq k \leq n - 1$. Then from the recursive definition of $Z_n$ and the assumed bounds, we have that
\[
\|Z_n\|_2 \leq M_n \sum_{k=1}^{n} \frac{C_0 C^k}{(k + Q)^2} \frac{R_0 R^{n-k}}{(n - k + Q)^2}
\leq \frac{R_0 R^n}{(n + Q)^2} (M_n C_0 N_Q(C/R)).
\]
If $R \geq M C_0 N_Q C$, then $M_n C_0 N_Q(C/R)$, and so $\|Z_n\|_2 \leq \frac{R_0 R^n}{(n + Q)^2}$. By mathematical induction, we have that $\|Z_n\|_2 \leq \frac{R_0 R^n}{(n + Q)^2}$ for all $n \in \mathbb{N} \cup \{0\}$. \qed

3.3.4 Left manifold. In this section, we describe how we obtain the parametrization of the unstable manifold of the fixed point zero of the system
\[
Z'(x; \lambda) = (A^{(2)}(x; \lambda) - \mu_L I)Z(x).
\]
We seek a solution of the same form as we did for the profile. That is, we seek a solution of the form
\[
Z(x) = \sum_{m,n=0}^{\infty} Z_{m,n} e^{\sigma_1 x} e^{\sigma_2 x}
\]
where $\sigma_1$ and $\sigma_2$ are the eigenvalues of the Jacobian of (3.3) evaluated at $(v_-, e_-)$. The coefficients $Z_{0,1}$ and $Z_{1,0}$ need to be either zero or an eigenvector of $A^{(2)} - \mu_L I$. We will rescale via $Z(x) = e^{-\sigma x}$ so that the problem to solve is
\[
Z'(x; \lambda) = (A^{(2)}(x; \lambda) - (\mu_L - \sigma_1) I)Z(x; \lambda)
\]
where $\sigma_1$ is one of the eigenvalues of the Jacobian of the profile (3.3) evaluated at the fixed point $(v_-, e_-)$. We note that an eigenvector of $A^{(2)}_+ - \mu_L I$ with corresponding eigenvalue zero is an eigenvector of $A^{(2)} - (\mu_L - \sigma_1) I$ with corresponding eigenvalue $\sigma_1$.

The recursion formula is given by
\[
Z_{mn} = \left( (m \sigma_1 + n \sigma_2) I - (A^{(2)}_+ - (\mu_L - \sigma_1) I) \right)^{-1} \sum_{j=0}^{m} \sum_{k=0}^{n} \delta_{j,k} A_{j,k} Z_{m-j,n-k},
\]
where $\delta_{j,k} = 0$ if $j = k = 0$, and otherwise, $\delta_{j,k} = 1$.

The proposition we use to show convergence of the series is as follows.
Proposition 25. Let $N \in \mathbb{N}$ and let $C_0, C, R_0, R, M > 0$. Let $Q \in \mathbb{N}$. Assume that

$$\|[(m-1)\sigma_1 + n\sigma_2 + \mu_L] - A_{0,0}]^{-1}\|_2 \leq M$$

for all $m + n \geq N$. Assume that

$$\|A_{m,n}\|_2 \leq \frac{C_0 C^{m+n}}{(m+Q)^2(n+Q)^2}$$

for all $m, n \in \mathbb{N} \cup \{0\}$ and that

$$\|Z_{m,n}\| \leq \frac{R_0 R^{m+n}}{(m+Q)^2(n+Q)^2}$$

for all $0 \leq m + n \leq N$. Then if $R \geq MC_0 N_2^2 C$, it holds that

$$\|Z_{m,n}\| \leq \frac{R_0 R^{m+n}}{(m+Q)^2(n+Q)^2}$$

for all $m, n \in \mathbb{N} \cup \{0\}$.

Proof. Let $N \in \mathbb{N}$ and let $C_0, C, R_0, R, M > 0$. Assume that the hypotheses of the proposition hold. Let $\tilde{N} > N$ be an integer and assume that

$$\|Z_{j,k}\| \leq \frac{R_0 R^{j+k}}{(j+Q)^2(k+Q)^2}$$

for all $j + k \leq \tilde{N}$. Let $m + n = \tilde{N} + 1$. Then from the recursive definition of $Z_{m,n}$ and the assumed bounds,

$$\|Z_{m,n}\|_2 \leq M \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{\delta_{j,k} C_0 C^k}{(j+Q)^2(k+Q)^2} \frac{R_0 R^{m-j+n-k}}{(m-j+Q)^2(n-k+Q)^2}$$

$$\leq \frac{R_0 R^{m+n}}{(m+Q)^2(n+Q)^2} (MC_0 N_2^2 (C/R)).$$

If $R \geq MC_0 N_2^2 C$, then $MC_0 N_2^2 (C/R) \leq 1$, and so $\|Z_{m,n}\|_2 \leq \frac{R_0 R^{m+n}}{(m+Q)^2(n+Q)^2}$. By the principle of mathematical induction,

$$\|Z_{m,n}\| \leq \frac{R_0 R^{m+n}}{(m+Q)^2(n+Q)^2}$$

holds for all $m, n \in \mathbb{N} \cup \{0\}$. \qed

Chapter 4. Conclusion
We have used computer assisted methods of proof to obtain a tight enclosure of the traveling wave profile for parameters that correspond to air. This computer assisted proof rigorously verifies both existence and uniqueness of the traveling wave profile and gives a tight, quantitative bound on the error of the numerical approximation of the profile. We have also developed error estimates for computer assisted methods of proof for solving the Evans function ODE system on the intervals \((-\infty, L], [R, +\infty),\) for some \(L < 0\) and \(R > 0\). Solving the Evans function ODE on these intervals is the part that requires specialized methods. Together, these results provide significant progress toward being able to prove the stability of traveling wave solutions of the compressible Navier-Stokes equations with an ideal gas equation of state.

4.1 Future directions

Proving the stability of traveling wave solutions of the one-dimensional compressible Navier-Stokes equations with an ideal gas equation of state is an interesting problem that we plan to pursue. The work of this thesis provides a significant step in that direction.

Appendix A. Appendix

For clarity of presentation, we place some results in this appendix.

A.1 A proposition used extensively in the proofs by induction.

We now provide the proof of Proposition 20.
Proof. Assume the hypothesis of the theorem and let \( a + b \leq n \).

\[
\sum_{k=a}^{n-b} \frac{1}{(k + Q)^2(n + Q - k)^2} = \int_{a}^{n-b+1} \frac{1}{(\lceil k \rceil + Q)^2(n + Q - \lceil k \rceil)^2} dk \\
\leq \int_{a}^{n-b+1} \frac{1}{(k - 1 + Q)^2(n + Q - k)^2} dk \\
\leq \frac{2}{(2Q + n - 1)^3} \int_{a}^{n-b+1} \frac{1}{k - 1 + Q} dk \\
+ \frac{1}{(2Q + n - 1)^2} \int_{a}^{n-b+1} \frac{1}{(k - 1 + Q)^2} dk \\
+ \frac{2}{(2Q + n - 1)^3} \int_{a}^{n-b+1} \frac{1}{n + Q - k} dk \\
+ \frac{1}{(2Q + n - 1)^2} \int_{a}^{n-b+1} \frac{1}{(n + Q - k)^2} dk \\
\leq \frac{2}{(2Q + n - 1)^3} \ln \left( \frac{n - b + Q}{a + Q - 1} \right) \\
+ \frac{1}{(2Q + n - 1)^2} \left( \frac{-1}{n - b + Q} + \frac{1}{a - 1 + Q} \right) \\
+ \frac{2}{(2Q + n - 1)^3} \ln \left( \frac{n - a + Q}{b + Q - 1} \right) \\
+ \frac{1}{(2Q + n - 1)^2} \left( \frac{-1}{n - a + Q} + \frac{1}{b - 1 + Q} \right) \\
\leq \frac{1}{(2Q + n - 1)^2(a - 1 + Q)} \left( 2 \frac{a + Q - 1}{2Q + n - 1} \ln \left( \frac{2Q + n - 1}{a + Q - 1} \right) + 1 \right) \\
+ \frac{1}{(2Q + n - 1)^2(b - 1 + Q)} \left( 2 \frac{b + Q - 1}{2Q + n - 1} \ln \left( \frac{2Q + n - 1}{b + Q - 1} \right) + 1 \right) \\
\leq \frac{1}{(2Q + n - 1)^2(a + Q - 1)} \left( \frac{2}{e} + 1 \right) \\
+ \frac{1}{(2Q + n - 1)^2(b + Q - 1)} \left( \frac{2}{e} + 1 \right) \\
\leq \frac{\kappa_{Q,a,b}}{(2Q + n - 1)^2}.
\]

\(\square\)
Bibliography


