2024-06-18

Zeros of Convex Combinations of Elementary Families of Harmonic Functions

Rebekah Ottinger
Brigham Young University

Zeros of Convex Combinations of Elementary Families of Harmonic Functions

Rebekah Ottinger

A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Science

Jennifer Brooks, Chair
Michael Dorff
Nickolas Andersen

Department of Mathematics

Brigham Young University

ABSTRACT

Zeros of Convex Combinations of Elementary Families of Harmonic Functions

Rebekah Ottinger
Department of Mathematics, BYU
Master of Science

Brilleslyper et al. analyzed a one-parameter family of harmonic trinomials, and Brooks and Lee analyzed a one-parameter family of harmonic functions with poles. Each family was explored to find the relationship between the size of the parameter and the number of zeros of the harmonic function. In this thesis, we examine convex combinations of members of these families. We determine conditions under which the critical curves separating the sense-preserving and sense-reversing regions are circular. We show that the number of zeros of a convex combination can be greater than the maximum number of zeros of either part.

Keywords: complex analysis, complex-valued harmonic function

Acknowledgements

I would like to express my deepest appreciation to Dr. Jennifer Brooks, my advisor, for the endless help and support throughout my coursework and research. Without her and Dr. Michael Dorff supporting me, I would not have been introduced to this fascinating research or created fast and strong bonds with my cohort. Dr. Nickolas Andersen has also provided invaluable support and been an amazing instrutor. I would like to thank my office mates and cohort for being my rocks during difficult coursework and injecting fun throughout the semester. I would also like to thank my roommates for making sure I took care of myself and just being amazing friends. Finally, I would like to thank my parents and siblings for pushing me to do my best and always being willing to provide words of wisdom and tasty food when I needed it.

Contents

List of Figures

Chapter 1. Introduction

The Fundamental Theorem of Algebra states that every analytic polynomial of degree $n \geq 1$ with complex coefficients has n zeros in the complex plane. A complex-valued harmonic polynomial can be represented as an analytic polynomial with some conjugate terms added. While the Fundamental Theorem of Algebra works for all analytic polynomials, it does not directly extend to harmonic polynomials. We consider all zeros of an analytic function to have a positive order whereas zeros of harmonic functions can be sensibly considered to have positive or negative order depending on the region in which they lie due to the addition of the conjugate terms. Therefore, the correct extension of the Fundamental Theorem of Algebra to harmonic polynomials counts the sum of the orders of the zeros, and harmonic polynomials are capable of having a total number of zeros that exceeds their degree.

There is a limit to how much a harmonic polynomial's total number of zeros can exceed its degree. Let f be a harmonic polynomial of degree n. Sheil-Small [6] conjectured and Wilmshurst [7] later proved that f can have at most n^2 zeros with this bound being a strict bound. Further studies have shown the number of zeros is closely related to the degrees of both the non conjugated terms and the conjugated terms. However, there is not a simple relationship between them since the coefficients of the polynomials also influence the number of zeros.

Consider the family studied by Brilleslyper, Brooks, Dorff, Howell, and Schaubroeck

$$p_c(z) = z^m + c\overline{z}^k - 1 \tag{1.1}$$

with $m, k \in \mathbb{N}$, $m > k$, and $c \in \mathbb{C} \setminus \{0\}$.

In Figure 1.1, we see that the number of zeros of $p_c(z) = z^5 + c\overline{z}^4 - 1$ can change depending on the value of c. The smallest number of zeros seems to be 5 and the largest seems to be 13. Although there exist numbers of zeros between 5 and 13 for other values of c, their presence is not pertinent to this thesis. The following is a corollary to the main theorem proved by Brilleslyper et al. about the number of zeros and its relationship to the parameter c:

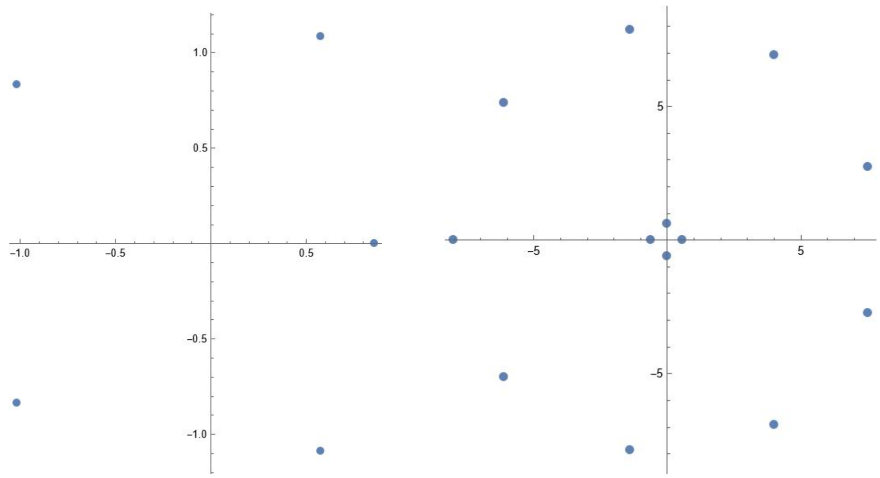

Figure 1.1: $p(z) = z^5 + c\overline{z}^4 - 1$ for $c = 1$ (left) and $c = 8$ (right).

Corollary 1.1 (BBDHS [1])**.** *Let p_c be as above. There exist $c_0, c_1 > 0$ such that*

(i) if $0 \leq |c| < c_0$, then p_c has m distinct zeros, and

(ii) if $|c| > c_1$, then p_c has $m + 2k$ distinct zeros.

In this thesis, we will examine convex combinations of previously studied elementary families of harmonic functions. For analytic polynomials, the number of zeros of a convex combination is the largest degree of its component parts. Since harmonic functions can vary in their number of zeros based on their coefficients, we will focus on when the component parts are obtaining their maximum number of zeros, and ask the question:

Question 1.2. How does the maximum number of zeros of the component parts of a convex combination of harmonic functions relate to the number of zeros of the convex combination?

The first *convex combination* we will consider is a combination of two members from the family (1.1):

$$f_{a,b,s}(z) = s(z^m + a\overline{z}^k - 1) + (1 - s)(z^n + b\overline{z}^\ell - 1). \tag{1.2}$$

where $m > n$, $m > k$, $n > \ell$, $s \in (0, 1)$, and $a, b > 0$.

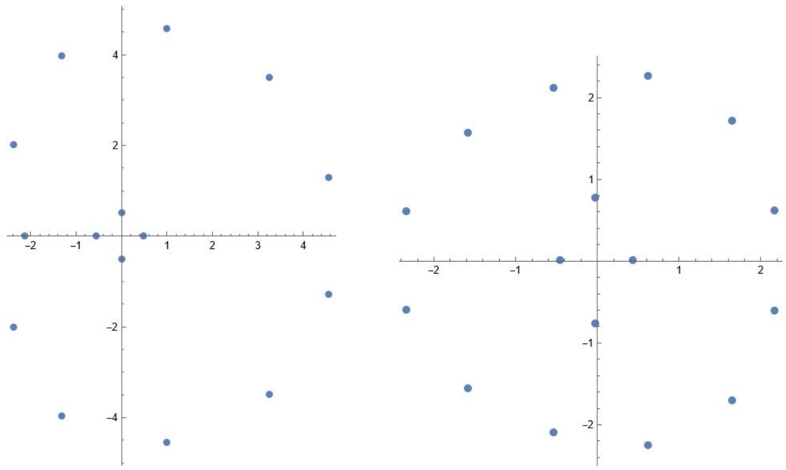

Figure 1.2: $\frac{3}{10}(z^7 + 15\overline{z}^5 - 1) + \frac{7}{10}(z^5 + 20\overline{z}^4 - 1)$ (left) and $\frac{3}{10}(z^8 + 11\overline{z}^2 - 1) + \frac{7}{10}(z^5 + 12\overline{z}^4 - 1)$ (right).

The maximum number of zeros for (1.1) occurs when c is sufficiently large. For the following examples, a and b will be chosen such that the component parts each have their maximum number of zeros.

In the left graph of Figure 1.2, the component parts have 17 and 13 zeros, respectively. The convex combination has 15 zeros, which sits between those two numbers and can give the impression that maybe taking the convex combination will result in an averaging of the two maximums. However, the right graph of Figure 1.2 has component parts with 12 and 13 zeros, respectively, and their convex combination has 16 zeros. This example shows that the convex combination can have more zeros than either component.

In order to better study the convex combination, we will need to understand the methods used in analyzing 1.1 and how they can be applied to the convex combination. There are harmonic analogues of the Argument Principle and Rouche's Theorem that count the sums of the orders of zeros instead of counting the number of zeros like their analytic counterparts. One of the motivations for exploring (1.1) comes from the *critical curves* of this family being circular. The critical curve of a harmonic function defines the boundary between the *sense-preserving* and *sense-reversing* regions. Zeros in the sense-preserving region are considered to have positive order whereas those in the sense-reversing region are considered to have negative order. Therefore, the harmonic Argument Principle or Rouché's Theorem applied

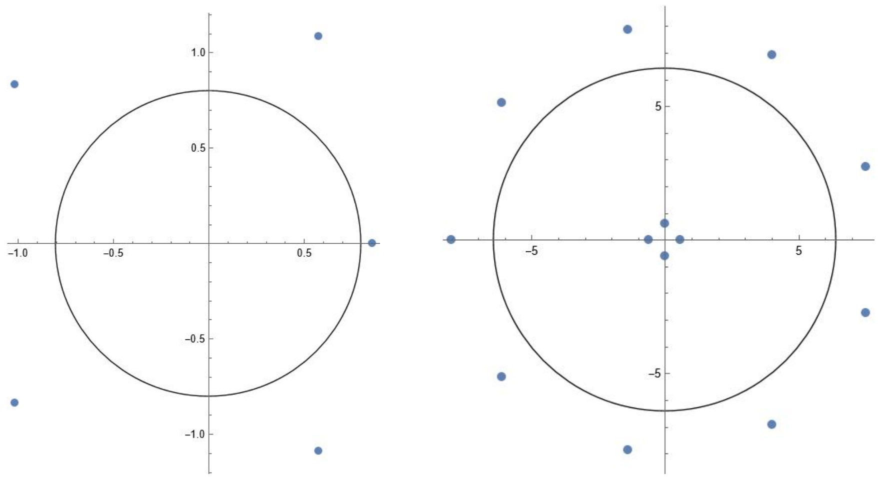

Figure 1.3: $p(z) = z^5 + c\overline{z}^4 - 1$ for $c = 1$ (left) and $c = 8$ (right) with their critical curves.

to a simple critical curve counts all of the zeros in the region within the critical curve. Then, the number of zeros outside the critical curve can be found using the number of zeros in the critical curve and the sum of the order of the zeros in the complex plane as found using the harmonic Argument Principle or Rouché's Theorem, as well. These two numbers sum to the total number of zeros for our harmonic function.

Figure 1.3 shows our examples of the original family from earlier with their circular critical curves. Since the sense-reversing region for the family (1.1) is inside the critical curve, the left graph has 5 zeros of positive order and 0 zeros of negative order. The right graph has 9 zeros of positive order and 4 zeros or negative order.

Now, let us examine the critical curves for the examples of convex combinations. In Figure 1.4, the critical curves are not circles. In fact, the graph on the right there are two small loops disjoint from the much larger components of the critical curve on the imaginary axis near $i/2$ and $-i/2$. These peculiar shapes and their proximity to the zeros makes a Rouché style argument difficult.

There are different ways the convex combination can be restricted so the critical curve is a circle. Chapter 3 provides the proofs of the following theorems about four different restrictions of (1.2) that yield circular critical curves. The second and fourth theorems show that those subfamilies have the same maximum number of zeros as the maximum number of

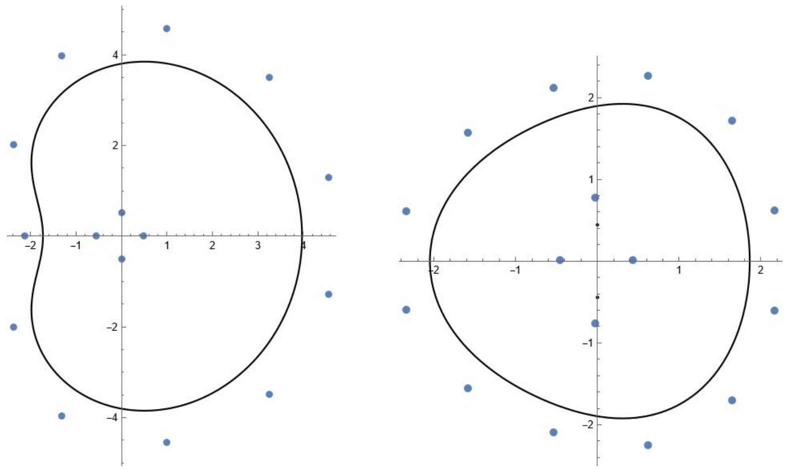

Figure 1.4: $f_{15,20,\frac{3}{10}}(z) = \frac{3}{10}(z^7 + 15\overline{z}^5 - 1) + \frac{7}{10}(z^5 + 20\overline{z}^4 - 1)$ (left) and $f_{11,12,\frac{3}{10}}(z)\frac{3}{10}(z^8 + 11\overline{z}^2 - 1) + \frac{7}{10}(z^5 + 12\overline{z}^4 - 1)$ (right) with their critical curves.

zeros of the parts. However, the first and third theorems show that the convex combination can have more zeros than either part since that is true for these subfamilies.

Theorem 1.3. *Let $f_{a,b,s}$ be as in* (1.2). *Let $m, k, n, \ell \in \mathbb{N}$ satisfy $m + k = n + \ell$ and $m > n > \ell > k$. Let $s = \frac{n}{m+n}$ and $a = \frac{mb\ell}{kn}$. Then, there exists b_0 such that for all $b > b_0$, $f_{a,b,s}$ has $m + 2\ell$ zeros.*

Theorem 1.4. *Let $f_{a,b,s}$ be as in* (1.2). *Let $m, k, n, \ell \in \mathbb{N}$ satisfy $m - k = n - \ell$, $m > n > \ell$, and $m > k > \ell$. Let $s = \frac{n}{m+n}$ and $a = \frac{mb\ell}{kn}$. Then, there exists b_0 such that for all $b > b_0$, $f_{a,b,s}$ has $m + 2k$ zeros.*

A slightly different method of restriction results in a larger family of polynomials that includes the first two without s restricted. While the first two theorems are subsumed by the third and fourth, we will include the first two theorems in this thesis to provide easier reading of the proof method.

Theorem 1.5. *Let $f_{a,b,s}$ be as in* (1.2). *Let $m, k, n, \ell \in \mathbb{N}$ satisfy $m + k = n + \ell$ and $m > n > \ell > k$. Let $a = \frac{(1-s)^2 bn\ell}{s^2 mk}$. Then, there exists b_0 such that for all $b > b_0$, $f_{a,b,s}$ has $m + 2\ell$ zeros.*

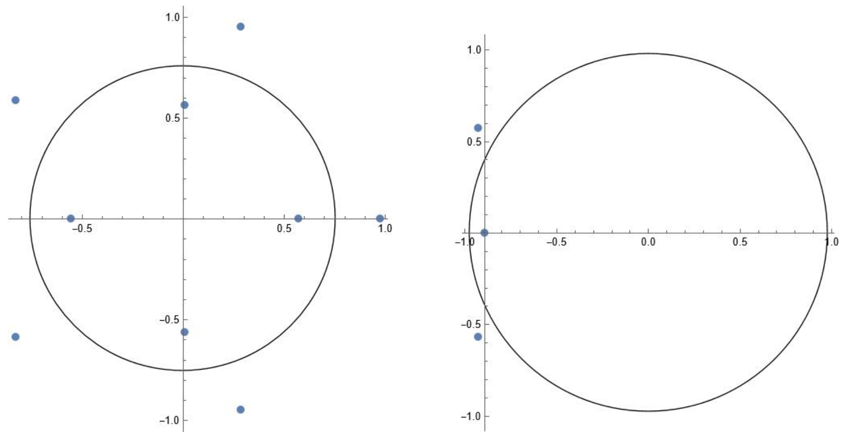

Figure 1.5: $p(z) = z^5 + \frac{c}{\bar{z}^4} - 1$ for $c = 0.1$ (left) and $c = 1$ (right) with their critical curves.

Theorem 1.6. *Let $f_{a,b,s}$ be as in* (1.2)*. Let $m, k, n, \ell \in \mathbb{N}$ satisfy $m - k = n - \ell$, $m > n > \ell$, and $m > k > \ell$. Let $a = \frac{mb\ell}{kn}$. Then, there exists b_0 such that for all $b > b_0$, $f_{a,b,s}$ has $m + 2k$ zeros.*

Brooks and Lee explored a variation of the original polynomial. They constructed a simple family of harmonic functions with poles for which an analysis similar to that of (1.1) applies. They consider

$$r_c(z) = z^m + \frac{c}{\bar{z}^k} - 1 \tag{1.3}$$

where $m > k$, $c \in \mathbb{C} \setminus \{0\}$, and $\gcd(m, k) = 1$.

Using similar methods to Brillesplyer et al., they proved a theorem that has the following corollary:

Corollary 1.7 (Brooks and Lee [3],[5])**.** *Let r_c be as above. There exist $c_0, c_1 > 0$ such that*

(i) if $0 < |c| < c_0$, then r_c has $m + k$ zeros, and

(ii) if $|c| > c_1$, then r_c has $m - k$ zeros.

This harmonic function also has a circular critical curve. Figure 1.5 shows examples of this function for c values for which it achieves its maximum and minimum number of zeros.

Consider a convex combination between (1.1) and (1.3)

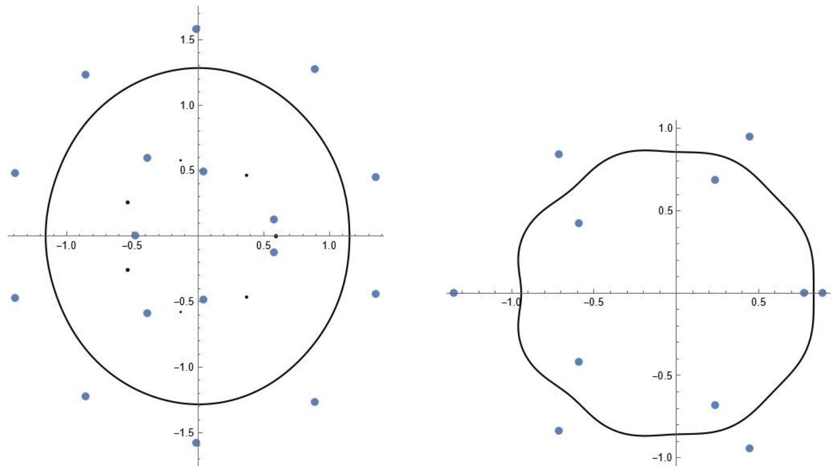

Figure 1.6: $r_{12,\frac{1}{10},\frac{3}{10}}(z) = \frac{3}{10}(z^5 + 12\overline{z}^3 - 1) + \frac{7}{10}(z^7 + \frac{1}{10\overline{z}^4} - 1)$ (left) and $r_{\frac{4}{9},\frac{1}{3},\frac{3}{7}}(z) = \frac{3}{7}(z^5 + \frac{4}{9}\overline{z}^3 - 1) + \frac{4}{7}(z^6 + \frac{1}{3\overline{z}^5} - 1)$ (right) with their critical curves.

$$r_{a,b,s}(z) = s(z^m + a\overline{z}^k - 1) + (1 - s)\left(z^n + \frac{b}{\overline{z}^\ell} - 1\right). \tag{1.4}$$

with $m > k$, $n > \ell$, $s \in (0, 1)$, and $a, b \in \mathbb{R} \setminus \{0\}$.

Once again, we want to know how the number of zeros of the component parts relates to the number of zeros of the convex combinations. However, unlike with the previous convex combination, (1.1) and (1.3) achieve their maximum and minimum number of zeros for different values of c. For sufficiently large c, (1.1) has its maximum of $m + 2k$ zeros, and (1.3) has its minimum of $n - \ell$ zeros. For sufficiently small c, (1.1) has its minimum of m zeros, and (1.3) has its maximum of $n + \ell$ zeros. As a result, we examine both sufficiently large and small values of a and b.

Figure 1.6 demonstrates that (1.4) can have complicated critical curves as well. The left figure has small pieces of the critical curve that are disjoint from the rest, and the right figure has a critical curve much too bumpy to be a circle. This figure also demonstrates that this convex combination can have more zeros than the maximum of either component since the left convex combination has 17 zeros while its components both have a maximum of 11.

As with the other convex combination, we want the critical curve to be circular, but it is not always circular. So, we restrict the parameters to ensure that we get a circular critical curve using the same method as earlier. We can restrict four ways as before with (1.2) to make four families with two of them subsumed by the others. However, we are not going to include the smaller families for this convex combination So, Chapter 4 will be spent proving the following two theorems that come from the less restrictive families with circular critical curves:

Theorem 1.8. *Let $r_{a,b,s}$ be as in (1.4). Let $m, k, n, \ell \in \mathbb{N}$ satisfy $m - k = n + \ell$, $m > k$, and $n > \ell$. Let $a = \frac{-mb\ell}{kn}$. Then, there exist b_0 and b_1 such that*

(1) for all $0 < |b| < b_0$, $r_{a,b,s}$ has $m + \ell$ zeros,

(2) for all $|b| > b_1$, $r_{a,b,s}$ has $m + \ell + 2k$ zeros.

Theorem 1.9. *Let $r_{a,b,s}$ be as in (1.4). Let $m, k, n, \ell \in \mathbb{N}$ satisfy $m + k = n - \ell$, $m > k$, and $n > \ell$. Let $a = \frac{-nb\ell(1-s)^2}{mks^2}$. Then, there exist b_0 and b_1 such that*

(1) for all $0 < |b| < b_0$, $r_{a,b,s}$ has $n + \ell$ zeros,

(2) for all $|b| > b_1$, $r_{a,b,s}$ has $n + \ell + 2k$ zeros.

Theorem 1 and its proof have already been published [2]. This paper also covers a theorem about a convex combination of (1.3) with itself under the same type of restrictions present in Theorem 1 which was proven by Lee and is therefore not in this thesis.

Chapter 2. Background and Definitions of the Subfamilies

In this chapter, we provide the definitions and theorems concerning complex-valued harmonic functions we need to analyze these convex combinations. We also restrict the convex combinations into subfamilies that are more easily analyzed using Rouché's Theorem for harmonic functions. The first section defines harmonic functions and their zeros with the second section building on that with a discussion about what happens when poles are added.

The third and fourth sections give the process used to restrict both convex combinations into their subfamilies and why these restrictions are easier to analyze using Rouché's Theorem for harmonic function.

2.1 Functions

In this thesis, we study complex-valued harmonic functions and their zeros. For a more detailed study, look at Duren's book [4].

First, we define what a complex-valued harmonic function is before we define the tools we use to analyze them.

Definition 2.1 (Harmonic Function)**.** A function $\phi : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is *harmonic* if it is twice continuously differentiable and satisfies Laplace's equation $\phi_{xx} + \phi_{yy} = 0$.

Definition 2.2 (Complex-valued Harmonic Function)**.** A function $f = u + iv : D \subseteq \mathbb{C} \to \mathbb{C}$ is a *complex-valued harmonic* function if both u and v are harmonic in D.

Any function written in the from $f = h + \overline{g}$ is complex-valued harmonic if h and g are analytic functions. In this form, we refer to h as the *analytic part* of f and $\overline{g}$ as the *co-analytic part* of f.

A function is *sense-preserving* on a domain D if the image of a simple, closed, positively oriented contour C in D whose interior lies entirely in D remains positively oriented. Similarly, a function is *sense-reversing* on a domain D if the image of such a contour C in D is negatively oriented. An analytic function is sense-preserving and a co-analytic function is sense-reversing. Since harmonic functions have both an analytic and a co-analytic part, they have both sense-preserving and sense-reversing regions. We can determine where these regions are using the complex dilatation.

Definition 2.3 (Complex Dilatation)**.** The *complex dilatation* of $f = h + \overline{g}$ is defined by

$$\omega(z) = \frac{g'(z)}{h'(z)}.$$

The complex dilatation gives us a way to measure whether the analytic part or co-analytic part of a harmonic function is dominant in a region. If the analytic part is dominant in a region, then the harmonic function is sense-preserving in that region. Similarly, if the co-analytic part of a harmonic function is dominant in a region, then the harmonic function is sense-reversing in that region. This relationship between the dilatation function and the sense-preserving and sense-reversing regions is demonstrated in the following proposition.

Proposition 2.4. *Let $f = h + \overline{g}$ be a harmonic function on D, with complex dilatation ω.*

- *If $|\omega(z)| < 1$ throughout some region $R \subseteq D$, then f is sense-preserving on R.*
- *If $|\omega(z)| > 1$ on $R \subseteq D$, then f is sense-reversing on R.*

Using these relations, we can define the boundary between the sense-preserving and sense-reversing regions.

Definition 2.5 (Critical Curve)**.** The *critical curve* of $f = h + \overline{g}$ is defined by

$$\{z \in \mathbb{C} : |\omega(z)| = 1\}.$$

Understanding where the sense-preserving and sense-reversing regions reside is important because they impact how zeros are counted using the methods present in the Argument Principle or Rouché's Theorem. For analytic functions, we define the order of the zero z_0 in terms of the lowest-order term in the Taylor expansion about z_0. For complex-valued harmonic functions, we can do the same thing except we need to expand the analytic and co-analytic parts separately to see whether this lowest order term is part of the analytic or co-analytic component of the function.

Definition 2.6 (Order of a Zero)**.** Let $f = h + \overline{g}$ be a complex-valued harmonic function and suppose $f(z_0) = 0$. Write

$$f(z) = a_0 + \sum_{j=r}^{\infty} a_j (z - z_0)^j + \overline{b_0 + \sum_{j=s}^{\infty} b_j (z - z_0)^j}.$$

If z_0 is in a sense-preserving region, we define the *order* of z_0 to be r and if z_0 is in a sense-reversing region, we define its order to be $-s$. If z_0 is on the critical curve, we call it a *singular zero* and its order is not defined.

Now, with this definition, we can describe how the Argument Principle and Rouché's Theorem extend to harmonic polynomials. The Argument Principle works the same in sense-preserving regions as it does for analytic polynomials. However, the Argument Principle yields a negative sign in sense-reversing regions due to the contours changing direction. Since we defined zeros in sense-reversing regions to have negative order, we are able to say that the Argument Principle for harmonic functions works the same as the analytic Argument Principle with two simple changes. It produces the sum of the orders of the zeros instead of the number of zeros, and there cannot be any singular zeros since their order is not defined.

The harmonic analogue to Rouché's Theorem is similar, and it is the theorem through which we will analyze the convex combinations in this thesis. For this theorem and the rest of the thesis, we will denote the sum of the orders of the zeros of p in C by $Z_{p,C}$.

Theorem 2.7 (Rouché's Theorem for Harmonic Functions)**.** *Let p and q be harmonic in a simply connected domain $D \subseteq \mathbb{C}$. Let C be a simple, closed curve contained in D. If $|p(z)| > |q(z)|$ at each point on C, and if p and q have no singular zeros in C, then $Z_{p,C} = Z_{p+q,C}$.*

Combining Rouché's Theorem with the critical curve gives a process to count the total number of zeros for a function with a simple critical curve. First, applying Rouché's Theorem to circles of sufficiently large radius gives the sum of the orders of the zeros in the plane. Then, applying Rouché's Theorem to the critical curve gives the sum of the orders of the zeros in the enclosed region. Since the orders of the zeros inside of the critical curve all have the same sign, the sum of the orders of the zeros is the same as the count of the zeros inside that region. Finally, with these two pieces of information, we can solve for the number of zeros outside of the critical curve which combined with the number of zeros inside gives us the total number of zeros.

2.2 Harmonic Functions with poles

Define z_0 to be a pole of f provided $\lim_{z\to z_0}|f(z)| = \infty$. The second convex combination (1.4) contains a $\frac{1}{\bar{z}^k}$ term which introduces a pole to the situation. Therefore, we need to consider the impact that adding a pole has on a harmonic function.

Conveniently, the definitions of sense-preserving and sense-reversing regions, dilatation function, and critical curve extend to harmonic functions with poles without any changes. Similar to how the sense-preserving and sense-reversing regions impact the way we think about the order of a zero for complex-valued harmonic polynomials, these regions also impact the way we think about the order of poles for complex-valued harmonic functions with poles. For meromorphic functions, we define the order of a pole z_0 in terms of the lowest-order term in the Laurent expansion about z_0. For complex-valued harmonic functions, we need to do the same thing as we did with the zeros and expand the analytic and co-analytic parts separately.

Definition 2.8. Let f be a harmonic function on a domain $D \subseteq \mathbb{C}$ except at a finite number of poles. Suppose that the local representation of f around a pole $z_0 \in D$ is

$$f(z) = \sum_{j=-r}^{\infty} a_j(z-z_0)^j + \overline{\sum_{j=-s}^{\infty} b_j(z-z_0)^j} + 2A\log|z-z_0|,$$

for some constant A, and where r and s are finite.

- If $a_{-r} \neq 0$ for some $r > 0$ and $r > s$, or $r = s$ with $|a_{-r}| > |b_{-s}|$, then f is sense-preserving near z_0 and f has a pole at z_0 of order r.

- If $b_{-s} \neq 0$ for some $s > 0$ and $r < s$, or $r = s$ with $|a_{-r}| < |b_{-s}|$, then f is sense-reversing near z_0 and f has a pole at z_0 of order $-s$.

This definition for the order of a pole enables us to write the Argument Principle and Rouché's Theorem for harmonic functions with poles using the sum of the orders of zeros minus the sum of the orders of the poles instead of the analytic analogue where we have the number of zeros minus the number of poles. It also includes a restriction that no zeros or poles can be on the critical curve since the order is not defined for those zeros or poles. Similarly to how we are denoting the sum of the orders of the zeros of p in C as $Z_{p,c}$ we will denote the sum of the orders of the poles of p in C as $P_{p,C}$ for the rest of this thesis.

Theorem 2.9 (Rouché's Theorem for Harmonic Functions with Poles)**.** *Let p and q be harmonic, except for a finite number of poles, in a simply connected domain $D \subseteq \mathbb{C}$. Let C be a simple, closed curve contained in D. If $|p(z)| > |q(z)|$ at each point on C, and if p and q have no poles on C and no singular zeros in C, then $Z_{p,C} - P_{p,C} = Z_{p+q,C} - P_{p+q,C}$.*

While the inclusion of a pole makes a Rouché style argument more difficult, we only have a single pole and it is located at the origin. So, the sums of the orders of the poles for our convex combination with a pole or any of its parts are found without hassle allowing us to focus on identifying the number of zeros.

2.3 Building the Restrictions on the First Convex Combination

The first convex combination we are looking at is

$$f_{a,b,s}(z) = s(z^m + a\overline{z}^k - 1) + (1-s)(z^n + b\overline{z}^\ell - 1). \tag{1.2}$$

where $a, b > 0$, $m > n$, $n > \ell$, and $m > k$. We want to restrict this convex combination in ways that guarantee a circular critical curve. The dilatation function of $f_{a,b,s}$ is

$$\omega_f(z) = \frac{ksaz^{k-1} + \ell(1-s)bz^{\ell-1}}{msz^{m-1} + n(1-s)z^{n-1}}. \tag{2.1}$$

If $\omega_f(z) = Cz^D$, then the critical curve will be a circle. We discuss two methods that restrict the dilatation function into this form. The first method consists of factoring the dilatation function to look like

$$Cz^D\left(\frac{A + z^B}{1 + Az^B}\right). \tag{2.2}$$

The hope was that since $\frac{z^B+A}{1+Az^B}$ is a composition of a Möbius transformation and a power function, this type of restriction would be enough to yield a critical curve of a circle. However,

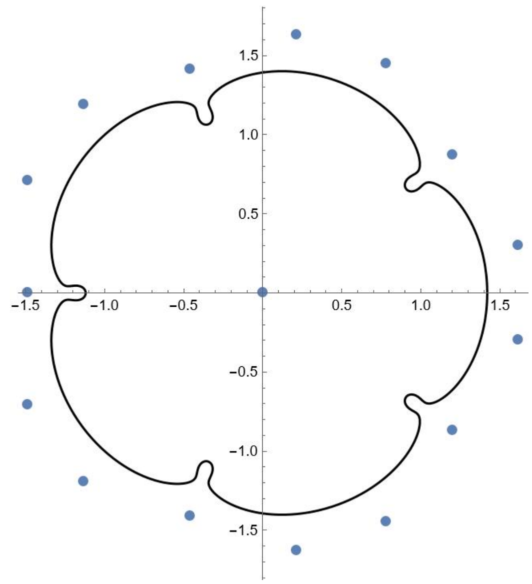

Figure 2.1: $f_{\frac{1120}{3},40,\frac{9}{16}}(z) = \frac{9}{16}(z^{14} + \frac{1120}{3}\overline{z} - 1) + \frac{7}{16}(z^9 + 40\overline{z}^6 - 1)$

$C \neq 1$ prevents a circular critical curve if $|A| \neq 1$, as seen in the example in figure 2.1 where $A = 2$. Therefore, we require $|A| = 1$. Given that A in the denominator for our convex combination will always be a positive real number, $|A| = 1$ fixes $A = 1$.

The second method factors the dilatation function slightly differently. So, it looks like

$$Cz^D \left(\frac{1 + Az^B}{1 + Az^B} \right). \tag{2.3}$$

This factorization gives A the freedom to be any number, which decreases the number of restrictions required for a circular critical curve.

2.3.1 Subfamily with $\ell > k$ and s Fixed. For our first restriction, we factor z^{k-1} from the numerator and z^{n-1} from the denominator of (2.1) and rearrange the coefficients to match (2.2) as follows:

$$\begin{aligned} w_f(z) &= \frac{ksaz^{k-1} + \ell(1-s)bz^{\ell-1}}{msz^{m-1} + n(1-s)z^{n-1}} \\ &= \frac{z^{k-1}}{z^{n-1}} \left(\frac{ksa + \ell(1-s)bz^{\ell-k}}{msz^{m-n} + n(1-s)} \right) \\ &= \left(\frac{b\ell}{n} \right) z^{k-n} \left(\frac{\frac{ksa}{b\ell(1-s)} + z^{\ell-k}}{\frac{ms}{n(1-s)} z^{m-n} + 1} \right). \end{aligned}$$

In order for the fraction on the right to simplify to 1, we add the following conditions:

- $\ell - k = m - n$
- $A = \frac{ksa}{b\ell(1-s)} = \frac{ms}{n(1-s)}$ which implies that $\frac{ka}{b\ell} = \frac{m}{n}$
- $|A| = 1$ which gives us that $\frac{ms}{n(1-s)} = 1$, and so $s = \frac{n}{m+n}$

Combining $\ell - k = m - n$ with $m > n$, $n > \ell$, and $m > k$ gives us that $m > n > \ell > k$. We can also rewrite the dilatation as

$$\omega_f(z) = \frac{b\ell}{n} \frac{1}{z^{n-k}}.$$

Proposition 2.10. *Let $f_{a,b,s}$ be as in* (1.2)*. Let $m, k, n, \ell \in \mathbb{N}$ satisfy $m + k = n + \ell$ and $m > n > \ell > k$. Let $s = \frac{n}{m+n}$ and $a = \frac{mb\ell}{kn}$. Then*

- *The critical curve is a circle centered at the origin with radius $\left(\frac{b\ell}{n}\right)^{\frac{1}{n-k}}$.*
- *The region inside the critical curve is sense-reversing.*
- *The region outside the critical curve is sense-preserving.*

2.3.2 Subfamily with $k > \ell$ and s Fixed. For our second restriction, we factor $z^{\ell-1}$ from the numerator and z^{n-1} from the denominator of (2.1) and rearrange the coefficients to match (2.2) as follows:

$$
\begin{aligned}
w_f(z) &= \frac{ksaz^{k-1} + b\ell(1-s)z^{\ell-1}}{msz^{m-1} + n(1-s)z^{n-1}} \\
&= \frac{z^{\ell-1}}{z^{n-1}} \left(\frac{ksaz^{k-\ell} + b\ell(1-s)}{msz^{m-n} + n(1-s)} \right) \\
&= \left(\frac{ksa}{n(1-s)} \right) z^{\ell-n} \left(\frac{z^{k-\ell} + b\ell(1-s)/(ksa)}{ms/(n(1-s))z^{m-n} + 1} \right).
\end{aligned}
$$

In order for the fraction on the right to simplify to 1, we add the following conditions:

- $k - \ell = m - n$
- $A = \frac{\ell b(1-s)}{ksa} = \frac{ms}{n(1-s)}$
- $|A| = 1$ which gives us that $\frac{ms}{n(1-s)} = 1$, and so $s = \frac{n}{m+n}$.

Combining $k - \ell = m - n$ with $m > n$, $n > \ell$, and $m > k$ gives us that $k > \ell$, and combining the second and third conditions gives that $a = \frac{mb\ell}{kn}$. We can also rewrite the dilatation as

$$\omega_f(z) = \frac{b\ell}{n} \frac{1}{z^{n-\ell}}.$$

Proposition 2.11. *Let $f_{a,b,s}$ be as in* (1.2). *Let $m, k, n, \ell \in \mathbb{N}$ satisfy $m - k = n - \ell$, $m > n > \ell$, and $m > k > \ell$. Let $s = \frac{n}{m+n}$ and $a = \frac{mb\ell}{kn}$. Then*

- *The critical curve is a circle centered at the origin with radius $\left(\frac{b\ell}{n}\right)^{\frac{1}{n-\ell}}$.*
- *The region inside the critical curve is sense-reversing.*
- *The region outside the critical curve is sense-preserving.*

2.3.3 Subfamily with $\ell > k$ and s Unrestricted. For our third restriction, we factor z^{k-1} from the numerator and z^{n-1} from the denominator of (2.1) and rearrange the coefficients to match (2.3) as follows:

$$\begin{aligned}
w_f(z) &= \frac{sakz^{k-1} + \ell(1-s)bz^{\ell-1}}{smz^{m-1} + (1-s)nz^{n-1}} \\
&= \frac{z^{k-1}}{z^{n-1}} \left(\frac{ksa + l(1-s)bz^{\ell-k}}{smz^{m-n} + (1-s)n} \right) \\
&= \left(\frac{sak}{(1-s)n} \right) z^{k-n} \left(\frac{1 + \frac{(1-s)b\ell}{sak} z^{\ell-k}}{\frac{sm}{(1-s)n} z^{m-n} + 1} \right).
\end{aligned}$$

In order for the fraction on the right to simplify to 1, we add the following conditions:

- $\ell - k = m - n$
- $A = \frac{(1-s)b\ell}{sak} = \frac{sm}{(1-s)n}$.

Unlike the conditions for the first subfamily, these conditions do not impose a restriction on s. Since the other conditions are the same, this subfamily contains the first subfamily, and preserves the consequences of the first two conditions like $m > n > \ell > k$. We can rewrite the dilatation as

$$\omega_f(z) = \frac{(1-s)b\ell}{sm} \frac{1}{z^{n-k}}.$$

Proposition 2.12. *Let $f_{a,b,s}$ be as in (1.2). Let $m, k, n, \ell \in \mathbb{N}$ satisfy $m + k = n + \ell$ and $m > n > \ell > k$. Let $a = \frac{(1-s)^2 bn\ell}{s^2 mk}$. Then*

- *The critical curve is a circle centered at the origin with radius $\left(\frac{(1-s)b\ell}{sm}\right)^{\frac{1}{n-k}}$.*
- *The region inside the critical curve is sense-reversing.*
- *The region outside the critical curve is sense-preserving.*

2.3.4 Subfamily with $k > \ell$ and s Unrestricted. For our fourth restriction, we factor $z^{\ell-1}$ from the numerator and z^{n-1} from the denominator of (2.1) and rearrange the coefficients to match (2.3) as follows:

$$\begin{aligned} w_f(z) &= \frac{ksaz^{k-1} + b\ell(1-s)z^{\ell-1}}{msz^{m-1} + n(1-s)z^{n-1}} \\ &= \frac{(1-s)b\ell z^{\ell-1}}{(1-s)nz^{n-1}} \left(\frac{\frac{ksa}{(1-s)b\ell} z^{k-\ell} + 1}{\frac{ms}{(1-s)n} z^{m-n} + 1} \right) \\ &= \frac{b\ell}{n} z^{\ell-n} \left(\frac{\frac{ksa}{(1-s)b\ell} z^{k-\ell} + 1}{\frac{ms}{(1-s)n} z^{m-n} + 1} \right). \end{aligned}$$

In order for the fraction on the right to simplify to 1, we add the following conditions:

- $k - \ell = m - n$
- $A = \frac{sak}{(1-s)b\ell} = \frac{sm}{(1-s)n}$.

Unlike the conditions for the second subfamily, these conditions does not impose a restriction on s. Since the other conditions are the same, this subfamily contains the second subfamily, and it preserves the consequences of the first two conditions like $k > \ell$. We can also rewrite the dilatation as

$$\omega_f(z) = \frac{b\ell}{n} \frac{1}{z^{n-\ell}}.$$

Proposition 2.13. *Let $f_{a,b,s}$ be as in (1.2). Let $m, k, n, \ell \in \mathbb{N}$ satisfy $m - k = n - \ell$, $m > n > \ell$, and $m > k > \ell$. Let $a = \frac{mb\ell}{kn}$. Then*

- *The critical curve is a circle centered at the origin with radius $\left(\frac{b\ell}{n}\right)^{\frac{1}{n-\ell}}$.*
- *The region inside the critical curve is sense-reversing.*
- *The region outside the critical curve is sense-preserving.*

2.4 Building Restrictions on Second Convex Combination

The second convex combination we are looking at is

$$r_{a,b,s}(z) = s(z^m + a\overline{z}^k - 1) + (1-s)\left(z^n + \frac{b}{\overline{z}^\ell} - 1\right). \tag{1.4}$$

with $m > k$, $n > \ell$, $s \in (0, 1)$, and $a, b \in \mathbb{R}$.

Similar to the first convex combination, we want a circular critical curve. The dilatation of (1.4) is

$$\omega_r(z) = \frac{sakz^{k-1} - (1-s)b\ell z^{-\ell-1}}{smz^{m-1} + (1-s)nz^{n-1}}. \tag{2.4}$$

We can restrict this dilatation so that the critical curve is a circle using either (2.2) or (2.3). However, much like the first combination, the subfamilies that result from restrictions using (2.2) are subsumed by subfamilies created from restrictions using (2.3). This time we are going to exclude the smaller families and focus on only the larger families.

2.4.1 Subfamily with $m > n$. By factoring the z^{k-1} term out of the numerator and the z^{m-1} term out of the denominator, we get that

$$\omega_r(z) = \frac{ak}{m} z^{k-m} \left(\frac{1 - \frac{(1-s)b\ell}{sak} z^{-\ell-k}}{1 + \frac{(1-s)n}{sm} z^{n-m}} \right). \tag{2.5}$$

Therefore, restrictions that yield a circular critical curve are

- $n - m = -(\ell + k)$
- $\frac{-b\ell}{ak} = \frac{n}{m}$.

Combining these restrictions with $m, n, \ell, k \in \mathbb{N}$, we know that $m > n$ and a and b have opposite signs. We also get

$$\omega_r(z) = \frac{b\ell}{n} \frac{1}{z^{m-k}}. \tag{2.6}$$

Proposition 2.14. *Let $r_{a,b,s}$ be as in (1.4). Let $m, k, n, \ell \in \mathbb{N}$ satisfy $m - k = n + \ell$, $m > k$, and $n > \ell$. Let $a = \frac{-mb\ell}{kn}$. Then*

- *The critical curve is a circle centered at the origin with radius $\left|\frac{b\ell}{n}\right|^{\frac{1}{m-k}}$.*
- *The region inside the critical curve is sense-reversing.*
- *The region outside the critical curve is sense-preserving.*

2.4.2 Subfamily with $n < m$.

By factoring the z^{k-1} term out of the numerator and the z^{n-1} term out of the denominator, we get that

$$\omega_r(z) = \frac{sak}{(1-s)n} z^{k-n} \frac{1 - \frac{(1-s)b\ell}{sak} z^{-\ell-k}}{1 + \frac{sm}{(1-s)n} z^{m-n}}. \tag{2.7}$$

Therefore, restrictions that yield a circular critical curve are

- $m - n = -(\ell + k)$
- $\frac{-(1-s)b\ell}{sak} = \frac{sm}{(1-s)n}$.

Combining these restrictions with $m, n, \ell, k \in \mathbb{N}$, we know that $n < m$ and a and b have opposite signs. We also get

$$\omega_r(z) = \frac{(1-s)b\ell}{sm} \frac{1}{z^{n-k}}. \tag{2.8}$$

Proposition 2.15. *Let $r_{a,b,s}$ be as in* (1.4). *Let $m, k, n, \ell \in \mathbb{N}$ satisfy $m + k = n - \ell$, $m > k$, and $n > \ell$. Let $a = \frac{-(1-s)^2 nb\ell}{s^2 mk}$. Then*

- *The critical curve is a circle centered at the origin with radius $\left|\frac{(1-s)b\ell}{sm}\right|^{\frac{1}{n-k}}$.*
- *The region inside the critical curve is sense-reversing.*
- *The region outside the critical curve is sense-preserving.*

Chapter 3. Proofs of Theorems 1.3, 1.4, 1.5, and 1.6

In this chapter, we prove zero-counting theorems for the subfamilies defined in Section 2.3 for sufficiently large b. For each of these proofs, we calculate the number of zeros inside of the critical curve and outside of the critical curve for each restriction using the sum of the orders of the zeros for each in the complex plane and inside of the critical curve. Since the sum of the orders of the zeros in the complex plane is dependent on the powers of the polynomial and not the coefficients, we find the sum of the orders of the zeros in the complex plane for the unrestricted convex combination.

Lemma 3.1. *The sum of the orders of the zeros of $f_{a,b,s}$ in $\mathbb{C}$ is m.*

Proof. Let C_R be a circle of radius R. We prove this lemma by showing that $Z_{f,C_R} = m$ for sufficiently large radius R. Since m is the largest power, the term $p(z) = sz^m$ is dominant on any sufficiently large circle, so by Rouché's Theorem, $Z_{f,C_R} = Z_{p,C_R} = m$. The details for showing that $p(z)$ is dominant for C_R with R sufficiently large are below.

If $|z| > 1$, then because $a, b, s, (1-s) > 0$, and because the natural number m exceeds n, k, and ℓ,

$$\begin{aligned}
|f_{a,b,s}(z) - p(z)| =& |sa\overline{z}^k + (1-s)z^n + (1-s)b\overline{z}^\ell - 1| \\
\leq& sa|z|^k + (1-s)|z|^n + (1-s)b|z|^\ell + 1 \\
<& sa|z|^{m-1} + (1-s)|z|^{m-1} + (1-s)b|z|^{m-1} + |z|^{m-1} \\
=& (sa + (1-s) + (1-s)b + 1)|z|^{m-1} \\
=& D|z|^{m-1}
\end{aligned}$$

where $D = sa + (1-s) + (1-s)b + 1$. If also $|z| \geq \frac{D}{s}$, then

$$|f_{a,b,s}(z) - p(z)| < D|z|^{m-1} \leq (s|z|)|z|^{m-1} = |sz^m| = |p(z)|.$$

It is easy to check that, because $0 < s < 1$, $\frac{D}{s} > 1$. Thus, if $R \geq \frac{D}{s}$, then for all $z \in \mathbb{C}$ with $|z| = R$

$$|f_{a,b,s}(z) - p(z)| < |p(z)|.$$

By Rouché's Theorem, $Z_{f,C_R} = Z_{p,C_R}$. Since p has a zero of order m at the origin, $Z_{f,C_R} = m$ for all sufficiently large R. □

With this lemma, we know the sum of the orders of the zeros for each subfamily, and each section focuses on the sum of the orders of the zeros inside the critical curve for each subfamily which must be handled case by case. Since the sum of the orders of the zeros inside the critical curve is just the sum of all the negatively ordered zeros, the total number of zeros is the sum of the orders of the zeros in the plane plus two times the number of negatively ordered zeros.

Proposition 3.2. *For $f_{a,b,s}$ and hence its subfamilies, if S is the number of zeros inside the critical curve, then the total number of zeros in the plane is $m + 2S$.*

3.1 Proof of Theorem 1.3

In this section, we prove the number of zeros the subfamily in Section 2.3.1 has for arbitrarily large b. In Section 2.3.1, we restricted $f_{a,b,s}$ by requiring

- $m + k = n + \ell$,
- $m > n > \ell > k$,
- $s = \frac{n}{m+n}$, and
- $a = \frac{mb\ell}{kn}$.

Since these restrictions fix s and make a a function of b, we will refer to $f_{a,b,s}$ with these restrictions as f_b for the rest of this section. Proposition 2.10 also gives us the critical curve

$$|z| = \left(\frac{b\ell}{n}\right)^{\frac{1}{n-k}}$$

for f_b with the sense-reversing region inside the critical curve and the sense-preserving region outside of the critical curve. We will denote the critical curve of f_b as Γ_b for the rest of the section.

Now, we find Z_{f_b,Γ_b}.

Lemma 3.3. *For sufficiently large b, $Z_{f_b,\Gamma_b} = -\ell$.*

Proof. We prove this lemma using Rouché's Theorem. We show that if $p(z) = (1 - s)b\bar{z}^\ell$, then $Z_{f_b,\Gamma_b} = Z_{p,\Gamma_b}$; that is, we show

$$|sz^m + sa\bar{z}^k + (1 - s)z^n - 1| < |(1 - s)b\bar{z}^\ell| \tag{3.1}$$

for all $z \in \Gamma_b$. Equation (3.1) will follow if we show

$$(1 - s)b|z|^\ell - (s|z|^m + sa|z|^k + (1 - s)|z|^n + 1) > 0 \tag{3.2}$$

for all $z \in \Gamma_b$. Since points on the critical curve satisfy $|z| = \left(\frac{b\ell}{n}\right)^{\frac{1}{n-k}}$, (3.2) is equivalent to

$$(1-s)b\left(\frac{b\ell}{n}\right)^{\frac{\ell}{n-k}} - s\left(\frac{b\ell}{n}\right)^{\frac{m}{n-k}} - sa\left(\frac{b\ell}{n}\right)^{\frac{k}{n-k}} - (1-s)\left(\frac{b\ell}{n}\right)^{\frac{n}{n-k}} - 1 > 0.$$

By factoring out $(1-s)b\left(\frac{b\ell}{n}\right)^{\frac{\ell}{n-k}}$ and using $m-\ell = n-k$ and $\frac{s\ell}{(1-s)n} = \frac{\ell}{m}$ to simplify the second term, the left-hand side of (3.2) becomes

$$(1-s)b\left(\frac{b\ell}{n}\right)^{\frac{\ell}{n-k}}\left[1 - \frac{\ell}{m} - \frac{sa}{(1-s)b}\left(\frac{b\ell}{n}\right)^{\frac{k-\ell}{n-k}} - \frac{1}{b}\left(\frac{b\ell}{n}\right)^{\frac{n-\ell}{n-k}} - \frac{1}{(1-s)b}\left(\frac{b\ell}{n}\right)^{\frac{-\ell}{n-k}}\right].$$

The third term can also be simplified using $\frac{sa}{(1-s)b} = \frac{\ell}{k}$, giving

$$(1-s)b\left(\frac{b\ell}{n}\right)^{\frac{\ell}{n-k}}\left[1 - \frac{\ell}{m} - \frac{\ell}{k}\left(\frac{b\ell}{n}\right)^{\frac{k-\ell}{n-k}} - \frac{1}{b}\left(\frac{b\ell}{n}\right)^{\frac{n-\ell}{n-k}} - \frac{1}{(1-s)b}\left(\frac{b\ell}{n}\right)^{\frac{-\ell}{n-k}}\right].$$

This expression is positive if and only if the expression in square brackets is positive. Due to the relation between m, n, ℓ, and k, we find that the total exponent on b in each term is negative. Indeed,

$$\frac{-\ell}{n-k} - 1 < \frac{k-\ell}{n-k} = \frac{n-\ell}{n-k} - 1 < 0.$$

Therefore, each term involving b tends to 0 as b tends to infinity. Thus, there exists b_0 such that for all $b > b_0$,

$$\frac{\ell}{k}\left(\frac{b\ell}{n}\right)^{\frac{k-\ell}{n-k}} + \frac{1}{b}\left(\frac{b\ell}{n}\right)^{\frac{n-\ell}{n-k}} + \frac{1}{(1-s)b}\left(\frac{b\ell}{n}\right)^{\frac{-\ell}{n-k}} < 1 - \frac{\ell}{m}.$$

Therefore, for such b,

$$0 < 1 - \frac{\ell}{m} - \frac{\ell}{k}\left(\frac{b\ell}{n}\right)^{\frac{k-\ell}{n-k}} - \frac{1}{b}\left(\frac{b\ell}{n}\right)^{\frac{n-\ell}{n-k}} - \frac{1}{(1-s)b}\left(\frac{b\ell}{n}\right)^{\frac{-\ell}{n-k}}.$$

It follows that p is the dominant term of f_b on the critical curve, and by Rouché's Theorem, $Z_{f_b,\Gamma_b} = Z_{p,\Gamma_b}$ for sufficiently large b. Since p only has a zero of order $-\ell$ located at the origin, $Z_{f_b,\Gamma_b} = -\ell$ for sufficiently large b. □

Now, we have everything needed to prove Theorem 1.3.

Proof of Theorem 1.3. By Lemma 3.3, $Z_{f_b,\Gamma_b} = -\ell$. Therefore, the number of zeros in the critical curve is ℓ, and Proposition 3.2 gives us the total number of zeros for f_b is $m + 2\ell$ for b sufficiently large. □

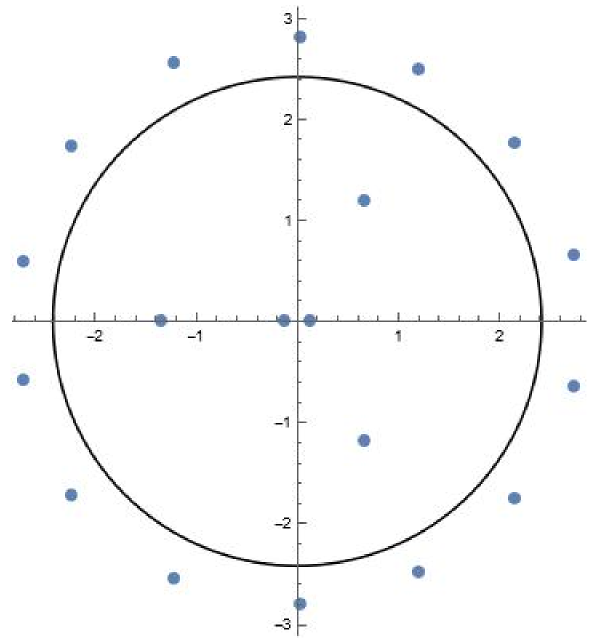

Figure 3.1: The zeros and critical curve of $f_{41} = \frac{6}{15}(z^9 + \frac{615}{4}\overline{z}^2 - 1) + \frac{9}{15}(z^6 + 41\overline{z}^5 - 1)$

Figure 3.1 displays an example of Theorem 1.3. In this example, the component parts of f_{41} have 13 and 16 as their maximum number of zeros. However, f_{41} has 19 zeros. For this subfamily, the number of zeros the convex combination has for sufficiently large b will always be larger than the maximum of the component parts since $m > n > \ell > k$.

3.2 Proof of Theorem 1.4

In this section, we prove the number of zeros the subfamily in Section 2.3.2 has for arbitrarily large b. In Section 2.3.2, we restricted $f_{a,b,s}$ by requiring

- $m - k = n - \ell$,
- $m > n > \ell$,
- $m > k > \ell$,
- $s = \frac{n}{m+n}$, and
- $a = \frac{mb\ell}{kn}$.

Since these restrictions fix s and make a a function of b, we will refer to $f_{a,b,s}$ with these restrictions as f_b for the rest of this section. Proposition 2.11 also gives us the critical curve

$$|z| = \left(\frac{b\ell}{n}\right)^{\frac{1}{n-\ell}}$$

for f_b with the sense-reversing region inside the critical curve and the sense-preserving region outside of the critical curve. We will denote the critical curve of f_b as Γ_b for the rest of the section.

Now, we find Z_{f_b,Γ_b}.

Lemma 3.4. *For sufficiently large b, $Z_{f_b,\Gamma_b} = -k$.*

Proof. We prove this lemma using Rouché's Theorem. We show that if $p(z) = saz^k$, then $Z_{f_b,\Gamma_b} = Z_{p,\Gamma_b}$; that is, we show

$$|sz^m + (1-s)b\overline{z}^\ell + (1-s)z^n - 1| < |sa\overline{z}^k| \tag{3.3}$$

for all $z \in \Gamma_b$. Equation (3.3) will follow if we show

$$sa|z|^k - (s|z|^m + (1-s)b|z|^\ell + (1-s)|z|^n + 1) > 0 \tag{3.4}$$

for all $z \in \Gamma_b$. Using the same process as for Lemma 3.3, we rearrange this expression into constants and powers of b and show the powers are negative. Since points on the critical curve satisfy $|z| = \left(\frac{b\ell}{n}\right)^{\frac{1}{n-\ell}}$ and $a = \frac{mb\ell}{kn}$, (3.4) is equivalent to

$$\frac{smb\ell}{kn}\left(\frac{b\ell}{n}\right)^{\frac{k}{n-\ell}} - \left(s\left(\frac{b\ell}{n}\right)^{\frac{m}{n-\ell}} + (1-s)b\left(\frac{b\ell}{n}\right)^{\frac{\ell}{n-\ell}} + (1-s)\left(\frac{b\ell}{n}\right)^{\frac{n}{n-\ell}} + 1\right) > 0.$$

By factoring out $\frac{smb\ell}{kn}\left(\frac{b\ell}{n}\right)^{\frac{k}{n-\ell}}$ and using $m-k = n-\ell$ to simplify the second term, the left-hand side of (3.4) becomes

$$\frac{smb\ell}{kn}\left(\frac{b\ell}{n}\right)^{\frac{k}{n-\ell}}\left[1 - \frac{kn}{mb\ell}\left(\frac{b\ell}{n}\right) - \frac{(1-s)kn}{sm\ell}\left(\frac{b\ell}{n}\right)^{\frac{\ell-k}{n-\ell}} - \frac{(1-s)kn}{smb\ell}\left(\frac{b\ell}{n}\right)^{\frac{n-k}{n-\ell}} - \frac{kn}{smb\ell}\left(\frac{b\ell}{n}\right)^{\frac{-k}{n-\ell}}\right]. \tag{3.5}$$

The third and fourth terms can also be simplified using $\frac{ms}{n(1-s)} = 1$, giving

$$\frac{smb\ell}{kn}\left(\frac{b\ell}{n}\right)^{\frac{k}{n-\ell}}\left[1 - \frac{k}{m} - \frac{k}{\ell}\left(\frac{b\ell}{n}\right)^{\frac{\ell-k}{n-\ell}} - \frac{k}{b\ell}\left(\frac{b\ell}{n}\right)^{\frac{n-k}{n-\ell}} - \frac{kn}{smb\ell}\left(\frac{b\ell}{n}\right)^{\frac{-k}{n-\ell}}\right].$$

This expression is positive if and only if the expression in square brackets is positive. Due to the relation between m, n, ℓ, and k, we find that the total exponent on b in each term is negative. Indeed,

$$\frac{-k}{n-\ell} - 1 < \frac{\ell - k}{n-\ell} = \frac{n-k}{n-\ell} - 1 < 0.$$

Therefore, each term involving b tends to 0 as b tends to infinity. Thus, there exists b_0 such that for all $b > b_0$,

$$\frac{k}{\ell}\left(\frac{b\ell}{n}\right)^{\frac{\ell-k}{n-\ell}} + \frac{k}{b\ell}\left(\frac{b\ell}{n}\right)^{\frac{n-k}{n-\ell}} + \frac{kn}{smb\ell}\left(\frac{b\ell}{n}\right)^{\frac{-k}{n-\ell}} < 1 - \frac{k}{m}.$$

Therefore, for such b,

$$0 < 1 - \frac{k}{m} - \frac{k}{\ell}\left(\frac{b\ell}{n}\right)^{\frac{\ell-k}{n-\ell}} - \frac{k}{b\ell}\left(\frac{b\ell}{n}\right)^{\frac{n-k}{n-\ell}} - \frac{kn}{smb\ell}\left(\frac{b\ell}{n}\right)^{\frac{-k}{n-\ell}}.$$

It follows that p is the dominant term of f_b on the critical curve, and by Rouché's Theorem, $Z_{f_b,\Gamma_b} = Z_{p,\Gamma_b}$. Since p only has a zero of order $-k$ located at the origin, $Z_{f_b,\Gamma_b} = -k$ for sufficiently large b. □

Now, we have everything needed to prove Theorem 1.4.

Proof of Theorem 1.4. By Lemma 3.4, $Z_{f_b,\Gamma_b} = -k$. Therefore, the number of zeros in the critical curve is k, and Proposition 3.2 gives us the total number of zeros for f_b is $m + 2k$ for b sufficiently large. □

Figure 3.2 displays an example of Theorem 1.4. In this example, the component parts of f_{50} have 17 and 11 as their maximum number of zeros. f_{50} has 17 zeros for b sufficiently large. For this subfamily, the number of zeros the convex combination has for sufficiently large b will be the same as first component which has the larger maximum number of zeros since $m > n$ and $k > \ell$. For both this subfamily and the subfamily in Section 3.1, the dominant term on the critical curve was the conjugate term with the larger power when the coefficients of those terms were arbitrarily large, so whether $\ell > k$ or $k > \ell$ determined the total number of zeros.

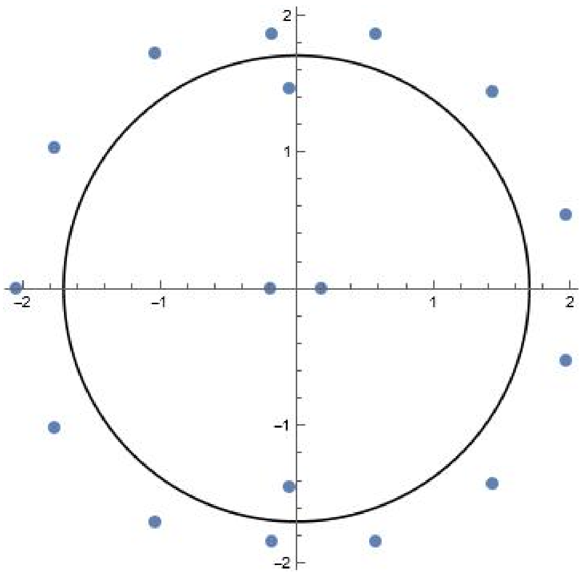

Figure 3.2: The zeros and critical curve of $f_{50} = \frac{7}{16}(z^9 + \frac{225}{7}\overline{z}^4 - 1) + \frac{9}{16}(z^7 + 50\overline{z}^2 - 1)$

3.3 Proof of Theorem 1.5

In this section, we prove the number of zeros the subfamily in Section 2.3.3 has for arbitrarily large b. In section 2.3.3, we restricted $f_{a,b,s}$ by requiring

- $m + k = n + \ell$,
- $m > n > \ell > k$, and
- $a = \frac{(1-s)^2 bn\ell}{s^2 mk}$.

Since these restrictions make a a function of b but do not restrict s, we will refer to $f_{a,b,s}$ with these requirements as $f_{b,s}$ for the rest of this section. Proposition 2.12 also gives us the critical curve

$$|z| = \left(\frac{(1-s)b\ell}{sm}\right)^{\frac{1}{n-k}}$$

for $f_{b,s}$ with the sense-reversing region inside the critical curve and the sense-preserving region outside of the critical curve. Since the expression $\frac{(1-s)\ell}{sm}$ is cumbersome and shows up frequently in proving the next lemma, we will rename it R_s, which will reduce the critical curve expression to $|z| = (bR_s)^{\frac{1}{n-k}}$. We denote the critical curve of $f_{b,s}$ as Γ_b for the rest of the section.

Now, we find $Z_{f_{b,s},\Gamma_b}$.

Lemma 3.5. *For sufficiently large b, $Z_{f_{b,s},\Gamma_b} = -\ell$.*

Proof. We prove this lemma using Rouché's Theorem. We show if $p(z) = (1-s)b\bar{z}^{\ell}$, then $Z_{f_{b,s},\Gamma_b} = Z_{p,\Gamma_b}$; that is, we show

$$|sz^m + sa\bar{z}^k + (1-s)z^n - 1| < |(1-s)b\bar{z}^{\ell}| \tag{3.6}$$

for all $z \in \Gamma_b$. Equation (3.6) will follow if we show

$$(1-s)b|z|^{\ell} - (s|z|^m + sa|z|^k + (1-s)|z|^n + 1) > 0 \tag{3.7}$$

for all $z \in \Gamma_b$. Since points on the critical curve satisfy $|z| = (bR_s)^{\frac{1}{n-k}}$, (3.7) is equivalent to

$$(1-s)b(bR_s)^{\frac{\ell}{n-k}} - s(bR_s)^{\frac{m}{n-k}} - sa(bR_s)^{\frac{k}{n-k}} - (1-s)(bR_s)^{\frac{n}{n-k}} - 1 > 0.$$

By factoring out $(1-s)b(bR_s)^{\frac{\ell}{n-k}}$ and using $m-\ell = n-k$ to simplify the second term, the left-hand side of (3.7) is equivalent to

$$(1-s)b(bR_s)^{\frac{\ell}{n-k}}\left[1 - \frac{\ell}{m} - \frac{sa}{(1-s)b}(bR_s)^{\frac{k-\ell}{n-k}} - \frac{1}{b}(bR_s)^{\frac{n-\ell}{n-k}} - \frac{1}{(1-s)b}(bR_s)^{\frac{-\ell}{n-k}}\right].$$

The third term can also be simplified using $a = \frac{(1-s)^2bn\ell}{s^2mk}$, giving

$$(1-s)b(bR_s)^{\frac{\ell}{n-k}}\left[1 - \frac{\ell}{m} - \frac{(1-s)n\ell}{smk}(bR_s)^{\frac{k-\ell}{n-k}} - \frac{1}{b}(bR_s)^{\frac{n-\ell}{n-k}} - \frac{1}{(1-s)b}(bR_s)^{\frac{-\ell}{n-k}}\right].$$

This expression is positive if and only if the expression in square brackets is positive. Due to the relation between m, n, ℓ, and k, we find that the total exponent on b in each term is negative. Indeed,

$$\frac{-\ell}{n-k} - 1 < \frac{k-\ell}{n-k} = \frac{n-\ell}{n-k} - 1 < 0.$$

Therefore, each term involving b tends to 0 as b tends to infinity. Thus, there exists b_0 such that for all $b > b_0$,

$$\frac{(1-s)n\ell}{smk}(bR_s)^{\frac{k-\ell}{n-k}} + \frac{1}{b}(bR_s)^{\frac{n-\ell}{n-k}} + \frac{1}{(1-s)b}(bR_s)^{\frac{-\ell}{n-k}} < 1 - \frac{\ell}{m}.$$

Therefore, for such b,

$$0 < 1 - \frac{\ell}{m} - \frac{(1-s)n\ell}{smk}(bR_s)^{\frac{k-\ell}{n-k}} - \frac{1}{b}(bR_s)^{\frac{n-\ell}{n-k}} - \frac{1}{(1-s)b}(bR_s)^{\frac{-\ell}{n-k}}.$$

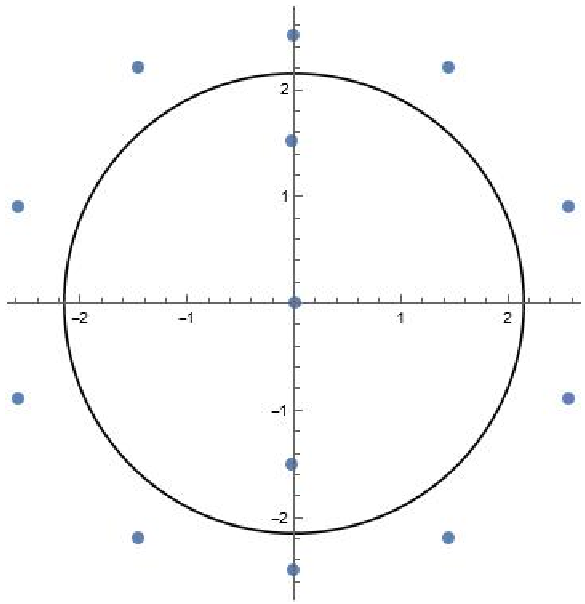

Figure 3.3: The zeros and critical curve of $f_{50,\frac{5}{12}} = \frac{5}{12}(z^7 + \frac{750}{7}\overline{z} - 1) + \frac{7}{12}(z^5 + 50\overline{z}^3 - 1)$

It follows that p is the dominant term of $f_{b,s}$ on the critical curve, and by Rouché's Theorem, $Z_{f_{b,s},\Gamma_b} = Z_{f_{b,s},\Gamma_b}$. Since p only has a zero of order $-\ell$ located at the origin, $Z_{f_{b,s},\Gamma_b} = -\ell$ for sufficiently large b. □

Now, we have everything needed to prove Theorem 1.5.

Proof of Theorem 1.5. By Lemma 3.5, $Z_{f_{b,s},\Gamma_b} = -\ell$. Therefore, the number of zeros in the critical curve is ℓ, and Proposition 3.2 gives us the total number of zeros for $f_{b,s}$ is $m + 2\ell$ for b sufficiently large. □

Figure 3.3 displays an example of Theorem 1.5. In this example, the component parts of $f_{50,\frac{5}{12}}$ have 9 and 11 as their maximum number of zeros. However, $f_{50,\frac{5}{12}}$ has 13 zeros for b sufficiently large. For this subfamily, the number of zeros the convex combination has for sufficiently large b will always be larger than the maximum of the component parts since $m > n > \ell > k$.

3.4 Proof of Theorem 1.6

In this section, we prove the number of zeros the subfamily in Section 2.3.4 has for arbitrarily large b. In section 2.3.4, we restricted $f_{a,b,s}$ by requiring

- $m - k = n - \ell$,
- $m > n > \ell$,
- $m > k > \ell$, and
- $a = \frac{mb\ell}{kn}$.

Since these restrictions make a a function of b and do not fix s, we will refer to $f_{a,b,s}$ with these restrictions as $f_{b,s}$ for the rest of this section. Proposition 2.13 also gives us the critical curve

$$|z| = \left(\frac{b\ell}{n}\right)^{\frac{1}{n-\ell}}$$

for $f_{b,s}$ with the sense-reversing region inside the critical curve and the sense-preserving region outside of the critical curve. We denote the critical curve of $f_{b,s}$ as Γ_b for the rest of the section.

Now, we find $Z_{f_{b,s},\Gamma_b}$.

Lemma 3.6. *For sufficiently large b, $Z_{f_{b,s},\Gamma_b} = -k$.*

Proof. The proof of this lemma is identical to the proof of Lemma 3.4 up through (3.5). For this proof, the expression (3.5) can not be simplified the same way since s is unrestricted. However, the powers on b in each term are unchanged and therefore negative. Thus, there exists b_0 such that for all $b > b_0$,

$$\frac{(1-s)kn}{sm\ell}\left(\frac{b\ell}{n}\right)^{\frac{\ell-k}{n-\ell}} + \frac{(1-s)kn}{smb\ell}\left(\frac{b\ell}{n}\right)^{\frac{n-k}{n-\ell}} + \frac{kn}{smb\ell}\left(\frac{b\ell}{n}\right)^{\frac{-k}{n-\ell}} < 1 - \frac{k}{m}.$$

Therefore, for such b,

$$0 < 1 - \frac{k}{m} - \frac{(1-s)kn}{sm\ell}\left(\frac{b\ell}{n}\right)^{\frac{\ell-k}{n-\ell}} - \frac{(1-s)kn}{smb\ell}\left(\frac{b\ell}{n}\right)^{\frac{n-k}{n-\ell}} - \frac{kn}{smb\ell}\left(\frac{b\ell}{n}\right)^{\frac{-k}{n-\ell}}.$$

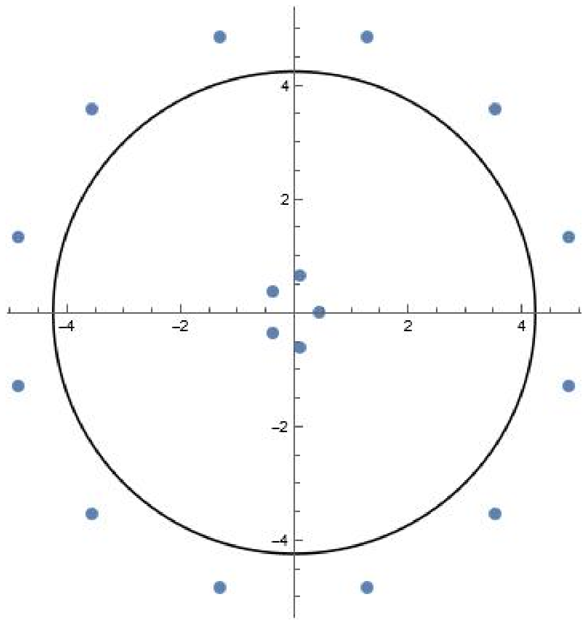

Figure 3.4: The zeros and critical curve of $f_{30,\frac{5}{12}} = \frac{5}{12}(z^7 + \frac{126}{5}\overline{z}^5 - 1) + \frac{7}{12}(z^5 + 30\overline{z}^3 - 1)$

It follows that p is the dominant term of $f_{b,s}$ on the critical curve, and by Rouché's Theorem, $Z_{f_{b,s},\Gamma_b} = Z_{p,\Gamma_b}$. Since p only has a zero of order $-k$ located at the origin, $Z_{f_{b,s},\Gamma_b} = -k$ for sufficiently large b. □

Now, we have everything needed to prove Theorem 1.6.

Proof of Theorem 1.6. By Lemma 3.6, $Z_{f_{b,s},\Gamma_b} = -k$. Therefore, the number of zeros in the critical curve is k, and Proposition 3.2 gives us the total number of zeros for $f_{b,s}$ is $m + 2k$ for b sufficiently large. □

Figure 3.4 displays an example of Theorem 1.6. In this example, the component parts of $f_{30,\frac{5}{12}}$ have 17 and 11 as their maximum number of zeros. $f_{30,\frac{5}{12}}$ has 17 zeros for b sufficiently large. For this subfamily, the number of zeros the convex combination has for sufficiently large b will be the same as its first component which has the larger maximum number of zeros since $m > n$ and $k > \ell$.

Chapter 4. Proofs of Theorems 1.8 and 1.9

In this chapter, we count the number of zeros each subfamily introduced in Section 2.4 has for arbitrarily small and large b. Since the largest power of $r_{a,b,s}$ is determined by the subfamily, the sum of the orders of the zeros in the complex plane needs to be found separately for each.

4.1 Proof of Theorem 1.8

In this section, we prove the number of zeros that the subfamily in Section 2.4.1 has for arbitrarily small and arbitrarily large b. In Section 2.4.1, we restricted $r_{a,b,s}$ by requiring

- $m - k = n + \ell$,
- $a = \frac{mb\ell}{kn}$.

Since these restrictions make a a function of b, we will refer to $r_{a,b,s}$ with these restrictions as $r_{b,s}$ for the rest of this section. Proposition 2.14 also gives us the critical curve

$$|z| = \left(\frac{b\ell}{n}\right)^{\frac{1}{m-k}}$$

for $r_{b,s}$ with the sense-reversing region inside the critical curve and the sense-preserving region outside of the critical curve. We denote the critical curve of $r_{b,s}$ as Γ_b for the rest of this section.

Lemma 4.1. $Z_{r_{b,s},\mathbb{C}} = m - \ell$.

Proof. We claim that if $p(z) = sz^m$, then $|p(z) - r_{b,s}(z)| < |p(z)|$. The proof of this is the same as the proof for the original combination with $-\ell$ replacing the ℓ. Using Rouché's Theorem, we get that

$$Z_{r_{b,s},\mathbb{C}} - P_{r_{b,s},\mathbb{C}} = Z_{p,\mathbb{C}} - P_{p,\mathbb{C}} = m - 0 = m.$$

Since $P_{r_{b,s},\mathbb{C}} = -\ell$; that is to say, the sum of the orders of the poles of $r_{b,s}$ in $\mathbb{C}$ is $-\ell$, we have that $Z_{r_{b,s},\mathbb{C}} = m - \ell$. □

Now, we find the sum of the orders of the zeros inside the critical curve for both small and large b.

Lemma 4.2. *For sufficiently small b, $Z_{r_{b,s},\Gamma_b} = -\ell$.*

Proof. On the critical curve, each term besides -1 has a b with a positive exponent. Since we are letting b be small, -1 should be the largest term for b small enough. So, we are going to let $p(z) = -1$ for our Rouché's Theorem argument below. On the critical curve,

$$\begin{aligned}
|p(z) - r_{b,s}(z)| &= |sz^m + saz^k + (1-s)z^n + (1-s)z^{-\ell}| \\
&\leq s|z|^m + s|a||z|^k + (1-s)|z|^n + (1-s)|b||z|^{-\ell} \\
&= s\left|\frac{b\ell}{n}\right|^{\frac{m}{m-k}} + s\left|\frac{mb\ell}{nk}\right|\left|\frac{b\ell}{n}\right|^{\frac{k}{m-k}} + (1-s)\left|\frac{b\ell}{n}\right|^{\frac{n}{m-k}} + (1-s)|b|\left|\frac{b\ell}{n}\right|^{\frac{-\ell}{m-k}} \\
&= s\left|\frac{b\ell}{n}\right|^{\frac{m}{m-k}}\left(1+\frac{m}{k}\right) + (1-s)\left|\frac{b\ell}{n}\right|^{\frac{n}{m-k}}\left(1+\left|\frac{\ell}{n}\right|^{\frac{-\ell}{m-k}}\right).
\end{aligned}$$

Since $\frac{m}{m-k} > 0$ and $\frac{n}{m-k} > 0$, there exists a b_0 such that for all b with $|b| < b_0$,

$$|p(z) - r_{b,s}(z)| < 1 = |p(z)|.$$

Therefore, by Rouché's Theorem, $Z_{r_{b,s},\Gamma_b} - P_{r_{b,s},\Gamma_b} = Z_{p,\Gamma_b} - P_{p,\Gamma_b} = 0$. Since $P_{r_{b,s},\Gamma_b} = -\ell$ as there is a pole of order $-\ell$ at the origin, $Z_{r_{b,s},\Gamma_b} = -\ell$. □

Since we have found the sum of the orders of the zeros inside of the critical curve for small b, we just need the sum for large b.

Lemma 4.3. *For sufficiently large b, $Z_{r_{b,s},\Gamma_b} = -(k+\ell)$.*

Proof. Let $b > 1$ and $p(z) = sz^m + sa\overline{z}^k$. Then, on the critical curve

$$\begin{aligned}
|p(z)| &= |sz^m + sa\overline{z}^k| \\
&\geq s(|a||z|^k - |z|^m) \\
&= s\left(|a|\left|\frac{b\ell}{n}\right|^{\frac{k}{m-k}} - \left|\frac{b\ell}{n}\right|^{\frac{m}{m-k}}\right) \\
&= s\left(\left|\frac{mb\ell}{kn}\right|\left|\frac{b\ell}{n}\right|^{\frac{k}{m-k}} - \left|\frac{b\ell}{n}\right|^{\frac{m}{m-k}}\right) \\
&= s\left|\frac{b\ell}{n}\right|^{\frac{m}{m-k}}\left(\frac{m}{k} - 1\right).
\end{aligned}$$

Also,

$$\begin{aligned}
|r_{b,s}(z) - p(z)| &= |(1-s)z^n + (1-s)b\overline{z}^{-\ell} - 1| \\
&\leq (1-s)|z|^n + (1-s)|b||z|^{-\ell} + 1 \\
&= (1-s)\left|\frac{b\ell}{n}\right|^{\frac{n}{m-k}} + (1-s)|b|\left|\frac{b\ell}{n}\right|^{\frac{-\ell}{m-k}} + 1 \\
&= (1-s)\left|\frac{b\ell}{n}\right|^{\frac{n}{m-k}} + (1-s)\left(\frac{\ell}{n}\right)^{\frac{-\ell}{m-k}}|b|^{\frac{n}{m-k}} + 1 \\
&\leq (1-s)\left|\frac{b\ell}{n}\right|^{\frac{n}{m-k}} + (1-s)\left(\frac{\ell}{n}\right)^{\frac{-\ell}{m-k}}|b|^{\frac{n}{m-k}} + |b|^{\frac{n}{m-k}} \\
&= \left[(1-s)\left(\frac{\ell}{n}\right)^{\frac{n}{m-k}} + (1-s)\left(\frac{\ell}{n}\right)^{\frac{-\ell}{m-k}} + 1\right]|b|^{\frac{n}{m-k}}.
\end{aligned}$$

Therefore, since $m > n$, there exists a $b_0 > 1$ such that for all $|b| > b_0$

$$\left[(1-s)\left(\frac{\ell}{n}\right)^{\frac{n}{m-k}} + (1-s)\left(\frac{\ell}{n}\right)^{\frac{-\ell}{m-k}} + 1\right]|b|^{\frac{n}{m-k}} \leq s\left|\frac{b\ell}{n}\right|^{\frac{m}{m-k}}\left(\frac{m}{k} - 1\right).$$

Therefore, $|p(z)| \geq |r_{b,s}(z) - p(z)|$. So, $Z_{r_{b,s},\Gamma_b} - P_{r_{b,s},\Gamma_b} = Z_{p,\Gamma_b} - P_{p,\Gamma_b}$.

The harmonic function p has no poles, so $P_{p,\Gamma_b} = 0$. Now, we just need to find Z_{p,Γ_b}. This is not as trivial as it is in Chapter 3. Since p has two terms, we need to find the locations of the zeros not at the origin in order to deduce if they are in the critical curve or not. Using Definition 2.6, we see that p has a zero of order $-k$ at the origin. If z_0 is a nonzero zero, then it satisfies

$$|a|^{\frac{1}{m-k}} = |z_0|$$
$$\implies \quad \left|\frac{mb\ell}{kn}\right|^{\frac{1}{m-k}} = |z_0|.$$

Therefore, the remaining zeros of p all lie on a circle of bigger radius than the critical curve. Therefore, $Z_{p,\Gamma_b} = -k$. So, $Z_{r_{b,s},\Gamma_b} - P_{r_{b,s},\Gamma_b} = -k$. Since $P_{r_{b,s},\Gamma_b} = -\ell$, $Z_{r_{b,s},\Gamma_b} = -(k+\ell)$. □

Now, we have everything we need to prove Theorem 1.8.

Proof of Theorem 1.8. For arbitrarily small b, Lemma 4.2 gives that $Z_{r_{b,s},\Gamma_b} = -\ell$. Since the sense-reversing region is inside of the critical circle, there are ℓ negatively ordered zeros. By Lemma 4.1, the $Z_{r_{b,s},\mathbb{C}} = m - \ell$, so the number of zeros in the sense-preserving region must be $(m-\ell)+\ell = m$. Therefore, the number of zeros for $r_{b,s}$ is $m+\ell$ when $|b| < b_0$ for some $b_0 > 0$.

For arbitrarily large b, Lemma 4.3 gives that $Z_{r_{b,s},\Gamma_b} = -(k+\ell)$. Since the sense-reversing region is inside of the critical circle, there are $k+\ell$ negatively ordered zeros. By Lemma 4.1, $Z_{r_{b,s},\mathbb{C}} = m-\ell$, so the number of zeros in the sense-preserving region must be $(m-\ell)+\ell+k = m+k$. Therefore, the number of zeros for $r_{b,s}$ is $m+\ell+2k$ when $|b| > b_1$ for some $b_1 > 0$.

□

Figure 4.1 displays an example of Theorem 1.8. In this example, $r_{b,\frac{3}{5}}$ has 8 zeros for b sufficiently small and 12 zeros for b sufficiently large. For this subfamily, the end behavior in the change the number of zeros the convex combination goes through from sufficiently small b to sufficiently large b follows the same pattern as Corollary 1.1 while the number of zeros for sufficiently small b matches the impact the pole had in Corollary 1.7.

4.2 Proof of Theorem 1.9

In this section, we prove the number of zeros that the subfamily in Section 2.4.2 has for arbitrarily small and arbitrarily large b. In Section 2.4.2, we restricted $r_{a,b,s}$ by requiring

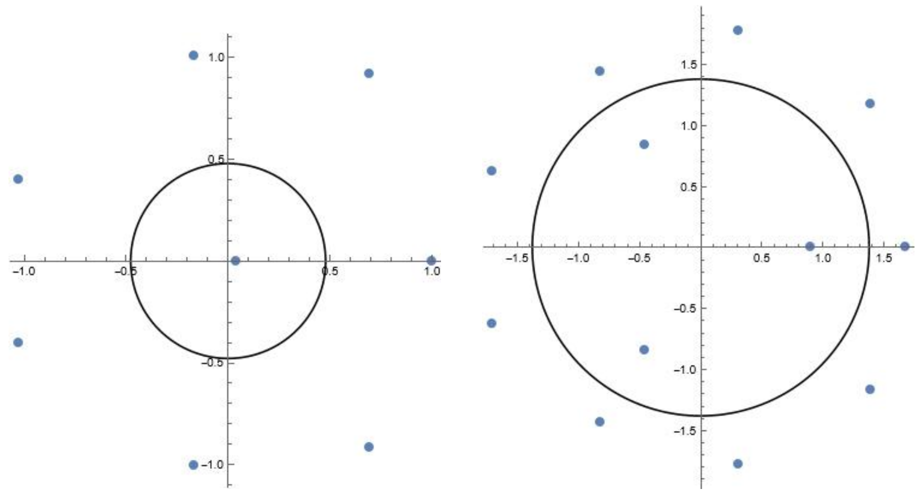

Figure 4.1: The zeros and critical curve of $r_{b,\frac{3}{5}} = \frac{3}{5}(z^7 - \frac{7}{8}b\overline{z}^2 - 1) + \frac{2}{5}(z^4 + \frac{b}{z} - 1)$ with $b = 0.1$ (left) and $b = 20$ (right)

- $m + k = n - \ell$,
- $a = \frac{-(1-s)^2 nb\ell}{s^2 mk}$.

Since these restrictions make a a function of b, we will refer to $r_{a,b,s}$ with these restrictions as $r_{b,s}$ for the rest of this section. Proposition 2.15 also gives us the critical curve

$$|z| = \left(\frac{(1-s)b\ell}{sm}\right)^{\frac{1}{n-k}}$$

for $r_{b,s}$ with the sense-reversing region inside the critical curve and the sense-preserving region outside of the critical curve. We denote the critical curve of $r_{b,s}$ as Γ_b for the rest of this section.

Lemma 4.4. $Z_{r_{b,s},\Gamma_b} = n - \ell$.

Proof. We claim that if $p(z) = sz^n$, then $|p(z) - f(z)| \leq |p(z)|$. The proof of this is the same as the proof for the original combination with $-\ell$ replacing the ℓ. Using Rouché's Theorem, we get that

$$Z_{f,\mathbb{C}} - P_{f,\mathbb{C}} = Z_{p,\mathbb{C}} - P_{p,\mathbb{C}} = n - 0 = n.$$

Since $P_{f,\mathbb{C}} = -\ell$, we have that $Z_{f,\mathbb{C}} = n - \ell$. In other words, the sum of the orders of the zeros of f for the entire complex plane is $n - \ell$. □

Now, we find the sum of the orders of the zeros inside the critical curve for both small and large b.

Lemma 4.5. *For sufficiently small b, $Z_{r_{b,s},\Gamma_b} = -\ell$.*

Proof. On the critical curve, each term besides -1 has a b with a positive exponent. Since we are letting b be small, -1 should be the largest term for b small enough. So, we are going to let $p(z) = -1$ for our Rouché's Theorem argument below. On the critical curve,

$$\begin{aligned}
|p(z) - r_{b,s}(z)| = & |sz^m + saz^k + (1-s)z^n + (1-s)z^{-\ell}| \\
\leq & s|z|^m + s|a||z|^k + (1-s)|z|^n + (1-s)|b||z|^{-\ell} \\
= & s\left|\frac{(1-s)b\ell}{sm}\right|^{\frac{m}{n-k}} + \left|\frac{(1-s)^2bn\ell}{s^2mk}\right|\left|\frac{(1-s)b\ell}{sm}\right|^{\frac{k}{n-k}} + (1-s)\left|\frac{(1-s)b\ell}{sm}\right|^{\frac{n}{n-k}} \\
& + (1-s)|b|\left|\frac{(1-s)b\ell}{sm}\right|^{\frac{-\ell}{n-k}} \\
= & s\left|\frac{(1-s)b\ell}{sm}\right|^{\frac{m}{n-k}} + \left|\frac{(1-s)^2b\ell}{s^2mk}\right|\left|\frac{(1-s)\ell}{sm}\right|^{\frac{k}{n-k}}|b|^{\frac{m}{n-k}} + (1-s)\left|\frac{(1-s)b\ell}{sm}\right|^{\frac{n}{n-k}} \\
& + (1-s)\left|\frac{(1-s)\ell}{sm}\right|^{\frac{-\ell}{n-k}}|b|^{\frac{m}{n-k}}.
\end{aligned}$$

Since $\frac{m}{n-k} > 0$ and $\frac{n}{n-k} > 0$, there exists a b_0 such that for all b with $|b| < b_0$,

$$|p(z) - f(z)| \leq 1 = |p(z)|.$$

Therefore, by Rouché's Theorem, $Z_{r_{b,s},\Gamma_b} - P_{r_{b,s},\Gamma_b} = Z_{p,\Gamma_b} - P_{p,\Gamma_b} = 0$. So, $Z_{r_{b,s},\Gamma_b} = -\ell$. $\square$

Since we have found the sum of the orders of the zeros inside of the critical curve for small b, we just need the sum for large b.

Lemma 4.6. *For sufficiently large b, $Z_{r_{b,s},\Gamma_b} = -(k+\ell)$.*

Proof. Let $b > 1$ and $p(z) = (1-s)z^n + sa\overline{z}^k$. Then, on the critical curve

$$\begin{aligned}
|p(z)| &= |(1-s)z^n + sa\overline{z}^k| \\
&\geq s|a||z|^k - (1-s)|z|^n \\
&= s\left|\frac{bn\ell(1-s)^2}{mks^2}\right|\left|\frac{b\ell(1-s)}{sm}\right|^{\frac{k}{n-k}} - (1-s)\left|\frac{b\ell(1-s)}{sm}\right|^{\frac{n}{n-k}} \\
&= (1-s)\left(\frac{n}{k}-1\right)\left(\frac{\ell(1-s)}{sm}\right)^{\frac{n}{n-k}}|b|^{\frac{n}{n-k}}.
\end{aligned}$$

Also,

$$\begin{aligned}
|r_{b,s}(z) - p(z)| &= |sz^m + (1-s)b\overline{z}^{-\ell} - 1| \\
&\leq s|z|^m + (1-s)|b||z|^{-\ell} + 1 \\
&= s\left|\frac{b\ell(1-s)}{sm}\right|^{\frac{m}{n-k}} + (1-s)|b|\left|\frac{b\ell(1-s)}{sm}\right|^{\frac{-\ell}{n-k}} + 1 \\
&= s\left|\frac{b\ell(1-s)}{sm}\right|^{\frac{m}{n-k}} + (1-s)\left(\frac{\ell(1-s)}{sm}\right)^{\frac{-\ell}{n-k}}|b|^{\frac{m}{n-k}} + 1 \\
&\leq \left[s\left(\frac{\ell(1-s)}{sm}\right)^{\frac{m}{n-k}} + (1-s)\left(\frac{\ell(1-s)}{sm}\right)^{\frac{-\ell}{n-k}} + 1\right]|b|^{\frac{m}{n-k}}.
\end{aligned}$$

Since $n > m$, there exists a $b_0 > 1$ such that for all $|b| > b_0$, $|p(z)| \geq |r_{b,s}(z) - p(z)|$. So, $Z_{r_{b,s},\Gamma_b} - P_{r_{b,s},\Gamma_b} = Z_{p,\Gamma_b} - P_{p,\Gamma_b}$.

The harmonic function p has no poles, so $P_{p,\Gamma_b} = 0$. Now, we just need to find Z_{p,Γ_b}. Similar to the previous section, p has two terms, so we need to find the locations of the zeros not on the origin in order to deduce if they are in the critical curve or not. Using Definition 2.6, p has a zero of order $-k$ at the origin. If z_0 is a nonzero zero, then it satisfies

$$\begin{aligned}
\left|\frac{sa}{1-s}\right|^{\frac{1}{n-k}} &= |z_0| \\
\Longrightarrow \quad \left|\frac{(1-s)nb\ell}{smk}\right|^{\frac{1}{n-k}} &= |z_0|
\end{aligned}$$

Therefore, the remaining zeros of p all lie on a circle of bigger radius than the critical curve. Therefore, $Z_{p,\Gamma_b} = -k$. So, $Z_{r_{b,s},\Gamma_b} - P_{r_{b,s},\Gamma_b} = -k$. Since $P_{r_{b,s},\Gamma_b} = -\ell$, $Z_{r_{b,s},\Gamma_b} = -(k+\ell)$. □

Now, we have everything we need to prove Theorem 1.8.

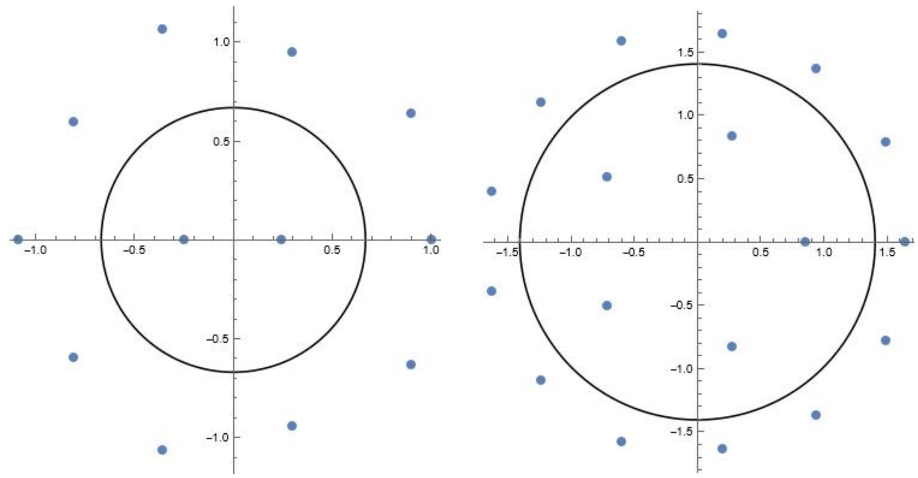

Figure 4.2: The zeros and critical curve of $r_{b,\frac{2}{5}} = \frac{2}{5}(z^5 - 3b\overline{z}^3 - 1) + \frac{3}{5}(z^{10} + \frac{b}{\overline{z}^2} - 1)$ with $b = 0.1$ (left) and $b = 18$ (right)

Proof of Theorem 1.8. For arbitrarily small b, Lemma 4.5 gives that $Z_{r_{b,s},\Gamma_b} = -\ell$. Since the sense-reversing region is inside of the critical circle, there are ℓ negatively ordered zeros. By Lemma 4.4, $Z_{r_{b,s},\mathbb{C}} = n - \ell$, so the number of zeros in the sense-preserving region must be $(n-\ell)+\ell = n$. Therefore, the number of zeros for $r_{b,s}$ is $n+\ell$ when $|b| < b_0$ for some $b_0 > 0$.

For arbitrarily large b, Lemma 4.6 gives that $Z_{r_{b,s},\Gamma_b} = -(k+\ell)$. Since the sense-reversing region is inside of the critical circle, there are $k+\ell$ negatively ordered zeros. By Lemma 4.4, $Z_{r_{b,s},\mathbb{C}} = n-\ell$, so the number of zeros in the sense-preserving region must be $(n-\ell)+\ell+k = n+k$. Therefore, the number of zeros for $r_{b,s}$ is $n+\ell+2k$ when $|b| > b_1$ for some $b_1 > 0$.

□

Figure 4.2 displays an example of Theorem 1.9. In this example, $r_{b,\frac{2}{5}}$ has 12 zeros for b sufficiently small and 18 zeros for b sufficiently large. This subfamily has the same end behavior as the other wherein the change the number of zeros the convex combination goes through from sufficiently small b to sufficiently large b follows the pattern as Corollary 1.1 while the number of zeros for sufficiently small b matches the impact the pole had in Corollary 1.7. However, $n > m$ is enough to replace the job of m in counting the number of zeros with n.

Chapter 5. Further Directions

Here are a list of questions that remain for future work:

- While this thesis does not contain an exploration of the convex combination of the Brooks and Lee family with itself, the paper [2] explores a subfamily of the Brooks and Lee convex combination using the same method of finding restrictions as the subfamily in Theorem 1.3. What subfamilies would be produced for the Brooks and Lee convex combination using the same methods for finding restrictions as the subfamilies in Theorems 1.4, 1.5, and 1.6? Would they have similar relationships to Theorems 1.3, 1.4, 1.5, and 1.6?

- In this thesis, we only explored two methods of restricting the dilatation function to get circular critical curves. Are there different restrictions of these convex combinations that could have circular critical curves?

- While a circular critical curve allows for a simpler Rouché's Theorem argument, a circular critical curve is not required to find the number of zeros. Are there any interesting results from analyzing subfamilies of these convex combinations with non-circular critical curves using annuli or other methods?

- Does increasing the number of functions in the convex combination, like

$$s_1(z^{m_1} + a_1\overline{z}^{k_1} - 1) + s_2(z^{m_2} + a_2\overline{z}^{k_2} - 1) + s_3(z^{m_3} + a_3\overline{z}^{k_3} - 1)$$

  with $s_1, s_2, s_3 \in (0, 1)$ and $s_1 + s_2 + s_3 = 1$, have any substantial consequences?

Bibliography

[1] Michael A. Brilleslyper, Jennifer L. Brooks, Michael John Dorff, Russell W. Howell, and Lisbeth E. Schaubroeck. Zeros of a one-parameter family of harmonic trinomials. *Proceedings of the American Mathematical Society, Series B*, 2020.

[2] Jennifer Brooks, Megan Dixon, Michael Dorff, Alexander Lee, and Rebekah Ottinger. Zeros of convex combinations of elementary families of harmonic functions. *Mathematics*, 11(19), 2023.

[3] Jennifer Brooks and Alexander Lee. Zeros of a family of complex-valued harmonic functions with poles. Submitted for Publication, 2024.

[4] Peter Duren. *Harmonic mappings in the plane*, volume 156. Cambridge university press, 2004.

[5] Alexander Lee. Zeros of a family of complex-valued harmonic rational functions. Master's thesis, Brigham Young University, Provo, UT, December 2022.

[6] T Sheil-Small. Tagesbericht. *Mathematisches Forschungsinstitut Oberwolfach, Funktionentheorie, 16bis*, 22(1992):19, 1992.

[7] Alan Wilmshurst. The valence of harmonic polynomials. *Proceedings of the American Mathematical Society*, 126(7):2077–2081, 1998.